Solutions: quiz 12, discussion section 10am

Math 226, Fall 2019, Prof. Mazel-Gee

1. Using Green's theorem, we compute that

$$\oint_{\partial D} \mathbf{F} \bullet d\mathbf{r} = \oint_{\partial D} \left\langle -\frac{1}{2}y^2x^2, y^3x \right\rangle \bullet d\mathbf{r} = \iint_D \left(\frac{\partial}{\partial x}(y^3x) - \frac{\partial}{\partial y}\left(-\frac{1}{2}y^2x^2\right)\right) \, dA$$
$$= \iint_D (y^3 + yx^2) \, dA = \int_{x=-1}^{x=-\frac{1}{2}} \int_{y=0}^{y=\sqrt{1-x^2}} (y^3 + yx^2) \, dy \, dx$$
$$= \int_{x=-1}^{x=-\frac{1}{2}} \left(\frac{1}{4}y^4 + \frac{1}{2}y^2x^2\right)\Big|_{y=0}^{y=\sqrt{1-x^2}} dx = \int_{x=-1}^{x=-\frac{1}{2}} \left((1-x^2)^2 + \frac{1}{2}(1-x^2)x^2\right) dx$$
$$= \int_{x=-1}^{x=-\frac{1}{2}} \left(\frac{1}{2}x^4 - \frac{3}{2}x^2 + 1\right) dx = \left(\frac{1}{10}x^5 - \frac{1}{2}x^3 + x\right)\Big|_{x=-1}^{x=-\frac{1}{2}} = \frac{371}{320} \, .$$

2. Recognizing that $\mathbf{F} = \nabla f$ where $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, we find that its curl is

$$\text{curl}(\mathbf{F}) = \nabla \times (\nabla f) = \vec{0} \, .$$

On the other hand, its divergence is

$$\text{div}(\mathbf{F}) = \nabla \bullet \mathbf{F} = \left\langle \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right), \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right), \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right\rangle$$
$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} \cdot \langle y^2+z^2, x^2+z^2, x^2+y^2 \rangle \, .$$