True or false. Briefly justify.

(1) In a metric space (M, d), any Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$ in M is convergent.

False. If M is not complete, then Cauchy $\not\Rightarrow$ convergent.

$M = \mathbb{Q}$. $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, $\longrightarrow \sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$,

(2) In a metric space (M, d), any convergent sequence $\{a_n\}_{n\in\mathbb{N}}$ in M is necessarily Cauchy.

Yes, $a+\varepsilon/2$, a, $a-\varepsilon/2$. For $\varepsilon/2$, $\exists N$. s.t. $d(a_n, a) < \varepsilon/2$

so for $m, n > N$.

$d(a_m, a_n) \le d(a_m, a) + d(a_n, a)$

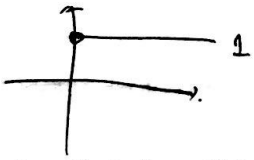
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\Rightarrow \{a_n\}_{n\in\mathbb{N}}$ is Cauchy in (M, d).

so there is not point in $\mathbb{Q}$ s.t. $\{a_n\} \to$ converge to

(3) Continuous function defined on $(0, \infty)$ can't be uniformly continuous.

False. $f(x) = 1$. on $(0, \infty)$



$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$.

(4) The set $A = \{x \in \mathbb{R} : 0 \le x < 1\}$ is uncountable.

True. $A = [0, 1)$. If ctble, then $A_2 = [1, 2)$ ctble.

$\Rightarrow \mathbb{R} = \bigcup [0,1) \cup [-1, 0) \cup [1, 2) \cdots$ ctble. $\not\in$.

$[-1, 0)$

(5) A is uncountable, B is countable, then $A \cap B$ must be countable.

True. $A \cap B$ is a subset of B.

ctble $\to A \cap B \subset B \leftarrow$ ctble.

(6) Assume $S \subset \mathbb{R}$ has a least upper bound, then $l.u.b.(S)$ may or may not be in S.

True. $S_1 = \{1\}$. $l.u.b(S_1) = 1 \in S_1$.

$S_2 = [0, 1)$. $l.u.b(S_2) = 1 \notin S_2$

(7) If there is a function $f : \mathbb{N} \to S$ is injective but not surjective, then $card(N) < card(S)$.

False. $f : \mathbb{N} \to S$ injective.

$\Rightarrow card(\mathbb{N}) \le card(S)$.

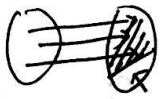
$f : \mathbb{N} \to \mathbb{Z}$

$i \mapsto i$ injective. but not surjective ($-1 \notin$ image(f))

$\Rightarrow$ but $card(\mathbb{N}) = card(\mathbb{Z})$

$f: \{1,2,3,4\} \to \{1,2\}$

If f is injective but not surjective,



(8) Every continuous function defined on $[a, b]$ is bounded.

True.

Thm 2f: Every conti real-valued function defined on $[a, b]$ is bdd, achieves minima/maxima, intermediate value / uniformly conti.

2. Prove the greatest lower bound of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is 0.

0 is a lower bd for $\{\frac{1}{n} : n\in\mathbb{N}\}$ bc $0 \le \frac{1}{n}$, $\forall n\in\mathbb{N}$.

If $\exists\ c>0$, also a lower bd. Then $\exists$ some $n > \frac{1}{c}$,

$\Rightarrow c > \frac{1}{n}$ $\not\in$ c is a lower bd.

So 0 is the greatest lower bd. ☒

3.. Let $f : X \to Y$ be a function, show that if $C, D \subset Y$, then $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

$\forall x \in f^{-1}(C\cup D)$, $f(x) \in C \cup D$.

$\Rightarrow f(x)\in C$ or $f(x)\in D$

$\Rightarrow x \in f^{-1}(C)$ or $x\in f^{-1}(D)$

$\Rightarrow x \in f^{-1}(C)\cup f^{-1}(D)$.

$\Rightarrow$ LHS $\subset$ RHS

conversely,

If $x \in f^{-1}(C) \cup f^{-1}(D)$

$\Rightarrow x\in f^{-1}(C)$ or $x\in f^{-1}(D)$

$\Rightarrow f(x)\in C$ or $f(x)\in D$

$\Rightarrow f(x) \in C\cup D$ ☒

$\Rightarrow x \in f^{-1}(C\cup D)$ $\Rightarrow$ RHS $\subset$ LHS

4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz continuous* if there exist a constant $K > 0$ such that $|f(x) - f(y)| < K|x - y|$, for any $x, y \in \mathbb{R}$. Prove every *Lipschitz continuous* function is uniformly continuous using $\epsilon - \delta$ principal.

$\forall \varepsilon > 0$, $\exists\ \delta = \frac{\varepsilon}{K}$, then s.t. $|x-y| < \delta = \frac{\varepsilon}{K}$.

Lipschitz

$\Rightarrow |f(x) - f(y)| < K \cdot \delta |x-y| \le K \cdot \frac{\varepsilon}{K} = \varepsilon$.

$\Rightarrow$ $f(x)$ is uniformly conti. ☒