SYLLABUS
ALGEBRAIC AND DIFFERENTIAL TOPOLOGY (MATH 151C)
SPRING 2022

Contents

1. General information

1.1. **Instructor.**

name: Aaron Mazel-Gee
email: mazelgee@caltech.edu
office hour: Mondays at 11am-12noon in 285 Linde[1]

1.2. **Teaching assistant (TA).**

name: Chi Zhang
email: czhang5@caltech.edu
office hour: Tuesdays at 5-6pm in 306 Linde

1.3. **Course logistics.**

lectures: MWF at 10-11am in 387 Linde
textbooks: We will follow *Characteristic classes* by Milnor & Stasheff, which is also available through the library's course reserves.
course website: etale.site/teaching/s22/
Canvas: yes, lightly (see §4.2)

1.4. **Homework and grades.** Homework will generally be assigned weekly. It will be due by the beginning of lecture (i.e. at 10am) on the due date, which will generally be Wednesdays. Your final grade in the course will be determined by your performance on the homework. Depending on how far into the material we get, there may also be a more open-ended final project (in which case the homework load would be correspondingly diminished).

last updated: March 28, 2022

[1] If my office hour conflicts with your schedule, I am also available to meet by appointment.

Please read the brief document Guidelines for Good Mathematical Writing, and adhere to its suggestions to the greatest extent possible. You will lose points for gross negligence.[2]

You are encouraged to write your homework in LaTeX (pronounced "LAH-teck"). This is a versatile and easy-to-use typesetting system, which is the industry standard in mathematics and adjacent fields. Some resources to help you get started writing in LaTeX are outlined in §4.5. In case you would like to include hand-drawn pictures in your homework submission, please embed these into the LaTeX file.

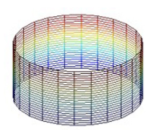
Homework will be submitted via Canvas. If you write out your homework by hand, please submit photos of it assembled as a single pdf (e.g. using an online photo-to-pdf converter).

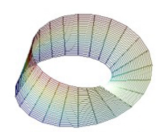
In order to account for unexpected circumstances, you are automatically granted a cumulative deadline extension of 72 hours.

1.5. **COVID-19 considerations.** Given the ongoing pandemic, it is essential that we all follow the guidance of Caltech and public health entities regarding measures such as wearing masks, maintaining an adequate distance, and quarantining. In particular, if you believe you may be contagious (e.g. due to symptoms or exposure to a known case), please let me know at your earliest convenience and do *not* attend lecture. It should be understood that the situation is subject to change at any time, and that we will all do our best to remain supportive and understanding as we navigate this unfamiliar and rapidly shifting terrain.

2. Course description

Roughly speaking, a ***vector bundle*** over a space X is an X-parametrized family $\{V_x\}_{x \in X}$ of (real or complex) vector spaces, which are called the *fibers* of the vector bundle. For example, the two real rank-1 vector bundles over the circle are the cylinder and the Möbius band.[3] The former is the *trivial* rank-1 vector bundle, while the latter is nontrivial; one says that it is *twisted*.





Vector bundles play two complementary roles in the study of spaces. On the one hand, a vector bundle may be thought of as providing a context for "generalized functions": a collection of vectors $\{v_x \in V_x\}_{x \in X}$ is called a (***cross-***)***section*** of the vector bundle, and a section of the trivial vector bundle with fiber $\mathbb{R}^1$ is the same thing as a function $X \to \mathbb{R}^1$.[4] On the other hand, a vector bundle can *itself* be thought of as a sort of "generalized function": it assigns to each point of X not a number but a vector space.

[2] In particular, you must write in complete sentences, and you should generally avoid shorthand symbols such as "$\because$" or "$\Rightarrow$" (although "iff" is acceptable).

[3] To be more precise, both of these spaces should be considered without their boundaries. Then, considering S^1 as the "core" circle, the fiber V_x corresponding to a point $x \in S^1$ is the open interval through x perpendicular to S^1 (considered as a copy of the vector space $\mathbb{R}^1$).

[4] One generally requires sections to be continuous (in an appropriate sense), or even smooth (in the context of a smooth vector bundle over a smooth manifold).

These two roles are related. For example, given a vector bundle, a basic question one can ask is whether it admits a *nonvanishing* section, i.e. a section $\{v_x \in V_x\}_{x \in X}$ such that $v_x \neq 0$ for all $x \in X$. As a trivial vector bundle always admits nonvanishing sections (e.g. those corresponding to nonvanishing constant functions), the failure of the existence of a nonvanishing section can be seen as an *obstruction* to the triviality of a given vector bundle. Indeed, as for the two rank-1 vector bundles over the circle, the cylinder (being a trivial vector bundle) admits a nonvanishing section, whereas the Möbius band does not – essentially by the intermediate value theorem.

In this course, we will study the theory of ***characteristic classes*** of vector bundles, which we overview presently.

The characteristic classes of a vector bundle over a space X are elements of the cohomology $H^*(X)$. These cohomology classes can be seen as measuring the nontriviality of the given vector bundle, and (relatedly) record topological information such as the nonexistence of nonvanishing sections. They are closely related to *universal* (or "maximally twisted") vector bundles, which live over Grassmannian manifolds.

Among all spaces, of central interest in algebraic topology are (smooth) manifolds. In contrast with an arbitrary space, a manifold M has a canonically associated vector bundle TM called its *tangent bundle*, whose fiber at a point $m \in M$ is the vector space of "tangent directions" in M through m. Applying the theory of characteristic classes, one can assign numerical invariants to manifolds, which provide a remarkable (though somewhat coarse) classification thereof: namely, up to an equivalence relation known as ***cobordism***.[5]

Restricting to manifolds often leads to a multitude of perspectives on a given invariant; for example, singular cohomology is isomorphic to de Rham cohomology. If time permits, we will see another manifestation of this principle: the definition of characteristic classes for vector bundles over manifolds in terms of ***Chern–Weil theory***. In short, one chooses a *connection* on the vector bundle – roughly, a notion of "differentiation" for its sections – and extracts de Rham forms therefrom, which give de Rham cohomology classes that turn out to be independent of the choice of connection.

Although the lectures will follow the textbook fairly closely, in the interest of time we will often have to omit details. Therefore, in order to succeed in this course, you should expect both to attend all lectures and to regularly read the book.

3. Resources

Collaboration with your fellow students can be a very effective way of learning, and is highly encouraged. Often, we may think we understand something very well, but in trying to

[5]Actually this only holds in the unoriented case; the oriented case is more difficult, although characteristic numbers provide strong partial results.

explain it to somebody else we find the gaps in our understanding.[6] However, *everyone must write up their own solutions separately*: you may not copy each other's work. Moreover, you must indicate any help that you received on your homework (e.g. from office hours, websites, collaboration with other students, etc.), either at the beginning of the entire document or as a footnote or marginalium to specific problems. For instance, at the top of your homework you might write "I worked on my homework with Alice and Bob.", or as a footnote to Problem N you might write "I found a hint/solution to this at `http://mathhelp4u.blogspot.com`.".

As a guideline for this collaboration policy, keep in mind that **you should be able to reproduce any solution you hand in without assistance**.

Here is a list of allowed and disallowed resources. This is intended to be reasonably comprehensive, but the fact that something is not on this list does *not* mean that it is permitted. If you are ever in doubt about whether something is allowed, it is **your responsibility** to ask.

you may consult:	
course textbooks	yes
other books	yes
solution manuals	no
internet	yes
notes from lecture (yours or others')	yes
course handouts	yes
your past homework	yes
homework and exams solutions posted on webpage	yes
homework and exams from previous years	no
communications (e.g. emails) from instructor or TA	yes
you may:	
discuss problems with others	yes
look at communal materials (e.g. (e-)whiteboard) when writing up solutions	yes
look at written work of others	no
post about problems online (e.g. at math.stackexchange)	yes
for computational aid, you may use:	

[6]An interesting research study illustrates this well. Subjects were asked to rate their understanding of some everyday mechanism (e.g. a ballpoint pen or the flushing mechanism of a toilet). Then, they were asked to explain it. Then, they were asked to rate their understanding again. As you might imagine, their own estimation of their understanding dropped significantly from the first round of rating to the second.

calculator	yes[7]
computer	yes[7]

4. Miscellanea

4.1. **Attendance and participation.** While they do not directly affect your grade, I would very much appreciate your regular attendance and participation in lectures. In my own experience, mathematics is most exciting and most rewarding when it is pursued as an interpersonal endeavor: as an interesting and unique way to interact with other human beings.

4.2. **Canvas.** Much of the relevant information for this course will be posted on the course website. However, we may use Canvas for a few purposes, e.g. for any material that may be considered as sensitive or unpolished – lecture video recordings, lecture notes, Zoom links, chat, discussion boards, grades, etc.

4.3. **Students with documented disabilities.** If you need an academic accommodation based on the impact of a disability, you must initiate the request with Caltech Accessibility Services for Students (CASS). Professional staff will evaluate your request with required documentation, recommend reasonable accommodations, and prepare an Accommodation Letter for the quarter in which the request is being made. In this case, you should contact CASS as soon as possible, since timely notice is needed to coordinate accommodations. Undergraduate students should contact Dr. Lesley Nye, Associate Dean of Undergraduate Students (administrative contact: Beth Larranaga) and graduate students should contact Dr. Kate McAnulty, Associate Dean of Graduate Studies (administrative contact: Jacob Dalton).

4.4. **Academic integrity.** You must abide by Caltech's honor code: *No member of the Caltech community shall take unfair advantage of any other member of the Caltech community.* General guidelines for how this applies to plagiarism are given here. Specific guidelines for this course are described in §3.

[7] It is not expected that computational aids will be helpful in this course (except possibly for visualization purposes). Nevertheless, you may use a calculator or computer while doing homework. However, you may not refer to this as a justification of your work. For example, "by Mathematica" is not an acceptable justification by itself.

4.5. **LaTeX resources.** LaTeX code is stored as plain text, and therefore can be written and saved with the filename extension `txt` (e.g. as `my-LaTeX-code.txt`). This raw code can then be *compiled* into a `pdf` file (such as the one you are reading now). You can compile LaTeX code directly on your computer after a somewhat lengthy download/installation process. Alternatively, you can write and compile LaTeX code online through the website Overleaf. I would recommend the latter option, especially if you are having difficulty with the former option (which is somewhat nontrivial).

A simple template that you can use to get started on your homework is available here; when compiled, that LaTeX code produces this file. Of course, you are also welcome to start from scratch. Along with occasional online searches, that template should suffice to get you started.[8] However, if you would like a further introduction to LaTeX you can check out some of the instructions provided by Overleaf: this shorter one or this longer one.

Of course, there are many online LaTeX resources. In particular, there are many "crib sheets", see e.g. here and here. Another great resource is Detexify: if you draw a mathematical symbol, it will attempt to suggest how to write it in LaTeX.

4.6. **Revisions to this syllabus.** This syllabus is subject to minor changes throughout the semester, as needed. In the interest of transparency, all versions will remain available on the course website.

[8]The diagram is drawn using the `tikz-cd` package, more information about which can be found here.