Solutions: quiz 12, discussion section 11am

Math 226, Fall 2019, Prof. Mazel-Gee

1. Let us parametrize the three oriented line segments comprising C as

$$C_1 \ : \ \mathbf{r}_1(t) = \langle t, 0 \rangle \ , \qquad C_2 \ : \ \mathbf{r}_2(t) = \langle 1, t \rangle \ , \qquad \text{and} \qquad C_3 \ : \ \mathbf{r}_3(t) = \langle 1-t, 1-t \rangle$$

all for $t \in [0,1]$. Then, we compute that

$$\begin{aligned}
\oint_C \mathbf{F} \bullet d\mathbf{r} &= \int_{C_1} \langle -1, xy \rangle \bullet d\mathbf{r} + \int_{C_2} \langle -1, xy \rangle \bullet d\mathbf{r} + \int_{C_3} \langle -1, xy \rangle \bullet d\mathbf{r} \\
&= \int_{t=0}^{t=1} \langle -1, 0 \rangle \bullet \langle 1, 0 \rangle \ dt + \int_{t=0}^{t=1} \langle -1, t \rangle \bullet \langle 0, 1 \rangle \ dt + \int_{t=0}^{t=1} \langle -1, (1-t)^2 \rangle \bullet \langle -1, -1 \rangle \ dt \\
&= \int_{t=0}^{t=1} (-1) \ dt + \int_{t=0}^{t=1} t \ dt + \int_{t=0}^{t=1} (2t - t^2) \ dt = (-1) + \frac{1}{2} + \left(1 - \frac{1}{3}\right) = \frac{1}{6} \ .
\end{aligned}$$

On the other hand, let $D \subset \mathbb{R}^2$ denote the domain bounded by C. Using Green's theorem, we compute that

$$\begin{aligned}
\oint_C \langle -1, xy \rangle \bullet d\mathbf{r} &= \iint_D \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(-1) \right) dA = \iint_D y \ dA = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=x} y \ dy \right) dx \\
&= \int_{x=0}^{x=1} \frac{1}{2} x^2 \ dx = \left. \frac{1}{6} x^3 \right|_{x=0}^{x=1} = \frac{1}{6} \ .
\end{aligned}$$

2. The vector field $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, so it is conservative if and only if $\text{curl}(\mathbf{F}) = \vec{0}$. We compute that

$$\begin{aligned}
\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle 1, \sin z, y \cos z \rangle \\
&= \left\langle \frac{\partial}{\partial y}(y \cos z) - \frac{\partial}{\partial z}(\sin z), \frac{\partial}{\partial z}(1) - \frac{\partial}{\partial x}(y \cos z), \frac{\partial}{\partial x}(\sin z) - \frac{\partial}{\partial y}(1) \right\rangle = \vec{0} \ ,
\end{aligned}$$

so indeed $\mathbf{F}$ is conservative. To find a potential function $f(x,y,z)$, we observe that in the first two slots the equation $\mathbf{F} = \nabla f$ requires that $f = x + y \sin z + C(z)$. Applying $\frac{\partial}{\partial z}$, we then find that $C'(z) = 0$. Hence, f is a potential function for $\mathbf{F}$ if and only if it takes the form $f = x + y \sin z + C$ for some constant $C \in \mathbb{R}$.