MATH 151A HOMEWORK 4

Hatcher §1.2: 1

(1) Suppose that two objects in a category $X, Y \in \mathcal{C}$ are either both initial or both final. Show that there exists a unique isomorphism $X \to Y$.

In light of (1), one may say that initial and final objects are "unique up to unique isomorphism". (In particular, it is essentially unambiguous to refer to *the* initial object or *the* final object of a category.)

(2) Fix groups G and H.

  (a) Describe the category $\mathsf{Fun}(\mathfrak{B}G, \mathfrak{B}H)$ in group-theoretic terms.

  (b) Describe the category $\mathsf{Fun}(\mathfrak{B}G, \mathfrak{B}H)$ more explicitly under the assumption that H is abelian.

(3) Fix a group G.

  (a) Describe the category $\mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})$ in group-theoretic terms.

  (b) Given a functor $\mathfrak{B}G \xrightarrow{F} \mathsf{Set}$, describe its limit and its colimit.

(4) Given positive integers $m, n \geq 1$, compute the pushout of the span $\mathbb{Z}/m \leftarrow \mathbb{Z} \rightarrow \mathbb{Z}/n$ in Grp (in which both morphisms are the defining quotient homomorphisms).

Let $\mathbb{N}$ denote the poset of natural numbers (equipped with its usual ordering), considered as a category. Let $\mathsf{Fin} \subset \mathsf{Set}$ denote the full subcategory of finite sets.

(5) Give an example (with proof) of a functor $\mathbb{N} \to \mathsf{Fin}$ that does not admit a colimit.

(6) Find necessary and sufficient conditions on a functor $\mathbb{N} \xrightarrow{\varphi} \mathbb{N}$ for the following condition to hold.

  ($*$) Choose any category $\mathcal{C}$ and any functor $\mathbb{N} \xrightarrow{F} \mathcal{C}$. If either $\mathsf{colim}(F)$ or $\mathsf{colim}(F\varphi)$ exists, then both exist and moreover there exists an isomorphism

$$\mathsf{colim}(F) \cong \mathsf{colim}(F\varphi) .$$

A simpler version of (6) is obtained by replacing condition (*) with the following.

($**$) For any functor $\mathbb{N} \xrightarrow{F} \mathsf{Set}$, there exists an isomorphism $\mathsf{colim}(F) \cong \mathsf{colim}(F\varphi)$.

For partial credit, you may solve (6) after replacing ($*$) by ($**$). Alternatively, you may take this simpler version as a hint: it turns out that conditions ($*$) and ($**$) are equivalent.

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