Math 425 worksheet#4

September 11th

1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$, and all $a_n's$ are integers. Show that there exists N such that $a_n = C$ for all $n \geq N$. where C is a constant.

By def, $\forall \varepsilon = \frac{1}{2}$, $\exists N$. s.t.

$0 \leq |a_m - a_n| < \frac{1}{2}$. But a_n's are integers

$\Rightarrow |a_m - a_n| = 0$. $\forall m, n \geq N$.

$\Rightarrow a_m = a_n$, $\forall m, n \geq N$. $\Rightarrow a_n = C$ for all $n \geq N$.

2. Consider the map $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined for $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ in $\mathbb{R}^2$ as

$$d_1(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$$

(a) Prove that $(\mathbb{R}^2, d_1)$ is a metric space.
(b) Define another map $d_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$d_\infty(\vec{x}, \vec{y}) = max\{|x_1 - y_1|, |x_2 - y_2|\}$$

show that $(\mathbb{R}^2, d_\infty)$ is also a metric space. also d_∞ and d_1 are equivalent.

(a) $d_1(\vec{x}, \vec{y}) \geq 0$, $\forall \vec{x}, \vec{y}$. also $d_1(\vec{x}, \vec{y}) = 0 \iff |x_1 - y_1| = 0$ & $|x_2 - y_2| = 0$

$\iff x_1 = y_1$, $x_2 = y_2 \iff \vec{x} = \vec{y}$.

$d_1(\vec{x}, \vec{y}) = d_2(\vec{y}, \vec{x})$. ✓

$|x_1 - y_1| + |x_2 - y_2| \overset{?}{\leq} (|x_1 - z_1| + |x_2 - z_2|) + (|y_1 - z_1| + |y_2 - z_2|)$.

Triangle inequality $\Rightarrow |x_1 - z_1| + |y_1 - z_1| \geq |x_1 - y_1|$

So $d_1(\vec{x}, \vec{y}) \leq d_1(\vec{x}, \vec{z}) + d_2(\vec{y}, \vec{z})$.

(b). $d_\infty(\vec{x}, \vec{y}) \geq 0$. $d_\infty(\vec{x}, \vec{y}) = 0 \iff \vec{x} = \vec{y}$.

$d_\infty(\vec{x}, \vec{y}) = max\{|\ |, |\ |\} \leq max\{|\ |, |\ |\} + max\{|\ |, |\ |\}$.

Triangle inequality. ✓ show $\frac{1}{2} d_1 \leq d_\infty \leq 1 \cdot d_1$ ✓

3. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two Cauchy sequences in $\mathbb{R}$. Show that $\{a_n + b_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Note: same argument can show that $\{a_n - b_n\}_{n\in\mathbb{N}}$ is Cauchy.

pick $\frac{\varepsilon}{2}$.

$\forall \varepsilon$, $\exists N_1$. $|a_n - a_m| < \varepsilon/2$ for $m, n \geq N_1$

$\exists N_2$ $|b_n - b_m| < \varepsilon/2$. for $m, n \geq N_2$.

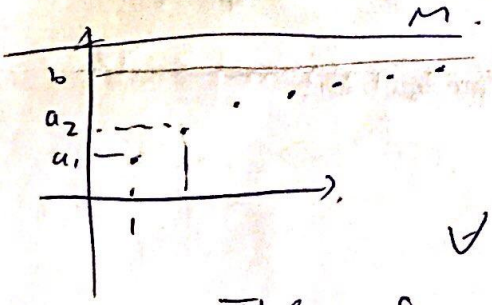
$\Rightarrow$ $|(a_n + b_n) - (a_m + b_m)| = |(a_n - a_m) + (b_n - b_m)|$

$\leq |a_n - a_m| + |b_n - b_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

So $\{a_n + b_n\}_{n\in\mathbb{N}}$ is Cauchy.

Monotone Convergence Thm

4. Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded increasing sequence in $\mathbb{R}$, i.e., $a_n \leq a_{n+1}$ and $|a_n| \leq M$ for all $n = 1, 2, \ldots$, show that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, therefore convergent. (Hint: use homework#2 13(a))



Let $S = \{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$.

S is bded from above so $\exists\, b = \sup(S)$ (l.u.b)

$\forall \varepsilon > 0$. $\exists\, n_0$ s.t. $b - \varepsilon \leq a_{n_0} \leq b$

Then for all $m \geq n_0$

$b - \varepsilon \leq a_{n_0} \leq a_m \leq b$.

$\Rightarrow$ Then $\forall m, n \geq n_0$. (wlog, $m \geq n$)

$|a_m - a_n| = a_m - a_n \leq b - (b - \varepsilon)$

$= \varepsilon$.

