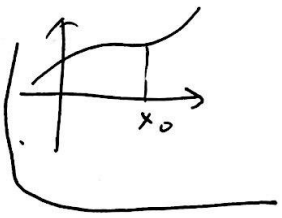
Math 425 worksheet

November 6, 2018

1. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, and suppose $f(x_0) \neq 0$, show that there exists $\delta > 0$ such that $f(x) \neq 0$ on $(x_0 - \delta, x_0 + \delta)$.
Note: use similar argument we can replace $' \neq'$ with $' >'$ or $' <'$ and it still holds true.



let $S = \{x : f(x) \neq 0\} = \{x : f(x) > 0\} \cup \{x : f(x) < 0\}$

$= f^{-1}((0,\infty)) \cup f^{-1}((-\infty, 0))$.

↑ open ↗ b/c f is (topologically) continuous

$x_0 \in S$

$\Rightarrow$ S is open in $\mathbb{R}$, $\exists \delta > 0$ s.t. $D_\delta(x_0) \subset S$.

$\Rightarrow$ $\forall x \in (x_0 - \delta, x_0 + \delta)$, $x \in S = \{x : f(x) \neq 0\}$. $\Rightarrow$ $f(x) \neq 0$ on $(x_0 - \delta, x_0 + \delta)$.

$\boxtimes$

2. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose there exists $R > 0$ such that $|f'(x)| \leq R$ for all $x \in (a, b)$. Show that f is uniformly continuous.

WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|t - x| < \delta \Rightarrow |f(t) - f(x)| < \varepsilon$

For any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{R}$, then for any $|t - x| < \delta$ use MVT $\exists\, 0 \in (t, x)$ s.t.

$R \geq |f'(0)| = \left| \frac{f(t) - f(x)}{t - x} \right| \Rightarrow |f(t) - f(x)| \leq R|t - x|$

$< R \cdot \delta = \varepsilon$.

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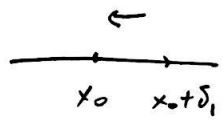
3. Let $f : (a, b) \to \mathbb{R}$ be twice differentiable. Suppose there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$ and $f''(x_0) < 0$. Show that there exists $\delta > 0$ so that

$$f(x_0) \geq f(x), \ \forall x \in (x_0 - \delta, x_0 + \delta)$$

That is, f attains a **local maximum** at x_0.

$$0 > f''(x_0) = \lim_{x \to x_0^+} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0^+} \frac{f'(x)}{x - x_0}$$

By definition, $\exists \delta_1 > 0$ s.t. when $0 < x - x_0 < \delta_1$



$$\frac{f'(x)}{x - x_0} < 0 \Rightarrow f'(x) < 0 \text{ for } x_0 < x < x_0 + \delta_1$$

then $\forall x \in (x_0, x_0 + \delta_1)$, $\exists \theta \in (x_0, x)$

s.t. $f'(\theta) = \dfrac{f(x) - f(x_0)}{x - x_0} < 0 \Rightarrow f(x) - f(x_0) < 0$

$\Rightarrow f(x_0) > f(x)$ for $x \in (x_0, x_0 + \delta_1)$.

Similarly, we can show $f(x_0) > f(x)$ for $x \in (x_0 - \delta_2, x$

so pick $\delta = \min\{|\delta_1|, |\delta_2|\}$, then

$f(x_0) \geq f(x)$, $\forall x \in (x_0 - \delta, x_0 + \delta)$.

$\Rightarrow$ f attains a local maximum.

$\square$