Math 425 worksheet#2

August 28th

1. Prove by induction that for any integer $n \geq 1$,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

factorize this,
$\Delta = 7^2 - 4 \times 2 \times 6$
$= 1$

for $n=1$, $1^2 = \frac{1 \times 2 \times 3}{6} = 1$ True

Assume true for n, i.e., $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

consider $n+1$ case, $1^2+2^2+\ldots+n^2+(n+1)^2$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6}$$

$$= \frac{(n+1)\left[2n^2+7n+6\right]}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)\left((n+1)+1\right)\left(2(n+1)+1\right)}{6}$$

True for $n+1$.
By induction, true $\forall n$. ☒

2. Prove the general **De Morgan' Law**,

$$(A_1 \cap A_2 \cap \ldots \cap A_n)^c = A_1^c \cup A_2^c \cup \ldots \cup A_n^c$$

and derive from it, the law

$$(A_1 \cup A_2 \cup \ldots \cup A_n)^c = A_1^c \cap A_2^c \cap \ldots \cap A_n^c$$

Show LHS $\subset$ RHS, and RHS $\supset$ LHS

$\forall x \in$ LHS, $x \in (A_1 \cap \cdots \cap A_n)^c$

$x \notin A_1 \cap \cdots \cap A_n$, $\exists j$ s.t. $x \notin A_j$

$\Rightarrow x \in A_j^c$, $\Rightarrow x \in \bigcup_{i=1}^{n} A_i^c \Rightarrow x \in$ RHS

$\forall x \in$ RHS, $x \in A_1^c \cup A_2^c \cup \cdots \cup A_n^c$

$\Rightarrow \exists j$ s.t. $x \in A_j^c \Rightarrow x \notin A_j \Rightarrow x \notin A_1 \cap A_2 \cdots \cap A_n$

$\Rightarrow x \in (A_1 \cap \cdots \cap A_n)^c$ ☒

Take $(A_1^c \cap A_2^c \cap \cdots A_n^c)^c = (A_1^c)^c \cup (A_2^c)^c \cdots \cup (A_n^c)^c$

$\Rightarrow A_1^c \cap \cdots A_n^c = (A_1 \cup A_2 \cup \ldots A_n)^c$ ☒

(e) $A_1 = a_{11} \to a_{12} \quad a_{13} \to \cdots$

$A_2 = a_{21} \quad a_{22} \quad (a_{23}) \cdots$

$A_3 = a_{31}$

$A = \bigcup_{i=1}^{\infty} A_i$ is countable

3. (a) Show that a subset of countable set is countable.
(b) Suppose that A and B are disjoint denumerable sets, show that $A \cup B$ is denumerable.
(c) Suppose that A and B are countable sets (not necessarily disjoint), then $A \cup B$ is countable.
(d) Suppose that A_1, A_2, ..., A_n are countable sets, show that $\bigcup_{i=1}^{n} A_i$ is countable.
(e) Suppose that A_1, A_2, ... are countable sets, show that $\bigcup_{i=1}^{\infty} A_i$ is countable.

(a) ① If the set is finite, it's obviously true

② If the set is denumerable, list the elements as

$S = \{a_1, a_2, a_3, \cdots\}$

Define $f(k)$ be the ith element of A that is in the subset. $f(1) = a_2$

$f: \mathbb{N} \to B$ bijection. If B has infinitely many ele

$f: \{1, 2, \cdots, n\} \to B$ bijection if B has finitely many

Anyway, B is countable.

(b) $a_1, b_1, a_2, b_2, a_3, b_3, \cdots$

element of A or B appears exactly once in the sequence.

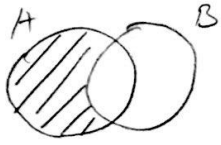
1 2 3 4 ...

(c) $(A - B)$. B are disjoint

countable countable.

$A \cup B = (A - B) \cup B$

disjoint countable.



$A - B$

Q: why $A - B$ is countable?

A: $A - B \subset A$ is a subset

(d) By induction, True for $n=2$, $A_1 \cup A_2$ is countable

Assume true for n, consider $(A_1 \cup A_2 \cdots \cup A_n) \cup A_{n+1}$

countable countable

By (c), it's countable.