MATH 641: FACTORIZATION HOMOLOGY
HOMEWORK PROBLEMS

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0. Preliminaries

0.1. **Instructions.** The homework problems assigned thus far below are listed below, organized by the lecture notes in which they appear. (This document will be updated regularly over the course of the semester.) Homework is due at the end of the semester. It must be written in TeX.

Each problem or sub-problem is labeled with a point value, corresponding to its difficulty. Partial answers will receive partial credit. However, sub-problems are linearly ordered: to earn points for sub-problem (n), you must have earned points for sub-problem $(n-1)$.

The current number of points necessary to receive an A is 25.

(The problems' point values may change slightly as this document continues to be updated (and certainly the running total will change), but they will not change by much.)

Being the first person to point out a typo in this document to me is worth 1 point. Being the first person to point out a mathematical error in this document to me is worth at least as many points as the problem or sub-problem itself, with the precise point value determined at my discretion.

As you probably know by now, learning mathematics requires active participation. For this reason, you are strongly encouraged to think about these problems as they are assigned (rather than waiting until the end of the semester). You are also encouraged to work on those problems that deal with the material with which you are *least* familiar. Please use me as a resource. You may also use any other resources, but you must write up your own solution, and also clearly indicate which resources you used (e.g. a fellow graduate student's name, the URL of a website).

0.2. **Notation and conventions.**

0.2.1. *Algebra.*

(1) All rings are unital.

(2) We write $\Bbbk$ for a fixed commutative ring.

(3) We write $\mathsf{Mod}_{\Bbbk}$ for the category of $\Bbbk$-modules, $\mathsf{Alg}_{\Bbbk}$ for the category of $\Bbbk$-algebras, and $\mathsf{CAlg}_{\Bbbk}$ for the category of commutative $\Bbbk$-algebras. We write $\otimes := \otimes_{\Bbbk}$ for the tensor product of $\Bbbk$-modules.

(4) We write $A \in \mathsf{Alg}_{\Bbbk}$ for a fixed $\Bbbk$-algebra.

(5) We write LMod_A for the category of left A-modules, and RMod_A for the category of right A-modules. For a (left or right) A-module M, we typically write α_M for the action map.

(6) For any $d \geq 1$, we write $\mathsf{M}_d(A) \in \mathsf{Alg}_{\Bbbk}$ for the algebra of $d \times d$ matrices with values in A.

0.2.2. *Topology.*

(1) We write $S^n = \{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\}$ for the n-sphere and $D^n = \{\vec{x} \in \mathbb{R}^n : |\vec{x}| \leq 1\}$ for the n-disk.

Date: February 27, 2019.

0.2.3. *Category theory.*

(1) For any categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ for the category
- whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and
- whose morphisms are natural transformations.

(2) For any (resp. symmetric) monoidal categories $\mathcal{V}$ and $\mathcal{W}$, we write $\mathsf{Fun}^{\otimes}(\mathcal{V},\mathcal{W})$ for the category
- whose objects are (resp. symmetric) monoidal functors from $\mathcal{V}$ to $\mathcal{W}$, and
- whose morphisms are natural transformations.

(3) For any category $\mathcal{C}$, we write $\mathsf{Yo} : \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^{\mathsf{op}},\mathsf{Set})$ for the Yoneda embedding, which takes an object $X \in \mathcal{C}$ to the functor $\mathsf{Yo}(X) := \mathsf{h}_X$ that it represents: $\mathsf{h}_X(Y) := \hom_{\mathcal{C}}(Y,X)$. We also write $\mathsf{h}^X \in \mathsf{Fun}(\mathcal{C},\mathsf{Set})$ for the functor that $X \in \mathcal{C}$ corepresents: $\mathsf{h}^X(Y) := \hom_{\mathcal{C}}(X,Y)$.

(4) Whenever they exist, we write $\varnothing_{\mathcal{C}}, \mathsf{pt}_{\mathcal{C}} \in \mathcal{C}$ for the initial and terminal objects in a category $\mathcal{C}$. For brevity, we may also simply write $\varnothing := \varnothing_{\mathcal{C}}$ and $\mathsf{pt} := \mathsf{pt}_{\mathcal{C}}$.

(5) We write Set for the category of sets, and $\mathsf{Fin} \subset \mathsf{Set}$ for the subcategory of finite sets.

(6) For any $p \geq 0$, we write $\underline{p} = \{1,2,\ldots,p\} \in \mathsf{Fin}$ for the standard p-element set. (In particular, $\underline{0} = \varnothing$.)

(7) We write Top for the category of topological spaces.

(8) We write Ab for the category of abelian groups. More generally, we write $\mathsf{Ab}(\mathcal{C})$ for the category of abelian group objects in a category $\mathcal{C}$. (So, $\mathsf{Ab} := \mathsf{Ab}(\mathsf{Set})$.)

0.2.4. *Factorization homology.*

(1) For a collection $\mathcal{B}$ of basic singularity types, we write $\mathsf{Disk}^{\mathcal{B}}$ for the category
- whose objects are finite disjoint unions of elements of $\mathcal{B}$, and
- whose morphisms are embeddings;

we consider $\mathsf{Disk}^{\mathcal{B}}$ as symmetric monoidal via disjoint union.

(2) For a collection $\mathcal{B}$ of basic singularity types and a symmetric monoidal category $\mathcal{V}$, we write $\mathsf{LCCS}^{\mathcal{B}}_{\mathcal{V}} \subset \mathsf{Fun}^{\otimes}(\mathsf{Disk}^{\mathcal{B}},\mathcal{V})$ for the full subcategory on the locally constant coefficient systems.

1. The idea of factorization homology

1.1. For any $A \in \mathsf{Alg}_{\Bbbk}$, construct an isomorphism [8 points]

$$\int_{S^1}^{\mathsf{Mod}_{\Bbbk}} \mathsf{M}_2(A) \xrightarrow{\cong} \int_{S^1}^{\mathsf{Mod}_{\Bbbk}} A\ .$$

Hint: Trace.

1.2. For any $A \in \mathsf{Alg}_{\Bbbk}$, $M \in \mathsf{RMod}_A$, and $N \in \mathsf{LMod}_A$, construct an isomorphism [5 points]

$$\int_{D^1}^{\mathsf{Mod}_{\Bbbk}} (M \curvearrowleft A \curvearrowright N) \xrightarrow{\cong} M \otimes_A N := \mathsf{coeq}\left(M \otimes A \otimes N \underset{\mathsf{id}_M \otimes \alpha_N}{\overset{\alpha_M \otimes \mathsf{id}_N}{\rightrightarrows}} M \otimes N \right)\ .$$

1.3. In this problem, you'll study locally constant coefficient systems in dimension 1, with respect to the collections $\mathcal{B} = \{\mathbb{R}^1\}$ and $\tilde{\mathcal{B}} = \{\mathbb{R}^1, \mathbb{R}^1_{\geq 0}, \mathbb{R}^1_{\leq 0}\}$ of basic singularity types (considered as oriented 1-manifolds, possibly with boundary).

1.3.1. In this sub-problem, you'll prove an equivalence of categories

$$\mathsf{LCCS}^{\mathcal{B}}_{\mathsf{Mod}_{\Bbbk}} \simeq \mathsf{Alg}_{\Bbbk} \ :$$

locally constant coefficient systems for boundaryless 1-manifolds valued in $\Bbbk$-modules are associative $\Bbbk$-algebras.

(1) Enumerate the isomorphism classes of objects of $\mathsf{Disk}^{\mathcal{B}}$. [1 point]

(2) Prove that $\mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(\varnothing, \mathbb{R}^1)$ has exactly one element, and hence exactly one isotopy class. [1 point]

(3) Prove that $\mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(\mathbb{R}^1, \mathbb{R}^1)$ has exactly one isotopy class (e.g. by exhibiting an isotopy between any element and the identity element). Hint: Use the identification [5 points]

$$\mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(\mathbb{R}^1, \mathbb{R}^1) = \{ f \in C^{\infty}(\mathbb{R}^1) : f'(x) > 0 \text{ for all } x \in \mathbb{R}^1 \} \ .$$

(4) Prove that $\mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(\mathbb{R}^1 \sqcup \mathbb{R}^1, \mathbb{R}^1)$ has exactly two isotopy classes. [2 points]

(5) For any $k \geq 0$, construct a surjective function [3 points]

$$\mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}\left(\bigsqcup_{k} \mathbb{R}^1, \mathbb{R}^1 \right) \longrightarrow \{\text{total orderings of the set } \underline{k}\}$$

and prove that it is precisely the quotient by the equivalence relation of isotopy.

(6) For any $k, l \geq 0$, describe the set [3 points]

$$\mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}\left(\bigsqcup_{k} \mathbb{R}^1, \bigsqcup_{l} \mathbb{R}^1 \right) \ ,$$

and describe its quotient by the equivalence relation of isotopy.

(7) Prove that the relation of isotopy is contagious under composition: [2 points]

(a) for any $i \in \mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(D_0, D_1)$, the morphisms $fi, gi \in \mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(D_0, D_2)$ are isotopic;

(b) for any $j \in \mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(D_2, D_3)$, the morphisms $jf, jg \in \mathsf{hom}_{\mathsf{Disk}^{\mathcal{B}}}(D_1, D_3)$ are isotopic.

(8) It follows that quotienting the hom-sets of $\mathsf{Disk}^{\mathcal{B}}$ defines a new category; for reasons that will become clear later, this category is called $\mathsf{ho}(\mathcal{D}\mathsf{isk}^{\mathcal{B}})$. It is also symmetric monoidal. There is a symmetric monoidal functor [5 points]

$$\mathsf{Disk}^{\mathcal{B}} \longrightarrow \mathsf{ho}(\mathcal{D}\mathsf{isk}^{\mathcal{B}})$$

which is the identity on objects, and which on hom-sets implements the equivalence relation of isotopy. Describe the symmetric monoidal category $\mathsf{ho}(\mathcal{D}\mathsf{isk}^{\mathcal{B}})$ in combinatorial terms.

(9) It follows that the subcategory $\mathsf{LCCS}^{\mathcal{B}}_{\mathsf{Mod}_{\Bbbk}} \subset \mathsf{Fun}^{\otimes}(\mathsf{Disk}^{\mathcal{B}}, \mathsf{Mod}_{\Bbbk})$ consists of precisely those symmetric monoidal functors F that admit a factorization [5 points]

$$\begin{array}{ccc} \mathsf{Disk}^{\mathcal{B}} & \xrightarrow{F} & \mathsf{Mod}_{\Bbbk} \\ \downarrow & \nearrow_{\dashrightarrow} & \\ \mathsf{ho}(\mathcal{D}\mathsf{isk}^{\mathcal{B}}) & & \end{array}$$

(which is necessarily unique (and in fact uniquely symmetric monoidal) if it exists). Use this observation to prove an equivalence $\mathsf{LCCS}^{\mathcal{B}}_{\mathsf{Mod}_{\Bbbk}} \simeq \mathsf{Alg}_{\Bbbk}$. Hint: The diagram

$$\begin{array}{ccc} \mathsf{LCCS}^{\mathcal{B}}_{\mathsf{Mod}_{\Bbbk}} & \xleftarrow{\ \sim\ }\!\!\!\rightarrow & \mathsf{Alg}_{\Bbbk} \\ & {\scriptstyle \mathsf{ev}_{\mathbb{R}^1}} \searrow \quad \swarrow {\scriptstyle \mathsf{fgt}} & \\ & \mathsf{Mod}_{\Bbbk} & \end{array}$$

commutes.

1.3.2. In this sub-problem, you'll prove that there is a pullback square

$$\begin{array}{ccc} (\mathsf{RMod}_A)_{A/} \times (\mathsf{LMod}_A)_{A/} & \longrightarrow & \mathsf{LCCS}^{\tilde{\mathcal{B}}}_{\mathsf{Mod}_{\Bbbk}} \\ \downarrow & & \downarrow \mathsf{fgt} \\ \mathsf{pt} & \xrightarrow[\langle A \rangle]{} & \mathsf{LCCS}^{\mathcal{B}}_{\mathsf{Mod}_{\Bbbk}} \end{array} \quad :$$

extending the locally constant coefficient system $A \in \mathsf{Alg}_{\Bbbk} \simeq \mathsf{LCCS}^{\mathcal{B}}_{\mathsf{Mod}_{\Bbbk}}$ for boundaryless 1-manifolds to 1-manifolds with boundary is equivalent to specifying a pointed right A-module and a pointed left A-module.[1]

(10) Enumerate the isomorphism classes of objects of $\mathsf{Disk}^{\tilde{\mathcal{B}}}$. [1 point]

(11) Specify conditions on objects $D_0, D_1 \in \mathsf{Disk}^{\tilde{\mathcal{B}}}$ that are necessary and sufficient to guarantee that $\mathsf{hom}_{\mathsf{Disk}^{\tilde{\mathcal{B}}}}(D_0, D_1) = \varnothing$. [3 points]

(12) Prove that $\mathsf{hom}_{\mathsf{Disk}^{\tilde{\mathcal{B}}}}(\mathbb{R}^1_{\geq 0}, \mathbb{R}^1_{\geq 0})$ has exactly one isotopy class. [5 points]

(13) Prove that $\mathsf{hom}_{\mathsf{Disk}^{\tilde{\mathcal{B}}}}(\mathbb{R}^1_{\geq 0} \sqcup \mathbb{R}^1, \mathbb{R}^1_{\geq 0})$ has exactly one isotopy class. [3 points]

(14) Prove that $\mathsf{hom}_{\mathsf{Disk}^{\tilde{\mathcal{B}}}}(\mathbb{R}^1_{\geq 0} \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^1, \mathbb{R}^1_{\geq 0})$ has exactly two isotopy classes. [3 points]

(15) For any $k, k', l, l' \geq 0$, describe the set [3 points]

$$\mathsf{hom}_{\mathsf{Disk}^{\tilde{\mathcal{B}}}} \left(\coprod_k \mathbb{R}^1_{\geq 0} \sqcup \coprod_{k'} \mathbb{R}^1 , \coprod_l \mathbb{R}^1_{\geq 0} \sqcup \coprod_{l'} \mathbb{R}^1 \right) ,$$

and describe its quotient by the equivalence relation of isotopy.

(16) Using again that the relation of isotopy is contagious under composition, it follows that quotienting the hom-sets of $\mathsf{Disk}^{\tilde{\mathcal{B}}}$ defines a new category; this will be called $\mathsf{ho}(\mathcal{D}\mathsf{isk}^{\tilde{\mathcal{B}}})$. It is also symmetric monoidal. There is a symmetric monoidal functor [5 points]

$$\mathsf{Disk}^{\tilde{\mathcal{B}}} \longrightarrow \mathsf{ho}(\mathcal{D}\mathsf{isk}^{\tilde{\mathcal{B}}})$$

which is the identity on objects, and which on hom-sets implements the equivalence relation of isotopy. Describe the symmetric monoidal category $\mathsf{ho}(\mathcal{D}\mathsf{isk}^{\tilde{\mathcal{B}}})$ in combinatorial terms.

(17) Use this combinatorial description to prove the asserted pullback square. [5 points]

2. The Dold–Thom theorem

2.1. Define a "free topological abelian group" functor [10 points]

$$\mathsf{Top} \underset{\mathsf{fgt}}{\overset{\mathbb{Z}\{-\}}{\underset{\longleftarrow}{\overset{\dashrightarrow}{\bot}}}} \mathsf{Ab}(\mathsf{Top})$$

[1] In general, the category of *pointed* objects in a category $\mathcal{C}$ admitting a terminal object $\mathsf{pt} := \mathsf{pt}_{\mathcal{C}} \in \mathcal{C}$ is the undercategory $\mathcal{C}_* := \mathcal{C}_{\mathsf{pt}/}$. However, in the context of A-modules, a *pointing* typically means a morphism from A itself. (Indeed, if $\mathcal{C}$ has a zero object (i.e. an object that is simultaneously initial and terminal (so that in particular the terminal object is initial)), then all objects are canonically pointed: the forgetful functor $\mathcal{C}_{\mathsf{pt}/} \xrightarrow{\sim} \mathcal{C}$ is an equivalence.

that is left adjoint to the forgetful functor. Hint: The diagram

$$\begin{array}{ccc} \mathsf{Top} & \overset{\mathbb{Z}\{-\}}{\dashrightarrow} & \mathsf{Ab}(\mathsf{Top}) \\ {\scriptstyle \mathsf{fgt}}\downarrow & & \downarrow{\scriptstyle \mathsf{fgt}} \\ \mathsf{Set} & \underset{\mathbb{Z}\{-\}}{\longrightarrow} & \mathsf{Ab}(\mathsf{Set}) \end{array}$$

commutes.

3. Singular homology

3.1. Using only the Eilenberg–Steenrod axioms, compute $H_*(S^n;\Bbbk)$ for any $n \geq 0$. Hint: $S^n \cong D^n/S^{n-1}$. [3 points]

4. Simplicial sets and geometric realization

4.1. Prove that the morphisms in $\mathbf{\Delta}$ are generated by [8 points]

- the coface maps $\delta^n_i : [n-1] \to [n]$ (the unique order-preserving injection that misses the element $i \in [n]$) and
- the codegeneracy maps $\sigma^n_i : [n+1] \to [n]$ (the unique order-preserving surjection under which the preimage of $i \in [n]$ has two elements),

subject to the following relations:

$$\begin{cases} \delta^{n+1}_j \circ \delta^n_i = \delta^{n+1}_i \circ \delta^n_{j-1} \ , & 0 \leq i < j \leq n \\ \sigma^n_j \circ \sigma^{n+1}_i = \sigma^n_i \circ \sigma^{n+1}_{j+1} \ , & 0 \leq i \leq j < n \\ \sigma^n_j \circ \delta^{n+1}_i = \begin{cases} \delta^n_i \circ \sigma^{n-1}_{j-1} \ , & 0 \leq i < j < n \\ \mathsf{id}_{[n]} \ , & 0 \leq j \leq i \leq j+1 \leq n \\ \delta^n_{i-1} \circ \sigma^{n-1}_j \ , & 0 \leq j < j+1 < i \leq n \ . \end{cases} \end{cases}$$

4.2. Prove the ("strong") Yoneda lemma, that for any category $\mathcal{C}$, any object $X \in \mathcal{C}$, and any functor $F \in \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set})$, there is a canonical isomorphism [10 points]

$$\begin{array}{ccc} \mathsf{hom}_{\mathsf{Fun}(\mathcal{C}^{\mathsf{op}},\mathsf{Set})}(\mathsf{h}_X, F) & \overset{\cong}{\longleftrightarrow} & F(X) \\ \alpha & \longmapsto & \alpha_X(\mathsf{id}_X) \\ \left(\mathsf{h}_X \xrightarrow{(Y \xrightarrow{f} X) \mapsto F(f)(\varphi)} F \right) & \longleftarrow\!\shortmid & \varphi \end{array} .$$

(In particular, make sense of this notation.) Formulate and prove an analogous statement for the contravariant Yoneda functor $\mathcal{C}^{\mathsf{op}} \xrightarrow{X \mapsto \mathsf{h}^X} \mathsf{Fun}(\mathcal{C}, \mathsf{Set})$. Hint: Use the first part to prove the second.

4.3. Let $\mathcal{A}$ be a category. Using the formula for left Kan extension, prove that for any cocomplete category $\mathcal{C}$, the left Kan extension [3 points]

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathsf{pt} & & \end{array}$$

selects the object $\mathsf{colim}_{\mathcal{A}}(F) \in \mathcal{C}$.

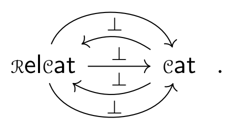
4.4. Let $\mathcal{A}$ be a category, and let $\mathcal{A}^{\triangleright}$ denote the *right cone* on $\mathcal{A}$: the category obtained from $\mathcal{A}$ by freely adjoining a terminal object ∞. Using the formula for left Kan extension, prove that for any cocomplete category $\mathcal{C}$, the left Kan extension [5 points]

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathcal{A}^{\triangleright} & & \end{array}$$

takes an object $a \in \mathcal{A} \subset \mathcal{A}^{\triangleright}$ to the object $F(a)$ and takes the object $\infty \in \mathcal{A}^{\triangleright}$ to the object $\mathsf{colim}_{\mathcal{A}}(F) \in \mathcal{C}$.

5. Model categories

5.1. Find a sequence of adjunctions [3 points]



5.2. Prove that if $\mathcal{M}$ is finitely bicomplete, then setting $\mathbf{C} = \mathbf{F} = \mathcal{M}$ makes $\min(\mathcal{M}) = (\mathcal{M}, \iota_0\mathcal{M})$ into a model category. [5 points]

5.3. Prove that $\max(\mathsf{Set}) = (\mathsf{Set}, \mathsf{Set})$ admits a model structure, whose cofibrations are the injections and whose fibrations are the surjections. [5 points]