Math 425 worksheet

October 9th

1. Given $(X, d_{discrete})$, and (Y, d) is any metric space. Show that any function $f : X \to Y$ is continuous.

$f: X \to Y$

Show (topologically) continuous.

$\forall$ U open in Y, $f^{-1}(U)$ is open in X b/c

any subset of X is open. (from last week).

$\Rightarrow$ f conti. $\boxtimes$

2. Suppose we have a metric space (M, d) and a Cauchy sequence $\{p_n\}_{n\in\mathbb{N}}$ in M. If there exists a subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ converging to a limit point $p \in M$, show that $\{p_n\}_{n\in\mathbb{N}} \to p$.

(WTS: $\forall \varepsilon > 0$. $\exists N$. s.t. $d(p_n, p) < \varepsilon$ for $n \geq N$)

$\wedge$

$d(p_n, p_{n_k}) + d(p_{n_k}, p)$

$\{p_n\}_{n\in\mathbb{N}}$ Cauchy, for $\varepsilon/2$. $\exists N_1 > 0$. s.t.

$d(p_n, p_m) < \varepsilon/2$ for $n, m \geq N_1$

$\{p_{n_k}\}_{k\in\mathbb{N}} \to p$. For $\varepsilon/2$, $\exists N_2 > 0$. s.t.

$d(p_{n_k}, p) < \varepsilon/2$ for $n_k \geq N_2$.

Now take $N = \max\{N_1, N_2\}$. Then for $n \geq N$.

with some $n_k \geq N$.

$d(p_n, p) \leq d(p_{n_k}, p) + d(p_{n_k}, p_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

↑ b/c $n_k \geq N \geq N_2$. ↑ b/c $n, n_k \geq N$

$\boxtimes$

3. Show two definitions of closure of a set S are equivalent, i.e., $S \subset M$,

$$\underline{lim(S)} = \{\text{limit points of S}\} = \bigcap_{\substack{V \supset S \\ V \text{closed}}} V$$

(both of them are denoted as $\overline{S}$ from now on)

Show: $\lim(S) \supset \bigcap_{\substack{V \supset S \\ V \text{closed}}} V$

$\lim(S)$ is closed b/c of a theorem in book.

$\lim(S) \supset S$ b/c every $p \in S$, $\exists \{p_n\} \equiv \{p\} \to p$

$\Rightarrow S \subset \lim(S)$.

↙ b/c this is intersection.

$\Rightarrow \lim(S) \supset \bigcap_{\substack{V \supset S \\ V \text{closed}}} V$.

Now: $\lim(S) \subset \bigcap_{\substack{V \supset S \\ V \text{closed}}} V$

$\forall p \in \lim(S)$. $\exists \{p_n\} \overset{\subset S}{\to} p$.

For every $V \supset S$, V closed. $p_n \in S \subset V$. b/c

V closed $\Rightarrow$ V containing the limit pt of $\{p_n\}$.

$\Rightarrow p \in V$. $\Rightarrow p \in \bigcap_{\substack{V \supset S \\ V \text{closed}}} V$.

$\Rightarrow$ LHS $\subset$ RHS. ☒