LOCALLY CONSTANT FACTORIZATION ALGEBRAS

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Lecture notes from *An invitation to factorization algebras*, co-taught with Peter Teichner at UC Berkeley in Spring 2016.

0. Overview

- baby case of factorization algebras: the *locally constant* ones
  - can say much more on the mathematical side
  - suggests refined (but as yet unrealized) structure on the physical side
- we'll cover three main results:
  1. ***homology theories for manifolds:*** factorization homology (i.e. global (co)sections of factorization algebras) defines a *homology theory for n-manifolds*, and these are all of them (recall Eilenberg–Steenrod).
  2. ***nonabelian Poincaré duality:*** when valued in $(\mathcal{S}, \times)$, this obeys a nonabelian form of Poincaré duality.
  3. ***Poincaré/Koszul duality:*** the general form of Poincaré duality for factorization homology of n-manifolds incorporates *Koszul duality* of n-disk algebras. for simplicity, let's restrict to M a compact n-manifold. then for A an augmented n-disk algebra, we have a diagram

$$\begin{array}{ccc} \left(\int_M A\right)^\vee & \longrightarrow & \int_M \mathbb{D}^n A \\ & \underset{\sim}{\searrow} \quad \nearrow & \\ & \int_M \mathrm{MC}_A & \end{array}$$

     in which:
     - the n-disk algebra $\mathbb{D}^n A$ is the *Koszul dual* to A;
     - the expression MC_A denotes the *Maurer–Cartan object* of A, a certain (generally non-affine) algebro-geometric object;
     - the upwards map is provided by an equivalence $\mathcal{O}(\mathrm{MC}_A) \simeq \mathbb{D}^n A$, i.e. by the corresponding "affinization" map $\mathrm{Spf}(\mathbb{D}^n A) \to \mathrm{MC}_A$ (or perhaps more like an "inclusion of a formal neighborhood"); note that factorization homology is covariant in the algebra variable.

     for an n-disk stack X, one can think of $\int_M X = \Gamma\left(X, \int_M \mathcal{O}_X\right)$ as something like "functions on $\mathrm{map}(M, X)$".
- in re (1), this also extends to *stratified spaces*, which can be modeled on other local structures besides just $\mathbb{R}^n$. a key example is when the "category of basics" consists of $(\varnothing \subset \mathbb{R}^3)$ and $(\mathbb{R}^1 \subset \mathbb{R}^3)$: we then obtain homology theories for knots/links embedded in 3-manifolds (somewhat akin to Khovanov homology). there's an explicit description of the coefficients for these, which makes use of Deligne's conjecture (it's $A \in \mathrm{Alg}_3$, $B \in \mathrm{Alg}_1$, and a map $\mathrm{HH}_*(A) \to \mathrm{HH}^*(B)$ in Alg_2).
- in re (3), recall that we view ordinary factorization homology as the global observables in a perturbative QFT; pursuing this, non-affine factorization homology computes global observables in a *nonperturbative* QFT. this should be thought of as an avatar of "S-duality" (a philosophy in QFT), which interchanges perturbative and nonperturbative phenomena.
- in re (3), Koszul duality and Maurer–Cartan objects come from the subject of *deformation theory*. (for instance, this explains the appearance of dgla's when considering cdga's (they "parametrize deformations" for one another – this should be familiar if you know the usual story of Maurer–Cartan elements of $\mathfrak{g} \otimes A$)). PKD is most naturally understood within this context, and so we'll spend some time on this as well.
- all of this will require a full embrace of homotopy-coherence, so we'll begin with that.

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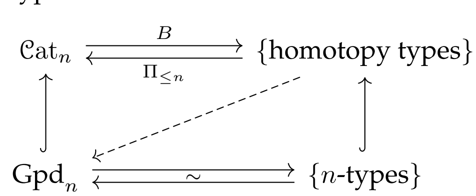
Contents

1. Background

1.1. Towards quasicategories.

- review of simplicial sets and their geometric realizations
  - Yoneda embedding as free cocompletion: formally allows for gluing simplices along specified maps (i.e. those in $\mathbf{\Delta}$)
  - geometric realization $|-| : s\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$
    - preserves products
      - illuminating example: $\Delta^1 \times \Delta^1$
- the Quillen equivalence $|-| : s\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}$
  - entire adjunction is uniquely determined by the cosimplicial object $\Delta^\bullet_{\mathrm{top}} \in c\mathcal{T}\mathrm{op}$
  - determines an equivalence of homotopy categories
  - Kan complexes $\approx$ singular complexes of topological spaces
    - these are the objects that "behave like topological spaces"
    - more precisely, $\hom_{s\mathcal{S}\mathrm{et}}(X,Y) \to \hom_{\mathrm{ho}(\mathcal{T}\mathrm{op})}(|X|,|Y|)$ is guaranteed to be surjective whenever Y is a Kan complex (and X is cofibrant, but this is automatic)
    - any sset has a *Kan complex replacement*
      - this is only unique "up to weak equivalence"...
      - but this choice can nevertheless be made to be functorial...
      - (though this functor is itself only unique "up to weak equivalence"...)
      - we'll denote any such by $\mathbb{R}_{\mathrm{KQ}} : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ (this comes equipped with a natural transformation $\mathrm{id}_{s\mathcal{S}\mathrm{et}} \to \mathbb{R}_{\mathrm{KQ}}$)
- the functor $\mathrm{N} : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$ and unique inner horn fillers
- Grothendieck's homotopy hypothesis:



- in particular, taking $n = \infty$ we should have an equivalence "$\Pi_{\leq\infty}$: {homotopy types} $\xrightarrow{\sim}$ Gpd_∞"
- quasicategories := LCM(Kan,N(cat))
- key examples:

– nerves of 1-categories
– Kan complexes
– the adjunction (in fact Quillen equivalence)

$$\mathfrak{C} : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows (\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}} : \mathrm{N}^{\mathrm{hc}}$$

determined by $\mathfrak{C}(\Delta^\bullet) \in c(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})$, which is defined by

$$\underline{\hom}_{\mathfrak{C}(\Delta^n)}(i,j) = \begin{cases} \varnothing, & j < i \\ \mathrm{N}(P_{i,j}), & i \leq j \end{cases}$$

where $P_{i,j}$ is the poset of paths in Δ^n from $\Delta^{\{i\}}$ to $\Delta^{\{j\}}$.
  ∗ examples of $\mathfrak{C}(\Delta^n)$ for $n = 0, 1, 2$
  ∗ in general, $\mathfrak{C}(\Delta^n) \xrightarrow{\approx} [n]$ is a cofibrant replacement (and in fact, so is $\mathfrak{C}(\Delta^\bullet) \to [\bullet]$)
  ∗ this restricts to the ordinary nerve on $\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$
  ∗ really, this should only be applied to *fibrant* objects, i.e. those whose hom-ssets are all Kan complexes (otherwise we shouldn't be mapping *into* them, but rather into Kan complex replacements)
– compose the above with the adjunction (also a Quillen equivalence)

$$|-| : \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \rightleftarrows \mathcal{C}\mathrm{at}_{\mathcal{T}\mathrm{op}} : \mathrm{Sing}$$

to get

$$|\mathfrak{C}(-)| : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows (\mathcal{C}\mathrm{at}_{\mathcal{T}\mathrm{op}})_{\mathrm{Bergner}} : \mathrm{N}^{\mathrm{hc}}(\mathrm{Sing}(-))$$

  ∗ example: suspension of a topological space as a homotopy-coherent diagram
  ∗ Sing takes all topological spaces to Kan complexes, so this always has the correct value
– for a relative category $(\mathcal{R}, \mathbf{W})$, the ∞-categorical pushout $\mathcal{R}[\![\mathbf{W}^{-1}]\!] := \mathrm{colim}(\mathcal{R} \leftarrow \mathbf{W} \rightarrow \mathbf{W}^{\mathrm{gpd}})$ (which will be more rigorously defined in a moment)
  ∗ in most cases where this overlaps with previous examples (namely: for compatibly $s\mathcal{S}\mathrm{et}/\mathcal{T}\mathrm{op}$-enriched model categories), the relevant notions of agree (up to weak equivalence of quasi-categories)

1.2. **Basic concepts in quasicategory theory.**

- from nerves of 1-cats, we know: vertices as "objects", edges as "1-morphisms"
- now, triangles as "*witnesses* to composition", i.e. paths in a "hom-space": given a qcat $\mathtt{C}$ and vertices $\mathtt{x}, \mathtt{y} \in \mathtt{C}_0$, we define

$$\mathtt{hom}_{\mathtt{C}}(\mathtt{x},\mathtt{y}) = \lim^{s\mathcal{S}\mathrm{et}} \left(\begin{array}{ccc} & & \underline{\hom}_{s\mathcal{S}\mathrm{et}}(\Delta^1, \mathtt{C}) \\ & & \downarrow \\ \{(\mathtt{x},\mathtt{y})\} & \longrightarrow & \underline{\hom}_{s\mathcal{S}\mathrm{et}}(\partial\Delta^1, \mathtt{C}) \end{array} \right),$$

which is a Kan complex (i.e. a "space")
  – this construction only admits composition "up to a contractible space of choices"
    ∗ more precisely, $\underline{\hom}_{s\mathcal{S}\mathrm{et}}(\Delta^2, \mathtt{C}) \to \underline{\hom}_{s\mathcal{S}\mathrm{et}}(\Lambda^2_1, \mathtt{C})$ is an acyclic fibration; in particular, every fiber is a contractible Kan complex
- degenerate edges are "identity 1-morphisms", and this leads to a definition of an edge being an "equivalence"
- equivalences of quasicategories: fully faithful and essentially surjective $\rightsquigarrow$ weak equivalences in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ (these are defined on *all* ssets)
- homotopy category
  – comes from the adjunction $\mathrm{ho} : s\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{C}\mathrm{at} : \mathrm{N}$
  – as expected, $\hom_{\mathrm{ho}(\mathtt{C})}(\mathtt{x},\mathtt{y}) \cong \pi_0(\mathtt{hom}_{\mathtt{C}}(\mathtt{x},\mathtt{y}))$ (note that π_0 preserves products)
- Kan complexes $\subset$ qcats are the "∞-groupoids"
  – there's a "maximal Kan complex" functor ι : qcats $\to$ Kan, right adjoint to the inclusion
  – this inclusion has no strict left adjoint, but we get a "groupoid completion" functor from a Quillen adjunction $\mathrm{id} : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} : \mathrm{id}$
- initial/terminal objects
- co/limits

- initial/terminal object in the quasicategory of co/cones extending the given diagram
- the homotopy-coherent suspension diagram in $\mathrm{N}^{\mathrm{hc}}(\mathrm{Sing}(\mathcal{T}\mathrm{op}))$ is actually an ∞-categorical colimit
- recall the funny business with "homotopy co/limits" previously (as left/right adjoint to $\mathrm{ho}(\mathcal{R}) \to \mathrm{ho}(\mathrm{Fun}(I, \mathcal{R}))$, for a relative category $\mathcal{R} = (\mathcal{R}, \mathbf{W})$, e.g. $(\mathrm{Ch}, \mathbf{W}_{\mathrm{qi}})$, $(s\mathcal{S}\mathrm{et}, \mathbf{W}_{\mathrm{KQ}})$, $(\mathcal{T}\mathrm{op}, \mathbf{W}_{\mathrm{whe}})$, $(\mathcal{T}\mathrm{op}, \mathbf{W}_{\mathrm{he}})$): the better definition is that they are just diagrams in a relative category that present ∞-categorical co/limits

- given a relative quasicategory $(\mathtt{C}, \mathbf{W})$ (e.g. the nerve of a relative 1-category), its *localization* is (a quasicategory replacement of) the pushout $\mathtt{C} \amalg_{\mathbf{W}} \mathbb{R}_{\mathrm{KQ}}(\mathbf{W})$ in $s\mathcal{S}\mathrm{et}$ (where $\mathbb{R}_{\mathrm{KQ}}$ is Kan complex replacement)
  - this is a homotopy colimit in $(s\mathcal{S}\mathrm{et}, \mathbf{W}_{\mathrm{Joyal}})$

1.3. From quasicategories to ∞-categories.

- henceforth, we'll work *model-independently*: by definition, an ***∞-category*** is an object of the quasicategory $\mathcal{C}\mathrm{at}_\infty := s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{\mathrm{Joyal}}^{-1}]\!] \simeq \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\![\mathbf{W}_{\mathrm{Bergner}}^{-1}]\!] \simeq \ldots$
  - slogan: red pill vs. blue pill, quasicats vs. ∞-cats
    - ∗ occasionally you need to pop your head out of the matrix to prove something [using qcats], but for the most part it's much more pleasant to just stay "plugged in" [to the model-independent world]
- not only are all these ∞-categories all equivalent, but the equivalences between them are all essentially unique (considered *inside of* any of these equivalent quasicategories – this is very false at the level of the model categories themselves)
- from now on, a ***space*** is an object of the ∞-category $\mathcal{S} := s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{\mathrm{KQ}}^{-1}]\!] \simeq \mathcal{T}\mathrm{op}[\![\mathbf{W}_{\mathrm{whe}}^{-1}]\!] \simeq \ldots$
  - if we mean "topological space", we'll say so

2. Homology theories for manifolds

2.1. Overview.

- a *homology theory for spaces* is a functor $\mathcal{F} : \mathcal{S}^{\mathrm{fin}} \to \mathrm{Ch} := \mathrm{Ch}[\![\mathbf{W}_{\mathrm{qi}}^{-1}]\!]$ satisfying:
  (1) ***additivity:*** for any finite set of finite spaces $\{X_j \in \mathcal{S}^{\mathrm{fin}}\}_{j \in J}$, the canonical map

$$\bigoplus_{j \in J} \mathcal{F}(X_j) \to \mathcal{F}\left(\coprod_{j \in J} X_j\right)$$

  is an equivalence; that is, $\mathcal{F} : (\mathcal{S}^{\mathrm{fin}})^{\amalg} \to \mathrm{Ch}^{\oplus}$ is symmetric monoidal.
  (2) ***excision:*** given any pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \coprod_X Z \end{array}$$

  in $\mathcal{S}^{\mathrm{fin}}$, the induced map in Ch in the diagram

$$\begin{array}{ccccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(Y) & & \\ \downarrow & & \downarrow & \searrow & \\ \mathcal{F}(Z) & \longrightarrow & \mathcal{F}(Y) \coprod_{\mathcal{F}(X)} \mathcal{F}(Z) & & \\ & \searrow & & \dashrightarrow & \\ & & & & \mathcal{F}(Y \coprod_X Z) \end{array}$$

  is an equivalence.
  - the usual definition of homology as a sequence of groups is recovered by taking homology of chain complexes; note that a pushout $D = B \amalg_A C$ in Ch give long exact sequence

$$\cdots \to H_{n+1}(D) \to H_n(A) \to H_n(B) \oplus H_n(C) \to H_n(D) \to H_{n-1}(A) \to \cdots$$

  in homology.

  - one can equivalently talk about functors $\mathcal{F} : \mathcal{S} \to \mathrm{Ch}$, but demand that they also commute with filtered colimits. (any space is canonically a filtered colimit of finite spaces.)
- these are characterized as follows. let us denote by $\mathcal{H}(\mathcal{S}^{\mathrm{fin}}, \mathrm{Ch}^{\oplus}) \subset \mathrm{Fun}^{\otimes}(\mathcal{S}^{\mathrm{fin}}, \mathrm{Ch})$ the full (∞-)subcategory on those symmetric monoidal functors satisfying excision. then, we recall the classical

  Theorem 0 (Eilenberg–Steenrod (reformulated))**.** *The functor*

  $$\mathrm{ev}_{\mathrm{pt}} : \mathcal{H}(\mathcal{S}^{\mathrm{fin}}, \mathrm{Ch}^{\oplus}) \to \mathrm{Ch}$$

  is an equivalence, with inverse given by taking $V \in \mathrm{Ch}$ to the functor $C_*(-;V)$ of singular chains with coefficients in V.

- now, if we want a "homology theory for manifolds", we could precompose the above with an "underlying space" functor (i.e. take singular homology). but this is invariant under (weak) homotopy equivalence and so is insensitive to a good deal of manifold theory (cf. e.g. lens spaces, which are homotopy equivalent but not homeomorphic – we'll be able to distinguish these).
- modifications:
  - replace $\mathcal{S}^{\mathrm{fin}}$ with a certain ∞-category $\mathcal{M}\mathrm{fld}_n$ of (suitably finitary) n-manifolds and embeddings.
  - replace $\mathrm{Ch}^{\oplus}$ by an arbitrary symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$.

  (the second is motivated by physical considerations, as we've seen before.)
- now, a ***homology theory for n-manifolds*** with values in $\mathcal{C} = \mathcal{C}^{\otimes}$ will be a functor $\mathcal{F} : \mathcal{M}\mathrm{fld}_n \to \mathcal{C}$ satisfying:
  (1) $\otimes$***-monoidality:*** for any finite set of n-manifolds $\{M_j \in \mathcal{M}\mathrm{fld}_n\}_{j \in J}$, we have an equivalence

  $$\bigotimes_{j \in J} \mathcal{F}(M_j) \xrightarrow{\sim} \mathcal{F}\left(\coprod_{j \in J} M_j\right).$$

  (2) $\otimes$***-excision:*** for any *collar gluing* among n-manifolds $M = M_0 \amalg_{N \times \mathbb{R}} M_1$, we have an equivalence

  $$\mathcal{F}(M_0) \bigotimes_{\mathcal{F}(N \times \mathbb{R})} \mathcal{F}(M_1) \xrightarrow{\sim} \mathcal{F}(M).$$

  (if axiom (1) is satisfied, the map in axiom (2) is determined.)
- our first main theorem will be that these are characterized as follows. let's write $\mathcal{H}(\mathcal{M}\mathrm{fld}_n, \mathcal{C}^{\otimes}) \subset \mathrm{Fun}^{\otimes}(\mathcal{M}\mathrm{fld}_n, \mathcal{C})$ for the full subcategory of homology theories for manifolds. then we have

  Theorem 1 (Ayala–Francis)**.** *Under mild hypotheses on $\mathcal{C}$, the functor*

  $$\mathrm{ev}_{\mathbb{R}^n} : \mathcal{H}(\mathcal{M}\mathrm{fld}_n, \mathcal{C}^{\otimes}) \to \mathrm{Alg}_n(\mathcal{C})$$

  *is an equivalence, with inverse given by **factorization homology**. (there are also versions for manifolds with other tangential structures.)*

- our current goal is to understand the players in this theorem – particularly the ∞-categories $\mathcal{M}\mathrm{fld}_n$ and $\mathrm{Alg}_n(\mathcal{C})$ – and to sketch a proof.

2.2. Connection with QFTs.

- if $\mathrm{Obs}^q : \mathrm{RiemMfld}_n \to \mathrm{Mod}_k$ is a QFT on Riemannian manifolds, then two things need to happen for it to fit into the "locally constant factorization algebras" framework.
  (1) ***factorization:*** this is generally not a cosheaf, but in good cases it's a *Weiss* cosheaf. writing $i : \mathrm{RiemDisks}_n^{\mathrm{fr}} \hookrightarrow \mathrm{RiemMfld}_n^{\mathrm{fr}}$, we have that Obs^q is in the image of $i_!^{\otimes} : \mathrm{Fun}(\mathrm{RiemDisks}_n^{\mathrm{fr}}, \mathrm{Mod}_k) \to \mathrm{Fun}(\mathrm{RiemMfld}_n^{\mathrm{fr}}, \mathrm{Mod}_k)$.
  (2) ***locally constant:*** for this we need for Obs^q to be a ***T***QFT, i.e. to be metric independent, i.e. to admit a factorization

  $$\begin{array}{ccc} \mathrm{RiemMfld}_n^{\mathrm{fr}} & \xrightarrow{\mathrm{Obs}^q} & \mathrm{Mod}_k \\ \downarrow & \nearrow & \\ \mathrm{Mfld}_n^{\mathrm{fr}} & & \end{array}$$

  (but note that this is generally *false* in physically meaningful situations).

assuming these, we obtain an $\mathcal{E}_n$-algebra $A = \mathrm{Obs}^q(\mathbb{R}^n)$ (by looking at the values on $\mathrm{Disks}^{\mathrm{fr}}_n \subset \mathrm{Mfld}^{\mathrm{fr}}_n$, which after all is just our ∞-category $\mathcal{D}\mathrm{isk}$ from before), and then we have $\mathrm{Obs}^q(M) = \int_M A$.

- in practice, this will be $\mathrm{Obs}^q(M) = \mathcal{O}(\mathrm{EL}^q(M))^\vee$, where the latter comes as follows:
  - recall the setup: there's an action functional $S : \mathcal{F}(M) \to \mathbb{R}$ for some sheaf $\mathcal{F} : \mathrm{RiemMfld} \to \mathrm{Mod}_k$, e.g. $\mathcal{F}(-) = \mathrm{map}^{C^\infty}(-, X)$.
  - from this,
    - ∗ the ***Euler–Lagrange equations*** are imposed by taking the intersection in $T^*\mathcal{F}(M)$ of its differential with the zero section:

$$\mathrm{EL}(M) = \lim \left(\begin{array}{ccc} & & \mathcal{F}(M) \\ & & \downarrow dS \\ \mathcal{F}(M) & \xrightarrow[0]{} & T^*\mathcal{F}(M) \end{array} \right);$$

    - ∗ the ***quantum Euler–Lagrange equations*** are imposed by replacing this with a *derived* intersection:

$$\mathrm{EL}^q(M) = \mathrm{holim} \left(\begin{array}{ccc} & & \mathcal{F}(M) \\ & & \downarrow dS \\ \mathcal{F}(M) & \xrightarrow[0]{} & T^*\mathcal{F}(M) \end{array} \right).$$

2.3. **Definitions.**

- write $\mathcal{M}\mathrm{fld}_n$ for the ∞-category underlying the $\mathcal{T}\mathrm{op}$-enriched category
  - whose objects are topological n-manifolds admitting a finite good cover, and
  - whose topological spaces of morphisms are given by embeddings (taken with the compact-open topology).

  equipped with the disjoint union $\sqcup$ (which is not a coproduct!), this is symmetric monoidal.
- write $\mathcal{E}\mathrm{uc}_n \subset \mathcal{M}\mathrm{fld}_n$ for the full subcategory on the object $\mathbb{R}^n$. **key fact** [Kister–Mazur]: the map $\mathrm{Aut}_{\mathcal{M}\mathrm{fld}_n}(\mathbb{R}^n) \to \mathrm{End}_{\mathcal{M}\mathrm{fld}_n}(\mathbb{R}^n)$ is a homotopy equivalence. in other words, $\mathcal{E}\mathrm{uc}_n \simeq B\mathrm{Top}(n)$, where $\mathrm{Top}(n) = \mathrm{Homeo}(\mathbb{R}^n)$ is the topological group of self-homeomorphisms (thought of as an ∞-group) and $B\mathrm{Top}(n)$ is the corresponding ∞-category (with a distinguished object).
- this allows us to define the ***tangent (microbundle) classifier*** as the composite

$$\tau : \mathcal{M}\mathrm{fld}_n \xrightarrow{\text{Yo}} \mathrm{Fun}((\mathcal{M}\mathrm{fld}_n)^{op}, \mathcal{S}) \xrightarrow{U} \mathrm{Fun}((\mathcal{E}\mathrm{uc}_n)^{op}, \mathcal{S}) \xrightarrow[\sim]{\text{K–M}} \mathrm{Fun}(B\mathrm{Top}(n), \mathcal{S}) \xrightarrow[\sim]{\mathrm{Gr}} \mathcal{S}_{/B\mathrm{Top}(n)}$$

  - the Grothendieck construction.
    - ∗ baby case: for a space X the isomorphism $\mathrm{Gr} : [X, \mathcal{S}\mathrm{et}^{\simeq}]_{\mathcal{S}} \xrightarrow{\sim} \mathcal{C}\mathrm{ov}(X)^{\simeq}$ to the set of covering spaces of X (up to iso)
      - · given $F : X \to \mathcal{S}\mathrm{et}^{\simeq}$, the fiber of $\mathrm{Gr}(F) \downarrow X$ over $x \in X$ is the set $F(x)$
      - · this is a sort of "gluing" construction
    - ∗ still true when allow natural transformations to get $\mathrm{Gr} : \mathrm{Fun}(X, \mathcal{S}\mathrm{et}) \xrightarrow{\sim} \mathcal{C}\mathrm{ov}(X)$
    - ∗ here, fibers are *spaces* instead of sets, so replace the target to get $\mathrm{Gr} : \mathrm{Fun}(X, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}_{/X}$
    - ∗ in fact, this "gluing" operation is just the colimit functor! the lift through $\mathcal{S}_{/X} \xrightarrow{U} \mathcal{S}$ is just recognizing the value on the terminal object: $\mathrm{const}_X(\mathrm{pt}_{\mathcal{S}}) \mapsto X$.
      - · fun fact: we can recover the limit of $F : X \to \mathcal{S}$ as $\Gamma(\mathrm{Gr}(F) \downarrow X)$
      - · special case: if $X = BG \to \mathcal{S}$ selects $G \curvearrowright Y$, then the colimit is $Y_G := Y_{hG}$ (the orbits) and the limit is $Y^G = Y^{hG}$ (the fixedpoints)
  - this picks up the "underlying space":

$$\mathcal{M}\mathrm{fld}_n \xrightarrow{\tau} \mathcal{S}_{/B\mathrm{Top}(n)} \xrightarrow{U} \mathcal{S}, \qquad U \circ \tau = \Pi_{\leq\infty}$$

    (implicitly precomposing composing with $\mathcal{M}\mathrm{fld}_n \xrightarrow{U} \mathcal{T}\mathrm{op}$)

- for any $(B \to B\mathrm{Top}(n)) \in \mathcal{S}_{/B\mathrm{Top}(n)}$, define the ∞-category of ***B-framed n-manifolds*** as the pullback

$$\begin{array}{ccc} \mathcal{M}\mathrm{fld}_n^B & \longrightarrow & \mathcal{S}_{/B} \\ \downarrow & & \downarrow U \\ \mathcal{M}\mathrm{fld}_n & \xrightarrow[\tau]{} & \mathcal{S}_{/B\mathrm{Top}(n)}. \end{array}$$

  - that is, an object of $\mathcal{M}\mathrm{fld}_n^B$ is a pair

$$\left(\text{an object } M \in \mathcal{M}\mathrm{fld}_n \text{ , a commutative triangle } \begin{array}{ccc} & & B \\ & \nearrow & \downarrow \\ \Pi_{\leq\infty}(M) & \xrightarrow[\tau(M)]{} & B\mathrm{Top}(n) \end{array} \text{ in } \mathcal{S} \right)$$

  (can present these data using $\mathcal{T}\mathrm{op} \in \mathcal{C}\mathrm{at}_{\mathcal{T}\mathrm{op}}$, which of course presents $\mathcal{S}$)
  - since $\mathrm{N}_\infty(-)_1 : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ commutes with limits, for any $M, N \in \mathcal{M}\mathrm{fld}_n^B$ we get a pullback square

$$\begin{array}{ccc} \hom_{\mathcal{M}\mathrm{fld}_n^B}(M,N) & \longrightarrow & \hom_{\mathcal{S}_{/B}}(\Pi_{\leq\infty}(M), \Pi_{\leq\infty}(N)) \\ \downarrow & & \downarrow \\ \hom_{\mathcal{M}\mathrm{fld}_n}(M,N) & \longrightarrow & \hom_{\mathcal{S}_{/B\mathrm{Top}(n)}}(\Pi_{\leq\infty}(M), \Pi_{\leq\infty}(N)) \end{array}$$

  - some key examples: $\mathrm{pt}_{\mathcal{S}} = B(\{e\}) \to B\mathrm{Spin}(n) \to B\mathrm{SO}(n) \to B\mathrm{O}(n) \simeq B\mathrm{GL}(n) \to B\mathrm{Top}(n)$
    - by smoothing theory [Kirby–Siebemann], $\mathcal{M}\mathrm{fld}_n^{B\mathrm{SO}(n)} \simeq \mathcal{M}\mathrm{fld}_n^{\mathrm{sm}}$ for $n \neq 4$
- define the ∞-category of ***B-framed n-disks*** to be the full subcategory $\mathcal{D}\mathrm{isk}_n^B \subset \mathcal{M}\mathrm{fld}_n^B$ on $\{\bigsqcup_k \mathbb{R}^n\}_{k\geq 0}$
  - space of B-framings on $\mathbb{R}^n$ is the fiber of $B \to B\mathrm{Top}(n)$, since $\Pi_{\leq\infty}(\mathbb{R}^n) \simeq \mathrm{pt}_{\mathcal{S}}$
    - for $B = BG \to B\mathrm{Top}(n)$, this fiber is $B(\mathrm{fib}(G \to \mathrm{Top}(n)))$
- define the ∞-category of ***B-framed n-disks in M*** to be

$$(\mathcal{D}\mathrm{isk}_n^B)_{/M} = \mathcal{D}\mathrm{isk}_n^B \underset{\mathcal{M}\mathrm{fld}_n^B}{\times} (\mathcal{M}\mathrm{fld}_n^B)_{/M}.$$

  so objects are embeddings $\bigsqcup_k \mathbb{R}^n \hookrightarrow M$, and morphisms are triangles of embeddings equipped with a distinguished isotopy
  - in fact, $((\mathcal{D}\mathrm{isk}_n^B)_{/M})^{\simeq} \simeq \coprod_{i\geq 0} \Pi_{\leq\infty}(\mathrm{Conf}_i(M)_{\Sigma_i}) \simeq \coprod_{i\geq 0} (\Pi_{\leq\infty}(\mathrm{Conf}_i(M)))_{\Sigma_i}$ (the coproduct of the unordered configuration spaces (note that the action in $\mathcal{T}\mathrm{op}$ is free, so the quotient computes the homotopy quotient))
- for $\mathcal{C} = (\mathcal{C}, \otimes)$ a symmetric monoidal ∞-category, define the ∞-category of ***B-framed n-disk algebras in*** $\mathcal{C}$ to be $\mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{C}) = \mathrm{Fun}^{\otimes}((\mathcal{D}\mathrm{isk}_n^B, \sqcup), (\mathcal{C}, \otimes))$
  - key examples:
    - $\mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^{\mathrm{fr}}}$ =: "$\mathcal{E}_n$-algebras"
    - $\mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n}$ =: "n-disk algebras" $\simeq$ $\mathcal{E}_n$-algebras equipped $\mathrm{Top}(n)$-action (for $n \neq 4$)
    - $\mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^{B\mathrm{O}(n)}}$ =: "ribbon $\mathcal{E}_n$–algebras"
    - when $n = 1$, get $\mathrm{Alg} := \mathrm{Alg}_{\mathcal{E}_1} := \mathrm{Alg}_{\mathcal{A}_\infty} := \mathrm{Alg}_{\mathcal{A}\mathrm{ss}}$
- now, our **key definition**: given $M \in \mathcal{M}\mathrm{fld}_n^B$ and $A \in \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{C})$, the ***factorization homology*** of A over M is the object of $\mathcal{C}$ given by

$$\int_M A := \mathrm{colim}\left((\mathcal{D}\mathrm{isk}_n^B)_{/M} \xrightarrow{U} \mathcal{D}\mathrm{isk}_n^B \xrightarrow{A} \mathcal{C} \right).$$

  - intuitively, a "point" is given by a configuration of points (really B-framed $\mathbb{R}^n$'s) in M, each labeled by "elements" of A. but points are allowed to move around continuously through embeddings, and when points collide we multiply their labels. (we also allow new points to appear on M, labeled by "the unit" of A.)
  - for $B \xrightarrow{\varphi} B' \to B\mathrm{Top}(n)$, we can take the factorization homology of $A \in \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^{B'}}(\mathcal{C})$ over $M \in \mathcal{M}\mathrm{fld}_n^B$ in two different ways, but they agree: $\int_{\varphi_* M} A \simeq \int_M \varphi^* A$
  - aside: localizing w/r/t isotopy equivalences.

* define the symmetric monoidal 1-categories $(\mathrm{Mfld}_n, \sqcup) \supset (\mathrm{Disk}_n, \sqcup)$. these admit symmetric monoidal functors

$$\begin{array}{ccc} \mathrm{Disk}_n & \longrightarrow & \mathcal{D}\mathrm{isk}_n \\ \big\downarrow & & \big\downarrow \\ \mathrm{Mfld}_n & \longrightarrow & \mathcal{M}\mathrm{fld}_n, \end{array}$$

and similarly for B-structured versions, defined by $\mathsf{X}_n^B := \mathsf{X}_n \times_{\mathcal{X}_n} \mathcal{X}_n^B$
* difficult fact: $(\mathrm{Disk}_n^B)_{/M} \to (\mathcal{D}\mathrm{isk}_n^B)_{/M}$ is a *localization* (in the ∞-categorical sense).
* corollary: it's *final*, i.e. colimits over either diagram (e.g. factorization homology!) are equivalent.
* in particular, the restriction $\mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Disk}_n^B}(\mathcal{C})$ is fully faithful, and surjects onto the *locally constant* algebras

2.4. **Homology theories for manifolds.**

- a variant: $\mathcal{M}\mathrm{fld}_n^\partial$ (now covered by $\mathbb{R}^n$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$), $\mathcal{D}\mathrm{isk}_n^\partial \subset \mathcal{M}\mathrm{fld}_n^\partial$ full on $\{\sqcup_i \mathbb{R}^n \coprod \sqcup_j \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}\}_{i,j \geq 0}$
  - further variant: *stratified spaces* (e.g. locally modeled on $\mathbb{R}^3$ or $\mathbb{R}^1 \subset \mathbb{R}^3 \rightsquigarrow$ "knots and links")
- factorization homology: identical definition.
  - example: $\mathcal{M}\mathrm{fld}_1^{\partial,\mathrm{or}}$
    * a s.m. functor $\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}} \to \mathcal{C}$ selects a triple $(A \in \mathrm{Alg}(\mathcal{C}), R \in \mathrm{RMod}_A(\mathcal{C}), L \in \mathrm{LMod}_A(\mathcal{C})$
    * most interesting manifold here: $M = I = [-1, 1]$
      · key fact: $\exists$ final functor $\mathbf{\Delta}^{op} \to (\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}})_{/I}$, an equivalence onto subcat of opens containing both endpoints (inverse is by "counting gaps" – the object $[n]^\circ \in \mathbf{\Delta}^{op}$ goes to an object with n *open* intervals)
      · the composite
      $$\mathbf{\Delta}^{op} \to (\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}})_{/I} \xrightarrow{U} \mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}} \xrightarrow{R \curvearrowleft A \curvearrowright L} \mathcal{C}$$
      selects the ***bar construction*** $\mathrm{Bar}(R, A, L)_\bullet = \{R \otimes A^{\otimes n} \otimes L\}_{n \geq 0}$, so
      $$\int_I (R \curvearrowleft A \curvearrowright L) \simeq |\mathrm{Bar}(R, A, L)_\bullet| =: R \otimes_A^{\mathbb{L}} L.$$
      · the actual *meaning* of this statement is somewhat subtle. in any 1-category, a geometric realization reduces to a coequalizer, so considered there this remains an ordinary tensor product (the only one possible among discrete A-modules, for A a discrete associative ring). but if we consider an abelian category $\mathcal{A}$ as a full subcategory $\mathcal{A} \subset \mathscr{D}(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\mathrm{qi}}^{-1}]\!]$ of its *derived ∞-category* (which also admits an abstract universal characterization not mentioning chain complexes, cf. [§1.3.3, HA]), then this geometric realization is an ∞-categorical colimit, and does indeed compute the "derived tensor product".
    * for arbitrary $M \in \mathcal{M}\mathrm{fld}_1^{\partial,\mathrm{or}}$: get $\otimes$ for disjoint unions, A for opens, R or L for half-opens
- a ***collar-gluing*** of any $M \in \mathcal{M}\mathrm{fld}_n^B$ is a continuous map $f : M \to I = [-1, 1]$ such that $f|_{(-1,1)}$ is a manifold bundle. write $M_- = f^{-1}([-1, 1))$, $M_+ = f^{-1}((-1, 1])$, and $M_0 = f^{-1}(\{0\})$, so this determines a decomposition $M \cong M_- \coprod_{M_0 \times \mathbb{R}} M_+$ (among B-framed manifolds, since we can just postcompose with the lift $\Pi_{\leq \infty}(M) \to B$, as the tangent microbundle is functorial for embeddings (by construction!))
- now, observe the functor $f^{-1} : (\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}})_{/I} \to (\mathcal{M}\mathrm{fld}_n^B)_{/M}$ (defined by *1-categorical* pullback). for any s.m. functor $\mathcal{F} : (\mathcal{M}\mathrm{fld}_n^B, \sqcup) \to (\mathcal{C}, \otimes)$, we get

$$\begin{array}{ccccc} (\mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}})_{/I} & \xrightarrow[\text{(not s.m. (meaningless!))}]{f^{-1}} & (\mathcal{M}\mathrm{fld}_n^B)_{/M} & \xrightarrow[\text{(not s.m. (meaningless!))}]{\mathcal{F}} & \mathcal{C}_{/\mathcal{F}(M)} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}} & \underset{\exists!\ \text{s.m. extn}}{\dashrightarrow} & \mathcal{M}\mathrm{fld}_n^B & \xrightarrow[\mathcal{F}]{} & \mathcal{C}. \end{array}$$

let's write $f_*\mathcal{F} : \mathcal{D}\mathrm{isk}_1^{\partial,\mathrm{or}} \to \mathcal{C}$ for the lower composite, a ***pushforward***. this selects:
  - $A := (f_*\mathcal{F})((-1, 1)) \simeq \mathcal{F}(M_0 \times \mathbb{R}) \in \mathrm{Alg}(\mathcal{C})$,

* note: $\mathbb{R} \in \mathrm{Alg}(\mathcal{M}\mathrm{fld}_1^{\mathrm{or}}) = \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}}(\mathcal{M}\mathrm{fld}_1^{\mathrm{or}})$ [tautologically] $\rightsquigarrow M_0 \times \mathbb{R} \in \mathrm{Alg}(\mathcal{M}\mathrm{fld}_n^B)$
  · really, use $B\mathrm{Top}(n-1) \to B\mathrm{Top}(n)$ to define $\mathcal{M}\mathrm{fld}_{n-1}^B = \mathcal{M}\mathrm{fld}_{n-1} \times_{\mathcal{S}_{B\mathrm{Top}(n)}} \mathcal{S}_{/B}$, and then have $\mathcal{M}\mathrm{fld}_{n-1}^B \times \mathcal{M}\mathrm{fld}_1^{\mathrm{or}} \xrightarrow{-\times-} \mathcal{M}\mathrm{fld}_n^B$
    for B-framings, note that $\Pi_{\leq\infty}(N \times (\sqcup_k \mathbb{R})) \simeq \coprod_k \Pi_{\leq\infty}(N)$
  · this is symmetric monoidal (for $\sqcup$) separately in each variable, so the composite

$$\mathcal{D}\mathrm{isk}_1^{\mathrm{or}} \xrightarrow{(\mathrm{const}(M_0),\iota)} \mathcal{M}\mathrm{fld}_{n-1}^B \times \mathcal{M}\mathrm{fld}_1^{\mathrm{or}} \xrightarrow{-\times-} \mathcal{M}\mathrm{fld}_n^B$$

  is symmetric monoidal
- $R := (f_*\mathcal{F})([-1,1)) \simeq \mathcal{F}(M_-) \in \mathrm{RMod}_A(\mathcal{C})$,
- $L := (f_*\mathcal{F})((-1,1]) \simeq \mathcal{F}(M_+) \in \mathrm{LMod}_A(\mathcal{C})$.

so we get $\int_I f_*\mathcal{F} \simeq R \otimes_A^{\mathbb{L}} L$. this is colim(down;right;right) = colim(right;right;down). but since $\mathcal{C}_{/\mathcal{F}(M)} \xrightarrow{U} \mathcal{C}$ commutes with colimits, we get a natural map

$$\int_I f_*\mathcal{F} \simeq \mathcal{F}(M_-) \bigotimes_{\mathcal{F}(M_0\times\mathbb{R})} \mathcal{F}(M_+) \to \mathcal{F}(M).$$

we say that $\mathcal{F} \in \mathrm{Fun}^{\otimes}((\mathcal{M}\mathrm{fld}_n^B, \sqcup), (\mathcal{C}, \otimes))$ is ***$\otimes$-excisive*** if this is an equivalence for all collar-gluings of B-framed n-manifolds. in this case, we also say that $\mathcal{F}$ is a ***homology theory***, and write $\mathcal{H}(\mathcal{M}\mathrm{fld}_n^B, \mathcal{C}) \subset \mathrm{Fun}^{\otimes}((\mathcal{M}\mathrm{fld}_n^B, \sqcup), (\mathcal{C}, \otimes))$ for the full subcategory of these.

- **main theorem**: *when* $\mathcal{C}$ *is* $\otimes$-presentable, *the inclusion* $\iota : \mathcal{D}\mathrm{isk}_n^B \hookrightarrow \mathcal{M}\mathrm{fld}_n^B$ *gives an equivalence*

$$\begin{array}{ccc} \int_{(=)}(-) \;:=\; \iota_! : \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{C}) & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathrm{Fun}^{\otimes}((\mathcal{M}\mathrm{fld}_n^B, \sqcup), (\mathcal{C}, \otimes)) : \iota^* \;=:\; \mathrm{ev}_{\mathbb{R}^n}. \\ & \overset{\sim}{\searrow} & \uparrow \\ & & \mathcal{H}(\mathcal{M}\mathrm{fld}_n^B, \mathcal{C}) \end{array}$$

with the (a posteriori) coreflective subcategory of homology theories.
  - *presentable* is a niceness condition meaning "presentable by generators and relations": it has small colimits and is generated under such by a small subcat $\mathcal{C}_0 \subset \mathcal{C}$
    * e.g., $\mathrm{PShv}(\mathrm{Ab}^{\mathrm{fgen}}) \rightleftarrows \mathrm{Ab}$: every abelian group is a colimit of finitely generated ones
  - $\otimes$-*presentable* (a/k/a *presentably symmetric monoidal*) means "presentable, symmetric monoidal, and the monoidal structure commutes with colimits separately in each variable"
    * e.g., Ab, $\mathrm{Ch}[\![\mathbf{W}_{\mathrm{qi}}]\!]$, $\mathcal{S}$, ... but *not* generally the opposite
- **proof**: breaks down into a few steps.
  (1) *[technical]* using that $\mathcal{C}$ is presentably symmetric monoidal, we get the functor $\iota_! : \mathrm{Alg} \to \mathrm{Fun}$, its factorization through $\mathrm{Fun}^{\otimes} \subset \mathrm{Fun}$, and the adjunction $\iota_! \dashv \iota^*$.
  (2) factorization homology is $\otimes$-excisive. for fixed $A \in \mathrm{Alg}$, write $\mathcal{F} = \int_{(-)} A : \mathcal{M}\mathrm{fld}_n^B \to \mathcal{C}$. then, given a collar-gluing $f : M \to I$, we get the composite

$$\int_{M_-} A \bigotimes_{\int_{M_0\times\mathbb{R}} A} \int_{M_+} A \to \int_I f_*\mathcal{F} \to \mathcal{F}(M) = \int_M A.$$

  - the first map is an equivalence by our previous computation of $\int_I$.
  - the second is an equivalence essentially because factorization homology is "pushforward to $\mathbb{R}^0$" (more generally, "pushforward" is just "fiberwise factorization homology"), so we're just using the factorization $M \to I \to \mathbb{R}^0$ (i.e., "colimits commute").
    * these "pushforward" maps are defined as follows. suppose we're given a map $f : M \to N$ where $M = M^n$ is B-framed and $N = N^k$ is B'-framed, and N might have boundary. then we get a composite

$$f^{-1} : (\mathrm{Disk}_k^{\partial,B'})_{/N} \to (\mathrm{Mfld}_n^B)_{/M} \to (\mathcal{M}\mathrm{fld}_n^B)_{/M}$$

      (after using the usual trick to assume that $B \xrightarrow{\equiv} B\mathrm{Top}(n)$ and $B' \xrightarrow{\equiv} B\mathrm{Top}(k)$) where the first functor is $(U \hookrightarrow N) \mapsto (U \times_N M \hookrightarrow M)$ and the second functor is the localization. assuming the restriction of f to $N^\circ = (N \backslash \partial N)$ and to ∂N are manifold bundles, this composite takes isotopy equivalences to equivalences, so descends to $f^{-1} : (\mathcal{D}\mathrm{isk}_k^{B'})_{/N} \to (\mathcal{M}\mathrm{fld}_n^B)_{/M}$.

* this **also** evidently works for collar-gluings, i.e. when $N = [-1, 1]$ but $f|_{\partial N}$ isn't necessarily a manifold bundle. this is intuitively clear, but the "real" reason might be that there are no obstructions to deforming such a thing to some f' with both $f'|_{N^\circ}$ and $f'|_{\partial N}$ be manifold bundles, and with homeomorphic preimages over the various objects of $(\mathrm{Disk}_k^{\partial,B'})_{/N}$.
* next, for such a map $f : M \to N$, define $\mathcal{D}\mathrm{isk}_f$ to be "some n-disks $(U \hookrightarrow M)$, some k-disks $(V \hookrightarrow N)$, and an embedding $U \hookrightarrow f^{-1}(V)$ over M (up to isotopy)" – the limit of $\mathcal{D}\mathrm{isk}_{/M} \hookrightarrow \mathcal{M}\mathrm{fld}_{/M} \xleftarrow{s} \mathrm{Fun}([1], \mathcal{M}\mathrm{fld}_{/M}) \xrightarrow{t} \mathcal{M}\mathrm{fld}_{/M} \xleftarrow{f^{-1}} \mathcal{D}\mathrm{isk}_{/N}^{\partial,\mathrm{or}}$. a **hard lemma** is that the "source" functor $\mathcal{D}\mathrm{isk}_f \to (\mathcal{D}\mathrm{isk}_n^B)_{/M}$ is final, and moreover that $(\mathcal{D}\mathrm{isk}_n^B)_{/M}$ is sifted.
* from here, for any $A \in \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{C})$ (for $\mathcal{C}$ presentably symmetric monoidal), assuming $B' = B\mathrm{Top}(n)^+$ (so asking for "oriented" structure), we get a canonical equivalence $\int_N f_* A \xrightarrow{\sim} \int_M A$.
  · by definition, we have
$$\begin{aligned}\int_N f_*A &\simeq \mathrm{colim}_{U \in (\mathcal{D}\mathrm{isk}_k^{\partial,\mathrm{or}})_{/N}} (f_*A)(U) \\ &\simeq \mathrm{colim}_U \int_{f^{-1}(U)} A \\ &\simeq \mathrm{colim}_U \left(\mathrm{colim}_{V \in (\mathcal{D}\mathrm{isk}_n^B)_{/f^{-1}(U)}} A(V) \right).\end{aligned}$$
  · we claim that this is $\simeq \mathrm{colim}_{(U,V) \in \mathcal{D}\mathrm{isk}_f} A(V)$. to see this, consider the left Kan extension diagram
$$\begin{array}{ccccccc} \mathcal{D}\mathrm{isk}_f & \xrightarrow{s} & (\mathcal{D}\mathrm{isk}_n^B)_{/M} & \longrightarrow & \mathcal{D}\mathrm{isk}_n^B & \xrightarrow{A} & \mathcal{C}. \\ {\scriptstyle t}\downarrow & & & & & \nearrow & \\ (\mathcal{D}\mathrm{isk}_k^{\partial,\mathrm{or}})_{/N} & & & & & & \end{array}$$
  the map t is a cocartesian fibration, so the fibers are final in the over-slices, so LKan can be computed by taking a colimit over the fiber, which is just $t^{-1}(U) = V \in (\mathcal{D}\mathrm{isk}_n^B)_{/f^{-1}(U)}$. so the above iterated colimit is colim(LKan), which is just $\mathrm{colim}_{\mathcal{D}\mathrm{isk}_f}$ (since a colimit is LKan along the terminal functor to pt).
  · we claim that this is $\simeq \mathrm{colim}_{V \in (\mathcal{D}\mathrm{isk}_n^B)_{/M}} A(V)$: this is precisely the previous finality.
  · finally, this is $\simeq \int_M A$ by definition.

(3) now, since ι is fully faithful, we get $\eta_A : A \xrightarrow{\sim} \iota^* \iota_! A$. so it remains to show that whenever $\mathcal{F} \in \mathcal{H}(\mathcal{M}\mathrm{fld}_n^B, \mathcal{C})$ (i.e. $\mathcal{F}$ is $\otimes$-excisive) then the counit map $\varepsilon : \int_{(-)} (\mathcal{F}|_{\mathbb{R}^n}) \to \mathcal{F}$ is also an equivalence. by definition, these are both symmetric monoidal ($\sqcup \mapsto \otimes$) and agree on $\mathbb{R}^n$, hence agree on $\sqcup_k \mathbb{R}^n$ for finite k. now, prove by induction for all $S^i \times \mathbb{R}^{n-i}$, with $i = 0$ done. the standard projection-along-an-axis collar-gluing $S^i \xrightarrow{f} I$ gives $S^i \times \mathbb{R}^{n-1} \xrightarrow{\mathrm{pr}_1} S^i \xrightarrow{f} I$, and writing $A = \mathcal{F}|_{\mathbb{R}^n}$ we get
$$\begin{aligned}\int_{S^i \times \mathbb{R}^i} A &\simeq \int_{\mathbb{R}^i_- \times \mathbb{R}^{n-i}} A \bigotimes_{\int_{S^{i-1} \times \mathbb{R}^{n-i+1}} A} \int_{\mathbb{R}^i_+ \times \mathbb{R}^{n-i}} A && \text{fact hlgy is } \otimes\text{-excisive} \\ &\simeq \mathcal{F}(\mathbb{R}^i_- \times \mathbb{R}^{n-i}) \bigotimes_{\mathcal{F}(S^{i-1} \times \mathbb{R}^{n-i+1})} \mathcal{F}(\mathbb{R}^i_+ \times \mathbb{R}^{n-i}) && \text{induction} \\ &\simeq \mathcal{F}(S^i \times \mathbb{R}^{n-i}). && \mathcal{F} \text{ is } \otimes\text{-excisive}\end{aligned}$$
   (a) *case* $n \neq 4$: all B-framed manifolds admit handlebody decompositions [Kirby–Siebemann for $n > 5$, Quinn for $n = 5$, Moise for $n = 3$].
   – recall handlebody decompositions in dimension 2:

        * index-0 adds a $\mathbb{R}^2$ "cup", attached along $\varnothing$ [note: this doesn't fit into usual "handle-gluing" picture, but no problem];
        * index-1 adds an ordinary handle: glue $\mathbb{R}^2$ along $S^0 \times \mathbb{R}^2$, embedded as the complement of a closed strip, e.g. $\{(x, y \in \mathbb{R}^2 : |x| \leq 1\})$;
        * index-2 adds a $\mathbb{R}^2$ "cap", glued along $S^1 \times \mathbb{R}^1$ (embedded as the complement of a closed disk).

      here, adding an index-$(i+1)$ handle to N gives $M \cong N \coprod_{S^i \times \mathbb{R}^{n-i}} \mathbb{R}^n$. so, follows from $\otimes$-excision by induction over handlebody decompositions.
    (b) *case* $n = 4$: can assume connected, since both are symmetric monoidal. for M connected, $x \in M$, there exists a smooth structure on $M \backslash \{x\}$. this then admits a Morse function and hence a handle decomposition, so as above we get $\int_{M\backslash\{x\}} A \simeq \mathcal{F}(M\backslash\{x\})$. then, apply $\otimes$-excision to $M \cong (M\backslash\{x\}) \coprod_{S^{n-1}\times\mathbb{R}} \mathbb{R}^n$.

- examples:
  - commutative coefficients: for any operad $\mathcal{O}$ we have a forgetful functor $\mathcal{C} \simeq \mathrm{CAlg}(\mathcal{C}) = \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ (e.g./i.e. $\mathcal{O} = \mathcal{D}\mathrm{isk}_n^B$ – in this case it's corepresented by the map $\pi_0 : \mathcal{D}\mathrm{isk}_n^B \to \mathcal{F}\mathrm{in}_* =:$ Comm $=:$ "$\mathcal{D}\mathrm{isk}_\infty$" (with $\sqcup$ as always))
    * example: any $(\mathcal{C}, \amalg)$ where $\mathcal{C}$ admits (finite) colimits, so that $\mathrm{CAlg}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$

    then, we just get $\int_{(=)}(-) : \mathcal{C} \to \mathcal{H}(\mathcal{M}\mathrm{fld}_n^B, \mathcal{C})$ is the *tensoring*, in the sense that the diagram

$$\begin{array}{ccccc}
\mathcal{M}\mathrm{fld}_n^B \times \mathrm{CAlg}(\mathcal{C}) & \xrightarrow{\Pi_{\leq\infty}\times \mathrm{id}} & \mathcal{S}^{\mathrm{fin}} \times \mathrm{CAlg}(\mathcal{C}) & \xrightarrow{-\odot-} & \mathrm{CAlg}(\mathcal{C}) \\
{\scriptstyle \mathrm{id}\times U}\downarrow & & & & \downarrow{\scriptstyle U} \\
\mathcal{M}\mathrm{fld}_n^B \times \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{C}) & & \xrightarrow[\int_{(-)}(-)]{\hspace{6em}} & & \mathcal{C}
\end{array}$$

    commutes. tensoring can be characterized as the unique colimit-preserving functor $\int_{(-)} A : \mathcal{S}^{\mathrm{fin}} \to \mathcal{C}$ with $\mathrm{pt}_{\mathcal{S}} \mapsto A$. ($\mathcal{S}^{\mathrm{fin}}$ is the free completion of $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$ for *finite* colimits.)
    * proof: true on disjoint unions (note that $\otimes$ is coproduct in $\mathrm{CAlg}(\mathcal{C})$), and $\otimes$-excisiveness follows since tensoring is a colimit and collar-gluings give pushouts in $\mathcal{S}$
    * for instance, when $\mathcal{C} = \mathrm{Ch}$ (so $\amalg = \oplus$), by this universal characterization, for any $A \in \mathrm{Ch}$ we have $\int_M A = C_*(M; A) := C_*(M; \mathbb{Z}) \otimes_{\mathbb{Z}} A$. [same deal for spectra $\approx$ chain complexes of sets, as long as we take $\oplus$.]
    * note: things are very different for $(\mathrm{Ch}, \otimes)$!
    * consequence 1: for $X \in \mathcal{C}$, $\int_M \mathrm{Sym}(X) \simeq \mathrm{Sym}(\Pi_{\leq\infty}(M) \odot X)$
    * consequence 2: factorization homology with commutative coefficients is fairly insensitive to manifold topology
  - for arbitrary $(\mathcal{C}, \otimes)$, the factorization homology $\int_{S^1}(-) : \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}}(\mathcal{C}) \to \mathcal{C}$ is called ***(topological) Hochschild homology***.
    * this agrees with usual definition by $\otimes$-excision on standard collar-gluing $f : S^1 \to I$
    * for commutative coefficients, can interpret $\mathrm{Spec}(\mathrm{HH}(A)) \simeq \mathcal{L}(\mathrm{Spec}(A))$ (or more generally $\mathrm{Spec}(\int_M(A)) \simeq \mathrm{map}(M, \mathrm{Spec}(A))$)
      · idea: tensoring in CAlg is cotensoring in Aff (which is reflective in Sch), describe (not so) toy picture "choose two points in $\mathrm{Spec}(A)$, set them equal twice"
      · see Ben-Zvi–Nadler's "loop spaces" paper
    * advantage: usual model for HH (as a chain complex) is combinatorial, and thus obscures functoriality arising from S^1
      · factorization homology is actually functorial for more than just embeddings, e.g. it's contravariantly functorial for proper fiber bundles
      · in a suitably sophisticated setting, this gives rise to *cyclotomic structure* on HH (connects to algebraic K-theory)

2.5. **Nonabelian Poincaré duality.**

- given a topological space $X \in \mathcal{T}\mathrm{op}$, let $\mathcal{K}_X$ denote its poset of compact subsets, and write X^+ for the one-point compactification

- then we get $(\mathcal{K}_X)^{op} \to (\mathcal{T}\mathrm{op}_*)_{X^+/}$ by $X_k \mapsto X/(X \backslash X_k)$; let's write $\mathrm{cs}(X) \in \mathrm{Ind}(\mathcal{S}_*)$ for the resulting composite $(\mathcal{K}_X)^{op} \to \mathcal{T}\mathrm{op}_* \to \mathcal{S}_*$ (which lifts to $(\mathcal{S}_*)_{\Pi_{\leq\infty}(X^+)/}$)
  - e.g. for $X = \mathbb{R}$ we have $X^+ = \mathbb{R}^+ = \mathbb{R} \cup \{\infty\} \cong S^1$, and for an inclusion of compact subsets $X_k = [a,b] \subset [a',b'] = X_{k'}$ we get a (pontrjagin–thom) collapse map $[a,b]/(a \sim b) \leftarrow [a',b']/(a' \sim b')$ under (the fixed copy of) S^1
- for a map in the ind-system $\mathrm{cs}(X)$, we might call precomposition with it ***extension by zero***
- for $Y \in \mathcal{S}_*$, the space of ***compactly-supported maps*** from X to Y is

$$\mathrm{map}^c(X,Y) = \mathrm{colim}_{k \in \mathcal{K}_X} \hom_{\mathcal{S}_*}(\Pi_{\leq\infty}(X_k), Y)$$

- this comes with a composite map

$$\mathrm{map}^c(X,Y) \to \hom_{\mathcal{S}_*}(\Pi_{\leq\infty}(X^+), Y) \to \hom_{\mathcal{S}}(\Pi_{\leq\infty}(X), Y)$$

  of spaces, called "forget that the map was compactly-supported"
  - if $X \in \mathcal{T}\mathrm{op}$ has "tame ends", this simplifies:
    * by definition, this means that there's a final subcategory $\mathcal{K}'_X \subset \mathcal{K}_X$ such that the composite $(\mathcal{K}'_X)^{op} \hookrightarrow (\mathcal{K}_X)^{op} \to \mathcal{T}\mathrm{op}_* \to \mathcal{S}_*$ is constant, with all structure maps to $\Pi_{\leq\infty}(X^+)$ being equivalences
    * then, we get an equivalence $\mathrm{map}^c(X,Y) \xrightarrow{\sim} \hom_{\mathcal{S}_*}(\Pi_{\leq\infty}(X^+), Y)$
    * **key example**: manifolds with tame ends, e.g. objects of $\mathcal{M}\mathrm{fld}_n^B$ (since we required them to be finitary, so e.g. we can't have a surface with infinite genus)
    * **particular example**: taking $\mathbb{R}^n$, any $X \in \mathcal{S}_*$ gives $\Omega^n X \in \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n}(\mathcal{S})$
- now, for any $X \downarrow B \downarrow B\mathrm{Top}(n)$ in $\mathcal{S}$, we get $\mathcal{M}\mathrm{fld}_n^B \xrightarrow{\Gamma_c(-;X)} \mathcal{S}$, which is symmetric monoidal: $\sqcup \rightsquigarrow \times$.
- in particular, any $X \in (\mathcal{S}_{/B})_* = \mathcal{S}_{B//B}$ defines $\Omega^n_B X \in \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{S})$ by $\mathcal{D}\mathrm{isk}_n^B \hookrightarrow \mathcal{M}\mathrm{fld}_n^B \xrightarrow{\Gamma_c(-;X)} \mathcal{S}$.
  - simpler example: if $X = (Y,y) \times B$, then the underlying object is just $\Omega^n_y Y$
  - more generally, think of $\Omega^n_B X$ as a *parametrized family* of n-fold loopspaces over B
- we now come to

  Theorem 2 (nonabelian Poincaré duality). *suppose $X \in (\mathcal{S}_{/B})_* = \mathcal{S}_{B//B}$ is fiberwise n-connected. then*

$$\int_M \Omega^n_B X \simeq \Gamma_c(M;X).$$

- the **proof** mostly just consists in checking that the right side is a homology theory: to check that the "coefficients" agree we just plug in $M = \mathbb{R}^n$, in which case the assertion reduces to the (definitional) equivalence $\Omega^n_B X \simeq \Gamma_c(\mathbb{R}^n;X)$
  - we already observed that the right side is symmetric monoidal, so what remains is excision
  - a collar-gluing $M \cong M_- \coprod_{M_0 \times \mathbb{R}} M_+$ induces a pushout diagram

$$\begin{array}{ccc} \mathrm{cs}(M_0) & \longrightarrow & \mathrm{cs}(M_+) \\ \downarrow & & \downarrow \\ \mathrm{cs}(M_-) & \longrightarrow & \mathrm{cs}(M) \end{array}$$

    in $\mathrm{Ind}(\mathcal{S}_*)$, and moreover $\mathrm{cs}(M_0) \simeq \mathrm{cs}(M_0 \times \mathbb{R})$
    * by our finiteness conditions, objects of $\mathcal{M}\mathrm{fld}_n^B$ have "tame ends" (as described previously), and hence for any $N \in \mathcal{M}\mathrm{fld}_n^B$ we get $\mathrm{cs}(N) \simeq \Pi_{\leq\infty}(N^+)$ (as objects of $\mathrm{Ind}(\mathcal{S}_*)$, where the latter is a constant diagram)
    * moreover, the existence of a collar neighborhood of M_0 in both M_- and M_+ implies that the maps $M_- \leftarrow M_0 \to M_+$ are cofibrations of topological spaces, so their pushout (or rather the pushout of their one-point compactifications) computes a pushout in $\mathcal{T}\mathrm{op}[\![\mathbf{W}_{\mathrm{whe}}^{-1}]\!] \simeq \mathcal{S}$
  - thus, we get a pullback

$$\begin{array}{ccc} \Gamma_c(M;X) & \longrightarrow & \Gamma_c(M_+;X) \\ \downarrow & & \downarrow \\ \Gamma_c(M_-;X) & \longrightarrow & \Gamma_c(M_0 \times \mathbb{R};X) \end{array}$$

    in $\mathcal{S}$

- given an arbitrary cospan $W \to Z \leftarrow Y$ in $\mathcal{S}$ and a point $z \in Z$, writing W_z and Y_z for the respective fibers, we get a canonical map
$$W_z \underset{\Omega_z Z}{\times} Y_z \to W \times_Z Y$$
(where in the source we're taking a balanced (monoidal=cartesian) product over $\Omega_z Z \in \mathrm{Alg}(\mathcal{S})$), and this is an equivalence if Z is connected
  - in essence, this results from the grothendieck construction
    - the (∞-)group $\Omega_z Z$ acts on the two modules W_z and Y_z (by "path lifting")
    - these actions are classified by maps $B(\Omega_z Z) \rightrightarrows \mathcal{S}$
    - just as the grothendieck construction over BG computes (homotopy) orbits, so does the "two-sided Grothendieck construction" (i.e. the fiber product of the two objects of $\mathcal{S}_{/BG}$) compute balanced tensor products
    - the canonical map $B(\Omega_z Z) \to Z$ is an equivalence if Z is connected, in which case these two induced objects are just $W, Y \in \mathcal{S}_{/Z}$
- in our case, we have $Z = \Gamma_c(M_0 \times \mathbb{R}; X)$, which is connected (essentially by cellular approximation, for maps out of the $(n-1)$-dimensional CW complex M_0 into the (fiberwise) n-connected object X)
- putting these facts together, we see that the functor $\Gamma_c(-;X) : (\mathcal{M}\mathrm{fld}_n^B)^{op} \to \mathcal{S}$ does indeed satisfy $\otimes$-excision in $(\mathcal{S}, \times)$

- special case: ordinary Poincaré duality
  - first, an identification of the left side:
    - this follows from Dold–Kan (a weak equivalence of relative categories $\mathrm{Ch}(R)_{\geq 0} \simeq \mathrm{Mod}_R(\mathcal{T}\mathrm{op})$ for R an ordinary associative ring), but there's a perhaps slightly more direct way to see it
    - considering $\mathbb{Z} \in \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{S}) \xrightarrow{U} \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^B}(\mathcal{S})$, we have $\int_M \mathbb{Z} = \mathrm{SP}^\infty(M^+;\mathbb{Z})$, the infinite symmetric power (or really its underlying space):
      - for $X \in \mathcal{T}\mathrm{op}_*$, set $\mathrm{SP}^n(X;\mathbb{Z}) = \mathrm{SP}^n(X) = X^{\wedge n}/\Sigma_n$, the space of unordered lists of n points in X modulo the relation that it all goes to the basepoint if a single one of them is the basepoint
      - when X is connected, this is the free abelian monoid in $(\mathcal{T}\mathrm{op}, \times)$ on X
      - the Dold–Thom theorem asserts that $\pi_*(\mathrm{SP}^\infty(X)) \cong \tilde{H}_*(X) = \tilde{H}_*(X;\mathbb{Z})$
      - more generally, setting $\mathrm{SP}^n(X;A) = \mathrm{SP}^n(X) \otimes_{\mathbb{Z}} A$ (so, "configurations of n points in X labeled by elements of A") and $\mathrm{SP}^\infty(X;A) = \mathrm{colim}_n\, \mathrm{SP}^\infty(X;A)$, we have $\pi_*(\mathrm{SP}^\infty(X;A)) \cong \tilde{H}_*(X;A)$
      - connection with Dold–Kan: this is really just modeling the restricted (∞-categorical) left adjoint
$$\mathcal{S}_*^{\geq 1} \hookrightarrow \mathcal{S}_* \rightleftarrows \mathrm{Mod}_{\mathbb{Z}}(\mathcal{S}_*) \simeq \mathrm{Ch}(\mathbb{Z})[\![\mathbf{W}_{\mathrm{qi}}^{-1}]\!]_{\geq 0}$$
and we've already seen that this "free" functor models "singular chains"
  - now, the right side:
    - in fact, $A \in \mathrm{Mod}_{\mathbb{Z}}(\mathcal{S}\mathrm{et}) \subset \mathrm{Mod}_{\mathbb{Z}}(\mathcal{S})$ is an infinite loopspace, with n-fold delooping $B^n A = K(A,n)$
    - classically $[-, B^n A] \cong H^n(-;A)$, but in fact also $[-, B^n A]^c \cong H_c^n(-;A)$
    - in other words, $\pi_0 \mathrm{map}^c(X, B^n A) \cong H_c^n(X;A)$; more generally, for $i \leq n$ we have the adjunction isomorphisms
$$\begin{aligned}\pi_i \mathrm{map}^c(X, B^n A) &\cong [S^i, \mathrm{map}^c(X, B^n A)]_{\mathcal{S}_*} \\ &\cong [S^0, \mathrm{map}^c(X, \Omega^i B^n A)]_{\mathcal{S}_*} \\ &\cong \pi_0 \mathrm{map}^c(X, B^{n-i} A) \\ &= H_c^{n-i}(X;A)\end{aligned}$$
(and in particular $\pi_{>n}\mathrm{map}^c(X, B^n A) = 0$)

  so in the end, taking $X = B^n A$ (so $\Omega^n X = A$) we get an equivalence
$$\mathrm{SP}^\infty(M^+;A) \simeq \mathrm{map}^c(M; B^n A)$$
in $\mathcal{S}_*$, and taking homotopy groups on both sides yields $\tilde{H}_*(M;A) \cong H_c^{n-*}(M;A)$

- incidentally, as mentioned earlier, we have

$$\mathrm{Alg}_{\mathcal{E}_n}(\mathcal{S}) \underset{U}{\overset{(-)^{\mathrm{gp}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{E}_n}(\mathcal{S}) \xleftarrow{U} \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{D}\mathrm{isk}_n}(\mathcal{S}) \xleftarrow{\Omega^n} \mathcal{S}^{\geq n}_*, \qquad \mathcal{S}^{\geq n}_* \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{E}_n}(\mathcal{S})$$

  - heuristically, n-fold loopspaces are the only possible (framed) coefficients up to group-completion
  - on the other hand, monoids are super difficult (cf. e.g. mcduff's theorem: for any connected space X there exists a (discrete) monoid M with $BM \simeq X$ (!!?!))
  - this all connects closely with configuration spaces and homological stability
- extraordinary homology theories can fit into this framework too, but this requires somewhat more care (working over $\mathbb{S}$ instead of $\mathbb{Z}$, and also nonconnectively)

3. POINCARÉ/KOSZUL DUALITY

3.1. **Zero-pointed manifolds, Koszul duality, and the Poincaré duality map.**

- poincare duality is about both homology and cohomology, which are resp. covariant and contravariant functors on spaces; to bring these into a common framework, we want to construct an ∞-category of manifolds that has both variances built in
- so, define the ∞-category $\mathcal{ZM}\mathrm{fld}_n$ of ***zero-pointed n-manifolds*** to be the symmetric monoidal ∞-category underlying the following symmetric monoidal $\mathcal{T}\mathrm{op}$-enriched category:
  - objects are locally-compact hausdorff based topological spaces $M_* \in \mathcal{T}\mathrm{op}_*$ such that $M = (M_*)\backslash\{*\}$ is a topological n-manifold
  - the topological space of morphisms from M_* to N_* is that of continuous based maps $f : M_* \to N_*$ such that the restriction $f|_N : f^{-1}(N) \to N$ is an open embedding (taken with the compact-open topology)
  - this has a symmetric monoidal structure given by the wedge sum $\vee$

  this has a zero object $\varnothing_*$ (i.e. it's both initial and terminal) – hence the name
- this has symmetric monoidal subcategories defined as the images of the (non-full!) inclusions

$$\mathcal{M}\mathrm{fld}_n \xrightarrow{(-)_+} \mathcal{ZM}\mathrm{fld}_n \xleftarrow{(-)^+} (\mathcal{M}\mathrm{fld}_n)^{op}$$

  given by adding a disjoint basepoint or one-point compactifying; denote these by

$$\mathcal{D}\mathrm{isk}_{n,+} \subset \mathcal{M}\mathrm{fld}_{n,+} \subset \mathcal{ZM}\mathrm{fld}_n \supset \mathcal{M}\mathrm{fld}_n^+ \supset \mathcal{D}\mathrm{isk}_n^+$$

  where the further full subcategories are resp. on $\{\bigvee_k(\mathbb{R}^n)_+\}_{k\geq 0}$ and $\{\bigvee_k(\mathbb{R}^n)^+\}_{k\geq 0}$
- another source of examples: if $\overline{M} \in \mathcal{M}\mathrm{fld}_n^\partial$ then $\overline{M} \amalg_{\partial\overline{M}} * \in \mathcal{ZM}\mathrm{fld}_n$
- this has a ***negation*** involution

$$(-)^\neg : (\mathcal{ZM}\mathrm{fld}_n)^{op} \xrightarrow{\sim} \mathcal{ZM}\mathrm{fld}_n$$

  given by $M_* \mapsto (M_*)^+\backslash\{*\}$
  - this restricts to $(-)^\neg : (\mathcal{M}\mathrm{fld}_{n,+})^{op} \xrightarrow{\sim} \mathcal{M}\mathrm{fld}_n^+$ and $(-)^\neg : (\mathcal{M}\mathrm{fld}_n^+)^{op} \xrightarrow{\sim} \mathcal{M}\mathrm{fld}_{n,+}$
  - these restrict further to $(-)^\neg : (\mathcal{D}\mathrm{isk}_{n,+})^{op} \xrightarrow{\sim} \mathcal{D}\mathrm{isk}_n^+$ and $(-)^\neg : (\mathcal{D}\mathrm{isk}_n^+)^{op} \xrightarrow{\sim} \mathcal{D}\mathrm{isk}_{n,+}$
- the restriction functor $\mathrm{Fun}^\otimes(\mathcal{M}\mathrm{fld}_{n,+}, \mathcal{C}) \to \mathrm{Fun}^\otimes(\mathcal{M}\mathrm{fld}_n, \mathcal{C})$ induces an equivalence $\mathrm{Fun}^\otimes(\mathcal{ZM}\mathrm{fld}_n, \mathcal{C}) \to \mathrm{Fun}^{\otimes,\mathrm{aug}}(\mathcal{M}\mathrm{fld}_n, \mathcal{C})$ to *augmented* symmetric monoidal functors $\mathcal{M}\mathrm{fld}_n \to \mathcal{C}$
  - augmentation: the functor $\mathrm{const}(\mathbb{1}_\mathcal{C})$ defines an initial object of $\mathrm{Fun}^\otimes(\mathcal{D}, \mathcal{C})$, and an augmentation is a map *to* that initial object
  - the augmentation is picked up by the terminal maps $M_+ \xrightarrow{!} \varnothing_+$ in $\mathcal{M}\mathrm{fld}_{n,+} \subset \mathcal{ZM}\mathrm{fld}_n$ (note that $\varnothing_+$ is the unit so this is sent to $\mathbb{1}_\mathcal{C}$)
  - so, it's consistent to define $\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{D}\mathrm{isk}_n}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_{n,+}}(\mathcal{C})$
  - since in general $\mathrm{coAlg}_\mathcal{O}(\mathcal{C}) := \mathrm{Fun}^\otimes(\mathcal{O}, \mathcal{C}^{op})$, it's also consistent to define $\mathrm{coAlg}^{\mathrm{aug}}_{\mathcal{D}\mathrm{isk}_n}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n^+}(\mathcal{C})$
- now, for any symmetric monoidal ∞-category $\mathcal{C}$, define ***factorization homology*** and ***factorization comology*** as the adjoints

$$\mathrm{Alg}_n^{\mathrm{aug}}(\mathcal{C}) \underset{(\iota_+)^*}{\overset{\int_{(=)}(-)}{\underset{\longleftarrow}{\overset{\dashrightarrow}{\bot}}}} \mathrm{Fun}^\otimes(\mathcal{ZM}\mathrm{fld}_n, \mathcal{C}) \underset{\int^{(=)^\neg}(-)}{\overset{(\iota^+)^*}{\underset{\dashleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{coAlg}_n^{\mathrm{aug}}(\mathcal{C})$$

(whenever they exist – factorization homology needs $\mathcal{C}$ to have sifted colimits (equivalently, to have geometric realizations and filtered colimits) and for $\otimes_{\mathcal{C}}$ to preserve these; factorization cohomology requires dual assumptions)

- in formulas,
$$\int_{M_*} A = \operatorname{colim}\left((\mathcal{D}\mathrm{isk}_{n,+})_{/M_*} \xrightarrow{U} \mathcal{D}\mathrm{isk}_{n,+} \xrightarrow{A} \mathcal{C}\right)$$
and
$$\int^{M_*} C = \lim\left((\mathcal{D}\mathrm{isk}_n^+)_{M_*^-/} \xrightarrow{U} \mathcal{D}\mathrm{isk}_n^+ \xrightarrow{C} \mathcal{C}\right)$$
- just for intuition, an example of factorization cohomology is that for $C \in \mathcal{S} \simeq \mathrm{coAlg}^{\mathrm{aug}}_{\mathrm{Comm}}(\mathcal{S}) \underset{\sim}{\xrightarrow{U}} \mathrm{coAlg}^{\mathrm{aug}}_n(\mathcal{S})$ (and M framed) we have $\int^{M_*} C = \mathrm{map}^c(M_*, C)$
  - ∗ note that this is exactly the right-hand side of the nonabelian poincare duality equivalence!
- more generally, in $(\mathcal{C}, \times)$ for any $C \in \mathcal{C} \simeq \mathrm{coAlg}^{\mathrm{aug}}_n(\mathcal{C})$ we have $\int^{M_*} C \simeq \Pi_{\leq\infty}(M) \pitchfork C$
- in fact, the pontrjagin–thom collapse gives an equivalence
$$((\mathcal{D}\mathrm{isk}_{n,+})_{/M_*})^{op} \xrightarrow{\sim} (\mathcal{D}\mathrm{isk}_n^+)_{M_*^-/},$$
which immediately implies that if $C \in \mathrm{coAlg}^{\mathrm{aug}}_n(\mathcal{C})$ corresponds to $A \in \mathrm{Alg}^{\mathrm{aug}}_n(\mathcal{C}^{op})$, then the object
$$\left(\int^{M_*} C\right) \in \mathcal{C}$$
corresponds to the object
$$\left(\int_{M_*} A\right) \in \mathcal{C}^{op}$$
  - ∗ so in particular, there's not a real handedness here: everything is cleanly interchanged by taking opposites

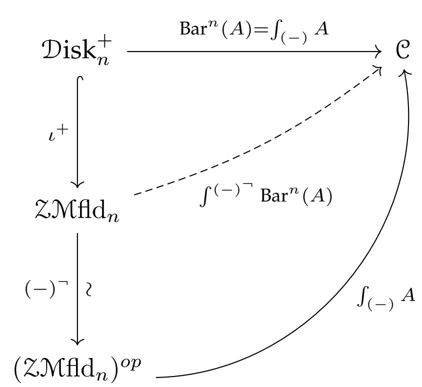
- now, the above adjunction might be called the ***Koszul duality adjunction***
$$\mathrm{Bar}^n : \mathrm{Alg}^{\mathrm{aug}}_n(\mathcal{C}) \underset{\left(\int^{(=)^-}(-)\right)\big|_{\mathcal{D}\mathrm{isk}_{n,+}}}{\overset{\left(\int_{(=)}(-)\right)\big|_{\mathcal{D}\mathrm{isk}_n^+}}{\rightleftarrows}} \mathrm{coAlg}^{\mathrm{aug}}_n(\mathcal{C}) : \mathrm{coBar}^n$$
  - in general, "Koszul duality" operates at a number of different categorical levels:
    - ∗ first of all, an operad O has a koszul dual cooperad $O^!$, which linearly dualizes to a koszul dual operad $O_! = (O^!)^\vee$
    - ∗ modulo finiteness conditions, there's an equivalence $\mathrm{coAlg}_{O^!} \simeq \mathrm{Alg}_{O_!}$ (via linear duality), and these sit in an adjunction $\mathrm{Alg}_O \rightleftarrows \mathrm{Alg}_{O_!}$ which restricts to an equivalence on suitably small/finite objects
      - · for $1 \leq n < \infty$ the $\mathcal{E}_n$ operad is koszul dual to itself, giving the above; on the other hand $\mathcal{E}_\infty = $ Comm is koszul dual to Lie
      - · we'll see these pairs again later when we talk about formal moduli problems; actually, this is *always* what koszul duality is about (deformation theory)
    - ∗ in fact, this can even be taken "one level higher": operads are themselves by definition $\mathcal{E}_1$-algebras in the ∞-category of symmetric sequences (with the "composition product" monoidal structure), and then the self koszul duality of the $\mathcal{E}_1$ operad is *itself* what's aligning these $\mathcal{E}_1$-algebras (i.e. operads) in koszul-dual pairs
      - · so this is a version of the "koszul duality at the level of algebras over an operad" story, but then at a higher level than even the operads themselves!
- these really are the n-fold iterates of the usual co/bar constructions:
  - classically, for $A \in \mathrm{Alg}^{\mathrm{aug}} = \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{E}_1}$ one defines $\mathrm{Bar}(A) = \mathbb{1}_{\mathcal{C}} \otimes_A \mathbb{1}_{\mathcal{C}}$ (via the augmentation), or really $\mathrm{Bar}(A) = |\mathrm{Bar}_\bullet(\mathbb{1}_{\mathcal{C}}, A, \mathbb{1}_{\mathcal{C}})|$
  - note that $\mathrm{Bar} = \int_{\mathbb{R}^+}$, and $\mathbb{R}^+$ is just the pointed circle, which is a coassociative coalgebra not just in $\mathcal{S}_*$ but in $\mathcal{ZM}\mathrm{fld}_1$, as is its thickening $\mathbb{R}^+ \wedge (\mathbb{R}^{n-1})_+$ in $\mathcal{ZM}\mathrm{fld}_n$
  - so, this actually lifts as $\mathrm{Bar} : \mathrm{Alg}^{\mathrm{aug}}_1 \to \mathrm{coAlg}^{\mathrm{aug}}_1$

- the above is assertion analogous, but just replaces 1 with n, or alternatively recognizes $(\mathbb{R}^n)^+ = (\mathbb{R}^+)^{\wedge n}$

- aside: a quick proof of (complex) ***bott periodicity*** using these techniques
  - consider the topological space $B\text{Vect} = B\text{Vect}_{\mathbb{C}} = \bigsqcup_{n\geq 0} BU(n)$, the topological groupoid of finite-dimensional complex vector spaces; this has a (symmetric) monoidal structure given by direct sum of vector spaces
  - we'll show that the delooping of this monoid is equivalent to U, i.e. that $B(B\text{Vect}) \simeq U$
  - since $\Omega \circ B(-)$ computes group-completion, then $\Omega(B(B\text{Vect})) \simeq BU \times \mathbb{Z}$, and so the above equivalence implies that $BU \times \mathbb{Z} \times \simeq \Omega U \simeq \Omega^2(BU \times \mathbb{Z})$, i.e. bott periodicity
  - now, we have
  $$B(B\text{Vect}) := \text{Bar}(B\text{Vect}) := \int_{\mathbb{R}^+} B\text{Vect},$$
  i.e. the space of configurations of non-basepoint points of $(\mathbb{R})^+ \cong S^1$ labeled by fdim vector spaces, where
    - $*$ we take $\oplus$ when points collide,
    - $*$ we forget points if they slide off to the basepoint, and
    - $*$ we allow points to appear labeled by a 0-dimensional vector space.
  - this space is equivalent to one where all the vector space labels are mutually orthogonal subspaces of $\mathbb{C}^\infty$, essentially by Gram–Schmidt (applied in a continuous sort of way)
  - this admits a map from U given by "spectrum" (i.e. given $X \in U$, label a point of $S^1 \subset \mathbb{C}$ by the corresponding eigenspace of X (besides the basepoint $1 \in S^1$)), and the spectral theorem asserts that this is a **homeomorphism**

- now, in the adjunction
$$\text{Fun}^{\otimes}(\mathcal{ZM}\text{fld}_n, \mathcal{C}) \underset{\int^{(=)^\neg}(-)}{\overset{(\iota^+)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{coAlg}_n^{\text{aug}}(\mathcal{C})$$
we have
$$\begin{array}{ccc} \int_{(-)} A & \longmapsto & \text{Bar}^n(A) \\ \int^{(-)^\neg} \text{Bar}^n(A), & \longleftarrow\!\shortmid & \text{Bar}^n(A) \end{array}$$
and so the unit (evaluated at M_*) gives a ***Poincaré duality map***
$$\int_{M_*} A \to \int^{M_*^\neg} \text{Bar}^n(A)$$

- another (equivalent) way of getting this map is as follows: by definition, for any $C \in \text{coAlg}_n(\mathcal{C})$ we have $\int^{(-)^\neg} C = (\iota^+)_* C$ (the right Kan extension), which fits into a (noncommutative) diagram



in which
  - on the one hand, the outer arrows commute by construction, while
  - on the other hand, the dotted arrow is a right Kan extension, i.e. it's the terminal functor $\mathcal{ZM}\text{fld}_n \xrightarrow{F} \mathcal{C}$ equipped with a map $F \circ \iota^+ \to \text{Bar}^n(A)$,

and so we get a universal map $\int_{(-)} A \to \int^{(-)^\neg} \mathrm{Bar}^n(A)$

- this is a (functorial!) version of the *scanning map*, which is easiest to describe when $(\mathcal{C}, \otimes) = (\mathcal{S}, \times)$, in which case it reduces to our previous nonabelian Poincaré duality equivalence

$$\int_M A \xrightarrow{\sim} \mathrm{map}^c(M, B^n A)$$

(say for $A \in \mathrm{Ab}$): then it takes a configuration of k points in M each labeled by an element of A to the corresponding composite $M \to \bigvee_k S^n \to B^n A = K(A, n)$ given by first collapsing to little neighborhoods of the points and then on each summand selecting the element of $\pi_n(K(A,n)) \cong A$ corresponding to the given label
- dual considerations give a map

$$\int_{M_*} \mathrm{coBar}^n(C) \to \int^{M_*^\neg} C,$$

but this is ultimately the exact same map after applying the appropriate involutions ($M_* \rightsquigarrow M_*^\neg$ and $\mathcal{C} \rightsquigarrow \mathcal{C}^{op}$)
- of course, a natural question is: when is the Poincaré duality map

$$\int_{M_*} A \to \int^{M_*^\neg} \mathrm{Bar}^n(A)$$

an equivalence?
- of course, there are a number of ways to slice this. let's quantify over all zero-pointed n-manifolds. then, it turns out that it's only guaranteed to be an equivalence in fairly restrictive situations. first of all, it's clearly *necessary* for $A \xrightarrow{\sim} \mathrm{coBar}^n(\mathrm{Bar}^n(A))$ (by evaluating on $M_* = (\mathbb{R}^n)_+$). but this is only *sufficient* for $(\mathcal{C}, \otimes) = (\mathcal{X}, \times)$, where $\mathcal{X}$ is an ∞-topos (i.e. either $\mathcal{S}$ or more generally pre/sheaves of spaces on any ∞-category / ∞-site) or else $\mathcal{X}$ stable and presentable (so that $\times = \oplus$, in which case as we've seen factorization homology essentially reduces to ordinary homology)
- in order to understand what's going on with the failure of the Poincaré duality map to be an equivalence, we'll need to talk about *deformation theory*

3.2. Formal moduli problems and deformation theory.

- all of this is due to Lurie (building on work of many others) – see his ICM address for an overview
- we work over a field k
- we begin with the commutative case first, so we'll work with $\mathrm{CAlg}_k = \mathrm{CAlg}(\mathcal{S}\mathrm{p})_{k/}$
  - if char(k) $= 0$ then CAlg_k is equivalent to (the underlying ∞-category of) cdga_k, but otherwise it's somewhat larger
- we define (***local***) ***Artin*** ***k-algebras*** to be the full subcategory $\mathrm{Artin} = \mathrm{Artin}_k \subset \mathrm{CAlg}_k$ on those $A \in \mathrm{CAlg}$ such that:
  - A is *perfect* (a/k/a "finite", i.e. generated as a k-module under finite colimits by k itself);
  - A is *connective* (i.e. $\pi_{<0}(A) = 0$);
  - *[locality]* the ring $\pi_0 A$ is local and the composite $k \to \pi_0 A \to (\pi_0 A)/\mathfrak{m}_A$ is an isomorphism

  in particular, note that Artin k-algebras are canonically augmented
- now, a ***moduli problem*** is a functor $X : \mathrm{CAlg} \to \mathcal{S}$
  - the main examples are derived schemes and derived stacks (so we're just missing a "geometricity" condition)
- given a moduli problem X and a k-point $x \in X(k)$, its ***completion*** at X is the functor $X^\wedge_x : \mathrm{Artin} \to \mathcal{S}_*$ given by

$$X^\wedge_x(A) = \lim \left(\begin{array}{ccc} & & X(A) \\ & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[x]{} & X(k) \end{array} \right)$$

  - this is identical to the definition in classical scheme theory
    - for $X = \mathrm{Spec}(A)$ and $x : \mathrm{Spec}(k) \to \mathrm{Spec}(A)$ selecting an augmentation $A \to A/\mathfrak{m} \cong k$, one often defines the formal completion at this point as $\mathrm{Spf}(A^\wedge_\mathfrak{m})$, the *formal* (*Zariski*) *spectrum* of the completion $A^\wedge_\mathfrak{m} = \lim(\cdots \to A/\mathfrak{m}^2 \to A/\mathfrak{m})$ (equipped with the "adic topology")

      · by definition, this evaluates on a(n untopologized, local) artin k-algebra R as

$$\hom^{\mathrm{cts}}(A^{\wedge}_{\mathfrak{m}}, R) \cong \mathrm{colim}_n \hom(A/\mathfrak{m}^n, R)$$

      · more generally, for $I \subset A$ an ideal, we get a "tubular neighborhood"

$$\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spf}(A^{\wedge}_I) \hookrightarrow \mathrm{Spec}(A)$$

      (suitably interpreted)

    * in the end, this is just a formula for exactly the same pullback as depicted above: the observation is that that pullback can be computed in these isomorphic ways

- axiomatizing the properties of $X^{\wedge}_x$, we define a ***formal moduli problem*** to be a functor $Y : \mathrm{Artin} \to \mathcal{S}$ such that:
  - $Y(k) \simeq \mathrm{pt}_{\mathcal{S}}$ (so that Y canonically factors through the forgetful functor $\mathcal{S}_* \to \mathcal{S}$);
  - if $A \to B \leftarrow C$ in Artin such that $\pi_0 A \to \pi_0 B \leftarrow \pi_0 C$ are surjections, then the canonical map

$$Y(A \times_B C) \to Y(A) \times_{Y(B)} Y(C)$$

    is an equivalence.

  we write Moduli $\subset \mathrm{Fun}(\mathrm{Artin}, \mathcal{S})$ for the full subcategory of formal moduli problems.
- not every $X^{\wedge}_x$ satisfies the second condition, but it does whenever X was defined "geometrically": the idea is that

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(C) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A \times_B C) \end{array}$$

  should be a pushout square (among these "sufficiently small" affine schemes)
- for a scheme X and a point $x \in X(k)$, classically one defines the tangent space as the fiber $T_x X = \mathrm{fib}(X(k[\varepsilon]/\varepsilon^2) \to X(k))$ over x
- here, for a formal moduli problem X we define its ***tangent complex*** as $T_X \in \mathrm{Mod}_k$ as a spectrum by $T_X(n) = X(k \oplus \Sigma^n k)$ (equipped with a square-zero multiplication); that this gives a spectrum comes from the pullback square

$$\begin{array}{ccc} k \oplus \Sigma^{n-1} k & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus \Sigma^n k \end{array}$$

  for $n \geq 1$ implying that we get $T_X(n-1) \simeq \Omega T_X(n)$
  - more precisely, we can construct this from the *excisive functors* picture of spectra / stabilization:
    * first of all, we have $\mathrm{Mod}_k \xrightarrow{\sim} \mathrm{Exc}_*(\mathrm{Mod}_k^{\mathrm{fin}}, \mathcal{S})$ by $W \mapsto (V \mapsto \hom_k(V^\vee, W))$
    * then, we define $T_X \in \mathrm{Exc}_*(\mathrm{Mod}_k^{\mathrm{fin}}, \mathcal{S})$ by $T_X(V) = X(k \oplus V)$
    * so, we have $T_X \in \mathrm{Mod}_k$ uniquely determined by the requirement that $\hom_k(V^\vee, T_X) \simeq X(k \oplus V)$
    * really, excisive functors are "globally" defined on $\mathcal{S}_*^{\mathrm{fin}}$, which might be thought of as the ∞-category of *universal Artin algebras*; this subcategory $\mathrm{Mod}_k^{\mathrm{fin}} \subset \mathrm{Mod}_k$ is just the image of it under the tensoring $\mathcal{S}_* \times \mathrm{Mod}_k \to \mathrm{Mod}_k$, i.e. they're exactly "chains on pointed finite spaces"
  - in the classical (i.e. discrete) case, T_X is (as expected) the dual of Quillen's cotangent complex, which is itself the derived functor of Kähler differentials (i.e. 1-forms)
- the tangent functor $T_{(-)}$: Moduli $\to \mathrm{Mod}_k$ is a very nice approximation to formal moduli problems, e.g. it's conservative (just by decomposing any Artin k-algebra as an iterated square-zero extension of k)
- this leads to a natural question: is it possible to put additional structure on T_X so that this association $X \mapsto T_X$ becomes an equivalence?
- **main theorem (commutative case)**: if $\mathrm{char}(k) = 0$, then

$$\begin{array}{ccc} \text{Moduli} & \underset{\mathrm{MC}_{(-)}}{\overset{\Phi}{\underset{\sim}{\rightleftarrows}}} & \mathrm{dgla}_k \\ & \searrow^{X \mapsto \Sigma^{-1} T_X} & \downarrow U \\ & & \mathrm{Mod}_k \end{array}$$

where $\mathrm{MC}_{\mathfrak{g}}$ is a derived enhancement of the classical "Maurer–Cartan solutions" functor $A \mapsto \{x \in \mathfrak{g} \otimes_k \mathfrak{m}_A : dx = [x, x]\}$.

  - the Lie algebra structure on $\Sigma^{-1}T_X$ should be thought of as "Lie bracket of vector fields"
  - this is an instance of (the deformation-theoretic content of) koszul duality: Lie algebras parametrize deformations of commutative algebras (and in fact conversely)
- roughly, the **proof** goes as follows:
  - define $\Omega X \in \text{Moduli}$ by $(\Omega X)(A) = \Omega(X(A))$
    * this again satisfies the hypotheses of a formal moduli problem since Ω commutes with pullbacks among based spaces
    * it follows immediately that $T_{\Omega X} \simeq \Sigma^{-1}T_X$
    * on the other hand, ΩX is naturally a *group object* in Moduli (coming from the fact that $S^1 \in \mathcal{S}_*$ is a cogroup object)
  - now, recall that an algebraic group G gives a Lie algebra structure on T_eG, and in characteristic 0 this suffices to reconstruct $G_e^\wedge$ via Baker–Campbell–Hausdorff
  - analogously, the group structure on ΩX gives a Lie algebra structure on $T_{\Omega X} \simeq \Sigma^{-1}T_X$, and our assumption that $\text{char}(k) = 0$ plus the fact that we're only working in a formal neighborhood anyways allows us to reconstruct ΩX [and from this we can eventually get X back too]
- now, the $\mathcal{E}_n$-case (for $1 \leq n < \infty$)
- definitions are all essentially the same; write $\text{Alg}_{\mathcal{E}_n} = \text{Alg}_{\mathcal{E}_n}(\mathcal{S}\text{p})_{k/}$
  - however, now for *any* ring k we have $\text{Alg}_{\mathcal{E}_1} \simeq \text{dga}_k$ (it doesn't have to be a field, let alone of characteristic 0), though for general n there's not quite such an easy algebraic description of $\text{Alg}_{\mathcal{E}_n}$
  - also note that this has the nice inductive characterization $\text{Alg}_{\mathcal{E}_n} \simeq \text{Alg}_{\mathcal{E}_1}(\text{Alg}_{\mathcal{E}_{n-1}})$ (this is *Dunn's additivity theorem*)
- write $\text{Moduli}_n \subset \text{Fun}(\text{Artin}_n, \mathcal{S})$ for the full subcategory of $\mathcal{E}_n$ moduli problems
- this has a functor $\text{Moduli}_n \xrightarrow{U_n} \text{Moduli}$ obtained by precomposing with the forgetful functor $\text{Artin} \to \text{Artin}_n$
  - thus, we should think of a formal $\mathcal{E}_n$ moduli problem as containing *more* data than a formal $\mathcal{E}_\infty$ moduli problem: it additionally knows how to evaluate on not-totally-commutative Artin k-algebras
- for $X \in \text{Moduli}_n$, define its ***tangent complex*** to be $T_X = T_{U_n(X)} \in \text{Mod}_k$
- again, it is natural to ask: what additional structure must we remember in order for this to be an equivalence?
- **main theorem ($\mathcal{E}_n$ case)**: for any field k,

$$\begin{array}{ccc} \text{Moduli}_n & \underset{\text{MC}_{(-)}}{\overset{\Phi_n}{\underset{\sim}{\rightleftarrows}}} & \text{Alg}^{\text{aug}}_{\mathcal{E}_n} \\ & {\scriptstyle X \mapsto \Sigma^{-n}T_X} \searrow & \downarrow {\scriptstyle A \mapsto \mathfrak{m}_A} \\ & & \text{Mod}_k, \end{array}$$

where $\text{MC}_A(R) = \hom_{\text{Alg}^{\text{aug}}_{\mathcal{E}_n}}((\mathbb{D}^n R)^\vee, A) \simeq \hom_{\text{Alg}^{\text{aug}}_{\mathcal{E}_n}}(k, A \otimes_k \mathbb{D}^n R)$

  - the dual here is appearing just to keep our conventions consistent: Lurie's version of Koszul duality is the linear dual of Ayala–Francis's (so e.g. he defines the koszul dual of an augmented $\mathcal{E}_1$-algebra A to be $\text{End}_A(k)$ instead of $k \otimes_A k$)
    * the definitions are actually consistent (i.e. there's no issue with dualizing infinite-dimensional stuff), because we're only applying this to Artin objects
  - again, this is about the koszul self duality of the $\mathcal{E}_n$ operad
- in particular, this is saying that $\Sigma^{-n}T_X$ admits a nonunital $\mathcal{E}_n$ structure; as $\text{Alg}^{\text{aug}}_{\mathcal{E}_n} \xrightarrow[\sim]{\ker(\varepsilon)} \text{Alg}^{\text{nu}}_{\mathcal{E}_n}$ is an equivalence, then this structure determines the formal $\mathcal{E}_n$ moduli problem X itself
- in particular, this asserts that again the tangent complex functor $\text{Moduli}_n \to \text{Mod}_k$ is conservative (since the forgetful functor $\text{Alg}^{\text{nu}}_{\mathcal{E}_n} \to \text{Mod}_k$ is)
- the functor Φ_n is ***the derived functor of Koszul duality*** (and Φ is similarly):
  - for

$$A \in (\text{Artin}_n)^{op} \overset{\text{Spec}}{\hookrightarrow} \text{Moduli}_n$$

we have

$$\Phi_n(\mathrm{Spec}(A)) = (\mathbb{D}^n A)^\vee$$

– for

$$A = \{A_\alpha\} \in (\mathrm{Pro}(\mathrm{Artin}_n))^{op} \xhookrightarrow{\mathrm{Spf}} \mathrm{Moduli}_n$$

(i.e. a formal filtered colimit, so $\mathrm{Spf}(A)(R) = \mathrm{colim}_\alpha \mathrm{Spec}(A_\alpha)(R)$), we have

$$\Phi_n(\mathrm{Spf}(A)) = \mathrm{colim}_\alpha \Phi_n(\mathrm{Spec}(A_\alpha))$$

 * in fact, $\mathrm{Spf} : \mathrm{Pro}(\mathrm{Artin}_n)^{op} \to \mathrm{Moduli}_n$ defines an equivalence onto the full subcategory of *left exact* reduced functors $\mathrm{Artin}_n \to \mathcal{S}$ (i.e. those preserving *all* finite limits, i.e. those which preserve not just pullbacks but also equalizers)

– more generally, any $X \in \mathrm{Moduli}_n$ admits a *smooth hypercovering* of the form

$$\{\mathrm{Spf}(A^n) \to X\}_{n \geq 0}$$

(a simplicial object in $(\mathrm{Moduli}_n)_{/X}$ satisfying certain iterated lifting conditions, coming from a cosimplicial object $A^\bullet \in c\mathrm{Pro}(\mathrm{Artin}_n)$), and we have

$$\Phi_n(X) \simeq |\Phi_n(\mathrm{Spf}(A^\bullet))|$$

3.3. Factorization homology of formal moduli problems.

- let's return to our n-disk algebra setting, still working over a field k [and also apparently this should actually just be for $\mathcal{E}_n$-algebras – there are some pretty subtle subtleties involving $\mathrm{Top}(n)$-module structures on chain complexes, and stuff]
- apologies for all the reassignment of notation (though it's overall pretty mild)
- we define ***Artin n-algebras*** to be the smallest subcategory $\mathrm{Artin}_n \subset \mathrm{Alg}_n^{\mathrm{nu},\geq 0}$ which is closed under finite extensions (but nonunital $\approx$ augmented, so basically "connective and augmented" plus finiteness)
- we define (***n-disk***) ***formal moduli problems*** to be the functor ∞-category

$$\mathrm{Moduli}_n = \mathrm{Fun}(\mathrm{Artin}_n, \mathcal{S})$$

  – gluing conditions will be unnecessary for us, though these could be imposed
- as a source of examples, we have the restricted yoneda embedding and its *global sections* right adjoint

$$\mathrm{Spf} : (\mathrm{Alg}_n^{\mathrm{aug}})^{op} \rightleftarrows \mathrm{Moduli}_n : \mathcal{O}$$

  – in other words, $\mathcal{O}$ is given as a right Kan extension

$$\begin{array}{ccc} (\mathrm{Artin}_n)^{op} & \hookrightarrow & (\mathrm{Alg}_n^{\mathrm{aug}})^{op} \\ {\scriptstyle \mathrm{Spf}}\downarrow & \overset{\mathcal{O}}{\nearrow} & \\ \mathrm{Moduli}_n & & \end{array}$$

  so in formulas,

$$\mathcal{O}(X) = \lim_{\substack{R \in (\mathrm{Artin}_n)^{op} \\ \mathrm{Spf}(R) \to X}} R = \lim \left(((\mathrm{Artin}_n^{op})_{/X})^{op} \xrightarrow{U} \mathrm{Artin}_n \hookrightarrow \mathrm{Alg}_n^{\mathrm{aug}} \right)$$

- to simplify things (i.e. avoid coalgebra(ic geometry)), we define the ***Koszul duality*** functor to be the composite

$$\mathbb{D}^n : (\mathrm{Alg}_n^{\mathrm{aug}})^{op} \xrightarrow{(\mathrm{Bar}^n)^{op}} (\mathrm{coAlg}_n^{\mathrm{aug}})^{op} \xrightarrow{(-)^\vee} \mathrm{Alg}_n^{\mathrm{aug}}$$

- we now define the ***Maurer–Cartan functor***, denoted

$$\mathrm{Alg}_n^{\mathrm{aug}} \xrightarrow{\mathrm{MC}_{(-)}} \mathrm{Moduli}_n,$$

to be given by

$$\mathrm{MC}_A(R) = \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(\mathbb{D}^n R, A)$$

- a fundamental calculation is that $\mathcal{O}(\mathrm{MC}_A) \simeq \mathbb{D}^n A$
  – for a test object $Z \in \mathrm{Alg}_n^{\mathrm{aug}}$, we have

$$\hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(Z, \mathcal{O}(\mathrm{MC}_A)) \simeq \lim_{\substack{R \in (\mathrm{Artin}_n)^{op} \\ \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(\mathbb{D}^n R, A)}} \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(Z, R)$$

- roughly, we want to "set $R = \mathbb{D}^n A$", equipped with its tautological map $\mathbb{D}^n(\mathbb{D}^n A)) \to A$ (out of a linear dual)
- however, $\mathbb{D}^n A$ needn't be Artin, so this isn't quite legal
- on the other hand, there's an an *initial subsystem* of this indexing ∞-category (dual to our previous notion of "final") that plucks out these data, and limiting just over this initial subcategory gets us to

$$\simeq \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(Z, \mathbb{D}^n A)$$

- so, we get our desired equivalence $\mathcal{O}(\mathrm{MC}_A) \xrightarrow{\sim} \mathbb{D}^n A$ by yoneda

- thus, by adjunction we obtain a universal *affinization* map

$$\mathrm{Spf}(\mathbb{D}^n A) \to \mathrm{MC}_A,$$

though we might slightly more profitably think of as "the inclusion of the completion at the distinguished point"
  - here, "affine" doesn't so much refer to global algebro-geometric objects (such as e.g. the non-affine scheme $\mathbb{P}^1$) as it refers to *stacky* phenomena
- now, for $X \in \mathrm{Moduli}_n$ we define its (***non-affine***) ***factorization homology*** over any zero-pointed n-manifold M_* to be

$$\int_{M_*} X = \lim_{\substack{R \in \mathrm{Artin}_n^{op} \\ \hom_{\mathrm{Moduli}_n}(\mathrm{Spf}(R), X)}} \int_{M_*} R = \lim \left(((\mathrm{Artin}_n^{op})_{/X})^{op} \xrightarrow{U} \mathrm{Artin}_n \xrightarrow{\int_{M_*}(-)} \mathcal{C} \right),$$

  - globally, this is computed by the right Kan extension

$$\begin{array}{ccc} \mathrm{Artin}_n & \xrightarrow{\int_{(=)}(-)} & \mathrm{Fun}(\mathcal{ZM}\mathrm{fld}_n^{\mathrm{fin}}, \mathcal{C}) \\ {\scriptstyle \mathrm{Spf}}\downarrow & \nearrow_{\int_{(=)}(-)} & \\ \mathrm{Moduli}_n^{op} & & \end{array}$$

  which is now *contravariant* in the formal moduli problem
  - heuristically (but fairly precisely), this can be thought of as the global sections

$$\int_{M_*} X = \Gamma\left(X; \int_{M_*} \mathcal{O}_X\right)$$

  of the "affine" (i.e. locally computed) factorization homology (taken over neighborhoods in X defined by (Spf of) Artin algebras)
- the affinization map $\mathrm{Spf}(\mathbb{D}^n A) \to \mathrm{MC}_A$ gives

$$\int_{M_*} \mathrm{MC}_A \to \int_{M_*} \mathbb{D}^n A$$

(slightly abusing notation)
  - this is generally *not* an equivalence: the source doesn't generally satisfy $\otimes$-excision...
- but this is a good thing! (it's *non-perturbative*, and) it leads us to

Theorem 3 (Poincaré/Koszul duality). *for any $A \in \mathrm{Alg}_n^{\mathrm{aug}}$ and any $M \in \mathcal{ZM}\mathrm{fld}_n^{\mathrm{fin}}$, we have an equivalence*

$$\left(\int_{M_*} A\right)^{\vee} \xrightarrow{\sim} \int_{M_*^{\neg}} \mathrm{MC}_A.$$

- the **proof** uses the following three ingredients:
  (1) we'll compute the source via a resolution of A by (a sifted diagram of) objects in $\mathrm{FPres}_n^{\leq(-n)}$
    - heuristically, the existence of such a resolution (i.e. by highly-coconnective objects) is akin to the presentation of a space by points (e.g. $S^1 \simeq \mathrm{colim}(\mathrm{pt} \leftarrow S^0 \to \mathrm{pt})$), i.e. colimits can raise coconnectivity
    - the importance of siftedness is that the forgetful right adjoint $\mathrm{Alg}_n^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ commutes with sifted colimits, so that whereas factorization homology is computed as a colimit in $\mathcal{C}$, it commutes with sifted colimits in the algebra variable

(2) the koszul duality adjunction

$$\mathbb{D}^n : (\mathrm{Alg}_n^{\mathrm{aug}})^{op} \rightleftarrows \mathrm{Alg}_n^{\mathrm{aug}} : (\mathbb{D}^n)^{op}$$

restricts to an equivalence

$$(\mathrm{Artin}_n)^{op} \simeq \mathrm{FPres}_n^{\leq(-n)}$$

(3) for $F \in \mathrm{FPres}_n^{\leq(-n)}$, we have not just the asserted equivalence but moreover the affineness result

$$\left(\int_{M_*} F\right)^\vee \xrightarrow{\sim} \int_{M_*^\neg} \mathrm{MC}_F \simeq \int_{M_*^\neg} \mathbb{D}^n F;$$

– the latter (affineness) equivalence follows from the above (anti)equivalence, as for $R \in \mathrm{Artin}_n$ we have

$$\begin{aligned}\mathrm{MC}_F(R) &= \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(\mathbb{D}^n R, F) = \hom_{\mathrm{Artin}_n}(\mathbb{D}^n R, F) \\ &\simeq \hom_{\mathrm{FPres}_n^{\leq(-n)}}(\mathbb{D}^n F, R) = \mathrm{Spf}(\mathbb{D}^n F)(R)\end{aligned}$$

putting everything together, we have

$$\begin{aligned}\int_{M_*^\neg} \mathrm{MC}_A &= \lim_{\substack{R \in \mathrm{Artin}_n^{op} \\ \hom_{\mathrm{Moduli}_n}(\mathrm{Spf}(R), \mathrm{MC}_A)}} \int_{M_*^\neg} R \\ &= \lim_{\substack{R \in \mathrm{Artin}_n^{op} \\ \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(\mathbb{D}^n R, A)}} \int_{M_*^\neg} R \\ &\overset{(2)}{\simeq} \lim_{\substack{F \in \mathrm{FPres}_n^{\leq(-n)} \\ \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(F, A)}} \int_{M_*^\neg} \mathbb{D}^n F \\ &\overset{(3)}{\simeq} \lim_{\substack{F \in \mathrm{FPres}_n^{\leq(-n)} \\ \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(F, A)}} \left(\int_{M_*} F\right)^\vee \\ &\simeq \left(\operatorname*{colim}_{\substack{F \in \mathrm{FPres}_n^{\leq(-n)} \\ \hom_{\mathrm{Alg}_n^{\mathrm{aug}}}(F, A)}} \int_{M_*} F \right)^\vee \\ &\overset{(1)}{\simeq} \left(\int_{M_*} A\right)^\vee .\end{aligned}$$