MATH 151A HOMEWORK 3

Hatcher §2.1: 17b

Problem (1) will lead you through a simplified proof of Proposition 2.21 in Hatcher. For completeness, we restate the assertion here, and give the relevant definitions.

Fix a topological space X and a set $\mathfrak{U} = \{U_j \subseteq X\}_{j \in J}$ of subsets of X whose interiors $\mathring{U}_j$ form an open cover of X. Let $C^{\mathfrak{U}}_\bullet(X) \subseteq C_\bullet(X)$ be the subcomplex given by the (levelwise) image of the homomorphism

$$\bigoplus_{j \in J} C_\bullet(U_j) \longrightarrow C_\bullet(X)$$

induced by the inclusions.[1] The assertion, then, is that the inclusion $C^{\mathfrak{U}}_\bullet(X) \hookrightarrow C_\bullet(X)$ is a quasi-isomorphism (i.e. that it induces an isomorphism on all homology groups).[2]

The proof will use (***barycentric***) ***subdivision***, a homomorphism

$$C_n(X) \xrightarrow{\mathsf{sd}} C_n(X) \ .$$

To define this, let us write $v_i \in \Delta^n$ for the i^{th} vertex (for $0 \leq i \leq n$). Let S_{n+1} denote the set of permutations of $\{0, 1, \ldots, n\}$. For each $\pi \in S_{n+1}$, we define the singular n-simplex $\Delta^n \xrightarrow{\tau_\pi} \Delta^n$ to be the unique linear function such that

$$\tau_\pi(v_i) = \frac{v_{\pi(0)} + \cdots + v_{\pi(i)}}{i+1}$$

for all $0 \leq i \leq n$ (using addition in $\mathbb{R}^{n+1}$). Using this, we define the subdivision of a generator $(\Delta^n \xrightarrow{\sigma} X) \in C_n(X)$ by the formula

$$\mathsf{sd}(\sigma) := \sum_{\pi \in S_{n+1}} \mathsf{sgn}(\pi) \cdot (\sigma \circ \tau_\pi) \ ,$$

where $S_{n+1} \xrightarrow{\mathsf{sgn}} \{\pm 1\}$ denotes the sign homomorphism.

(1) Carry out the following steps for $n = 0, 1, 2$.[3,4]

   (a) Prove that $\partial_n(\mathsf{sd}(\sigma)) = \sum_{i=0}^n (-1)^i \cdot \mathsf{sd}(d_i^n(\sigma))$.[5]

   (b) For $m \leq n$, inductively define a homomorphism $C_m(X) \xrightarrow{P_m} C_{m+1}(X)$ such that $\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = \mathsf{sd} - \mathsf{id}_{C_m(X)}$.[6]

Date: October 16, 2021.

[1]Explicitly, $C^{\mathfrak{U}}_n(X) \subseteq C_n(X)$ consists of singular n-chains in X whose constituent singular n-simplices in X each land in one of the subsets U_j.

[2]Hatcher proves a stronger claim, namely that this inclusion is a homotopy equivalence; see Problem (2).

[3]You may wish to consult Hatcher's proof for inspiration.

[4]You may do the general case for extra credit. However, you must do all parts of the general case in order to receive any extra credit (since some of them are no harder in the general case).

[5]When done for all $n \in \mathbb{Z}$, this shows that subdivision defines a chain map $C_\bullet(X) \xrightarrow{\mathsf{sd}} C_\bullet(X)$.

[6]When done compatibly for all $n \in \mathbb{Z}$, this defines a chain homotopy $\mathsf{id}_{C_\bullet(X)} \Rightarrow \mathsf{sd}$.

(c) Prove that for every n-chain $c \in C_n(X)$ there exists some $K \geq 0$ such that $\mathsf{sd}^{\circ k}(c) \in C_n^{\mathfrak{U}}(X)$ for all $k \geq K$.[7]

(d) Prove that for every n-cycle $z \in Z_n(C_\bullet^{\mathfrak{U}}(X))$, the difference $z - \mathsf{sd}(z)$ lies in $B_n(C_\bullet^{\mathfrak{U}}(X))$, and hence so does the difference $z - \mathsf{sd}^{\circ k}(z)$ for any $k \geq 0$.[8]

(e) Apply part (d) in a special case to prove that for every n-cycle $z \in Z_n(C_\bullet(X))$ we have an equality $[z] = [\mathsf{sd}(z)]$ in $H_n(X) := H_n(C_\bullet(X))$, and hence also an equality $[z] = [\mathsf{sd}^{\circ k}(z)]$ for any $k \geq 0$.

(f) Use the above facts to prove that the inclusion $C_\bullet^{\mathfrak{U}}(X) \hookrightarrow C_\bullet(X)$ induces an isomorphism $H_n(C_\bullet^{\mathfrak{U}}(X)) \xrightarrow{\cong} H_n(C_\bullet(X)) =: H_n(X)$.

(2) A chain map $C_\bullet \xrightarrow{f_\bullet} D_\bullet$ is called a ***chain homotopy equivalence*** if there exists a chain map $C_\bullet \xleftarrow{g_\bullet} D_\bullet$ as well as chain homotopies $\mathsf{id}_{C_\bullet} \Rightarrow g_\bullet \circ f_\bullet$ and $f_\bullet \circ g_\bullet \Rightarrow \mathsf{id}_{D_\bullet}$.

(a) Show that every chain homotopy equivalence is a quasi-isomorphism.

(b) Find a quasi-isomorphism that is not a homotopy equivalence.[9]

[7]Hint: First do this in the case that $c = (\Delta^n \xrightarrow{\sigma} X)$ is a singular n-simplex.

[8]This should use the *naturality* of your construction of the homomorphisms $\{P_m\}$, which implies their independence under considering chains as lying in some U_j or in the larger space X. (It is essentially impossible to have defined them unnaturally, because essentially nothing is known about the space X.)

[9]Hint: This can be done entirely with two-term chain complexes.