MATH 151A HOMEWORK 2

May §1: 2

Hatcher §1.1: 3, 6

Let $X \xrightarrow{f} Y$ be a morphism in a category $\mathcal{C}$. Recall that a morphism $Y \xrightarrow{g} X$ is called

- a ***left inverse*** of f if $gf = \mathsf{id}_X$,
- a ***right inverse*** of f if $fg = \mathsf{id}_Y$, and
- a (***two-sided***) ***inverse*** to f if it is both a left inverse of f and a right inverse of f.

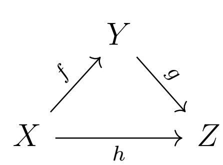
We say that f is an ***isomorphism*** if it admits an inverse.

(1) Show that inverses are unique when they exist, i.e. if g and h are inverses to f then $g = h$.

It follows from (1) that we may unambiguously denote an inverse of a morphism f by f^{-1}.

(2) Show that if a morphism $X \xrightarrow{f} Y$ in a category $\mathcal{C}$ admits both a left inverse and a right inverse, then it is an isomorphism.

(3) Show that isomorphisms obey the following ***two-out-of-three property***: if



is a commutative triangle in a category $\mathcal{C}$ and any two of the morphisms f, g, and h are isomorphisms, then so is the third.

(4) Show that taking hom into or out of an isomorphism in a category $\mathcal{C}$ gives an isomorphism in Set: i.e., that if $X \xrightarrow{f} Y$ is an isomorphism in $\mathcal{C}$, then for any object $Z \in \mathcal{C}$ the functions

$$\mathsf{hom}_{\mathcal{C}}(Z, X) \xrightarrow{f \circ (-)} \mathsf{hom}_{\mathcal{C}}(Z, Y) \qquad \text{and} \qquad \mathsf{hom}_{\mathcal{C}}(Y, Z) \xrightarrow{(-) \circ f} \mathsf{hom}_{\mathcal{C}}(X, Z)$$

are isomorphisms in Set. (It is a basic fact that the isomorphisms in Set are precisely the bijections, but you should not need to use this in your proof.)

Date: October 5, 2020.