HIGHER ALGEBRA, CHAPTER 0

AARON MAZEL-GEE

ABSTRACT. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the spring 2023 quarter (building off of those from the winter 2021 iteration, which are available here). They are **under construction**, and will be updated at the course website at the end of each lecture.

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0. MISCELLANEA

0.1. **Exercises.**

0.1.1. In order to obtain a grade of 100% on the homework by the end of the quarter, you will need to earn **TBD** points.

0.1.2. Partial solutions may be submitted for partial credit.

0.1.3. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a "yes or no" question).

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0.1.4. In preparing your homework, please copy down the problem statement, since it is possible that the numbering may change.

0.2. **Conventions.**

0.2.1. As this document is a work in progress, I will sometimes want to make changes to existing material (in addition to adding material as we cover it during lecture). Of course, it would be quite difficult for a reader to spot such changes, especially as the document grows. Therefore, any substantial such changes will be temporarily flagged by their change date (for easy searching) and additionally will be colored as follows: changes that have occurred since the most recent lecture are **red**, changes made between the past two lectures are **blue**, and changes made between two and three lectures ago are **green**. Older changes are no longer colored or dated.

0.2.2. We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and Set denotes the category of sets. However, notation will very often only be "local": the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.3. The term "natural number" (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.4. We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We consider posets as categories without comment. We write e.g. $\mathbb{N}^{\leqslant}$ and $\mathbb{Z}^{\leqslant}$ for the usual poset structures on the natural numbers and the integers. Given some datum in a category (e.g. an object or morphism), we may use the superscript $(-)^{\circ}$ to denote the corresponding datum in the opposite category, although we may also omit this superscript when our meaning is sufficiently clear. We mostly ignore set-theoretic issues.[1] We refer the reader to [Shua] for a thorough discussion of the role of set theory in category theory.

0.2.5. Especially in later sections, we will frequently give references to Lurie's books [Lur09] and [Lur]. Our exposition is nevertheless intended to be self-contained, with these references merely providing the reader with entry points for exploring those books further. For brevity, we will use the abbreviations "T" and "A" to refer to these works, and moreover we will omit environment names (except for the section symbol §). So for instance, we will refer to [Lur09, Theorem 4.1.3.1] simply as [T.4.1.3.1] and to [Lur, §1.3.3] simply as [§A.1.3.3].

0.2.6. The term "(commutative) ring" means "associative unital (resp. commutative) ring". Likewise, modules are always unital (meaning that the unit element acts as the identity).

[1] Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

0.2.7. To simplify language, universal quantifiers will often be dropped. For instance, an assertion involving an integer n should generally be understood to refer to *all* integers n unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

0.2.8. For brevity, we will often use a slash to make multiple statements at once. This idiom has two possible meanings; the specific meaning should always be clear from context. On the one hand, we will write e.g. "homotopy co/kernel sequence" as a stand-in for "sequences which are simultaneously homotopy cokernel sequences and homotopy kernel sequences – and let us not forget that it suffices to check either condition in order to deduce both". On the other hand, we will write e.g. "co/limits" as a stand in for "both colimits and limits".

Part I. Homological algebra

1. Some motivation for homological algebra

1.1. **Intersection theory.** A basic endeavor in geometry is to understand *intersections*. For example, given a (smooth) manifold M and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} \ .$$

If N_0 and N_1 intersect *transversely* (i.e. $T_p N_0 + T_p N_1 = T_p M$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

However, if the intersection of N_0 and N_1 is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either N_0 or N_1 that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.[2] However, this approach has a number of (related) drawbacks.

(1) Perturbations are noncanonical.
(2) Perturbations will generally destroy the inherent symmetries of the situation.[3]
(3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally only transcendental.[4]

[2] A good introduction to these ideas is [GP74].

[3] For instance, perturbations to transverse intersections need not exist in the equivariant context.

[4] It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, which is fundamental in the classical approach to intersection theory in algebraic geometry [EH16]. (In that context, "perturb" means "change to a new but rationally equivalent algebraic cycle".)

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. **Tensor products of modules.** We first recall the notion of tensor product.

Let R be a commutative ring, and let M and N be R-modules. The (***relative***) ***tensor product*** of M and N over R,[5] denoted $M \otimes_R N$, is the universal abelian group equipped with an R-balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N \ ,$$

i.e. a function satisfying the following axioms:

(1) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$ and $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$;
(2) $\varphi(m \cdot r, n) = \varphi(m, r \cdot n)$.[6]

In other words, for any abelian group A, precomposition with φ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \to A\} \xleftarrow{\cong} \{\text{abelian group homomorphisms } M \otimes_R N \to A\} \ .$$

In the case that R is understood (and particularly when $R = \mathbb{Z}$ or when R is a field), we may simply write $\otimes := \otimes_R$.

The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

(1) $(m + m', n) \sim (m, n) + (m', n)$ and $(m, n + n') \sim (m, n) + (m, n')$;
(2) $(m \cdot r, n) \sim (m, r \cdot n)$.

Exercise 1.1 (2 points)**.** For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\mathsf{gcd}(m, n)$.

1.3. **Basic principles of algebraic geometry.** In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §I.1].

1.3.1. The polynomial functions on $\mathbb{R}^n$ are the n-variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving n implicit).

[5]The word "relative" here is meant to emphasize that R is an arbitrary commutative ring. By contrast, the term "absolute tensor product" would emphasize that $R = \mathbb{Z}$.

[6]The notation here stems from the fact that more generally, we can define the relative tensor product when R is merely an associative ring, M is a right R-module, and N is a left R-module.

1.3.2. By definition, an ***algebraic subset*** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.[7] In this case we write $Z = V(S)$, and we say that Z is the ***vanishing locus*** of the elements of S. If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.[8]

1.3.3. We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along Z. Then, the ring of polynomial functions on Z is

$$\mathcal{O}(Z) = R/I(Z) \ .$$

1.3.4. Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in I\} \subseteq \mathbb{R}^n \ .$$

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{ideals in } R\} \quad .[9]$$

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 \ .$$

1.4. **Intersections via tensor products.** We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x - 1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x, y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \qquad \text{and} \qquad Z_1 = \{(x, y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \qquad \text{and} \qquad I_1 = I(Z_1) = (y - x) \subseteq R \ .$$

[7] By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

[8] Since R is noetherian, any ideal is finitely generated. In other words, we may always take S to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

[9] Over an algebraically closed field $\Bbbk$, this construction restricts to a bijection between *closed* subsets of $\Bbbk^n$ (with respect to the Zariski topology) and *radical* ideals of $\Bbbk[x_1, \ldots, x_n]$. The composite $V \circ I$ carries a subset $Y \subseteq \Bbbk^n$ to its closure $\overline{Y} \subseteq \Bbbk^n$, while the composite $I \circ V$ carries an ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ to its radical $\sqrt{I} = \{f \in \Bbbk[x_1, \ldots, x_n] : \exists\ n > 0 \text{ s.t. } f^n \in I\} \subseteq R$. By contrast, over $\mathbb{R}$ the function V fails to be injective, e.g. $V(\mathbb{R}[x]) = V(x^2 + 1) = \varnothing$.

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned}\mathcal{O}(Z_0 \cap Z_1) &\cong R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y-x^2, y-x) \cong \mathbb{R}[x]/(x-x^2) \\ &= \mathbb{R}[x]/(x\cdot(1-x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1-x) \cong \mathbb{R}\times\mathbb{R}\ ,\end{aligned}$$

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a PID, in fact it is a Euclidean domain).[10] The fact that this is a 2-dimensional $\mathbb{R}$-algebra corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements, which can in good situations detect the correct multiplicity of a non-transverse intersection point.

Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in R. These correspond to the curves $y = x^2$ and $y = 0$. These intersect "twice" at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y-x^2, y) \cong \mathbb{R}[x]/(x^2)\ .$$

The 2-dimensionality of this $\mathbb{R}$-algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect "with multiplicity two". Namely, this $\mathbb{R}$-algebra corresponds to "the origin along with infinitesimal fuzz in the direction of the x-axis". This is in contrast with the previous example, where the tensor product split as a cartesian product.

These techniques are quite robust.

Exercise 1.2 (4 points)**.** Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x - a) \subseteq R$ and $I_1 = (x - b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x-a, x-b) \cong \mathbb{R}/(a-b) \cong \begin{cases} \mathbb{R}\ , & a = b \\ 0\ , & a \neq b \end{cases}\ .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$-algebra.

[10] An explicit inverse is given by carrying the pair $(a, b) \in \mathbb{R}\times\mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b-a)\cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x\cdot(1-x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0),g(1)}$ lies in the ideal generated by $x\cdot(x-1)$. Observe that $g - f_{g(0),g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

Using homological algebra, namely the notion of ***derived tensor products***, we will be able to obtain the expected intersection number of 0 even when $a = b$.

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

Exercise 1.3 (6 points)**.** Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

(a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines "just barely avoid intersecting": morally they should intersect "at infinity".[11] This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3\backslash\{0\})/\mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.[12]

(b) Show that a *homogenous* polynomial $g \in \mathbb{R}[x, y, z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.

(c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogenous polynomials $g_1, g_2 \in \mathbb{R}[x, y, z]$ such that $g_i([x : y : 1]) = f_i(x, y)$.

(d) Compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. As we have seen in §1.4.4, given two lines in $\mathbb{R}^2$, we are more likely to get the expected number if we intersect them (or rather their closures) in $\mathbb{RP}^2$: namely, this gives the correct answer even when the lines are parallel. However, this fails to give the correct answer when the two lines are *equal*. Derived tensor products repair this failure. Namely, the derived tensor product of a (projective) line with itself in $\mathbb{RP}^2$ is "a line, but with cardinality equal to that of a single point".

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

[11] A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \to 1^+$ their intersection point has $x \to -\infty$, while as $t \to 1^-$ their intersection point has $x \to +\infty$. This suggests that there should be a *single* point "at infinity" where they intersect in the case that $t = 1$.

[12] So, the "points at infinity" are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2\backslash\{0\})/\mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1\backslash\mathbb{R}^1$).

2. Chain complexes, homology, and tensor products

We now proceed to introduce the basic objects of study in homological algebra.

2.1. **Algebra conventions.** For concreteness, we work in the context of ordinary algebra. Namely, we fix a commutative ring $\Bbbk$ and a $\Bbbk$-algebra R. For the most part, we will work in Mod_R, the category of (right) R-modules, and one may take $\Bbbk$ to be $\mathbb{Z}$. However, at times we will want to specialize to a commutative ring, and for this it is convenient for $\Bbbk$ to be arbitrary.[13] Moreover, we will study some interactions between $\Bbbk$-modules and R-modules. At the level of ordinary (i.e. non-homological) algebra, these are encapsulated by the following facts.

(1) The category $\mathsf{Mod}_\Bbbk$ is symmetric monoidal via the tensor product, which we denote by $\otimes := \otimes_\Bbbk$; its unit object is $\Bbbk$.
(2) The category Mod_R is naturally enriched in $\mathsf{Mod}_\Bbbk$. In other words, for any R-modules $M, N \in \mathsf{Mod}_R$, the set $\mathsf{hom}_{\mathsf{Mod}_R}(M, N)$ of R-linear homomorphisms carries the natural structure of a $\Bbbk$-module, and moreover composition in Mod_R is $\Bbbk$-multilinear.
(3) Moreover, $\Bbbk$-modules naturally act on R-modules in two different ways: for any $\Bbbk$-module $T \in \mathsf{Mod}_\Bbbk$ and any R-modules $M, N \in \mathsf{Mod}_R$ we have R-modules

$$T \otimes_\Bbbk M \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N) \ ,$$

where the (right) R-actions are induced from those on M and N, and these constructions participate in natural isomorphisms

$$\mathsf{hom}_{\mathsf{Mod}_R}(T \otimes_\Bbbk M, N) \cong \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, \mathsf{hom}_{\mathsf{Mod}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Mod}_R}(M, \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N)) \ . \tag{1}$$

Of course, one may take $R = \Bbbk$ as a special case.[14] As a result, the notions that we will develop relating to the interactions between $\Bbbk$-modules and R-modules will all be *generalizations* of the notions that we develop relating to $\Bbbk$-modules alone.

As we will see later, most of the theory works equally well for a general abelian category, although there will be some additional hiccups that do not arise when studying modules.[15]

Exercise 2.1 (4 points)**.**

(a) Given a $\Bbbk$-module $T \in \mathsf{Mod}_\Bbbk$ and R-modules $M, N \in \mathsf{Mod}_R$, verify the isomorphisms (1).
(b) Verify the naturality of the isomorphisms (1) in all three variables.

[13] Of course, we will apply results developed for R-modules to $\Bbbk$-modules without comment.

[14] The above facts then reduce to the assertion that $\mathsf{Mod}_\Bbbk$ is a *closed* symmetric monoidal category, i.e. that it carries a self-enrichment that is compatible with its symmetric monoidal structure. (More generally, facts 2 and 3 can be simultaneously articulated as the assertion that Mod_R is naturally a *closed module* over the closed symmetric monoidal category $\mathsf{Mod}_\Bbbk$.)

[15] On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into Mod_R for some ring R (although the choice of such a ring R is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

2.2. **Chain complexes.** A ***chain complex*** of R-modules is a diagram

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of R-modules such that for all $n \in \mathbb{Z}$ we have the equality

$$d_n \circ d_{n+1} = 0$$

in $\mathsf{hom}_{\mathsf{Mod}_R}(M_{n+1}, M_{n-1})$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that "all indices are being referred to at once". To emphasize the differentials, one may write $(M_\bullet, d_\bullet)$. Also, one may simply refer to a chain complex as a "complex".[16] On the other hand, we also may omit the bullet and simply write $M := M_\bullet$ for simplicity. The integer n is called the ***degree*** or the ***dimension***. For an element $m \in M_n$, we may write $\deg(m) := n$.

The morphisms d_n are called the ***differentials*** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of d_n is M_n). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as d_n^M.

In this notation, a *morphism* of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is by definition a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of R-modules such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^M_{n+2}} & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow[d^N_{n+2}]{} & N_{n+1} & \xrightarrow[d^N_{n+1}]{} & N_n & \xrightarrow[d^N_n]{} & N_{n-1} & \xrightarrow[d^N_{n-1}]{} & \cdots
\end{array}$$

commutes.[17] These morphisms are often referred to as ***chain maps***. We write Ch_R for the category of chain complexes of R-modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline whenever it appears explicitly.

Any R-module $M \in \mathsf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \underset{\sim}{M} \longrightarrow 0 \longrightarrow \cdots \ .$$

This construction determines a fully faithful embedding

$$\mathsf{Mod}_R \hookrightarrow \mathsf{Ch}_R \ .$$

[16] The word "chain" here is historical. Namely, given a simplicial complex X, the first example of a chain complex was given in degree n by the "n^{th} chain group" of X, i.e. the group of chains (i.e. formal linear combinations) of n-simplices of X, with differentials given by taking an n-simplex to the signed sum of its boundary $(n-1)$-simplices (for which reason the morphisms d_n of a chain complex are also sometimes referred to as its "boundary maps"); see **??** (and **??**) for a closely related discussion. It was only later realized that chain complexes are worth studying in their own right.

[17] For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

As a result, we may not notationally distinguish between data in Mod_R and its image in Ch_R.

When a complex only has a few nonzero terms, for brevity one may omit the terms that are zero. For instance, the above complex may also be written as $\underset{\sim}{M}$.

References

[AF] David Ayala and John Francis, Zero-pointed manifolds, available at `arXiv:1409.2857`, v4.

[AF15] ———, Factorization homology of topological manifolds, J. Topol. **8** (2015), no. 4, 1045–1084.

[AF20] ———, Fibrations of ∞-categories, High. Struct. **4** (2020), no. 1, 168–265.

[AFR18] David Ayala, John Francis, and Nick Rozenblyum, Factorization homology I: Higher categories, Adv. Math. **333** (2018), 1042–1177.

[AFR19] ———, A stratified homotopy hypothesis, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 4, 1071–1178.

[AFT17] David Ayala, John Francis, and Hiro Lee Tanaka, Factorization homology of stratified spaces, Selecta Math. (N.S.) **23** (2017), no. 1, 293–362.

[AMGR] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum, Stratified noncommutative geometry, available at `arXiv:1910.14602`, v2.

[AR94] Jiří Adámek and Jiří Rosický, Locally presentable and accessible categories, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.

[BCKW] Ulrich Bunke, Denis-Charles Cisinski, Daniel Kaprowski, and Christoph Winges, Controlled objects in left-exact ∞-categories and the Novikov conjecture, available at `arXiv:1911.02338`, v2.

[BGN18] Clark Barwick, Saul Glasman, and Denis Nardin, Dualizing cartesian and cocartesian fibrations, Theory Appl. Categ. **33** (2018), Paper No. 4, 67–94.

[BV73] J. M. Boardman and R. M. Vogt, Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973.

[Cis19] Denis-Charles Cisinski, Higher categories and homotopical algebra, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019.

[EH16] David Eisenbud and Joe Harris, 3264 and all that—a second course in algebraic geometry, Cambridge University Press, Cambridge, 2016, available here.

[GH15] David Gepner and Rune Haugseng, Enriched ∞-categories via non-symmetric ∞-operads, Adv. Math. **279** (2015), 575–716.

[GHN17] David Gepner, Rune Haugseng, and Thomas Nikolaus, Lax colimits and free fibrations in ∞-categories, Doc. Math. **22** (2017), 1225–1266. MR 3690268

[Goo90] Thomas G. Goodwillie, Calculus. I. The first derivative of pseudoisotopy theory, K-Theory **4** (1990), no. 1, 1–27.

[Goo03] ———, Calculus. III. Taylor series, Geom. Topol. **7** (2003), 645–711.

[Goo92] ———, Calculus. II. Analytic functors, K-Theory **5** (1991/92), no. 4, 295–332.

[GP74] Victor Guillemin and Alan Pollack, Differential topology, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.

[GR17] Dennis Gaitsgory and Nick Rozenblyum, A study in derived algebraic geometry. Vol. I. Correspondences and duality, Mathematical Surveys and Monographs, vol. 221, American Mathematical Society, Providence, RI, 2017.

[Har77] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[HKR62] G. Hochschild, Bertram Kostant, and Alex Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. **102** (1962), 383–408.

[Hov99] Mark Hovey, Model categories, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[JT07] André Joyal and Myles Tierney, Quasi-categories vs Segal spaces, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326.

[Lod98] Jean-Louis Loday, Cyclic homology, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1998, Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[Low] Zhen Lin Low, Universes for category theory, available at `arXiv:1304.5227`, v2.

[Lur] Jacob Lurie, Higher algebra, available from the author's website (version dated 9/18/2017).

[Lur09] ———, Higher topos theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Mac71] Saunders MacLane, Categories for the working mathematician, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 5.

[May72] J. P. May, The geometry of iterated loop spaces, Springer-Verlag, Berlin-New York, 1972, Lectures Notes in Mathematics, Vol. 271.

[McD79] Dusa McDuff, On the classifying spaces of discrete monoids, Topology **18** (1979), no. 4, 313–320.

[MG] Aaron Mazel-Gee, The zen of ∞-categories, available here.

[MG19a] ———, On the Grothendieck construction for ∞-categories, J. Pure Appl. Algebra **223** (2019), no. 11, 4602–4651.

[MG19b] ———, A user's guide to co/cartesian fibrations, Grad. J. Math. **4** (2019), no. 1, 42–53.

[ML63] Saunders Mac Lane, Natural associativity and commutativity, Rice Univ. Stud. **49** (1963), no. 4, 28–46.

[Pro13] The Univalent Foundations Program, Homotopy type theory—univalent foundations of mathematics, The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ, 2013.

[Rez01] Charles Rezk, A model for the homotopy theory of homotopy theory, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007.

[Rez10] ———, A Cartesian presentation of weak n-categories, Geom. Topol. **14** (2010), no. 1, 521–571.

[Saf] Pavel Safronov, Category theory [lecture notes], available here.

[Shua] Michael Shulman, Set theory for category theory, available at `arXiv:0810.1279`.

[Shub] Michael A. Shulman, Set theory for category theory, available at `arXiv:0810.1279`, v2.

[Spa88] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Math. **65** (1988), no. 2, 121–154.

[Sta63] James Dillon Stasheff, Homotopy associativity of H-spaces. I, II, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. **108** (1963), 293–312. MR 0158400

[TV11] Bertrand Toën and Gabriele Vezzosi, Algèbres simpliciales S^1-équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs, Compos. Math. **147** (2011), no. 6, 1979–2000.

[Wal65] C. T. C. Wall, Finiteness conditions for CW-complexes, Ann. of Math. (2) **81** (1965), 56–69.

[Wei94] Charles A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wei13] ———, The K-book, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory.