Math 226 – Midterm 2 Name: ______________________

Show all your work, and indicate clearly if you continue on the back. The entire exam is worth 50 points.

(4 points) 1. Find an equation of the tangent plane to the surface $z = \ln(x - 2y)$ at the point $(3, 1, 0)$, and use it to approximate the z-value of the surface at which $(x, y) = (2, \frac{1}{2})$.

We compute the partial derivatives of z to be $z_x = \frac{1}{x-2y}$ and $z_y = \frac{-2}{x-2y}$, so that $z_x(3,1) = 1$ and $z_y(3,1) = -2$. Hence, the tangent plane is the graph of the function $L(x,y) = (x-3)-2(y-1)$. This yields the approximation $z(2, \frac{1}{2}) \approx L(2, \frac{1}{2}) = (2-3) - 2(\frac{1}{2} - 1) = 0$.[1]

(8 points) 2. If $z = f(x, y)$ where $x = r^2$ and $y = 2rs$, find $\frac{\partial^2 z}{\partial r \partial s}$.

We use the chain rule to compute first that

$$\frac{\partial z}{\partial s} = f_x \cdot \frac{\partial x}{\partial s} + f_y \cdot \frac{\partial y}{\partial s} = 2r \cdot f_y \ ,$$

and thereafter that

$$\frac{\partial^2 z}{\partial r \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial s}\right) = \frac{\partial}{\partial r}(2r \cdot f_y) = 2 \cdot f_y + 2r \cdot \left(f_{yx} \cdot \frac{\partial x}{\partial r} + f_{yy} \cdot \frac{\partial y}{\partial r}\right) = 2 \cdot f_y + 4r^2 \cdot f_{yx} + 4rs \cdot f_{yy} \ .$$

(8 points) 3. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$. Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

The temperature function is of the form

$$T(x, y, z) = \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}}$$

for some constant α. We solve for the constant α by setting $T(1, 2, 2) = 120$ (leaving the units implicit), which implies that $\alpha = 360$. So, we have

$$T(x, y, z) = \frac{360}{\sqrt{x^2 + y^2 + z^2}} = 360(x^2 + y^2 + z^2)^{-1/2} \ .$$

This implies that

$$\nabla T = -180(x^2 + y^2 + z^2)^{-3/2} \cdot \langle 2x, 2y, 2z \rangle \ ,$$

and so

$$\nabla T(1, 2, 2) = -\frac{20}{3} \cdot \langle 2, 4, 4 \rangle \ .$$

Now, the displacement vector from the point $(1, 2, 2)$ to the point $(2, 1, 3)$ is $\langle 1, -1, 1 \rangle$. The unit vector in this same direction is $\mathbf{u} = \frac{1}{\sqrt{3}} \cdot \langle 1, -1, 1 \rangle$. So, the desired directional derivative is

$$D_{\mathbf{u}} T(1, 2, 2) = \nabla T(1, 2, 2) \bullet \mathbf{u} = -\frac{20}{3\sqrt{3}} \cdot (2 - 4 + 4) = -\frac{40}{3\sqrt{3}} \ .$$

(8 points) 4. Find the absolute extrema of the function $f(x, y) = x^2 + y^2 + x^2 y + 4$ over the domain

$$D = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \ , \ |y| \leq 1\} \ .$$

We begin by finding the critical points of f that lie in D.

- *On the one hand, we have that $f_x = 2x + 2xy = 2x(1 + y)$, which is zero precisely when $x = 0$ or $y = -1$.*
- *On the other hand, we have that $f_y = 2y + x^2$, which is zero precisely when $y = -\frac{1}{2}x^2$.*

Hence, the only critical point of f lying in the domain D is $(0, 0)$. We compute that $f(0, 0) = 4$.

We now extremize f along the boundary of the square.

[1] *In fact, this approximation happens to be exactly correct: $z(2, \frac{1}{2}) = \ln(2 - 2 \cdot \frac{1}{2}) = \ln(1) = 0$.*

- *Because* $f(x,y) = f(-x,y)$, *its values along both the left and right edges are* $g(y) = f(\pm 1, y) = y^2 + y + 5$, *for* $y \in [-1,1]$. *To extremize this function of* y, *we compute that* $g'(y) = 2y + 1$, *which is zero precisely when* $y = -\frac{1}{2}$. *We compute that* $g(-\frac{1}{2}) = \frac{19}{4}$.
- *Along the top edge of the square,* f *takes the values* $h(x) = f(x,1) = 2x^2 + 5$ *for* $x \in [-1,1]$, *which has maximum value* 7 *when* $x = \pm 1$ *and has minimum value* 5 *when* $x = 0$.
- *Along the bottom edge of the square,* f *takes the values* $i(x) = f(x,-1) = 5$, *which has both maximum and minimum value* 5.

So, the absolute extrema of f *over* D *are* $4 = f(0,0)$ *and* $7 = f(\pm 1, 1)$.

(8 points) 5. Using Lagrange multipliers, find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

We would like to find the constrained minima of the function $D(x,y,z) = x^2+y^2+z^2$, *subject to the constraint that* $g(x,y,z) = y^2 - 9 - xz = 0$. *In order to verify that we can use the method of Lagrange multipliers, we compute that* $\nabla g = \langle -z, 2y, -x \rangle$, *and so* $\nabla g(x,y,z) = \vec{0}$ *only at the origin, which does not satisfy the constraint that* $g = 0$. *Hence, every constrained extremum must be a solution to the system of equations*

$$\begin{cases} \nabla D = \lambda \cdot \nabla g \\ g = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \langle 2x, 2y, 2z \rangle = \lambda \cdot \langle -z, 2y, -x \rangle \\ y^2 - 9 - xz = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 2x = \lambda \cdot (-z) \\ 2y = \lambda \cdot 2y \\ 2z = \lambda \cdot (-x) \\ y^2 = 9 + xz \end{cases} .$$

The second equation implies that either $\lambda = 1$ *or* $y = 0$. *If* $\lambda = 1$, *then we have* $-z = 2x$ *and* $-x = 2z$ *so it must be that* $x = z = 0$, *so* $y = \pm 3$. *On the other hand, if* $y = 0$ *then it must be that* $(x,z) = (\pm 3, \mp 3)$. *Finally, we compute that*

$$D(0, \pm 3, 0) = 9 \qquad \textit{and} \qquad D(\pm 3, 0, \mp 3) = 18 ,$$

so the constrained minimum of D *is* 9 *and it occurs at the points* $(0, \pm 3, 0)$. *That is, the points* $(0, \pm 3, 0)$ *are the points on the given surface that are closest to the origin.*

(6 points) 6. Consider the solid cube in $\mathbb{R}^3$ defined by $0 \le x, y, z \le 4$. If the cube has charge density $\rho(x,y,z) = xy$, use the midpoint rule with $l = m = n = 2$ to approximate its total charge.

In all three coordinates, we partition the interval $[0,4]$ *into two equal subintervals of width* 2, *i.e. with* $x_0 = y_0 = z_0 = 0$, $x_1 = y_1 = z_1 = 2$, *and* $x_2 = y_2 = z_2 = 4$. *These have midpoints* $\overline{x}_1 = \overline{y}_1 = \overline{z}_1 = 1$ *and* $\overline{x}_2 = \overline{y}_2 = \overline{z}_2 = 3$, *and the corresponding little cubes have volumes* $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z = 2 \cdot 2 \cdot 2 = 8$. *So, we approximate the total charge to be*

$$\begin{aligned} Q &= \iiint_{[0,4]\times[0,4]\times[0,4]} \rho(x,y,z)\ dV \approx \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \rho(\overline{x}_i, \overline{y}_j, \overline{z}_k) \cdot \Delta V \\ &= 8 \cdot (\rho(1,1,1) + \rho(1,1,3) + \rho(1,3,1) + \rho(3,1,1) + \rho(1,3,3) + \rho(3,1,3) + \rho(3,3,1) + \rho(3,3,3)) \\ &= 8 \cdot (1 + 1 + 3 + 3 + 3 + 3 + 9 + 9) = 8 \cdot 32 = 256 . \end{aligned}$$ [2]

(8 points) 7. Consider the solid cylinder $E \subset \mathbb{R}^3$ defined by $x^2 + y^2 \le 1$ and $0 \le z \le 1$. If the cylinder has density function $\rho(x,y,z) = (x^2 + y^2)z$, find its center of mass.

We use cylindrical coordinates. We first compute the mass to be

$$\begin{aligned} m &= \iiint_E \rho(x,y,z)\ dV = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=1} \rho(r\cos\theta, r\sin\theta, z) \cdot r\ dz\ dr\ d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=1} r^3 z\ dz\ dr\ d\theta = \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \cdot \left(\int_{r=0}^{r=1} r^3\ dr \right) \cdot \left(\int_{z=0}^{z=1} z\ dz \right) \\ &= 2\pi \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{\pi}{4} . \end{aligned}$$

[2] *In fact, this approximation happens to be exactly correct: using Fubini's theorem, we compute that*

$$Q = \iiint_{[0,4]\times[0,4]\times[0,4]} \rho(x,y,z)\ dV = \int_{x=0}^{x=4} \int_{y=0}^{y=4} \int_{z=0}^{z=4} xy\ dz\ dy\ dx = \left(\int_{x=0}^{x=4} x\ dx \right) \cdot \left(\int_{y=0}^{y=4} y\ dy \right) \cdot \left(\int_{z=0}^{z=4} dz \right) = 8 \cdot 8 \cdot 4 = 256 .$$

Because the cylinder itself as well as the density function $\rho(x, y, z)$ are rotationally symmetric about the z-axis, the center of mass $(\overline{x}, \overline{y}, \overline{z})$ must have $\overline{x} = \overline{y} = 0$, i.e. it must be located on the z-axis. So it remains to compute that

$$\begin{aligned}
\overline{z} &= \frac{1}{m} \cdot \iiint_E z \cdot \rho(x, y, z)\ dV = \frac{1}{m} \cdot \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=1} z \cdot \rho(r\cos\theta, r\sin\theta, z) \cdot r\ dz\ dr\ d\theta \\
&= \frac{4}{\pi} \cdot \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=1} r^3 z^2\ dz\ dr\ d\theta = \frac{4}{\pi} \cdot \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \cdot \left(\int_{r=0}^{r=1} r^3\ dr \right) \cdot \left(\int_{z=0}^{z=1} z^2\ dz \right) \\
&= \frac{4}{\pi} \cdot 2\pi \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{2}{3}\ .
\end{aligned}$$