Solutions: quiz 6, discussion section 11am

Math 226, Fall 2019, Prof. Mazel-Gee

1. Let us write $f(x,y,z) = xyz - 1$. Then we have

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle = \langle yz, xz, xy \rangle \ .$$

Thus the vector $\nabla f(1,1,1) = \langle 1,1,1 \rangle$ is perpendicular to the desired tangent plane, and so it has equation

$$\langle 1,1,1 \rangle \bullet (\langle x,y,z \rangle - \langle 1,1,1 \rangle) = 0 \ ,$$

or equivalently $x + y + z = 3$. The desired normal line then has symmetric equations

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1} \ ,$$

or equivalently $x = y = z$.

2. We compute that

$$f_x(x,y) = y - 2xy - y^2 = y(1-2x-y) \qquad \text{and that} \qquad f_y(x,y) = x - x^2 - 2xy = x(1-x-2y) \ .$$

These functions both exist for all $(x,y) \in \mathbb{R}^2$, so the critical points of f are precisely those points $(x,y) \in \mathbb{R}^2$ for which $f_x(x,y) = 0 = f_y(x,y)$. In order to have $f_x(x,y) = 0$ it must be that $y = 0$ or that $1 - 2x - y = 0$; in order to have $f_y(x,y) = 0$, it must be that $x = 0$ or that $1 - x - 2y = 0$. We therefore have the following critical points: $(0,0)$, $(1,0)$, $(0,1)$, and $(\frac{1}{3}, \frac{1}{3})$.[1] In order to classify each critical point $(a,b) \in \mathbb{R}^2$, we compute the corresponding discriminant

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2 \ .$$

For this, we first compute that

$$f_{xx}(x,y) = -2y \ , \qquad f_{yy}(x,y) = -2x \ , \qquad \text{and} \qquad f_{xy}(x,y) = 1 - 2x - 2y = f_{yx}(x,y) \ ,$$

from which we find that $D(0,0) = D(1,0) = D(0,1) = -1 < 0$ and $D(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3} > 0$. So the critical points $(0,0)$, $(1,0)$, and $(0,1)$ are all saddle points, while the critical point $(\frac{1}{3}, \frac{1}{3})$ is a local extremum. In light of the fact that $f_{xx}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3} < 0$ (or that $f_{yy}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3} < 0$), we see that $(\frac{1}{3}, \frac{1}{3})$ is a local maximum.

[1] In the following graph, the solution set of the equation $f_x(x,y) = 0$ is plotted in blue and the solution set of the equation $f_y(x,y) = 0$ is plotted in red.