MATH 151A HOMEWORK 8

Hatcher §1.3: 12, 18, 19

Through the following sequence of problems, we will generalize our classification of covering spaces, removing all basepoint and connectedness assumptions.[1]

Fix a topological space $X \in \mathsf{Top}$. We write $\mathsf{Cov}(X)$ for the category of covering spaces of X.

Fix a covering space $(\tilde{X} \xrightarrow{p} X) \in \mathsf{Cov}(X)$. For any point $x \in X$, the ***fiber*** of p over x is the preimage $F(p)(x) := p^{-1}(x) \subseteq \tilde{X}$.

(1) Prove that the fiber $F(p)(x) \subseteq \tilde{X}$ is always a discrete topological space (when equipped with the subspace topology).

As a result of (1), it is no loss to consider the fiber as a set (instead of as a topological space).

Any path $f \in PX(x, y)$ determines a function

$$F(p)(x) \xrightarrow{F(p)(f)} F(p)(y) ,$$

obtained by path-lifting: for each $\tilde{x} \in F(p)(x)$, we take the unique lift $\tilde{f}$ of f beginning at $\tilde{x}$ and then define $F(p)(f)(\tilde{x}) := \tilde{f}(1) \in F(p)(y)$. Moreover, if two paths $f, g \in PX(x, y)$ are homotopic rel endpoints, then the homotopy lifting property guarantees that $F(p)(f) = F(p)(g)$. In other words, an equivalence class of path $[f] \in \Pi X(x, y)$ determines a well-defined function

$$F(p)(x) \xrightarrow{F(p)([f])} F(p)(y) .$$

From here, it is easy to see that this construction assembles into a functor

$$\Pi X \xrightarrow{F(p)} \mathsf{Set} ,$$

which carries each object $x \in \Pi X$ to the fiber $F(p)(x)$. Thereafter, it is easy to see that this construction assembles into a functor

$$\mathsf{Cov}(X) \xrightarrow{F} \mathsf{Fun}(\Pi X, \mathsf{Set}) .$$

Our main goal will be to prove the following.

Theorem A. *Assume that X is locally path-connected and semilocally simply-connected.*[2] *Then, the functor*

$$\mathsf{Cov}(X) \xrightarrow{F} \mathsf{Fun}(\Pi X, \mathsf{Set})$$

is an equivalence of categories.

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[1] This will be closely related to the discussion on pp. 68-70 of Hatcher, as well as that of §3 of May.

[2] Note that these are *point-set* conditions on X, in contrast with the *algebraic* conditions of being connected and equipped with a basepoint. In particular, this theorem applies to any locally contractible space (e.g. any cell complex).

Going forward, let us assume that X is pointed and connected (in addition to being locally path-connected and semilocally simply-connected), denoting the basepoint by $x \in X$. For simplicity, we write $G := \pi_1(X, x)$.[3] Then, the bulk of the proof of Theorem A will consist in establishing a commutative diagram

$$
\begin{array}{ccccc}
\mathsf{Cov}(X)^{\mathsf{conn},\mathsf{ptd},e} & \xrightarrow[\sim]{F_4} & \mathsf{Sub}(G)^e & \xrightarrow[\sim]{G/(-)} & \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans},\mathsf{ptd},e} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathsf{Cov}(X)^{\mathsf{conn},\mathsf{ptd}} & \xrightarrow[\sim]{F_3} & \mathsf{Sub}(G) & \xrightarrow[\sim]{G/(-)} & \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans},\mathsf{ptd}} \\
\wr\big\downarrow & & & & \big\downarrow\wr \\
\mathsf{Cov}(X)^{\mathsf{conn}} & & \xrightarrow[\sim]{F_2} & & \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans}} \\
\big\downarrow & & & & \big\downarrow \\
\mathsf{Cov}(X) & & \xrightarrow[\sim]{F_1} & & \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set}) \\
 & \searrow^{\sim}_{F} & & & \wr\big\uparrow i^* \\
 & & & & \mathsf{Fun}(\Pi X, \mathsf{Set})
\end{array}
\quad , \tag{1}
$$

which is partially explained as follows (the rest will be explained over the course of the proof).

- Recall that $\mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})$ may be identified with the category of (left) G-sets: its objects are sets equipped with G-actions, and its morphisms are G-equivariant functions.
- The superscript conn denotes the restriction to connected covering spaces, while the superscript trans denotes the restriction to transitive G-sets. Here, we take the parallel conventions that the empty topological space is *not* connected and that the unique G-action on the empty set is *not* transitive.
- The superscript ptd denotes the requirement that objects come equipped with distinguished basepoints, and the further superscript e denotes the restriction to morphisms that respect basepoints. (Note that the basepoint of a G-set is *not* required to be fixed by the G-action.)
- The category $\mathsf{Sub}(G)$ is defined as follows.
  - Its objects are the subgroups of G.
  - Given subgroups $H, K \subseteq G$, we define the hom-set
  $$\mathsf{hom}_{\mathsf{Sub}(G)}(H, K) := \{g \in G : g^{-1}Hg \subseteq K\}/\sim$$
  to be the set of elements of G whose inverses subconjugate H into K, modulo the equivalence relation that $g \sim h$ iff $gh^{-1} \in K$.

[3] The problems involving only the group G (and not the space X) will be entirely group-theoretic; indeed, recall that any group arises as the fundamental group of a pointed connected space (even of a pointed connected cell complex).

– Given a pair of composable morphisms

$$H \xrightarrow{[g_1]} K \xrightarrow{[g_2]} L \ ,$$

their composite is defined to be the morphism

$$H \xrightarrow{[g_1 g_2]} L \ .$$ [4]

- We define the subcategory $\mathsf{Sub}(G)^e \subseteq \mathsf{Sub}(G)$ by restricting to those morphisms of the form $[e]$; in other words, $\mathsf{Sub}(G)^e$ is the poset whose objects are subgroups of G and whose morphisms are inclusions.
- Given a subgroup $H \subseteq G$ and an element $\gamma \in G$, recall that we may form the (left) coset $\gamma H := \{\gamma h \in G : h \in H\}$. We write $G/H := \{\gamma H : \gamma \in G\}$ for the set of cosets of H.[5] This comes equipped with a canonical basepoint $eH \in G/H$.
- The functor F_4 is defined by

$$\begin{array}{ccc} \mathsf{Cov}(X)^{\mathsf{ptd,conn},e} & \xrightarrow{F_4} & \mathsf{Sub}(G)^e \\ \uplus & & \uplus \\ ((\tilde{X},\tilde{x}) \xrightarrow{p} (X,x)) & \longmapsto & \mathsf{im}(\pi_1(\tilde{X},\tilde{x})) \end{array} ;$$

this is an equivalence by Theorem 1.38 in Hatcher.

- We write $\mathfrak{B}G := \mathfrak{B}\pi_1(X,x) \overset{i}{\hookrightarrow} \Pi X$ for the inclusion of the full subcategory on the object $x \in \Pi X$, and we write i^* for the functor given by precomposition with i.
- All unlabeled functors are the evident forgetful/inclusion functors.

(2) Prove that the formula $g \cdot (\gamma H) := (g\gamma)H$ gives a well-defined left G-action on G/H, that an inclusion $H \subseteq K$ of subgroups of G gives a basepoint-preserving G-equivariant function $G/H \to G/K$, and that this construction defines an equivalence of categories

$$\mathsf{Sub}(G)^e \xrightarrow{G/(-)} \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{ptd,trans},e} \ .$$

(3) Fix a subgroup $H \subseteq G$ and an element $g \in G$, and write $H' := gHg^{-1} \subseteq G$ for the conjugate subgroup. Show that the formula

$$\begin{array}{ccc} G/H & \longrightarrow & G/H' \\ \uplus & & \uplus \\ \gamma H & \longmapsto & (\gamma g^{-1})H' \end{array}$$

gives a well-defined G-equivariant bijection (i.e. an isomorphism of G-sets). Use this to construct an equivalence of categories

$$\mathsf{Sub}(G) \xrightarrow{G/(-)} \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{ptd,trans}}$$

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[4]To see that this composition rule is well-defined, note that if $g_1 \sim h_1$ and $g_2 \sim h_2$, then $(g_1g_2)(h_1h_2)^{-1} = g_1(g_2h_2^{-1})h_1^{-1} \in L$. Thereafter, it is associative because multiplication in G is.

[5]Note that we may have $\gamma_1 H = \gamma_2 H$ even if $\gamma_1 \neq \gamma_2$. Indeed, $\gamma_1 H = \gamma_2 H$ iff $\gamma_1^{-1}\gamma_2 \in H$.

such that the upper right square in diagram (1) commutes.[6]

(4) Construct an equivalence of categories

$$\mathsf{Cov}(X)^{\mathsf{ptd,conn}} \xrightarrow{F_3} \mathsf{Sub}(G) \ ,$$

such that the upper left square in diagram (1) commutes.

(5) Prove that the forgetful functors

$$\mathsf{Cov}(X)^{\mathsf{conn,ptd}} \longrightarrow \mathsf{Cov}(X)^{\mathsf{conn}} \qquad \text{and} \qquad \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans,ptd}} \longrightarrow \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans}}$$

are equivalences of categories.

(6) Describe the composite functor

$$\begin{array}{ccc} \mathsf{Cov}(X) & \overset{F_1}{\dashrightarrow} & \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set}) \\ & \searrow^{F} & \uparrow i^* \\ & & \mathsf{Fun}(\Pi X, \mathsf{Set}) \end{array}$$

in concrete terms, and verify that it satisfies the following conditions.

(i) The functor F_1 carries disjoint unions of covering spaces to disjoint unions of G-sets.

(ii) There exists a (necessarily unique) factorization

$$\begin{array}{ccc} \mathsf{Cov}(X)^{\mathsf{conn}} & \overset{F_2}{\dashrightarrow} & \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans}} \\ \downarrow & & \downarrow \\ \mathsf{Cov}(X) & \xrightarrow[F_1]{} & \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set}) \end{array} ,$$

i.e. F_1 carries connected covering spaces to transitive G-sets.

(iii) Using this factorization F_2, the upper wide rectangle in diagram (1) commutes.[7]

As equivalences of categories satisfy the two-out-of-three property, it follows that the functor F_2 is an equivalence.

A common feature of the full subcategories

$$\mathsf{Cov}(X)^{\mathsf{conn}} \subseteq \mathsf{Cov}(X) \qquad \text{and} \qquad \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans}} \subseteq \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})$$

[6]Here and below, "construct an equivalence of categories" should be taken to mean: first carefully define a functor (including checking that it respects composition), and then prove that it is an equivalence of categories. Also, here and below, you should explain why the asserted diagram of categories commutes.

[7]Actually, this rectangle will only commute "up to natural isomorphism". A natural isomorphism is simply a natural transformation whose components are isomorphisms. You may simply indicate the components of this natural isomorphism (i.e. for each object of $\mathsf{Cov}(X)^{\mathsf{conn,ptd}}$ an isomorphism in $\mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans}}$ between its two images), without verify their naturality.

is that they both *freely generate under coproducts* (note that coproducts are given by disjoint union in both $\mathsf{Cov}(X)$ and $\mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})$), so that the functor F_1 is an equivalence by conditions (i) and (ii) of (6).[8]

(7) Given any categories $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathsf{Cat}$ and an equivalence of categories $\mathcal{C} \xrightarrow{j} \mathcal{D}$, prove that the functor
$$\mathsf{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{j^*} \mathsf{Fun}(\mathcal{C}, \mathcal{E})$$
is an equivalence of categories.

The functor $\mathfrak{B}G := \mathfrak{B}\pi_1(X, x) \xrightarrow{i} \Pi X$ is an equivalence of categories: it is fully faithful by definition, and it is essentially surjective because X is path-connected (because it is connected and locally path-connected). Hence, the functor i^* is an equivalence, and so the functor F is an equivalence because equivalences of categories satisfy the two-out-of-three property.

To prove Theorem A in full generality, it suffices to observe that both constructions $\mathsf{Cov}(-)$ and $\mathsf{Fun}(\Pi(-), \mathsf{Set})$ carry disjoint unions of topological spaces to products of categories. More precisely, suppose now that X is an arbitrary locally path-connected and semilocally simply-connected space (i.e. it is no longer assumed to be pointed and connected), and let us write $X \cong \coprod_{\alpha \in A} X_\alpha$ for its unique decomposition as a disjoint union of its connected components (which are also its path components, since it is locally path-connected). Then, we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Cov}(X) & \xrightarrow{F_X} & \mathsf{Fun}(\Pi X, \mathsf{Set}) \\ \wr\downarrow & & \downarrow\wr \\ \prod_{\alpha \in A} \mathsf{Cov}(X_\alpha) & \xrightarrow[\prod_{\alpha \in A} F_{X_\alpha}]{\sim} & \prod_{\alpha \in A} \mathsf{Fun}(\Pi X_\alpha, \mathsf{Set}) \end{array} ,$$

in which

- we subscript the various instances of F for disambiguation,
- the left vertical functor is given in each factor by pullback of covering spaces,
- the right vertical functor is given in each factor by precomposition of functors, and

[8] This may be explained in slightly more detail as follows. First of all, for any category $\mathcal{C}$ we may form its ***free coproduct-completion***, denoted $\mathcal{C}^{\amalg}$. (This may be constructed as follows: an object is given by a pair $(S \in \mathsf{Set}, S \xrightarrow{X_\bullet} \mathcal{C})$ (which may be thought of as the formal (i.e. freely adjoined) coproduct $\coprod_{s \in S} X_s$), and a morphism $(S, X_\bullet) \to (T, Y_\bullet)$ is given by a function $S \xrightarrow{f} T$ along with a set $\{X_s \xrightarrow{\varphi_s} T_{f(s)}\}_{s \in S}$ of morphisms in $\mathcal{C}$.) This construction defines a functor
$$\mathsf{Cat} \xrightarrow{(-)^{\amalg}} \mathsf{Cat}^{\mathsf{coprod}}$$
to the category whose objects are categories admitting all coproducts and whose morphisms are functors that preserve coproducts. The above facts imply that
$$\left(\mathsf{Cov}(X)^{\mathsf{conn}} \xrightarrow{F_2} \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})^{\mathsf{trans}}\right)^{\amalg} = \left(\mathsf{Cov}(X) \xrightarrow{F_1} \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})\right) .$$
So if the morphism F_2 in Cat is an equivalence, then the morphism $F_1 = (F_2)^{\amalg}$ in $\mathsf{Cat}^{\mathsf{coprod}}$ is also an equivalence.

- the lower functor is an equivalence because it is the product of the functors
$$\mathsf{Cov}(X_\alpha) \xrightarrow{F_{X_\alpha}} \mathsf{Fun}(\Pi X_\alpha, \mathsf{Set}) \ ,$$
each of which is an equivalence because X_α is connected, locally path-connected, and semilocally simply-connected.

So, the functor F_X is an equivalence because equivalences of categories satisfy the two-out-of-three property.

(8) Give an example of a topological space X for which the functor $\mathsf{Cov}(X) \xrightarrow{F} \mathsf{Fun}(\Pi X, \mathsf{Set})$ is not an equivalence.

(9) For any subgroup $H \subseteq G$, describe the group of automorphisms of the transitive G-set $G/H \in \mathsf{Fun}(\mathfrak{B}G, \mathsf{Set})$ in algebraic terms, and prove that every endomorphism is an automorphism (i.e. that all G-equivariant functions $G/H \to G/H$ are isomorphisms).