Math 226 – Midterm 1 Name: ________________________

Show all your work, and indicate clearly if you continue on the back. The entire exam is worth 50 points.

(2 points ea.) 1. Indicate whether each statement is true or false. (You do not need to show your work here.)

(i) _____ The equation $(x-1)^2 - 2y^2 - (3z+1)^2 = 0$ defines a cone.
TRUE. *This cone is centered at the point* $(1, 0, -\frac{1}{3})$, *and is rotationally symmetric about the line* $y = z + \frac{1}{3} = 0$; *its traces through the planes* $x = x_0$ *are ellipses.*

(ii) _____ Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any function. Suppose that $\lim_{t\to 0} f(\mathbf{r}(t)) = L$ for all straight lines $\mathbf{r}(t)$ with $\mathbf{r}(t) = (0,0)$. Then, $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

FALSE. *A counterexample is given by the function* $f(x,y) = \frac{xy^2}{x^2+y^4}$ *(which is Example 3 of §11.2 in the textbook).*

(iii) _____ If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ with $\mathbf{v} \bullet \mathbf{w} = 0$, then $\mathbf{v} \times \mathbf{w} = \vec{0}$.
FALSE. *For example, taking* $\mathbf{v} = \mathbf{i}$ *and* $\mathbf{w} = \mathbf{j}$ *we have* $\mathbf{i} \bullet \mathbf{j} = 0$ *and* $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

(iv) _____ The level sets of the function $f(x,y) = \frac{e^y}{x}$ are hyperbolas.
FALSE. *The* k^{th} *level set of this function is the locus* $\{(x,y) \in \mathbb{R}^2 | k = \frac{e^y}{x}\}$, *which is the graph of the function* $g(x) = \ln(kx)$.

(v) _____ There exists a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $\mathbf{v} \times \langle 0,0,1\rangle = \langle 1,0,0\rangle$ and $\mathbf{v} \times \langle 1,0,0\rangle = -\langle 0,0,1\rangle$.
TRUE. *For example, take* $\mathbf{v} = \mathbf{j}$.

(vi) _____ The function $f(x,y) = e^{2xy}$ satisfies the partial differential equation $xf_x + yf_y = 2xyf$.
FALSE. *We compute that* $f_x = 2ye^{2xy} = 2yf$ *and* $f_y = 2xe^{2xy} = 2xf$, *so that* $xf_x + yf_y = x \cdot 2yf + y \cdot 2xf = 4xyf \neq 2xyf$.

(vii) _____ If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors in $\mathbb{R}^3$, then $\|\mathbf{v}\| = \|\mathbf{w}\|$.
FALSE. *For example, take* $\mathbf{v} = \mathbf{i}$ *and* $\mathbf{w} = 2\mathbf{j}$.

(4 points) 2. A wrench 10 cm long lies along the positive x-axis and grips a bolt at the origin. A force is applied in the direction of $\langle 0,0,-1\rangle$ at the end of the wrench. Find the magnitude of force necessary to exert 1 N · m of torque to the bolt.

The torque is computed by the formula $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, *where* $\mathbf{r}$ *is the displacement vector from the bolt to the end of the wrench and* $\mathbf{F}$ *is the force vector. In this case* $\mathbf{r} \perp \mathbf{F}$ *and so* $\|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\| \cdot \|\mathbf{F}\|$, *and so setting*

$$1 \text{ N} \cdot \text{m} = 100 \text{ N} \cdot \text{cm} \overset{\text{set}}{=} \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\| \cdot \|\mathbf{F}\| = 10 \text{ cm} \cdot \|\mathbf{F}\|$$

we find that $\|\mathbf{F}\| = 10$ N.

(4 points) 3. A plane in $\mathbb{R}^3$ contains the origin and is normal to the vector $\langle 3,0,1\rangle$. Find all values of t such that the space curve $\mathbf{r}(t) = \langle e^t, \cos t, -e^{3t}\rangle$ intersects the plane.

In general, a plane with normal vector $\mathbf{n}$ *containing the reference point* $\mathbf{r}_0$ *is described by the equation* $\mathbf{n} \bullet (\mathbf{r} - \mathbf{r}_0) = 0$, *where* $\mathbf{r} = \langle x,y,z\rangle$. *Here we obtain the equation* $3x + z = 0$. *The space curve* $\mathbf{r}(t)$ *lies in this plane precisely at those* t*-values for which its coordinates satisfy the equation of the plane. So we obtain the equation* $3 \cdot e^t + (-e^{3t}) = 0$, *which has the unique solution* $t = \frac{1}{2}\ln 3$.

(4 points) 4. A parallelepiped in $\mathbb{R}^3$ has one of its vertices at the point $(2,0,4)$, and its three adjacent vertices are at the points $(3,0,5)$, $(1,2,3)$, and $(0,0,-1)$. Find its volume.

The edges of the parallelepiped are spanned by the displacement vectors $\mathbf{a} = \langle 3-2, 0-0, 5-4\rangle = \langle 1,0,1\rangle$, $\mathbf{b} = \langle 1-2, 2-0, 3-4\rangle = \langle -1,2,-1\rangle$, *and* $\mathbf{c} = \langle 0-2, 0-0, (-1)-4\rangle = \langle -2,0,-5\rangle$. *So, its volume is computed by the absolute value of their scalar triple product,* $|(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}|$. *We compute that*

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ -1 & 2 & -1 \end{vmatrix} = \langle 0\cdot(-1) - 1\cdot 2, 1\cdot(-1) - 1\cdot(-1), 1\cdot 2 - 0\cdot(-1)\rangle = \langle -2,0,2\rangle \ ,$$

and thereafter that

$$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = \langle -2,0,2\rangle \bullet \langle -2,0,-5\rangle = (-2)\cdot(-2) + 0\cdot 0 + 2\cdot(-5) = -6 \ ,$$

so that the volume of the parallelepiped is

$$|(\mathbf{a}\times\mathbf{b})\bullet(\mathbf{c})| = |-6| = 6 \ .$$

(6 points) 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function assigning to a point $(x,y)\in\mathbb{R}^2$ the square of its distance from the line $\mathbf{r}(t) = \langle t,t\rangle$. Evaluate $f_{xy}(2,7)$.

The line $\mathbf{r}(t)$ *contains the vector* $\mathbf{v} = \langle 1,1\rangle$. *The distance from an arbitrary point* (x,y) *to this line is the magnitude of the vector* $\langle x,y\rangle - \mathrm{proj}_{\mathbf{v}}(\langle x,y\rangle)$. *We compute this projection to be*

$$\mathrm{proj}_{\mathbf{v}}(\langle x,y\rangle) = \frac{\mathbf{v}\bullet\langle x,y\rangle}{\|\mathbf{v}\|^2}\cdot\mathbf{v} = \frac{x+y}{2}\cdot\langle 1,1\rangle \ ,$$

so that

$$f(x,y) = \left\| \langle x,y\rangle - \frac{x+y}{2}\cdot\langle 1,1\rangle \right\|^2 = \left\| \left\langle \frac{x-y}{2}, \frac{y-x}{2} \right\rangle \right\|^2 = \left(\frac{x-y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2 = \frac{x^2-2xy+y^2}{2} \ .$$

From here, we compute that $f_x(x,y) = x - y$, *and then that* $f_{xy}(x,y) = -1$. *So* $f_{xy}(2,7) = -1$.

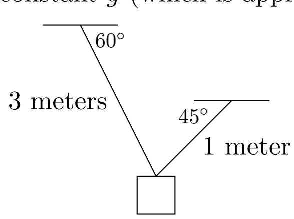
(6 points) 6. Find an equation for the set of points in $\mathbb{R}^3$ that are equidistant to the lines $x = y+1 = 0$ and $y - 1 = z = 0$, put it in standard form, and identify the quadric surface that it defines.

The first line runs parallel to the z*-axis and through the point* $(0,-1,0)$, *so the distance to it from an arbitrary point* (x,y,z) *is* $\sqrt{x^2+(y+1)^2}$. *Meanwhile, the second line runs parallel to the* x*-axis and through the point* $(0,1,0)$, *so the distance to it from an arbitrary point* (x,y,z) *is* $\sqrt{(y-1)^2+z^2}$. *Because distances are nonnegative, equating these two distances is equivalent to equating their squares, and so we obtain the equation*

$$x^2 + (y+1)^2 = (y-1)^2 + z^2 \ .$$

This simplifies to the equation $4y = z^2 - x^2$, *which defines a hyperbolic paraboloid.*

(6 points) 7. A box is hanging from two ropes as indicated in the figure. The mass of the box is 100 kilograms. Find the magnitude of the tension in the shorter rope. (You may write your answer in terms of the gravitational constant g (which is approximately $9.81\frac{\mathrm{m}}{\mathrm{s}^2}$).)



Let T_1 *denote the tension in the longer rope and* T_2 *denote the tension in the shorter rope, both measured in Newtons. Then, arranging our coordinate system in the standard way (so that* x *increases in the rightwards direction while* y *increases in the upwards direction), the three force vectors acting on the box can be identified as follows:*

- *the force exerted on the box by the longer rope is* $T_1 \cdot \langle \cos(120°), \sin(120°) \rangle = \langle \frac{-1}{2} T_1, \frac{\sqrt{3}}{2} T_1 \rangle$,
- *the force exerted on the box by the shorter rope is* $T_2 \cdot \langle \cos(45°), \sin(45°) \rangle = \langle \frac{1}{\sqrt{2}} T_2, \frac{1}{\sqrt{2}} T_2 \rangle$, *and*
- *the force exerted on the box by gravity is* $\langle 0, -100g \rangle$.

Because the box is at rest, these forces must all cancel out:

$$\left\langle \frac{-1}{2} T_1, \frac{\sqrt{3}}{2} T_1 \right\rangle + \left\langle \frac{1}{\sqrt{2}} T_2, \frac{1}{\sqrt{2}} T_2 \right\rangle + \langle 0, -100g \rangle = \langle 0, 0 \rangle \ .$$

Because vectors are equal precisely when their components are equal, this equality is equivalent to the following system of two equations in the two unknowns T_1 *and* T_2*:*

$$\begin{cases} \frac{-1}{2} T_1 + \frac{1}{\sqrt{2}} T_2 + 0 = 0 \\ \frac{\sqrt{3}}{2} T_1 + \frac{1}{\sqrt{2}} T_2 - 100g = 0 \end{cases} ,$$

whose unique solution has

$$T_2 = \frac{200g}{\sqrt{6} + \sqrt{2}} \ .$$

(6 points) 8. Determine the set of points at which the function

$$f(x, y) = \begin{cases} \dfrac{e^{-x^2-y^2} - 1}{x^2 + y^2} \ , & (x, y) \neq (0, 0) \\ 0 \ , & (x, y) = (0, 0) \end{cases}$$

is continuous.

This function is continuous at all $(x, y) \neq (0, 0)$ *because it is a ratio of continuous functions whose denominator does not vanish. To determine whether it is continuous at* $(0, 0)$*, we must find the limit of* $f(x, y)$ *at* $(0, 0)$ *and determine whether it equals* $f(0, 0) = 0$*. Using polar coordinates, we have that*

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} = \lim_{r\to 0^+} \frac{e^{-r^2} - 1}{r^2} \ .$$

This last limit is that of a ratio of functions that both approach 0, and so we apply l'Hôpital's rule to find that it equals

$$\lim_{r\to 0^+} \frac{-2r \cdot e^{-r^2} - 0}{2r} = \lim_{r\to 0^+} -e^{-2r} = -e^{-2\cdot 0} = -1 \ .$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x, y) = -1 \neq 0 = f(0, 0)$*, and so* f *is not continuous at* $(0, 0)$*. So the set of points at which* f *is continuous is* $\mathbb{R}^2 \setminus \{(0, 0)\} = \{(x, y) \in R^2 | (x, y) \neq (0, 0)\}$.