COURSE OUTLINE: FACTORIZATION HOMOLOGY

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Abstract. This is an outline of a topics course at USC for the spring 2019 semester, which uses factorization homology as an excuse to introduce and showcase the fundamentals of abstract homotopy theory. Topics to be covered: abstract homotopy theory; homology theories for manifolds; nonabelian Poincaré duality. Possible further topics: Poincaré/Koszul duality; homology theories for knots and links; factorization homology of (∞, n)-categories; trace methods in algebraic K-theory; connections with quantum field theory.

Contents

1. Overview

Factorization homology is a powerful formalism that merges manifolds and *higher algebra*. First, an $\mathbb{E}_n$-*algebra* is a higher-algebraic generalization of an associative ring (which is recovered as a special case when $n = 1$), which has multiplications indexed by configurations of n-disks in $\mathbb{R}^n$ and their collisions. Then, in its most primitive form, factorization homology takes as input an $\mathbb{E}_n$-algebra A and an n-manifold M and returns a chain complex

$$\int_M A$$

obtained by "integrating" A over configurations of n-disks in M [Lur, AF15].

(1) By fixing A and varying M, this can be seen as a *homology theory for manifolds*; a mild variant gives a homology theory for knots and links.
(2) By fixing M and varying A, this can be seen as a *homology theory for* $\mathbb{E}_n$-*algebras*.

Both of these directions of focus arose independently in their respective fields.

(1) Factorization homology is intimately related to the study of configuration spaces of manifolds, and to mapping spaces more generally. In particular, a central theorem in factorization homology is *nonabelian Poincaré duality* [Sal01, Seg10, Lur, AF15], an equivalence

$$\int_M A \simeq \mathsf{map}(M, \mathsf{B}^n A) \quad (*)$$

with a certain mapping space, which recovers the classical Poincaré duality isomorphism

$$H_i(M; A) \cong H^{n-i}(M; A) \quad (**)$$

in the special case that A is an abelian group.

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(2) In the case that $n = 1$, factorization homology over the circle recovers *Hochschild homology*, a central construction in the study of associative rings and their deformation theory. In particular, another core result in factorization homology (generalizing nonabelian Poincaré duality) is *Poincaré/Koszul duality* [AFb]. This intertwines Poincaré duality for manifolds with *Koszul duality* for $\mathbb{E}_n$-algebras, generalizing a classical result in Hochschild homology. But it is better, even when $n = 1$: the classical result requires certain "connectivity" hypotheses, whereas Poincaré/Koszul duality does not. This added generality stems from recognizing the failure of the naive guess

$$\left(\int_M A \right)^\vee \simeq \int_M \mathbb{D}^n A$$

(which only holds under connectivity hypotheses) as arising from deformation theory [Lur10]: the Koszul dual $\mathbb{E}_n$-algebra $\mathbb{D}^n A$ is the *affinization* of a more primitive object (which is generally non-affine) built from A.

2. Syllabus

2.1. **Abstract homotopy theory.** As indicated in §1, factorization homology belongs to the realm of "higher" – also called "derived" or "homotopy-coherent" – mathematics. Thus, we will begin the course with a thorough introduction to this area. Keywords to be defined and discussed include: chain complex; derived category; derived functor; model category; ∞-category; $\mathbb{E}_n$-algebra [Wei94, Hir03, MG, Lur09, Lur].

2.2. **Homology theories for manifolds.** In modern language, the Eilenberg–Steenrod axioms for a "homology theory for spaces" lead to an equivalence

$$\mathsf{ev}_{\mathsf{pt}} : \mathscr{H}(\mathsf{Spaces}, \mathsf{Ch}) \xrightarrow{\sim} \mathsf{Ch}$$

between chain complex-valued homology theories for spaces and chain complexes themselves, given by evaluation on the one-point space pt, and with inverse given by singular homology. Analogously, the Ayala–Francis axioms for a "homology theory for n-manifolds" lead to an equivalence

$$\mathsf{ev}_{\mathbb{R}^n} : \mathscr{H}(\mathsf{Mfld}_n, \mathcal{V}) \xrightarrow{\sim} \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{V})$$

between homology theories for n-manifolds valued in a symmetric monoidal ∞-category $\mathcal{V}$ and $\mathbb{E}_n$-algebras in $\mathcal{V}$, given by evaluation on the n-manifold $\mathbb{R}^n$, and with inverse given by factorization homology.

2.3. **Nonabelian Poincaré duality.** A primary example of an $\mathbb{E}_n$-algebra in spaces is an n-fold loopspace $A = \Omega^n X := \mathsf{map}_*(S^n, X)$ of a based space X; we write $\mathsf{B}^n A$ for the "universal" based space X with $A \simeq \Omega^n X$, and refer to it as the *n-fold delooping* of A. For such $\mathbb{E}_n$-algebras, the nonabelian Poincaré duality equivalence (*) holds. In particular, an abelian group is an *infinite loopspace*, i.e. an n-fold loopspace for all $n \geq 0$, and taking homotopy groups on both sides of the equivalence (*) yields the isomorphisms (**) for all $0 \leq i \leq n$.

2.4. **Possible further topic 1: Poincaré/Koszul duality.** The nonabelian Poincaré duality result of §2.3 relies crucially on the fact that $(\mathsf{Spaces}, \times)$ is a *cartesian* symmetric monoidal ∞-category, i.e. its monoidal structure is the cartesian product. Generalizing to an arbitrary symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes)$ (such as $(\mathsf{Ch}, \otimes)$, chain complexes with tensor product), we must also generalize the passage from A to $\mathsf{B}^n A$: this becomes an instance of *Koszul duality*. However, there is a twist: the passage from an $\mathbb{E}_n$-algebra A to its Koszul dual $\mathbb{D}^n A$ is still not quite enough to yield a general version of nonabelian Poincaré duality. Instead, we must consider the *Maurer–Cartan functor* of A, a construction which uses A to parametrize deformations of $\mathbb{E}_n$-algebras. This is in general a non-affine object (in the sense of algebraic geometry), whose affinization is $\mathbb{D}^n A$.

2.5. **Possible further topic 2: Homology theories for knots and links.** The reason that $\mathbb{E}_n$-algebras can be integrated over n-manifolds is that n-manifolds are locally modeled on $\mathbb{R}^n$. One can expand the list of local models to include $(\mathbb{R}^d \subset \mathbb{R}^n)$, giving homology theories for n-manifolds equipped with d-dimensional submanifolds [AFT17]. In particular, taking $d = 1$ and $n = 3$ gives homology theories for knots and links. Possibly in conjunction with the theory described in §2.6, these are expected to recover existing knot homology theories of interest.

2.6. **Possible further topic 3: Factorization homology of (∞, n)-categories.** A monoid is equivalent to a one-object category, just as an associative ring is equivalent to a one-object Ab-enriched category. More generally, an $\mathbb{E}_n$-algebra is equivalent to a one-object (∞, n)-category. One can extend the definition of factorization homology to take as coefficients not just $\mathbb{E}_n$-algebras but arbitrary (∞, n)-categories [AFR]. This is the context for a proof-in-progress of the *cobordism hypothesis* [AFa], which is arguably the best theorem in all of algebraic topology. A conjectural generalization to *enriched* (∞, n)-categories (see [AMGRa] for the case where $n = 1$) is expected to recover existing TQFTs of interest (see §2.8).

2.7. **Possible further topic 4: Trace methods in algebraic K-theory.** Algebraic K-theory is a means of studying varieties and schemes through their vector bundles, though it has deep connections throughout a wide range of areas of mathematics [FG05]. It is very difficult to compute, and essentially all known computations arise through a certain "trace" map $\mathsf{K} \to \mathsf{TC}$ to *topological cyclic homology*. Though TC was invented over two decades ago, its algebro-geometric meaning has only become fully understood recently [AMGRc, AMGRa, AMGRb]. This is because TC was originally built from topological Hochschild homology, i.e. spectrally-enriched factorization homology over S^1, using certain constructions which remain mysterious from an algebro-geometric point of view. The paper [AMGRb] provides a new construction of TC by analyzing the factorization homology of *spectrally-enriched* $(\infty, 1)$*-categories* [AMGRa] (recall §2.6), which purely through universal properties acquires a "cyclotomic structure" from an analogous structure on the factorization homology over S^1 of *unenriched* $(\infty, 1)$-categories [AMGRa], which in turn arises directly from the geometry of compact framed 1-manifolds.

2.8. **Possible further topic 5: Connections with quantum field theory.** Factorization homology was first conceived as an analog for manifolds of Beilinson–Drinfeld's algebro-geometric theory of chiral homology, which computes "conformal blocks" in conformal field theory [BD04]; it is also the basis for Costello–Gwilliam's approach to perturbative quantum field theory [CG17, CG]. In these settings, it may be viewed as a form of *categorified integration*. In the desire to piece together many local computations into a single global one, one confronts the question: Where do these computations live? Factorization homology provides an answer: Each local computation lives in its own canonically-defined local vector space, and these piece together into an element of a globally-defined vector space obtained through factorization homology.

3. Suggested prerequisites

Ideally, participants in the course would have some familiarity with the following concepts. However, any of these can be reviewed as appropriate.

- basic algebra: associative and commutative rings, modules, tensor products.
- basic category theory: categories, functors, limits and colimits, adjunctions, the Yoneda lemma.
- basic homotopy theory: homology and cohomology, fundamental groups.

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