SYLLABUS
ALGEBRAIC AND DIFFERENTIAL TOPOLOGY (MATH 151A)
FALL 2020

Contents

1. General information

1.1. Instructor.

name: Aaron Mazel-Gee

email: mazelgee@caltech.edu

office hour: Mondays at 1-2pm in 285 Linde[1]

1.2. Teaching assistant (TA).

name: Chi Zhang

email: czhang5@caltech.edu

office hour: Tuesdays at 5-6pm in 306 Linde

1.3. Course logistics.

lectures: MWF at 10-11am in 387 Linde

textbooks: We will primarily follow *Algebraic Topology* by Hatcher. Other recommended references include *Homotopical Topology* by Fomenko & Fuchs,[2] *Topology and Geometry* by Bredon, and *A Concise Course in Algebraic Topology* by May. These are all available either online or through the library's course reserves.

course website: etale.site/teaching/f21/

Canvas: yes, lightly (see §4.2)

last updated: October 10, 2021

[1]If my office hour conflicts with your schedule, I am also available to meet by appointment.

[2]The pictures in this book (which can also be found online) of various concepts in algebraic topology are required viewing.

1.4. **Homework and grades.** Homework will generally be assigned weekly. It will be due by the beginning of lecture (i.e. at 10am) on the due date, which will generally be Wednesdays. Your final grade in the course will be determined by your performance on the homework. Depending on how far into the material we get, there may also be a more open-ended final project (in which case the homework load would be correspondingly diminished).

Please read the brief document Guidelines for Good Mathematical Writing, and adhere to its suggestions to the greatest extent possible. You will lose points for gross negligence.[3]

You are encouraged to write your homework in LaTeX (pronounced "LAH-teck"). This is a versatile and easy-to-use typesetting system, which is the industry standard in mathematics and adjacent fields. Some resources to help you get started writing in LaTeX are outlined in §4.5. In case you would like to include hand-drawn pictures in your homework submission, please embed these into the LaTeX file.

Homework will be submitted via Canvas. If you write out your homework by hand, please submit photos of it assembled as a single pdf (e.g. using an online photo-to-pdf converter).

In order to account for unexpected circumstances, you are automatically granted a cumulative deadline extension of 72 hours.

1.5. **COVID-19 considerations.** Given the ongoing pandemic, it is essential that we all follow the guidance of Caltech and public health entities, e.g. wearing masks, maintaining an adequate distance, and quarantining as appropriate. In particular, if you believe you may be contagious (e.g. due to symptoms or exposure to a known case), please let me know at your earliest convenience and do *not* attend lecture. It should be understood that the situation is subject to change at any time, and that we will all do our best to remain supportive and understanding as we navigate this unfamiliar and rapidly shifting terrain.

2. Course description

Broadly speaking, the fundamental aim of topology is to understand manifolds. What we generally mean by this is that we attach *invariants* to manifolds. The purpose of doing so is twofold:

(1) to distinguish between manifolds based on their values; and
(2) to encode topological information.

As a basic example of (1), the Euler characteristic can distinguish between the torus and the sphere: we have the inequality

$$\chi(T^2) = 0 \neq 2 = \chi(S^2) \ ,$$

and so it must be the case that $T^2 \not\cong S^2$. However, from the point of (2), the Euler characteristic is rather deficient: it does not admit a uniform geometric description, and it

[3] In particular, you must write in complete sentences, and you should generally avoid shorthand symbols such as "$\because$" or "$\Rightarrow$" (although "iff" is acceptable).

tells us little about the manifold itself – for instance, it cannot tell us its number of path components or its dimension.

This situation may be remedied by passing from the Euler characteristic to *homology*. Given a manifold M, its homology is a sequence of vector spaces $\{H_n(M)\}_{n\geq 0}$ with the following two features.

(1) For each $n \geq 0$, the dimension of the vector space $H_n(M)$ reflects the number of "n-dimensional holes" in M.
(2) We can recover the Euler characteristic of M as the alternating sum

$$\chi(M) = \sum_{n\geq 0} (-1)^n \dim(H_n(M)) \ .$$ [4]

As a result of (2), homology is a strictly stronger invariant than the Euler characteristic. Of course, it is also necessarily at least as difficult to compute. However, as it turns out, homology is not much harder to compute than the Euler characteristic. It also enjoys an additional feature known as *functoriality*, which is helpful both for computation and because it makes homology into a more sensitive invariant. Hence, homology provides a striking improvement over the Euler characteristic as an invariant of manifolds.

Taking linear duals leads to a closely related invariant called *cohomology*, which is likewise a sequence of vector spaces $\{H^n(M)\}_{n\geq 0}$. Homology and cohomology can be defined for any topological space, but for manifolds these bear a remarkable relationship called *Poincaré duality*, which in its simplest form gives isomorphisms

$$H^i(M) \cong H_{d-i}(M)$$

(when M is a closed oriented manifold of dimension d). Poincaré duality reflects a deep symmetry that is present in the topology of manifolds; for example, taking $M = (S^1)^{\times d}$ to be the d-dimensional torus, comparing dimensions yields the identity $\binom{d}{i} = \binom{d}{d-i}$ between binomial coefficients.

In this course, we will primarily study homology and cohomology, with a main goal of proving Poincaré duality. If time permits (or as final projects), we may also study various applications or related topics. Notably, an important and recurring theme in topology is the interplay between differential and algebraic techniques, and the cohomology of manifolds can be described in a number of different ways (including de Rham theory, Morse theory, and Hodge theory).

The material covered in the lectures is intended to be complementary to the material contained in the textbook. In order to succeed in this course, you should expect both to attend (or asynchronously view) all lectures and to regularly read the book.

[4] This sum will always be finite, because $H_n(M) = 0$ whenever $n > \dim(M)$.

3. Resources

Collaboration with your fellow students can be a very effective way of learning, and is highly encouraged. Often, we may think we understand something very well, but in trying to explain it to somebody else we find the gaps in our understanding.[5] However, *everyone must write up their own solutions separately*: you may not copy each other's work. Moreover, you must indicate any help that you received on your homework (e.g. from office hours, websites, collaboration with other students, etc.), either at the beginning of the entire document or as a footnote or marginalium to specific problems. For instance, at the top of your homework you might write "I worked on my homework with Alice and Bob.", or as a footnote to Problem N you might write "I found a hint/solution to this at `http://mathhelp4u.blogspot.com`.".

As a guideline for this collaboration policy, keep in mind that **you should be able to reproduce any solution you hand in without assistance**.

Here is a list of allowed and disallowed resources. This is intended to be reasonably comprehensive, but the fact that something is not on this list does *not* mean that it is permitted. If you are ever in doubt about whether something is allowed, it is **your responsibility** to ask.

you may consult:	
course textbooks	yes
other books	yes
solution manuals	no
internet	yes
notes from lecture (yours or others')	yes
course handouts	yes
your past homework	yes
homework and exams solutions posted on webpage	yes
homework and exams from previous years	no
communications (e.g. emails) from instructor or TA	yes
you may:	
discuss problems with others	yes
look at communal materials (e.g. (e-)whiteboard) when writing up solutions	yes
look at written work of others	no

[5] There was some research that came out about this recently. Subjects were asked to rate their understanding of some everyday mechanism (e.g. a ballpoint pen or the flushing mechanism of a toilet). Then, they were asked to explain it. Then, they were asked to rate their understanding again. As you might imagine, their own estimation of their understanding dropped significantly from the first round of rating to the second.

post about problems online (e.g. at math.stackexchange)	yes
for computational aid, you may use:	
calculator	yes[6]
computer	yes[6]

4. Miscellanea

4.1. **Attendance and participation.** While they do not directly affect your grade, I would very much appreciate your regular attendance and participation in lectures. In my own experience, mathematics is most exciting and most rewarding when it is pursued as an interpersonal endeavor: as an interesting and unique way to interact with other human beings.

4.2. **Canvas.** Much of the relevant information for this course will be posted on the course website. However, we may use Canvas for a few purposes, e.g. for any material that may be considered as sensitive or unpolished – lecture video recordings, lecture notes, Zoom links, chat, discussion boards, grades, etc.

4.3. **Students with documented disabilities.** If you need an academic accommodation based on the impact of a disability, you must initiate the request with Caltech Accessibility Services for Students (CASS). Professional staff will evaluate your request with required documentation, recommend reasonable accommodations, and prepare an Accommodation Letter for the quarter in which the request is being made. In this case, you should contact CASS as soon as possible, since timely notice is needed to coordinate accommodations. Undergraduate students should contact Dr. Lesley Nye, Associate Dean of Undergraduate Students (administrative contact: Beth Larranaga) and graduate students should contact Dr. Kate McAnulty, Associate Dean of Graduate Studies (administrative contact: Jacob Dalton).

4.4. **Academic integrity.** You must abide by Caltech's honor code: *No member of the Caltech community shall take unfair advantage of any other member of the Caltech community.* General guidelines for how this applies to plagiarism are given here. Specific guidelines for this course are described in §3.

[6] It is not expected that computational aids will be helpful in this course (except possibly for visualization purposes). Nevertheless, you may use a calculator or computer while doing homework. However, you may not refer to this as a justification of your work. For example, "by Mathematica" is not an acceptable justification by itself.

4.5. **LaTeX resources.** LaTeX code is stored as plain text, and therefore can be written and saved with the filename extension `txt` (e.g. as `my-LaTeX-code.txt`). This raw code can then be *compiled* into a `pdf` file (such as the one you are reading now). You can compile LaTeX code directly on your computer after a somewhat lengthy download/installation process. Alternatively, you can write and compile LaTeX code online through the website Overleaf. I would recommend the latter option, especially if you are having difficulty with the former option (which is somewhat nontrivial).

A simple template that you can use to get started on your homework is available here; when compiled, that LaTeX code produces this file. Of course, you are also welcome to start from scratch. Along with occasional online searches, that template should suffice to get you started.[7] However, if you would like a further introduction to LaTeX you can check out some of the instructions provided by Overleaf: this shorter one or this longer one.

Of course, there are many online LaTeX resources. In particular, there are many "crib sheets", see e.g. here and here. Another great resource is Detexify: if you draw a mathematical symbol, it will attempt to suggest how to write it in LaTeX.

4.6. **Revisions to this syllabus.** This syllabus is subject to minor changes throughout the semester, as needed. In the interest of transparency, all versions will remain available on the course website.

[7]The diagram is drawn using the `tikz-cd` package, more information about which can be found here.