

13.2

§ 13.2: Line integrals
really "curve integrals": $[a, b] \xrightarrow{\text{plane curve } C} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

idea: sum values of f over C , but weighted by arc/length in plane (instead of standard length in $[a, b]$)

e.g. given $C: [0, 2\pi] \rightarrow \mathbb{R}^2$
 $t \mapsto \langle \cos t, \sin t \rangle$

$C': [0, \pi] \rightarrow \mathbb{R}^2$
 $t \mapsto \langle \cos 2t, \sin 2t \rangle$

we should have $\int_C f(x, y) ds = \int_{C'} f(x, y) ds$

$$\star \int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

(e.g. arclength of C when $f(x, y) = 1$)

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Curves in $\mathbb{R}^3$: exact same idea.

Ex.: C the line segment from $(1, 0, 0)$ to $(2, 3, -1)$, $f(x, y, z) = x^2 + y - z$,

$$\begin{aligned} \int_C f(x, y, z) ds &= \int_0^1 f(x(t), y(t), z(t)) \sqrt{x'^2 + y'^2 + z'^2} dt \\ &= \int_0^1 (1+t)^2 + 3t - (-t) \cdot \sqrt{1^2 + 3^2 + (-1)^2} dt = \sqrt{11} \cdot \int_0^1 (t^2 + 6t + 1) dt = \sqrt{11} \cdot \left(\frac{1}{3} + 3 + 1\right) = \frac{13\sqrt{11}}{3} \end{aligned}$$

Physics ex.: wire shaped like curve C , with (linear) density $\rho(x, y)$ at any (x, y) where it sits $\rightarrow m = \int_C \rho(x, y) ds = \text{total mass}$; moreover, center of mass @ $(\bar{x}, \bar{y}) = \left(\frac{1}{m} \cdot \int_C x \cdot \rho(x, y) ds, \frac{1}{m} \cdot \int_C y \cdot \rho(x, y) ds\right)$.

Variation: line integrals of along C w.r.t. x or y (instead of $s = \text{arclength}$):

$$\int_C f(x,y) dx = \int_a^b f(x(t), y(t)) \cdot \frac{dx}{dt} dt \quad \text{or } z$$

$$\int_C f(x,y) dy = \int_a^b f(x(t), y(t)) \cdot \frac{dy}{dt} \cdot dt$$

Ex.: C the part of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $(1,1,1)$ to $(2,4,8)$,

$$\int_C (x^2 + y) dz = \int_1^2 (t^2 + t^2) \cdot 3t^2 dt = \frac{6}{5} t^5 \Big|_{t=1}^{t=2} = \boxed{\frac{6 \cdot 31}{5}}$$

Rmk.: Writing $-C$ for same curve w/ reversed orientation, we have

$$\int_{-C} f ds = - \int_C f ds, \text{ but still } \int_{-C} f dx = - \int_C f dx, \int_{-C} f dy = - \int_C f dy, \text{ (Eq. 2)}$$

Δs always positive, by definition

Line integrals of vector fields

idea: sum contributions of a vector field $\vec{F}$ in the direction of the curve C (e.g. work done on an object w/ trajectory $\vec{r}(t)$)

$$\star \text{ line integral of } \vec{F} \text{ along } C = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$d\vec{r} = \vec{r}'(t) dt$$

Ex.: force field $\vec{F}(x,y) = \langle x^2, -xy \rangle$

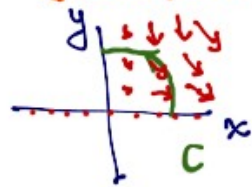
C = quarter circle $\vec{r}(t) = \langle \cos t, \sin t \rangle, t \in [0, \pi/2]$

$$\vec{F}(\vec{r}(t)) = \langle \cos^2 t, -\cos t \sin t \rangle, \vec{r}'(t) = \langle -\sin t, \cos t \rangle$$

$$\begin{aligned} \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt &= \int_0^{\pi/2} (-\sin t \cdot \cos^2 t - \sin t \cdot \cos^2 t) dt = -2 \cdot \int_0^{\pi/2} \cos^2 t \sin t dt \\ &= \frac{2}{3} \cos^3 t \Big|_{t=0}^{t=\pi/2} = \frac{2}{3} \cdot (0 - 1) = \boxed{-\frac{2}{3}} \end{aligned}$$

negative: $\vec{F}$ working against movement

Rmk.: $\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$ (on $-C$, $\vec{r}'(t)$ switches signs)



Remark: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, then $\int_C \vec{F} \cdot d\vec{r} = \int_C (P dx + Q dy)$.

★ all same in 3d

Connection: writing $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ for unit tangent vector, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds \quad (\|\vec{r}'(t)\| dt = ds)$$