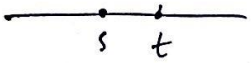
Math 425 worksheet#3

September 4th



1. Consider $s, t \in \mathbb{Q}$. Show
(i) $s \le t$ if and only if $s^* \subseteq t^*$;
(ii) $s = t$ if and only if $s^* = t^*$;
(iii) $(s+t)^* = s^* + t^*$. Note that $s^* + t^*$ is a sum of Dedekind cuts. It suffices to show

(i) $s^* = \{r \in \mathbb{Q} : r < s\}$ | rest of $\mathbb{Q}$

$t^* = \{r \in \mathbb{Q} \mid r < t\}$ | rest of $\mathbb{Q}$

$s < t \Leftrightarrow \{r \in \mathbb{Q} \mid r < s\} \subset \{r \in \mathbb{Q} \mid r < t\}$

$\Leftrightarrow s^* \le t^*$.

(ii) $s = t \Leftrightarrow \{r \in \mathbb{Q} : r < s\} = \{r \in \mathbb{Q} \mid r < t\}$

$\Leftrightarrow \{\quad\}$ | rest of $\mathbb{Q}$ $= \{\quad\}$ | rest of $\mathbb{Q}$

(iii) $\{r \in \mathbb{Q} : r < s+t\}$

$= \{a+b : a, b \in \mathbb{Q}, a < s, b < t\}$.

RHS $\subset$ LHS: b/c
$\forall\, a+b$ with $a < s$, $b < t$
$a + b < s + t$, $a + b \in \mathbb{Q}$.
$\Rightarrow a + b \in$ LHS.

$\forall\, r \in \mathbb{Q}$ with $r < s+t$

$\left(s - \frac{s+t-r}{2}\right)$ $\quad$ $+\left(t - \frac{s+t-r}{2}\right)$.

$= s + t - (s+t-r) = r$.

$s - \frac{(s+t-r)}{2} > 0$

$s - \frac{s+t-r}{2} < s$, rational. $t - \frac{s+t-r}{2} < t$, rational

2. (a) Prove that if $S \subset \mathbb{R}$ is non-empty and bounded below, then it has greatest lower bound, denoted as $g.l.b.(S)$.
(b) Prove that if $S \subset \mathbb{R}$ is non-empty and bounded below, then

$$g.l.b.(S) = -l.u.b.(-S)$$

(a) Consider $-S \subset \mathbb{R}$, non-empty, S bdd below so $\exists\, x \in \mathbb{R}$.

$x \le s$, $\forall s \in S$, $\Rightarrow -x \ge -s$, $\forall s \in S$. $-x$ is an upper bdd for $-S$. So $(-S)$ has an l.u.b. $b = l.u.b.(-S)$.

Want to show $(-b)$ is the great lower bdd for S.

$b \ge -s$, $\forall$ a $s \in S$, so $s \ge -b$ $\forall s \in S$. $(-b)$ is an lower bdd for S. Suppose b' is another l.b for S, same argument show $-b'$ is another upper bdd for $(-S)$. but $b \in -b'$

$\Rightarrow -b) \ge b'$. some $(-b)$ is the greatest lower bound.

(b) part(a) shows $b = l.u.b(-S)$. $(-b) = g.l.b(S)$

$\Rightarrow g.l.b(S) = -l.u.b(-S)$.

$\square$

3. **Let** $S.T \subset \mathbb{R}$ be non-empty subsets that are bounded above, and define $S+T=\{s+t : s\in S, t\in T\}$. Use homework#2 13(*a*) to show

$$sup(S+T) = sup(S) + sup(T)$$

Then, use this to prove that $\forall x \in \mathbb{R}$ and $S+x$ is the set $\{s+x : s\in S\}$. then $sup(S+x) = sup(S)+x$.

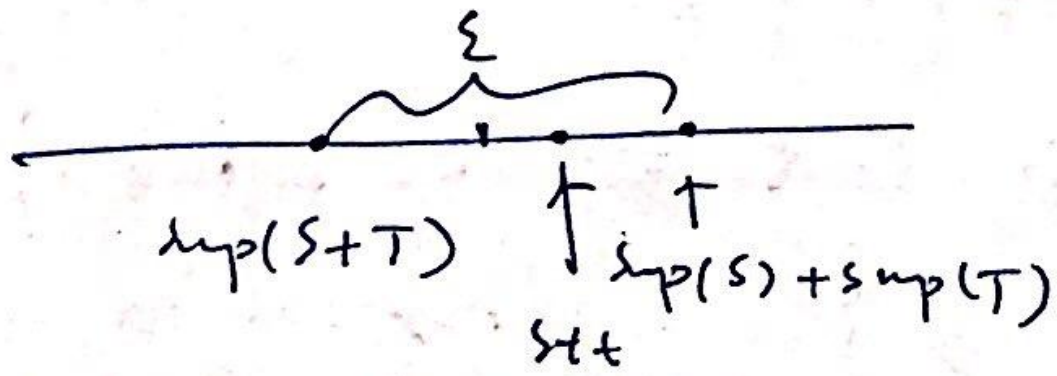
1) $\sup(S) + \sup(T)$ is an upper bdd for $S+T$.

$\forall$ $s+t$ with $s\in S$, $t\in T$,

$s+t \le \sup(S)+\sup(T)$. So $\sup(S)+\sup(T)$ is an upper bdd,

but $\sup(S+T)$ is the l.u.b. So $\sup(S)+\sup(T) \ge \sup(S+T)$.

② Suppose $\sup(S)+\sup(T) > \sup(S+T)$.



$\forall \frac{\varepsilon}{4}$, $\exists$ $s\in S$ with

$\sup(S)-\frac{\varepsilon}{4} \le s \le \sup(S)$

$\sup(T)-\frac{\varepsilon}{4} \le t \le \sup(T)$

$\Rightarrow \sup(S)+\sup(T) - \frac{\varepsilon}{2} \le s+t \le \sup(S)+\sup(T)$

$\sup(S+T)+\frac{\varepsilon}{2} > \sup(S+T)$

$\Rightarrow \exists$ $s\in S$, $t\in T$ s.t. $s+t > \sup(S+T)$. ⊥.

So $\sup(S)+\sup(T) \le \sup(S+T)$. ☒

let $T=\{x\}$. $\sup(S+T) = \sup(S)+\sup(T)$.

$\sup(S+x) = \sup(S)+x$.

sup is invariant under constant translation.

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