Solutions: quiz 6, discussion section 10am

Math 226, Fall 2019, Prof. Mazel-Gee

1. We use the chain rule to compute that

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial z}{\partial x} \cdot \cos\theta + \frac{\partial z}{\partial y} \cdot \sin\theta$$

and that

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x} \cdot (-r\sin\theta) + \frac{\partial z}{\partial y} \cdot (r\cos\theta) \ .$$

It follows that when $r \neq 0$, we have

$$\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \cdot \left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x} \cdot \cos\theta + \frac{\partial z}{\partial y} \cdot \sin\theta\right)^2 + \frac{1}{r^2} \cdot \left(\frac{\partial z}{\partial x} \cdot (-r\sin\theta) + \frac{\partial z}{\partial y} \cdot (r\cos\theta)\right)^2 \\
&= \left(\frac{\partial z}{\partial x}\right)^2 \cdot (\cos^2\theta + \sin^2\theta) \\
&\quad + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot (2\cos\theta\sin\theta - 2\sin\theta\cos\theta) \\
&\quad + \left(\frac{\partial z}{\partial y}\right)^2 \cdot (\sin^2\theta + \cos^2\theta) \\
&= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \ .
\end{aligned}$$

2. We compute that

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \left\langle \frac{(x+y)y - xy \cdot 1}{(x+y)^2}, \frac{(x+y)x - xy \cdot 1}{(x+y)^2} \right\rangle = \left\langle \frac{y^2}{(x+y)^2}, \frac{x^2}{(x+y)^2} \right\rangle \ .$$

Writing $\mathbf{u} = \frac{1}{\sqrt{5}}\langle 2,1\rangle$ for the unit vector in the direction of the vector $\langle 2,1\rangle$, we then compute that

$$D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \bullet \mathbf{u} = \frac{1}{9\sqrt{5}} \cdot \langle 4,1\rangle \bullet \langle 2,1\rangle = \frac{1}{\sqrt{5}} \ .$$