Math 425 worksheet

October 16th

1. (a) Show that the intersection of denumerably many compact sets is compact.
(b) Show that the union of finitely many compact sets is compact.

(a) $\bigcap_{n=1}^{\infty} K_n = K$ is compact.

For every $\{p_n\}_{n\in\mathbb{N}}$ in K,

$\{p_n\}_{n\in\mathbb{N}} \subset K_1$, K_1 compact.

$\Rightarrow \exists \{p_{n_j}\}_{j\in\mathbb{N}} \to p \in K_1$,

but $\{p_{n_j}\}_{j\in\mathbb{N}} \subset \{p_n\}_{n\in\mathbb{N}} \subset K$.

K closed, $\Rightarrow$ contain limit point of $\{p_{n_j}\}_{j\in\mathbb{N}}$. $\Rightarrow p \in K$. $\Rightarrow K$ compact.

(b) $\bigcup_{i=1}^{n} K_i = K$ is compact.

For every $\{p_n\}_{n\in\mathbb{N}}$ in K,

$\{p_n\}_{n\in\mathbb{N}}$ has infinitely many terms,

$K_1, \dots, K_n$ are finite.

$\Rightarrow \exists$ infinitely many terms in one K_ℓ.

Take the subsequence $\{p_{n_j}\}_{j\in\mathbb{N}} \subset K_\ell$ for some ℓ.

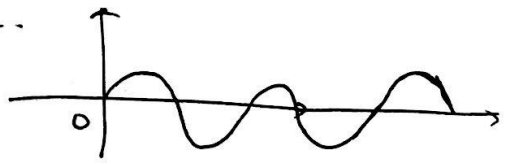
K_ℓ compact $\Rightarrow \{p_{n_{j_k}}\}_{k\in\mathbb{N}} \to$ some p in $K_\ell \subset K$

$\Rightarrow K$ is compact. ☒

2. Determine whether the following sets are compact or not.
(a) $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y = sin(x)\}$. Namely, the graph of the function $y = sinx$ when $x \geq 0$.
(b) $Y = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, a sphere in $\mathbb{R}^3$.
(c) $Z = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}...\} \subset \mathbb{R}$.
(d) $V = Z\backslash\{0\} \subset \mathbb{R}$.

(a) In $\mathbb{R}^2$, compact $(\Leftrightarrow)$ closed and bdd.

X is unbdd. $\Rightarrow$ not compact.



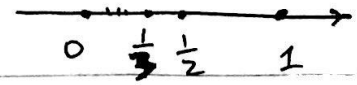
(b) Y is bdd by any ball with larger radius.

Y is closed b/c $Y = f^{-1}(\{1\})$. $f: \mathbb{R}^3 \to \mathbb{R}$.

$(x,y,z) \mapsto x^2+y^2+z^2$.

$\{1\}$ is closed $\Rightarrow Y$ is closed. $\Rightarrow Y$ compact

(c) Z is compact.

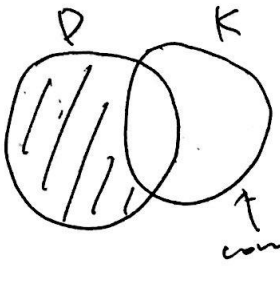


~~bdd. closed b/c~~

$\forall x \in Z^c$. can find a ball s.t. $B_r(x) \subset Z^c$.

(d) V is not compact. b/c $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \to 0$ but $0 \notin V$.

3. If K is compact, D is closed and $D \cap K^c$ is compact, prove that D is compact.



$D \cap K^c = D \setminus K$ compact.

$D = (D \setminus K) \cup (D \cap K)$ $\Rightarrow$ compact.

compact. compact.

$D \cap K$ is closed s/c D, K are closed.

$\Rightarrow$ $D \cap K \subset K$ a closed subset of compact se

$\Rightarrow$ $D \cap K$ compact. ☒

4. Construct a homeomorphism from $(-1,1)\backslash\{0\}$ to $\mathbb{R}\backslash[0,1]$. After you learn connectedness, show that there isn't homeomorphism from $(-1,1)$ to $\mathbb{R}\backslash[0,1]$.

ϕ $(-1,1)\setminus\{0\} \cong \mathbb{R}\setminus[0,1]$.

$f: (-1,1)\setminus\{0\} \longrightarrow \mathbb{R}\setminus[-1,1]$

$x \longmapsto \frac{1}{x}$.

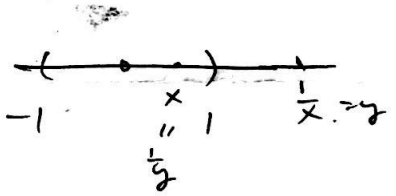
bijection. conti.

$f^{-1}: \mathbb{R}\setminus[-1,1] \longrightarrow (-1,1)\setminus\{0\}$.

$y \longmapsto \frac{1}{y}$.

conti.

$\Rightarrow$ f is homeomorphism.



Assume there is homeo from $(-1,1)$ to $\mathbb{R}\setminus[-1,1]$.

$(-1,1) \cong \mathbb{R}\setminus[-1,1] \cong (-1,1)\setminus$

↑ connected ↑ disconnect

$\rightarrow\leftarrow$.

☒