HIGHER ALGEBRA, CHAPTER 0

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ABSTRACT. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the spring 2023 quarter (building off of those from the winter 2021 iteration, which are available here). They are **under construction**, and will be updated at the course website at the end of each lecture.

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last updated: May 12, 2023

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0. Miscellanea

0.1. **Exercises.**

0.1.1. [changed 5/8] The number of points needed to achieve various letter grades for your homework score is recorded in the syllabus.

0.1.2. Partial solutions may be submitted for partial credit.

0.1.3. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a "yes or no" question).

0.1.4. In preparing your homework, please copy down the problem statement, since it is possible that the numbering may change.

0.2. **Conventions.**

0.2.1. As this document is a work in progress, I will sometimes want to make changes to existing material (in addition to adding material as we cover it during lecture). Of course, it would be quite difficult for a reader to spot such changes, especially as the document grows. Therefore, any substantial such changes will be temporarily flagged by their change date (for easy searching) and additionally will be colored as follows: changes that have occurred since the most recent lecture are **red**, changes made between the past two lectures are **blue**, and changes made between two and three lectures ago are **green**. Older changes are no longer colored or dated.

0.2.2. We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and Set denotes the category of sets. However, notation will very often only be "local": the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.3. The term "natural number" (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.4. We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We consider posets as categories without comment. We write e.g. $\mathbb{N}^{\leqslant}$ and $\mathbb{Z}^{\leqslant}$ for the usual poset structures on the natural numbers and the integers. Given some datum in a category (e.g. an object or morphism), we may use the superscript $(-)^{\circ}$ to denote the corresponding datum in the opposite category, although we may also omit this superscript when our meaning is sufficiently clear. We mostly ignore set-theoretic issues.[1] We refer the reader to [Shua] for a thorough discussion of the role of set theory in category theory.

0.2.5. Especially in later sections, we will frequently give references to Lurie's books [Lur09] and [Lur]. Our exposition is nevertheless intended to be self-contained, with these references merely providing the reader with entry points for exploring those books further. For brevity, we will use the abbreviations "T" and "A" to refer to these works, and moreover we will omit environment names (except for the section symbol §). So for instance, we will refer to [Lur09, Theorem 4.1.3.1] simply as [T.4.1.3.1] and to [Lur, §1.3.3] simply as [§A.1.3.3].

0.2.6. The term "(commutative) ring" means "associative unital (resp. commutative) ring". Likewise, modules are always unital (meaning that the unit element acts as the identity).

0.2.7. To simplify language, universal quantifiers will often be dropped. For instance, an assertion involving an integer n should generally be understood to refer to *all* integers n unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

0.2.8. For brevity, we will often use a slash to make multiple statements at once. This idiom has two possible meanings; the specific meaning should always be clear from context. On the one hand, we will write e.g. "homotopy co/kernel sequence" as a stand-in for "sequences which are simultaneously homotopy cokernel sequences and homotopy kernel sequences – and let us not forget that it suffices to check either condition in order to deduce both". On the other hand, we will write e.g. "co/limits" as a stand in for "both colimits and limits".

[1] Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

Part I. Homological algebra

1. Some motivation for homological algebra

1.1. **Intersection theory.** A basic endeavor in geometry is to understand *intersections*. For example, given a (smooth) manifold M and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} \ .$$

If N_0 and N_1 intersect *transversely* (i.e. $T_pN_0 + T_pN_1 = T_pM$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

However, if the intersection of N_0 and N_1 is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either N_0 or N_1 that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.[2] However, this approach has a number of (related) drawbacks.

(1) Perturbations are noncanonical.
(2) Perturbations will generally destroy the inherent symmetries of the situation.[3]
(3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally only transcendental.[4]

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. **Tensor products of modules.** We first recall the notion of tensor product.

Let R be a commutative ring, and let M and N be R-modules. The (***relative***) ***tensor product*** of M and N over R,[5] denoted $M \otimes_R N$, is the universal abelian group equipped with an R-balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N \ ,$$

i.e. a function satisfying the following axioms:

(1) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$ and $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$;

[2] A good introduction to these ideas is [GP74].

[3] For instance, perturbations to transverse intersections need not exist in the equivariant context.

[4] It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, which is fundamental in the classical approach to intersection theory in algebraic geometry [EH16]. (In that context, "perturb" means "change to a new but rationally equivalent algebraic cycle".)

[5] The word "relative" here is meant to emphasize that R is an arbitrary commutative ring. By contrast, the term "absolute tensor product" would emphasize that $R = \mathbb{Z}$.

(2) $\varphi(m \cdot r, n) = \varphi(m, r \cdot n)$.[6]

In other words, for any abelian group A, precomposition with φ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \to A\} \xleftarrow{\cong} \{\text{abelian group homomorphisms } M \otimes_R N \to A\} .$$

In the case that R is understood (and particularly when $R = \mathbb{Z}$ or when R is a field), we may simply write $\otimes := \otimes_R$.

The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

(1) $(m + m', n) \sim (m, n) + (m', n)$ and $(m, n + n') \sim (m, n) + (m, n')$;
(2) $(m \cdot r, n) \sim (m, r \cdot n)$.

Exercise 1.1 (2 points)**.** For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\mathsf{gcd}(m, n)$.

1.3. **Basic principles of algebraic geometry.** In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §I.1].

1.3.1. The polynomial functions on $\mathbb{R}^n$ are the n-variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving n implicit).

1.3.2. By definition, an ***algebraic subset*** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.[7] In this case we write $Z = V(S)$, and we say that Z is the ***vanishing locus*** of the elements of S. If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.[8]

1.3.3. We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along Z. Then, the ring of polynomial functions on Z is

$$\mathcal{O}(Z) = R/I(Z) .$$

1.3.4. Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in J\} \subseteq \mathbb{R}^n .$$

[6]The notation here stems from the fact that more generally, we can define the relative tensor product when R is merely an associative ring, M is a right R-module, and N is a left R-module.

[7]By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

[8]Since R is noetherian, any ideal is finitely generated. In other words, we may always take S to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{ideals in } R\} \quad .^{9}$$

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 \ .$$

1.4. **Intersections via tensor products.** We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x-1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \qquad \text{and} \qquad Z_1 = \{(x,y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \qquad \text{and} \qquad I_1 = I(Z_1) = (y - x) \subseteq R \ .$$

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned} \mathcal{O}(Z_0 \cap Z_1) &\cong R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y - x^2, y - x) \cong \mathbb{R}[x]/(x - x^2) \\ &= \mathbb{R}[x]/(x \cdot (1-x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1-x) \cong \mathbb{R} \times \mathbb{R} \ , \end{aligned}$$

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a PID, in fact it is a Euclidean domain).[10] The fact that this is the cartesian product of two copies of $\mathbb{R}$ corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

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[9] Over an algebraically closed field $\Bbbk$, this construction restricts to a bijection between *closed* subsets of $\Bbbk^n$ (with respect to the Zariski topology) and *radical* ideals of $\Bbbk[x_1, \ldots, x_n]$. The composite $V \circ I$ carries a subset $Y \subseteq \Bbbk^n$ to its closure $\overline{Y} \subseteq \Bbbk^n$, while the composite $I \circ V$ carries an ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ to its radical $\sqrt{I} = \{f \in \Bbbk[x_1, \ldots, x_n] : \exists\ n > 0 \text{ s.t. } f^n \in I\} \subseteq R$. By contrast, over $\mathbb{R}$ the function V fails to be injective, e.g. $V(\mathbb{R}[x]) = V(x^2 + 1) = \varnothing$.

[10] An explicit inverse is given by carrying the pair $(a,b) \in \mathbb{R} \times \mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b-a) \cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x \cdot (1-x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0),g(1)}$ lies in the ideal generated by $x \cdot (x-1)$. Observe that $g - f_{g(0),g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements, which can in good situations detect the correct multiplicity of a non-transverse intersection point.

Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in R. These correspond to the curves $y = x^2$ and $y = 0$. These intersect "twice" at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x, y]/(y - x^2, y) \cong \mathbb{R}[x]/(x^2) \ .$$

The 2-dimensionality of this $\mathbb{R}$-algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect "with multiplicity two". Namely, this $\mathbb{R}$-algebra corresponds to "the origin along with infinitesimal fuzz in the direction of the x-axis". This is in contrast with the previous example, where the tensor product split as a cartesian product.

These techniques are quite robust.

Exercise 1.2 (4 points)**.** Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x - a) \subseteq R$ and $I_1 = (x - b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x - a, x - b) \cong \mathbb{R}/(a - b) \cong \begin{cases} \mathbb{R} \ , & a = b \\ 0 \ , & a \neq b \end{cases} \ .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$-algebra.

Using homological algebra, namely the notion of ***derived tensor products***, we will be able to obtain the expected intersection number of 0 even when $a = b$.

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

Exercise 1.3 (6 points)**.** Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

(a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines "just barely avoid intersecting": morally they should intersect "at infinity".[11] This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3\backslash\{0\})/\mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.[12]

(b) Show that a *homogeneous* polynomial $g \in \mathbb{R}[x, y, z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.

(c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogeneous polynomials $g_1, g_2 \in \mathbb{R}[x, y, z]$ such that $g_i([x : y : 1]) = f_i(x, y)$.

(d) Working in an appropriate affine chart, use tensor products to compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. As we have seen in §1.4.4, given two lines in $\mathbb{R}^2$, we are more likely to get the expected number if we intersect them (or rather their closures) in $\mathbb{RP}^2$: namely, this gives the correct answer even when the lines are parallel. However, this fails to give the correct answer when the two lines are *equal*. Derived tensor products repair this failure. Namely, the derived tensor product of a (projective) line with itself in $\mathbb{RP}^2$ is "a line, but with cardinality equal to that of a single point".

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

2. Chain complexes, homology, and tensor products

We now proceed to introduce the basic objects of study in homological algebra.

2.1. **Algebra conventions.** For concreteness, we work in the context of ordinary algebra. Namely, we fix a commutative ring $\Bbbk$ and a $\Bbbk$-algebra R. For the most part, we will work in Mod_R, the category of (right) R-modules, and one may take $\Bbbk$ to be $\mathbb{Z}$. However, at times we will want to specialize to a commutative ring, and for this it is convenient for $\Bbbk$

[11] A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \to 1^+$ their intersection point has $x \to -\infty$, while as $t \to 1^-$ their intersection point has $x \to +\infty$. This suggests that there should be a *single* point "at infinity" where they intersect in the case that $t = 1$.

[12] So, the "points at infinity" are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2\backslash\{0\})/\mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1\backslash\mathbb{R}^1$).

to be arbitrary.[13] Moreover, we will study some interactions between $\Bbbk$-modules and R-modules. At the level of ordinary (i.e. non-homological) algebra, these are encapsulated by the following facts.

(1) The category $\mathsf{Mod}_\Bbbk$ is symmetric monoidal via the tensor product, which we denote by $\otimes := \otimes_\Bbbk$; its unit object is $\Bbbk$.
(2) The category Mod_R is naturally enriched in $\mathsf{Mod}_\Bbbk$. In other words, for any R-modules $M, N \in \mathsf{Mod}_R$, the set $\mathsf{hom}_{\mathsf{Mod}_R}(M, N)$ of R-linear homomorphisms carries the natural structure of a $\Bbbk$-module, and moreover composition in Mod_R is $\Bbbk$-multilinear.
(3) Moreover, $\Bbbk$-modules naturally act on R-modules in two different ways: for any $\Bbbk$-module $T \in \mathsf{Mod}_\Bbbk$ and any R-modules $M, N \in \mathsf{Mod}_R$ we have R-modules

$$T \otimes_\Bbbk M \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N) \ ,$$

where the (right) R-actions are induced from those on M and N, and these constructions participate in natural isomorphisms

$$\mathsf{hom}_{\mathsf{Mod}_R}(T \otimes_\Bbbk M, N) \cong \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, \mathsf{hom}_{\mathsf{Mod}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Mod}_R}(M, \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N)) \ . \tag{1}$$

Of course, one may take $R = \Bbbk$ as a special case.[14] As a result, the notions that we will develop relating to the interactions between $\Bbbk$-modules and R-modules will all be *generalizations* of the notions that we develop relating to $\Bbbk$-modules alone.

As we will see later, most of the theory works equally well for a general abelian category, although there will be some additional hiccups that do not arise when studying modules.[15]

Exercise 2.1 (5 points)**.**
(a) Given a $\Bbbk$-module $T \in \mathsf{Mod}_\Bbbk$ and R-modules $M, N \in \mathsf{Mod}_R$, verify the isomorphisms (1).
(b) Verify the naturality of the isomorphisms (1) in all three variables.

2.2. **Chain complexes.** A ***chain complex*** of R-modules is a diagram

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of R-modules such that for all $n \in \mathbb{Z}$ we have the equality

$$d_n \circ d_{n+1} = 0$$

[13] Of course, we will apply results developed for R-modules to $\Bbbk$-modules without comment.

[14] The above facts then reduce to the assertion that $\mathsf{Mod}_\Bbbk$ is a *closed* symmetric monoidal category, i.e. that it carries a self-enrichment that is compatible with its symmetric monoidal structure. (More generally, facts 2 and 3 can be simultaneously articulated as the assertion that Mod_R is naturally a *closed module* over the closed symmetric monoidal category $\mathsf{Mod}_\Bbbk$.)

[15] On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into Mod_R for some ring R (although the choice of such a ring R is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

in $\mathsf{hom}_{\mathsf{Mod}_R}(M_{n+1}, M_{n-1})$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that "all indices are being referred to at once". To emphasize the differentials, one may write $(M_\bullet, d_\bullet)$. Also, one may simply refer to a chain complex as a "complex".[16] On the other hand, we also may omit the bullet and simply write $M := M_\bullet$ for simplicity. The integer n is called the ***degree*** or the ***dimension***. For an element $m \in M_n$, we may write $\deg(m) := n$.

The morphisms d_n are called the ***differentials*** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of d_n is M_n). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as d_n^M.

In this notation, a *morphism* of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is by definition a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of R-modules such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^M_{n+2}} & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow[d^N_{n+2}]{} & N_{n+1} & \xrightarrow[d^N_{n+1}]{} & N_n & \xrightarrow[d^N_n]{} & N_{n-1} & \xrightarrow[d^N_{n-1}]{} & \cdots
\end{array}$$

commutes.[17] These morphisms are often referred to as ***chain maps***. We write Ch_R for the category of chain complexes of R-modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline whenever it appears explicitly.

Any R-module $M \in \mathsf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \underset{\sim}{M} \longrightarrow 0 \longrightarrow \cdots \ .$$

This construction determines a fully faithful embedding

$$\mathsf{Mod}_R \hookrightarrow \mathsf{Ch}_R \ .$$

As a result, we may not notationally distinguish between data in Mod_R and its image in Ch_R.

When a complex only has a few nonzero terms, for brevity one may omit the terms that are zero. For instance, the above complex may also be written as $\underset{\sim}{M}$.

[16] The word "chain" here is historical. Namely, given a simplicial complex X, the first example of a chain complex was given in degree n by the "n^{th} chain group" of X, i.e. the group of chains (i.e. formal linear combinations) of n-simplices of X, with differentials given by taking an n-simplex to the signed sum of its boundary $(n-1)$-simplices (for which reason the morphisms d_n of a chain complex are also sometimes referred to as its "boundary maps"); see **??** (and **??**) for a closely related discussion. It was only later realized that chain complexes are worth studying in their own right.

[17] For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

2.3. **Homology.** Fix a chain complex $M_\bullet$. Its ***n*-cycles** and ***n*-boundaries** are the submodules

$$\mathsf{Z}_n(M_\bullet) := \mathsf{ker}(d_n) \subseteq M_n \qquad \text{and} \qquad \mathsf{B}_n(M_\bullet) := \mathsf{im}(d_{n+1}) \subseteq M_n \ .^{18}$$

Note that $\mathsf{B}_n(M_\bullet) \subseteq \mathsf{Z}_n(M_\bullet)$ because $d^2 = 0$. Then, the $n^{\textbf{th}}$ ***homology*** of $M_\bullet$ is the quotient R-module

$$\mathsf{H}_n(M_\bullet) := \frac{\mathsf{Z}_n(M_\bullet)}{\mathsf{B}_n(M_\bullet)} := \frac{\mathsf{ker}(d_n)}{\mathsf{im}(d_{n+1})} \ .$$

Although $\mathsf{H}_n(M_\bullet)$ is an R-module, it is common to refer to it merely as a *homology group*.

Exercise 2.2 (3 points)**.** Verify that the constructions Z_n, B_n, and H_n define functors

$$\mathsf{Ch}_R \longrightarrow \mathsf{Mod}_R \ .$$

A morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R is called a ***quasi-isomorphism*** if the induced morphisms $H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We may indicate that a morphism is a quasi-isomorphism by decorating the arrow as $\xrightarrow{\approx}$.

We say that a complex $M_\bullet$ is ***acyclic*** if $\mathsf{H}_n(M_\bullet) = 0$ for all n. So, $M_\bullet$ is acyclic if and only if either (and hence both) of the unique maps $0 \to M_\bullet$ and $M_\bullet \to 0$ is a quasi-isomorphism.

2.4. The derived category of R-modules.

2.4.1. By and large, we would like to think of quasi-isomorphic chain complexes as "essentially interchangeable", with some representatives of a given quasi-isomorphism class (namely the *projective* and *injective* complexes introduced in §5) being "well-adapted" for certain purposes.[19,20] In other words, one should think of quasi-isomorphisms as if they are actual isomorphisms.

This can be made literally true by ***localizing*** the category Ch_R at the quasi-isomorphisms, i.e. by adjoining formal inverses for them. This yields a category that (for reasons that will be explained shortly in §2.4.3) we will denote by $\mathsf{H}_0(\mathbf{D}_R)$ and refer to as ***the derived category***

[18]The German words for "cycle" and "boundary" respectively begin with the letters "Z" and "B".

[19]This is very closely akin to how one should think of equivalent categories as "essentially interchangeable", even when they are not isomorphic. However, in a precise sense, *all* categories are "equally well-adapted" for all purposes (in contrast with chain complexes).

[20]While quasi-isomorphic complexes have isomorphic homology groups, the converse is generally false: the obstruction is encoded by ***k*-invariants**, as explained in **??**. (This same name is given to the (closely analogous) obstructions to spaces with the same homotopy groups being weak homotopy equivalent.) That is, a quasi-isomorphism class of complexes is equivalent data to its homology groups along with all of its k-invariants. For this reason, we will generally consider quasi-isomorphism classes of complexes themselves as the "true" mathematical objects of lasting interest, while their homology groups are merely algebraic invariants that can be extracted therefrom.

of R-modules; its objects are called ***derived R-modules***.[21] So by definition, there is a canonical functor

$$\mathsf{Ch}_R \longrightarrow \mathsf{H}_0(\mathbf{D}_R)$$

that carries all quasi-isomorphisms to isomorphisms, and moreover it is universal with respect to this requirement. Indeed, for any category $\mathcal{C}$, the restriction functor

$$\mathsf{Fun}(\mathsf{Ch}_R, \mathcal{C}) \longleftarrow \mathsf{Fun}(\mathsf{H}_0(\mathbf{D}_R), \mathcal{C}) \tag{2}$$

is a fully faithful inclusion, whose image consists of those functors $\mathsf{Ch}_R \to \mathcal{C}$ that carry quasi-isomorphisms to isomorphisms.[22]

Note that a derived R-module is a "purely homotopical" object: while it can by definition be presented by a chain complex of R-modules, one cannot speak e.g. of its underlying R-module in dimension 0, as this notion is not preserved under quasi-isomorphisms.[23] On the other hand, one *can* speak e.g. of its n^{th} homology, since by definition a quasi-isomorphism induces an isomorphism on n^{th} homology. Thus, the n^{th} homology functor admits a factorization

$$\begin{array}{ccc} \mathsf{Ch}_R & \xrightarrow{\mathsf{H}_n} & \mathsf{Mod}_R \\ \downarrow & \nearrow_{\mathsf{H}_n} & \\ \mathsf{H}_0(\mathbf{D}_R) & & \end{array},$$

which is unique up to unique isomorphism (and which is denoted using the same notation, as is justified by the fully faithful inclusion (2)).

2.4.2. Essentially by construction, given two complexes $M, N \in \mathsf{Ch}_R$, morphisms from M to N in the derived category $\mathsf{H}_0(\mathbf{D}_R)$ are given by equivalence classes of zigzags

$$M \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \cdots \longrightarrow \bullet \xleftarrow{\approx} N$$

[21] The placement of the word "derived" in the term "the derived category of R-modules" is admittedly slightly unfortunate, but this term is quite common. (It arises from the fact that this term is a shortening of "the derived category of the category of R-modules".)

[22] This should be thought of as a "many-object" version of the localization of the $\Bbbk$-algebra R with respect to a subset $S \subset R$: this yields a $\Bbbk$-algebra $R[S^{-1}]$ equipped with a $\Bbbk$-algebra homomorphism $R \to R[S^{-1}]$ such that for any $\Bbbk$-algebra T, the restriction map $\mathsf{hom}_{\mathsf{Alg}_\Bbbk}(R, T) \hookleftarrow \mathsf{hom}_{\mathsf{Alg}_\Bbbk}(R[S^{-1}], T)$ is the inclusion of those homomorphisms $R \to T$ that carry all elements of S to invertible elements of T. (Namely, the latter is obtained by considering $\Bbbk$-algebras as one-object $\Bbbk$-linear categories.)

[23] Likewise, one cannot speak of the underlying set of a weak homotopy equivalence class of topological spaces, nor can one speak of the set of objects of an equivalence class of categories.

(in which all backwards maps are quasi-isomorphisms).[24] Thankfully, it will turn out that every equivalence class contains representatives of the forms

$$M \xleftarrow{\approx} \bullet \longrightarrow N \qquad \text{and} \qquad M \longrightarrow \bullet \xleftarrow{\approx} N \ ,$$

which makes the situation substantially more manageable.[25]

2.4.3. Although we introduce the derived category now, we will not have much use for it: it contains too little information. The richer and more primitive object is $\mathbf{D}_R$, the ***derived*** ∞***-category*** of R. This is a mathematical entity whose objects are still the derived R-modules, but whose hom-objects are more elaborate: namely, they are *derived* $\Bbbk$*-modules*.[26] As the notation suggests, passing from $\mathbf{D}_R$ to $\mathsf{H}_0(\mathbf{D}_R)$ amounts to extracting the 0^{th} homology groups of these hom-objects.

In fact, the relationship between the derived ∞-categories $\mathbf{D}_R$ and $\mathbf{D}_\Bbbk$ precisely mirrors that between the categories Mod_R and $\mathsf{Mod}_\Bbbk$ laid out in §2.1. However, it will take us a while to see this, since the likewise relationship between the categories Ch_R and $\mathsf{Ch}_\Bbbk$ will have to be "homotopically corrected" in order to descend to the corresponding derived ∞-categories. We discuss tensor products of chain complexes presently, but defer the discussion of hom-complexes to §4 in order to highlight more fundamental homotopical notions in §3.[27] We discuss "homotopical corrections" in §5, and finally return to the derived ∞-category in §6.

2.5. **Tensor products of chain complexes.** Given complexes $M, N \in \mathsf{Ch}_\Bbbk$ of $\Bbbk$-modules, we define their ***tensor product*** complex

$$(M \otimes N)_\bullet := (M \otimes_\Bbbk N)_\bullet$$

as follows. First of all, we define its k^{th} term to be

$$(M \otimes N)_k := \bigoplus_{i+j=k} (M_i \otimes N_j) := \bigoplus_{i+j=k} (M_i \otimes_\Bbbk N_j) \ .$$

Then, the differential is characterized by the fact that it carries a pure tensor

$$m \otimes n \in (M_i \otimes N_j) \subseteq (M \otimes N)_k$$

[24] Returning to the context of Footnote 22, this zigzag of morphisms in Ch_R is analogous to a formal product $s_1^{-1} \cdot r_1 \cdot s_2^{-1} \cdot r_2 \cdot \ldots \cdot s_n^{-1}$ in which the elements s_i^{-1} are formally adjoined inverses of the elements $s_i \in S$.

[25] These reductions are guaranteed by the existence of two different *model structures* on Ch_R, which respectively have the features that all objects are fibrant and that all objects are cofibrant; see **??**.

[26] So to be more precise, $\mathbf{D}_R$ is a $\Bbbk$-*linear* ∞-category (see §6.2).

[27] This ordering of topics is due to the fact that the discussion of §3 makes (light) use of the tensor product of complexes.

to the sum of pure tensors

$$d(m \otimes n) := d(m) \otimes n + (-1)^i \cdot m \otimes d(n) \ ,^{28}$$

(an element of $((M_{i-1} \otimes N_j) \oplus (M_i \otimes N_{j-1})) \subseteq (M \otimes N)_{k-1}$). More elaborately, this may be written as

$$d_k^{M \otimes N}(m \otimes n) := d_i^M(m) \otimes n + (-1)^i \cdot m \otimes d_j^N(n) \ .$$

Exercise 2.3 (2 points)**.** Verify that this formula defines a complex.

In particular, in solving Exercise 2.3 you will see why the signs are necessary in the definition of the tensor product of complexes. In fact, many sign conventions are possible (and all give equivalent symmetric monoidal categories), but it is impossible to remove all signs from the theory (unless one works over $\mathbb{F}_2$).

From here, it is straightforward to see that the above construction defines a monoidal structure

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_{\Bbbk} \xrightarrow{\otimes} \mathsf{Ch}_{\Bbbk} \ ,$$

with unit object $\Bbbk := \underset{\sim}{\Bbbk} \in \mathsf{Ch}_{\Bbbk}$. In fact, this is a *symmetric* monoidal structure, with symmetry isomorphisms

$$M \otimes N \xrightarrow{\cong} N \otimes M$$

determined by the formula

$$m \otimes n \longmapsto (-1)^{\deg(m) \cdot \deg(n)} \cdot n \otimes m \ .^{29}$$

More generally, this same construction defines an action

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{\otimes} \mathsf{Ch}_R$$

of the symmetric monoidal category $(\mathsf{Ch}_{\Bbbk}, \otimes, \Bbbk)$ on the category Ch_R.

Exercise 2.4 (8 points)**.** Fix two complexes $M, N \in \mathsf{Ch}_{\Bbbk}$.

(a) Verify that the formula $[m] \otimes [n] \mapsto [m \otimes n]$ determines a morphism

$$H_i(M) \otimes H_j(N) \longrightarrow H_{i+j}(M \otimes N)$$

of $\Bbbk$-modules.

It follows that we obtain a morphism

$$\bigoplus_{i+j=k} (H_i(M) \otimes H_j(N)) \longrightarrow H_k(M \otimes N)$$

of $\Bbbk$-modules.

(b) Prove that this is an isomorphism under the assumption that $\Bbbk$ is a field.

[28]The factor $(-1)^i$ is determined by the *Koszul sign rule*, which is a general principle asserting that in commuting graded quantities past one another of degrees $\alpha, \beta \in \mathbb{Z}$ one should pick up a factor of $(-1)^{\alpha \cdot \beta}$. (We consider the symbol "d" as an expression of degree -1 since it lowers degrees by 1.)

[29]This formula is another instance of the Koszul sign rule.

(c) Find an example where this is not an isomorphism.

3. Homotopies, homotopy co/kernels, and exact sequences

3.1. **Homotopies.**

3.1.1. Let $M_1 \xrightarrow{f} M_0$ be a morphism of R-modules. This gives us a complex $M_\bullet := (M_1 \xrightarrow{f} M_0)$. Observe that this has a canonical morphism

$$\begin{array}{c} M_\bullet \\ \downarrow \\ \mathsf{coker}(f) \end{array} \quad := \quad \begin{array}{ccc} M_1 & \xrightarrow{f} & M_0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{coker}(f) \end{array}$$

to the cokernel of f (considered as a complex in degree 0). Observe further that

$$\mathsf{H}_n(M_\bullet) \cong \begin{cases} \mathsf{coker}(f)\ , & n = 0 \\ \mathsf{ker}(f)\ , & n = 1 \\ 0\ , & \text{otherwise} \end{cases} .$$

Hence, the above map is a quasi-isomorphism iff f is an injection. One might think of $M_\bullet$ as a "presentation" of the underlying R-module $\mathsf{H}_0(M_\bullet) \cong \mathsf{coker}(f)$: the generators are M_0, the relations are M_1 (i.e. each $m \in M_1$ gives a relation $d(m) \sim 0$), but then $\mathsf{H}_1(M_\bullet)$ furthermore measures the "redundancy" of the relations. Said differently, $M_\bullet$ is a "homotopically correct" version of the cokernel of f, which remembers not only the literal cokernel but also the extent to which the relations are overdetermined. Indeed, it will be the *homotopy cokernel* of the morphism f.

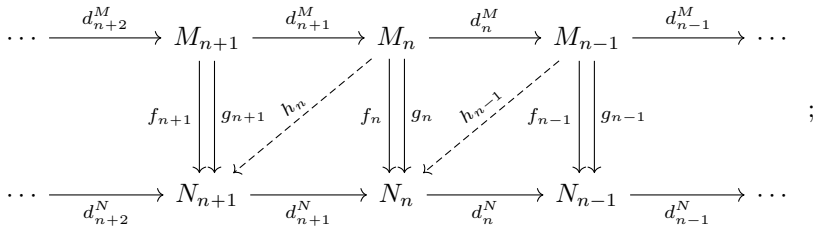
3.1.2. Let $M_\bullet, N_\bullet \in \mathsf{Ch}_R$ be complexes and let $f_\bullet, g_\bullet \in \mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ be morphisms. A (***chain***) ***homotopy*** from $f_\bullet$ to $g_\bullet$ is a set of morphisms

$$M_n \xrightarrow{h_n} N_{n+1}$$

satisfying the condition that

$$g_n - f_n = d^N_{n+1} \circ h_n + h_{n-1} \circ d^M_n \ .$$[30]

[30]Chain homotopies are sometimes depicted as in the diagram



this may be helpful as a mnemonic device, but note that it does *not* commute in any reasonable sense.

We may write this as $f_\bullet \overset{h_\bullet}{\Rightarrow} g_\bullet$. A ***nullhomotopy*** of $g_\bullet$ is a homotopy $0 \overset{h_\bullet}{\Rightarrow} g_\bullet$ from the zero map. A ***contraction*** of a complex is a nullhomotopy of its identity map. If a complex admits a contraction, we say that it is ***contractible***.

Exercise 3.1 (2 points)**.** Show that the relation of homotopy on $\mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ is an equivalence relation.

In fact, the argument of Exercise 3.1 easily upgrades to imply that we can enhance Ch_R from an ordinary category to a category enriched in groupoids (a.k.a. a $(2,1)$-category):[31] its objects are chain complexes, its 1-morphisms are chain maps, and its 2-morphisms are chain homotopies.[32] Indeed, the arguments for transitivity, reflexivity, and symmetry of the relation of homotopy respectively endow these hom-categories with their composition laws, identity morphisms, and inverses (so that they are indeed hom-groupoids). It is moreover clear that homotopies may be composed appropriately, either by definition or using Exercise 3.2 below.

3.1.3. To give a more geometric perspective on chain homotopies, let us introduce the complex $\mathbb{I} \in \mathsf{Ch}_\Bbbk$ and the morphisms $i_0, i_1 \in \mathsf{hom}_{\mathsf{Ch}_\Bbbk}(\Bbbk, \mathbb{I})$ according to the diagram

$$\begin{array}{ccc} \Bbbk & & 0 \longrightarrow \Bbbk \\ {\scriptstyle i_0}\downarrow & & \downarrow \quad \downarrow{\scriptstyle (\mathsf{id}_\Bbbk, 0)} \\ \mathbb{I} & := & \Bbbk \xrightarrow{(-\mathsf{id}_\Bbbk, \mathsf{id}_\Bbbk)} \Bbbk \oplus \Bbbk \\ {\scriptstyle i_1}\uparrow & & \uparrow \quad \uparrow{\scriptstyle (0, \mathsf{id}_\Bbbk)} \\ \Bbbk & & 0 \longrightarrow \Bbbk \end{array} .$$

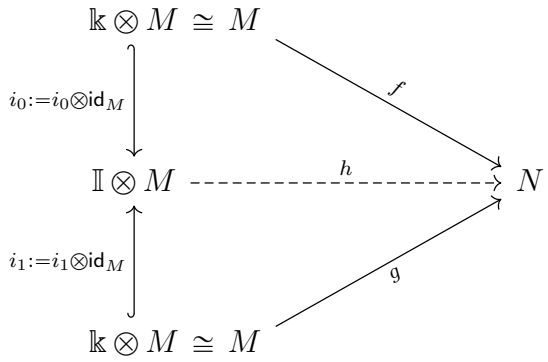
This object $\mathbb{I} \in \mathsf{Ch}_\Bbbk$ (along with the two maps i_0 and i_1) is an ***interval object*** for the homotopy theory of chain complexes (which explains the notation).[33] The general definition of an interval object is suggested by the discussion of **??**. In the present setting, this assertion amounts to the following result.

[31] As we will see, this is the homotopy $(2,1)$-category of a more fundamental object, namely an ∞-category (meaning an $(\infty,1)$-category).

[32] This explains the notation $f_\bullet \Rightarrow g_\bullet$ just introduced.

[33] This is a particularly natural choice of an interval object: as we will see, it is the (nondegenerate) simplicial chains (with coefficients in $\Bbbk$) on the 1-simplex Δ^1 (whose underlying topological space is a closed interval), and the two maps i_0 and i_1 are the simplicial chains on the inclusions of its two 0-simplices (i.e. the endpoints of the closed interval).

Exercise 3.2 (4 points)**.** Given maps $f, g \in \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$, prove that a homotopy $f \Rightarrow g$ is equivalent data to a map $\mathbb{I} \otimes M \xrightarrow{h} N$ that makes the diagram



commute.

So, a map $\mathbb{I} \otimes M \to N$ is the same thing as two maps $M \to N$ and a homotopy between them. Said differently, $\mathbb{I} \otimes M \in \mathsf{Ch}_R$ is the universal object equipped with two maps from M and a homotopy between them. Of course, the two universal maps are i_0 and i_1.

Exercise 3.3 (2 points)**.** Give a formula for the universal homotopy $i_0 \Rightarrow i_1$ between two maps from M.

Similarly, it follows from Exercise 3.2 that a map

$$C := \mathsf{colim} \left(\begin{array}{ccc} M & \xhookrightarrow{i_1} & \mathbb{I} \otimes M \\ {\scriptstyle i_0}\downarrow & & \\ \mathbb{I} \otimes M & & \end{array} \right) \longrightarrow N$$

is equivalent data to a triple of morphisms $f_0, f_1, f_2 \in \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$ as well as a pair of homotopies $f_0 \Rightarrow f_1 \Rightarrow f_2$.[34] Said differently, $C \in \mathsf{Ch}_R$ is the universal object equipped with three maps from M and two homotopies as indicated, say $j_0 \Rightarrow j_1 \Rightarrow j_2$. On the other hand, Exercise 3.1 implies that these two homotopies can be composed to a single homotopy $j_0 \Rightarrow j_2$.

Exercise 3.4 (6 points)**.** Give a formula for the map $\mathbb{I} \otimes M \to C$ corresponding to the universal composite homotopy $j_0 \Rightarrow j_2$ of two composable homotopies between maps from M.[35]

[34]This colimit is an algebraic analog of the topological construction taking a space X to the space

$$(X \times [0,1]) \coprod_{X \times \{1\}} (X \times [0,2]) \ .$$

[35]Note that co/limits in chain complexes are computed levelwise. Indeed, Ch_R is a full subcategory of the functor category $\mathsf{Fun}((\mathbb{Z}^{\leqslant})^{\mathsf{op}}, \mathsf{Mod}_R)$ (where $\mathbb{Z}^{\leqslant} := \{\cdots \to -1 \to 0 \to 1 \to \cdots\}$), and co/limits in any functor category are computed pointwise. So, it suffices to see that the subcategory $\mathsf{Ch}_R \subseteq \mathsf{Fun}((\mathbb{Z}^{\leqslant})^{\mathsf{op}}, \mathsf{Mod}_R)$ is

3.1.4. A morphism $M \xrightarrow{f} N$ in Ch_R is a ***homotopy equivalence*** if there exists a morphism $N \xrightarrow{g} M$ and homotopies $\mathsf{id}_M \Rightarrow g \circ f$ and $f \circ g \Rightarrow \mathsf{id}_N$.[36] We may indicate that a morphism is a homotopy equivalence by decorating it as $\xrightarrow{\sim}$.

As a special case, a complex $M_\bullet$ is contractible if and only if either unique map $0 \to M_\bullet$ or $M_\bullet \to 0$ is a homotopy equivalence.

Exercise 3.5 (2 points)**.** Show that homotopic maps on complexes give equal maps on homology.

It follows immediately from Exercise 3.5 that homotopy equivalences are quasi-isomorphisms, and in particular that contractible complexes are acyclic. However, the converse is false.

Exercise 3.6 (6 points)**.**

(a) Show that the complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots$$

of $\mathbb{Z}/4$-modules is acyclic but not contractible.

(b) Show that the complex

$$\cdots \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \cdots$$

of $\Bbbk[x]/x^2$-modules is acyclic. Show that it is contractible as a complex of $\Bbbk$-modules, but not as a complex of $\Bbbk[x]/x^2$-modules (assuming that $\Bbbk \neq 0$).

3.1.5. We will generally consider homotopic maps as "essentially interchangeable". On the other hand, rather than merely positing the *existence* of a homotopy between two maps, we will always want to *keep track* of the homotopy that witnesses them as being homotopic. In fact, this will be accomplished by hom-complexes, which will also keep track of homotopies between homotopies, homotopies between those, ad infinitum (see §4.1).

The idea of keeping track of data instead of merely remembering its existence is a fundamental one in (higher) category theory, which has already appeared in §3.1.1 and will continue to feature prominently throughout this text.

3.2. **Homotopy cokernels.**

stable under co/limits. This follows from the more general fact that the full subcategory of the arrow category $\mathsf{Ar}(\mathsf{Mod}_R) := \mathsf{Fun}([1], \mathsf{Mod}_R)$ on those morphisms that factor through $0 \in \mathsf{Mod}_R$ is stable under co/limits; as co/limits in $\mathsf{Ar}(\mathsf{Mod}_R)$ are of course also computed pointwise, this follows from the fact that the zero object $0 \in \mathsf{Mod}_R$ is stable under co/limits (which follows from the fact that it is both initial and terminal).

[36]The directions of these homotopies are intended to be suggestive of adjunctions (with f functioning as the left adjoint), but by Exercise 3.1 (specifically the symmetry of the relation of homotopy) they are irrelevant. (Likewise, equivalences of categories are both left adjoints and right adjoints.)

3.2.1. Recall that the cokernel of a morphism $M \xrightarrow{f} N$ in Mod_R is by definition an R-module $\mathsf{coker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$N \xrightarrow{u} \mathsf{coker}(f)$$

satisfying the universal property that precomposition with u determines a natural bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(\mathsf{coker}(f), T) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N \to T \text{ such that the} \\ \text{composite } M \xrightarrow{f} N \to T \text{ is zero} \end{array} \right\} .^{37}$$

From here, the principles indicated in §3.1.5 lead directly to the definition of a ***homotopy cokernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R: this is an object $\mathsf{hcoker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$N_\bullet \xrightarrow{u_\bullet} \mathsf{hcoker}(f_\bullet)$$

satisfying the universal property that precomposition with $u_\bullet$ determines a natural bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f_\bullet), T_\bullet) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N_\bullet \to T_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } M_\bullet \xrightarrow{f_\bullet} N_\bullet \to T_\bullet \end{array} \right\} .^{38}$$

[37] Said differently, the cokernel of f is the pushout

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{coker}(f) \end{array} ,$$

i.e. the initial cocone under the diagram $0 \leftarrow M \xrightarrow{f} N$.

[38] Similarly, the homotopy cokernel of f may be characterized as the *homotopy* pushout

$$\begin{array}{ccc} M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \\ \downarrow & \nRightarrow & \downarrow \\ 0 & \longrightarrow & \mathsf{hcoker}(f_\bullet) \end{array} ,$$

i.e. the *initial homotopy-coherent cocone* under the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$ (though we'll refrain from defining this notion precisely here). Here, the symbol $\Rightarrow$ indicates that the square only commutes up to a specified homotopy. (This homotopy arises from the image of the identity map $\mathsf{id}_{\mathsf{hcoker}(f_\bullet)} \in \mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f_\bullet), \mathsf{hcoker}(f_\bullet))$ under the above natural bijection (as it must by the Yoneda lemma).) This is an algebraic analog of the topological construction taking a map $X \to Y$ of spaces to the space

$$\mathsf{hcoker}(X \to Y) := Y \coprod_X \mathsf{cone}(X) ,$$

where $\mathsf{cone}(X) := (X \times [0,1])/(X \times \{0\})$ denotes the *cone* on the space X (and we consider X to be embedded as the subspace $X \times \{1\}$ thereof). This topological construction is often called the (*mapping*) *cone*, and as a result it is common to the refer its algebraic analog using this same term. However, we prefer the term "homotopy cokernel" as it is more directly suggestive of the object's "role in life" (a.k.a. its *raison d'être*). (It is also common to refer to this object as the *homotopy cofiber*, but that term deviates from the more standard terminology of "cokernel" (as opposed to "cofiber") in the context of an abelian category.)

In fact, we have already seen an example of a homotopy cokernel: namely, for any morphism $M \xrightarrow{f} N$ in Mod_R, the complex $(M \xrightarrow{f} \underset{\sim}{N}) \in \mathsf{Ch}_R$ equipped with the map

$$\begin{array}{ccc} 0 & \longrightarrow & \underset{\sim}{N} \\ \downarrow & & \downarrow{\scriptstyle \mathsf{id}_N} \\ M & \xrightarrow[f]{} & \underset{\sim}{N} \end{array}$$

is a homotopy cokernel of f (when considered in Ch_R). More generally, given a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R, consider the diagram

$$\cdots \xrightarrow{\begin{pmatrix} -d_1^M & 0 \\ f_1 & d_2^N \end{pmatrix}} \begin{matrix} M_0 \\ \oplus \\ N_1 \end{matrix} \xrightarrow{\begin{pmatrix} -d_0^M & 0 \\ f_0 & d_1^N \end{pmatrix}} \underset{\sim}{\begin{matrix} M_{-1} \\ \oplus \\ N_0 \end{matrix}} \xrightarrow{\begin{pmatrix} -d_{-1}^M & 0 \\ f_{-1} & d_0^N \end{pmatrix}} \begin{matrix} M_{-2} \\ \oplus \\ N_{-1} \end{matrix} \xrightarrow{\begin{pmatrix} -d_{-2}^M & 0 \\ f_{-2} & d_{-1}^N \end{pmatrix}} \cdots$$

of R-modules.

Exercise 3.7 (6 points)**.** Verify that the above diagram indeed defines a complex and moreover is a homotopy cokernel of $f_\bullet$.

Exercise 3.8 (6 points)**.** Verify that the homotopy cokernel of f is the colimit of the diagram

$$\begin{array}{ccccc} & & M & \longrightarrow & 0 \\ & & {\scriptstyle i_0}\downarrow & & \\ M & \xrightarrow{i_1} & \mathbb{I} \otimes M & & \\ {\scriptstyle f}\downarrow & & & & \\ N & & & & \end{array} .$$

In particular, we find that for a morphism $M \xrightarrow{f} N$ in Mod_R, we have a canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f)$, and this is a quasi-isomorphism iff f is injective. This is an instance of the general principle that homotopically sensitive constructions are suitably equivalent (in this case, quasi-isomorphic) to their ordinary variants in "simple" situations (in this case, when there is no redundancy in the relations). More generally, we have the following.

Exercise 3.9 (6 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R that is injective in each dimension.

(a) Show that the canonical morphism

$$\mathsf{hcoker}(f) \longrightarrow \mathsf{coker}(f)$$

to the levelwise cokernel is a quasi-isomorphism.

(b) Give an example showing that this map need not be a homotopy equivalence.

As the following exercise illustrates, homotopy cokernels give us a way of translating conditions on morphisms between chain complexes to conditions on chain complexes themselves.

Exercise 3.10 (10 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R.

(a) Prove that f is a quasi-isomorphism iff $\mathsf{hcoker}(f)$ is acyclic.[39]

(b) Prove that f is a homotopy equivalence iff $\mathsf{hcoker}(f)$ is contractible.

3.2.2. As a special case of a homotopy cokernel, we simply write

$$\Sigma M_\bullet := \mathsf{hcoker}(M_\bullet \longrightarrow 0) \ ,^{40}$$

and refer to this as the ***suspension*** of $M_\bullet$.[41] So by definition, giving a chain map $\Sigma M_\bullet \to T_\bullet$ is equivalent to giving a nullhomotopy of the composite map $M_\bullet \to 0 \to T_\bullet$.

It is evident from the construction that suspension defines an autoequivalence

$$\mathsf{Ch}_R \xrightarrow{\Sigma} \mathsf{Ch}_R \ .$$

Namely,

$$\Sigma\left(\cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} \underset{\sim}{M_0} \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots\right) \cong \left(\cdots \xrightarrow{-d_1} M_0 \xrightarrow{-d_0} \underset{\sim}{M_{-1}} \xrightarrow{-d_{-1}} M_{-2} \xrightarrow{-d_{-2}} \cdots\right) :$$

the operation of suspension simply shifts all terms up by one and negates all differentials. We write Σ^{-1} for its inverse, which we refer to as ***desuspension***. More generally, for any $k \in \mathbb{N}$ we write $\Sigma^k := \Sigma^{\circ k}$ and $\Sigma^{-k} := (\Sigma^{-1})^{\circ k}$. We also note for future reference the evident natural isomorphisms

$$\mathsf{H}_n \circ \Sigma^k \cong \mathsf{H}_{n-k}$$

for all $n, k \in \mathbb{Z}$.

Exercise 3.11 (6 points)**.** Given a morphism $M \xrightarrow{f} N$ in Ch_R, establish a canonical homotopy equivalence

$$\Sigma M \simeq \mathsf{hcoker}(N \xrightarrow{u} \mathsf{hcoker}(f)) \ .^{42}$$

3.3. **Homotopy kernels.**

[39] The corresponding statement fails e.g. for spaces: there exist spaces X such that the morphism $X \to \mathsf{pt}$ is not a weak homotopy equivalence and yet $\Sigma X := \mathsf{hcoker}(X \to \mathsf{pt})$ is weakly contractible. Such spaces are called *acyclic*, as they are characterized by having the integral homology of a point.

[40] Not coincidentally, this admits a natural isomorphism

$$\Sigma M_\bullet \cong M \otimes_R (R \to \underset{\sim}{0})$$

with the R-balanced tensor product of $M_\bullet$ and the R-valued reduced simplicial chains on the simplicial circle $\Delta^1/\partial\Delta^1$ (using the fact that the latter is actually a complex of R-bimodules).

[41] This is more often denoted $M_\bullet[1]$ and referred to as the *shift* of M, but we prefer the more blatantly topological notation and terminology.

[42] That these are quasi-isomorphic follows from Exercise 3.9(a).

3.3.1. Dually, recall that the kernel of a morphism $M \xrightarrow{f} N$ is Mod_R is by definition an R-module $\mathsf{ker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$\mathsf{ker}(f) \xrightarrow{v} M$$

satisfying the universal property that postcomposition with v determines a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(T, \mathsf{ker}(f)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T \to M \text{ such that the} \\ \text{composite } T \to M \xrightarrow{f} N \text{ is zero} \end{array} \right\}.^{43}$$

This leads to the dual notion of a ***homotopy kernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R: this is an object $\mathsf{hker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$\mathsf{hker}(f_\bullet) \xrightarrow{v_\bullet} M_\bullet$$

satisfying the universal property that postcomposition with $v_\bullet$ determines a bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(T_\bullet, \mathsf{hker}(f_\bullet)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T_\bullet \to M_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } T_\bullet \to M_\bullet \xrightarrow{f_\bullet} N_\bullet \end{array} \right\}.^{44}$$

Exercise 3.12 (4 points)**.** Given a morphism $M \xrightarrow{f} N$ in Ch_R, prove that the evident levelwise projection map

$$\Sigma^{-1}\mathsf{hcoker}(f) \longrightarrow M$$

is a homotopy kernel of f.[45]

By combining Exercises 3.9(a) and 3.12, it follows that if a morphism $M \xrightarrow{f} N$ in Ch_R is surjective in each dimension, then the canonical morphism

$$\mathsf{ker}(f) \longrightarrow \mathsf{hker}(f)$$

from the levelwise kernel is a quasi-isomorphism (though again it is not necessarily a homotopy equivalence).

[43]Said differently, the kernel of f is the pullback

$$\begin{array}{ccc} \mathsf{ker}(f) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array}.$$

[44]Similarly, the homotopy kernel of f may be characterized as the *homotopy* pullback

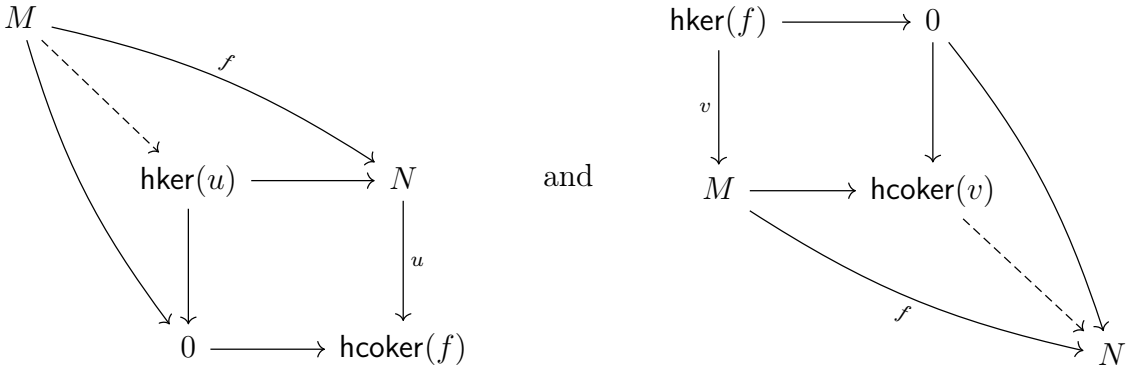
$$\begin{array}{ccc} \mathsf{hker}(f_\bullet) & \longrightarrow & 0 \\ \downarrow & \swarrow & \downarrow \\ M_\bullet & \xrightarrow[f_\bullet]{} & N_\bullet \end{array},$$

i.e. the *terminal homotopy-coherent cone* over the diagram $M_\bullet \xrightarrow{f_\bullet} N_\bullet \leftarrow 0$.

[45]There are unfortunately some signs that should arise here. They could be removed by tweaking the construction of hker (giving a different but isomorphic formula), but they would then arise elsewhere.

3.3.2. Although it will take some time to see why, the following feature is in some sense the *fundamental advantage* of working in the ∞-category $\mathbf{D}_R$ instead of in the ordinary category Mod_R.

Exercise 3.13 (6 points)**.** Prove that for any morphism $M \xrightarrow{f} N$ in Ch_R the dashed canonical morphisms



are homotopy equivalences.[46]

Namely, this implies that up to homotopy equivalence, every homotopy cokernel sequence is a homotopy kernel sequence, and conversely. This feature – along with the existence of a zero object and homotopy co/kernels in the first place – is succinctly expressed in the assertion that $\mathbf{D}_R$ is in fact a ***stable*** ∞-category: its fundamental advantage over Mod_R. By contrast, the corresponding assertion in Mod_R fails in a rather dramatic way: a cokernel sequence is a kernel sequence iff the original map is injective, and a kernel sequence is a cokernel sequence iff the original map is surjective.[47,48]

3.4. **Exact sequences.**

3.4.1. A complex $M_\bullet \in \mathsf{Ch}_R$ is said to be ***exact*** at M_n if $\mathsf{H}_n(M_\bullet) = 0$. In particular, an acyclic complex is also called an ***exact sequence*** in Mod_R, or sometimes a ***long exact sequence*** to emphasize that it is (potentially) infinite in one or both directions.

As a special case, a ***short exact sequence*** in Mod_R is a three-term acyclic complex

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \ .$$

So, a short exact sequence is such a diagram satisfying the conditions that f is injective, g is surjective, and $\mathsf{ker}(g) = \mathsf{im}(f)$. In this case, one may also say that M is an ***extension*** of

[46]Note that these diagrams are only *homotopy-coherently* commutative (indeed, the canonical morphisms are induced by homotopy-coherent universal properties).

[47]As Mod_R has no intrinsic notion of "homotopy", the only reasonable definition of "homotopy co/kernel" therein simply reduces to the usual notion of co/kernel.

[48]In fact, ordinary categories are almost never stable (when considered as ∞-categories): an ordinary category is stable if and only if it is trivial (i.e. it has only a zero object (up to equivalence)).

N by L. For instance,

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules, which expresses $\mathbb{Z}/4$ as an extension of $\mathbb{Z}/2$ by itself.

More generally, we may refer to any (possibly finite) sequence of composable morphisms in Mod_R as an ***exact sequence*** if it is exact at all *interior* terms. So for example, one may refer to the diagram

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

as an exact sequence.[49]

3.4.2. The following is the source of almost every single long exact sequence in mathematics.[50]

Exercise 3.14 (2 points)**.** For any morphism $M \xrightarrow{f} N$ in Ch_R, show that the sequence

$$\mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f))$$

is exact.

Namely, from Exercises 3.11, 3.12, and 3.13 we obtain an infinite sequence

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\Sigma^{-1}v} & \Sigma^{-1}M & \xrightarrow{\Sigma^{-1}f} & \Sigma^{-1}N & \xrightarrow{\Sigma^{-1}u} & \Sigma^{-1}\mathsf{hcoker}(f) & & & & \\
 & & & & & & \wr\| & & & & \\
 & & & & & & \mathsf{hker}(f) & \xrightarrow{v} & M & \xrightarrow{f} & N \xrightarrow{u} \mathsf{hcoker}(f) \\
 & & & & & & & & & & \wr\| \\
 & & & & & & & & & & \Sigma\mathsf{hker}(f) \xrightarrow{\Sigma v} \Sigma M \xrightarrow{\Sigma f} \Sigma N \xrightarrow{\Sigma u} \cdots
\end{array}$$

of morphisms in Ch_R in which every composable pair of morphisms is a homotopy co/kernel sequence up to homotopy equivalence.[51] Thereafter, by Exercise 3.14, applying H_0 yields a long exact sequence

(3)

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow \mathsf{H}_1(M) & \longrightarrow & \mathsf{H}_1(N) & \longrightarrow & \mathsf{H}_1(\mathsf{hcoker}(f)) & & & & & & \\
 & & & & \wr\| & & & & & & \\
 & & & & \mathsf{H}_0(\mathsf{hker}(f)) & \longrightarrow & \mathsf{H}_0(M) & \longrightarrow & \mathsf{H}_0(N) & \longrightarrow & \mathsf{H}_0(\mathsf{hcoker}(f)) \\
 & & & & & & & & & & \wr\| \\
 & & & & & & & & & & \mathsf{H}_{-1}(\mathsf{hker}(f)) \longrightarrow \mathsf{H}_{-1}(M) \longrightarrow \mathsf{H}_{-1}(N) \longrightarrow \cdots
\end{array}$$

in Mod_R.

[49] Note that here we are *not* implicitly extending the sequence by zero.

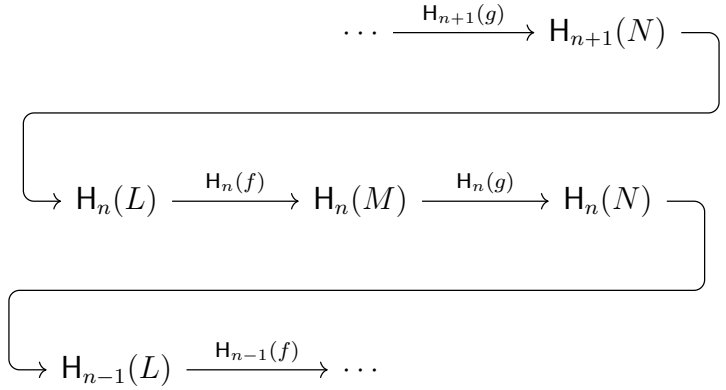
[50] A notable exception is the long exact sequence on (stable) homotopy groups, of which this is actually a special case.

[51] To be precise, here we are using the fact that homotopy co/kernels are invariant up to homotopy equivalence under homotopy equivalence (i.e. componentwise homotopy equivalence in $\mathsf{Fun}([1], \mathsf{Ch}_R)$).

Just as (quasi-isomorphism classes of) complexes encode more information than their homology groups (recall Footnote 20), so do (quasi-isomorphism classes of) morphisms in Ch_R encode more information than the corresponding long exact sequences. Therefore, we view the homotopy co/kernel sequence as the more fundamental notion.

The following exercise illustrates the more classical approach to constructing long exact sequences.

Exercise 3.15 (6 points)**.** Suppose that $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a sequence of morphisms in Ch_R that is exact in each dimension.[52] Construct a long exact sequence



in Mod_R.

Note that Exercise 3.15 gives a special case of the long exact sequence (3), because the sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a homotopy co/kernel sequence up to quasi-isomorphism by Exercise 3.9(a).[53]

4. The dg-category of complexes

4.1. **Hom-complexes.** Given complexes $M, N \in \mathsf{Ch}_R$, we have seen that morphisms $M \to N$ may be related by *homotopies*. In fact, these homotopies can be related by *higher homotopies*, ad infinitum. Altogether, these data organize neatly into the ***hom-complex*** from M to N, denoted

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \in \mathsf{Ch}_{\Bbbk} \ .^{54}$$

Namely, in dimension n the hom-complex is given by

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)_n := \prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \ ,$$

[52] Recall from Exercise 3.9(a) that this implies that the canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f) \cong N$ is a quasi-isomorphism.

[53] On the other hand, in a sense the long exact sequence of Exercise 3.15 is no less general: up to quasi-isomorphism, every homotopy co/kernel sequence is of this form. (However, we view homotopy co/kernel sequences as the more fundamental notion.)

[54] In fact, the hom-complex also contains "lower homotopies", in the sense that complexes are "like spaces, but with the possibility of cells in negative dimensions".

and its differential

$$\prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \xrightarrow{d_n} \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1})$$

carries a tuple $\{f_i \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n})\}_{i\in\mathbb{Z}}$ to the tuple

$$\left\{(d^N_{i+n} \circ f_i + (-1)^{n-1} \cdot f_{i-1} \circ d^M_i) \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1})\right\}_{i\in\mathbb{Z}} .$$

Exercise 4.1 (2 points)**.** Verify that this formula defines a complex.

Directly from the definition, we have that $\mathsf{Z}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$. Moreover, given any morphism $M \xrightarrow{f} N$ in Ch_R, a nullhomotopy $0 \overset{h}{\Rightarrow} f$ is equivalent data to an element $h \in \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)_1$ such that $d_1(h) = f$. So, $\mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N))$ is canonically isomorphic to the abelian group of *homotopy classes* of morphisms $M \to N$. Note too that

$$\Sigma^i \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(\Sigma^{-i}M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, \Sigma^i N) ;$$

this gives an analogous description of all homology groups of the hom-complex, as $\mathsf{H}_n \cong \mathsf{H}_0 \circ \Sigma^{-n}$. On the other hand, these homology groups can also be understood in a more homotopical (although closely related) manner.

Exercise 4.2 (4 points)**.** Given a chain map $M \xrightarrow{f} N$ and two nullhomotopies $0 \overset{h_0}{\Rightarrow} f$ and $0 \overset{h_1}{\Rightarrow} f$, define a notion of a homotopy $h_0 \Rightarrow h_1$ between homotopies, and prove that such a higher homotopy always exists precisely when $\mathsf{H}_1(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) = 0$.

If one views the complexes M and N as 0-cells, the maps $M \xrightarrow{0} N$ and $M \xrightarrow{f} N$ as 1-cells, and the homotopies h_i as 2-cells, then such a higher homotopy should be viewed as a 3-cell. Of course, there are analogs of this same interpretation for all the homology groups of $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$.

For brevity, we may simply write

$$\underline{\mathsf{hom}}(M, N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) .$$

4.2. **The dg-category of complexes.** These hom-complexes, in turn, can be naturally assembled into a single object.

Exercise 4.3 (8 points)**.** Fix any complexes $L, M, N, O \in \mathsf{Ch}_R$.

(a) Construct a map

$$\underline{\mathsf{hom}}(M, N) \otimes \underline{\mathsf{hom}}(L, M) \xrightarrow{\chi_{L,M,N}} \underline{\mathsf{hom}}(L, N)$$

in $\mathsf{Ch}_{\Bbbk}$ that encodes the composition of morphisms of complexes.[55]

[55] [changed 5/8] Note the ordering of the hom-complexes, which adheres to standard notation for composition. Of course, we could also consider this as a map $\underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, N) \to \underline{\mathsf{hom}}(L, N)$, but note that this would introduce unnatural signs in the composition map. (Recall from §2.5 that the symmetry isomorphism of the symmetric monoidal category $\mathsf{Ch}_{\Bbbk}$ follows the Koszul sign rule.)

(b) Verify that these composition morphisms are associative in the sense that the diagram

$$\begin{array}{ccc} \underline{\mathsf{hom}}(N,O)\otimes\underline{\mathsf{hom}}(M,N)\otimes\underline{\mathsf{hom}}(L,M) & \xrightarrow{\mathsf{id}\otimes\chi_{L,M,N}} & \underline{\mathsf{hom}}(N,O)\otimes\underline{\mathsf{hom}}(L,N) \\ \downarrow{\scriptstyle \chi_{M,N,O}\otimes\mathsf{id}} & & \downarrow{\scriptstyle \chi_{L,N,O}} \\ \underline{\mathsf{hom}}(M,O)\otimes\underline{\mathsf{hom}}(L,M) & \xrightarrow[\chi_{L,M,O}]{} & \underline{\mathsf{hom}}(L,O) \end{array}$$

commutes.[56]

(c) Construct a map

$$\Bbbk \xrightarrow{\iota_M} \underline{\mathsf{hom}}(M,M)$$

using the identity morphism of $M \in \mathsf{Ch}_R$, and verify that it defines a two-sided identity for the above composition in the sense that the diagrams

$$\begin{array}{ccc} \Bbbk\otimes\underline{\mathsf{hom}}(L,M) & \xrightarrow{\iota_M\otimes\mathsf{id}} & \underline{\mathsf{hom}}(M,M)\otimes\underline{\mathsf{hom}}(L,M) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \chi_{L,M,M}} \\ & & \underline{\mathsf{hom}}(L,M) \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathsf{hom}}(M,N)\otimes\Bbbk & \xrightarrow{\mathsf{id}\otimes\iota_M} & \underline{\mathsf{hom}}(M,N)\otimes\underline{\mathsf{hom}}(M,M) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \chi_{M,M,N}} \\ & & \underline{\mathsf{hom}}(M,N) \end{array}$$

commute (where the isomorphisms are the canonical ones coming from the fact that $\Bbbk \in \mathsf{Ch}_\Bbbk$ is the unit object).

Altogether, Exercise 4.3 yields a ($\Bbbk$-***linear***) ***dg-category***,[57] i.e. a category enriched in the symmetric monoidal category $(\mathsf{Ch}_\Bbbk, \otimes, \Bbbk)$:[58] its objects are the chain complexes of R-modules and its hom-objects are the hom-complexes between them. We denote this dg-category by

[56] This implicitly uses the associativity of the operation $\otimes$ in $\mathsf{Ch}_\Bbbk$.

[57] Here, "dg" is short for "differential graded". (In general, it is common to refer to a chain complex of R-modules as a *dg-R-module*.)

[58] Of course, a dg-category is a particular instance of a more general notion. Namely, given a monoidal category $\mathcal{V} := (\mathcal{V}, \otimes_\mathcal{V}, \mathbb{1}_\mathcal{V})$, there is a natural notion of a ***category enriched in*** $\mathcal{V}$, or simply a ***$\mathcal{V}$-enriched category***: a $\mathcal{V}$-enriched category $\mathcal{C}$ consists of a set of objects, the data of hom-objects $\underline{\mathsf{hom}}_\mathcal{C}(X,Y) \in \mathcal{V}$ for all $X, Y \in \mathcal{C}$, and the data of composition and identity morphisms

$$\underline{\mathsf{hom}}_\mathcal{C}(Y,Z)\otimes_\mathcal{V}\underline{\mathsf{hom}}_\mathcal{C}(X,Y) \xrightarrow{\chi_{X,Y,Z}} \underline{\mathsf{hom}}_\mathcal{C}(X,Z) \qquad \text{and} \qquad \mathbb{1}_\mathcal{V} \xrightarrow{\iota_Y} \underline{\mathsf{hom}}_\mathcal{C}(Y,Y)$$

in $\mathcal{V}$ for all $X, Y, Z \in \mathcal{C}$, subject to the evident associativity and unitality conditions. (As indicated, one sometimes uses an underline to emphasize that these are *enriched* hom-objects (as opposed to mere hom-sets).) On the other hand, note that the notation $\mathsf{hom}_\mathcal{C}(X,Y)$ is already unambiguous, as $\mathcal{C}$ is a $\mathcal{V}$-enriched category.

$\mathbf{K}_R$ and refer to it as ***the dg-category of complexes of*** R***-modules***.[59] In particular, we may also write $\mathsf{hom}_{\mathbf{K}_R}(M,N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)$; our chosen notation will depend on our desired emphasis.

The fundamental purpose of the dg-category $\mathbf{K}_R$ is to assemble the data of chain maps, homotopies, and higher (and lower) homotopies into a single object. In particular, it is the homology groups of the hom-complexes in $\mathbf{K}_R$ that are relevant. Therefore, it will be natural to view the hom-complexes in $\mathbf{K}_R$ as "only being important up to quasi-isomorphism".

As a special case, note that for any complex $M \in \mathsf{Ch}_R$, composition in $\mathbf{K}_R$ makes the complex $\mathsf{hom}_{\mathbf{K}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ into an *associative algebra object*, i.e. it is a ***dg-algebra*** (or ***dga*** for short) over $\Bbbk$.[60]

Exercise 4.4 (4 points)**.** Show that the following conditions are equivalent.

(i) The complex $M \in \mathbf{K}_R$ is contractible.
(ii) The complex $\mathsf{hom}_{\mathbf{K}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ is acyclic.
(iii) The complex $\mathsf{hom}_{\mathbf{K}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ has that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(M,M)) = 0$.
(iv) The element $[\mathsf{id}_M] \in \mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(M,M))$ has that $[\mathsf{id}_M] = 0$.

Among the morphisms in the dg-category $\mathbf{K}_R$, the homotopy equivalences play a distinguished role.

Given a $\mathcal{V}$-enriched category $\mathcal{C}$, we can obtain an ordinary category $\mathsf{fgt}(\mathcal{C})$ – its "underlying unenriched category" – by defining $\mathsf{fgt}(\mathcal{C})$ to have the same objects as $\mathcal{C}$ and defining

$$\mathsf{hom}_{\mathsf{fgt}(\mathcal{C})}(X,Y) := \mathsf{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \mathsf{hom}_{\mathcal{C}}(X,Y)) \ .$$

(Then, e.g. the identity element of $Y \in \mathsf{fgt}(\mathcal{C})$ is simply the element $\iota_Y \in \mathsf{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \underline{\mathsf{hom}}_{\mathcal{C}}(Y,Y))$. More broadly and more formally, this construction yields a $(\mathsf{Set},\times)$-enriched category (e.g. with the expected composition maps as well as their associativity and unitality) because $\mathsf{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}},-)$ defines a *right-laxly monoidal* functor $(\mathcal{V},\otimes_{\mathcal{V}}) \to (\mathsf{Set},\times)$ (see e.g. **??**).) So, a "$\mathcal{V}$-enrichment" of an ordinary category $\mathcal{D}$ may be defined to be a $\mathcal{V}$-enriched category $\tilde{\mathcal{D}}$ as well as an identification $\mathcal{D} \cong \mathsf{fgt}(\tilde{\mathcal{D}})$.

It is worth acknowledging that the notion of a $\mathsf{Ch}_{\Bbbk}$-enrichment of an ordinary category is somewhat more exotic than that of e.g. a $\mathsf{Mod}_{\Bbbk}$- or Top-enrichment. On the one hand, the functors $\mathsf{hom}_{\mathsf{Mod}_{\Bbbk}}(\Bbbk,-)$ and $\mathsf{hom}_{\mathsf{Top}}(\mathsf{pt},-)$ are simply the respective "underlying set" functors, so a $\mathsf{Mod}_{\Bbbk}$- or Top-enrichment of an ordinary category amounts to endowing its hom-sets with the indicated structures (satisfying appropriate conditions, e.g. multilinearity or continuity respectively). By contrast, the functor $\mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(\Bbbk,-)$ can be identified as the functor $\mathsf{Ch}_{\Bbbk} \xrightarrow{\mathsf{Z}_0} \mathsf{Mod}_{\Bbbk} \xrightarrow{\mathsf{fgt}} \mathsf{Set}$ of 0-cycles, so obtaining a $\mathsf{Ch}_{\Bbbk}$-enrichment of an ordinary category involves defining hom-complexes whose 0-cycles are the original hom-sets. (Recall our trivial observation that $\mathsf{Z}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M,N)$.)

[59] The German word for "complex" begins with the letter "K".

[60] Of course, this construct can be reversed. Namely, given a dg-algebra $A \in \mathsf{Alg}(\mathsf{Ch}_{\Bbbk})$, we can form a dg-category $\mathfrak{B}A \in \mathsf{cat}^{\mathsf{dg}}$ as follows: it has a single object $*$, and we declare that $\mathsf{hom}_{\mathfrak{B}A}(*,*) := A$ (with composition and identity respectively defined as the multiplication and unit in A).

Exercise 4.5 (4 points)**.** Given a homotopy equivalence $M \to N$ in $\mathbf{K}_R$, show that the natural transformations

$$\mathsf{hom}_{\mathbf{K}_R}(-, M) \longrightarrow \mathsf{hom}_{\mathbf{K}_R}(-, N) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(M, -) \longleftarrow \mathsf{hom}_{\mathbf{K}_R}(N, -)$$

are natural homotopy equivalences.

On the other hand, the quasi-isomorphisms cannot play any such distinguished role in $\mathbf{K}_R$. For instance, given any acyclic but noncontractible complex $A \in \mathbf{K}_R$ (recall Exercise 3.6), by Exercise 4.4 the morphism

$$0 \cong \mathsf{hom}_{\mathbf{K}_R}(A, 0) \longrightarrow \mathsf{hom}_{\mathbf{K}_R}(A, A)$$

cannot be a quasi-isomorphism (let alone a homotopy equivalence).

4.3. **Key features of the dg-category of complexes.**

4.3.1. As we will discuss in §5.1, the basic operations of hom and tensor product among ordinary modules do *not* generally respect co/kernels. We will repair this failure in two steps. First, we will pass to the world of complexes, and from ordinary co/kernels to *homotopy* co/kernels; the latter are respected by hom and tensor product, as we will see in §4.3.3. However, these constructions fail to preserve quasi-isomorphisms. So, we will further correct them by studying *resolutions*, which are the topic of §5.

4.3.2. In order to prove that hom and tensor product among complexes preserve homotopy co/kernels, it will be convenient to use tensor-hom adjunctions among complexes, which we now establish. For any complex $T \in \mathsf{Ch}_{\Bbbk}$ of $\Bbbk$-modules and any complexes $M, N \in \mathsf{Ch}_R$ of R-modules, we may form the complexes

$$T \otimes_{\Bbbk} M \qquad \text{and} \qquad \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, N)$$

of R-modules, where the (right) R-actions are induced from those on M and N. These satisfy the universal properties that

$$\mathsf{hom}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \cong \mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, N))\ .^{61}$$

In fact, these satisfy *enriched* universal properties.

Exercise 4.6 (6 points)**.** Prove that for any $T \in \mathsf{Ch}_{\Bbbk}$ and any $M, N \in \mathsf{Ch}_R$ we have natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, N))$$

in $\mathsf{Ch}_{\Bbbk}$.

Of course, these generalize the natural isomorphisms (1) in the context of ordinary modules (instead of chain complexes of modules).

[61] In the general context of enriched category theory, one says that $T \otimes M$ is the *tensoring* of M by T and that $\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, N)$ is the *cotensoring* of N by T.

4.3.3. We can now show that hom and tensor product among complexes preserve homotopy co/kernels.

Exercise 4.7 (8 points)**.** Choose any complexes $M, N, T \in \mathsf{Ch}_R$ and any morphism $M \xrightarrow{f} N$. Using Exercise 4.6 and the universal property of the homotopy co/kernel, construct natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(T, \mathsf{hker}\left(M \xrightarrow{f} N\right)\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, M) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T,f)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, N)\right)$$

and

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(\mathsf{hcoker}\left(M \xrightarrow{f} N\right), T\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(N, T) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(f,T)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, T)\right)$$

in $\mathsf{Ch}_{\Bbbk}$.

Of course, essentially identical reasoning shows that the hom-complex bifunctor

$$\mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.[62]

Exercise 4.8 (4 points)**.** Show that the tensor product bifunctor

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.

5. Projective and injective resolutions

5.1. **Motivation for resolutions.** The original motivation for homological algebra is the fact that many natural functors on ordinary modules do not preserve exact sequences. Equivalently but more fundamentally, they do not preserve short exact sequences, i.e. they do not respect both kernels and cokernels.[63]

Exercise 5.1 (6 points)**.**

[62] Alternatively, this follows from Exercise 4.7, the commutative square

$$\begin{array}{ccc} \mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_R & \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} & \mathsf{Ch}_R \\ \downarrow{\scriptstyle \mathsf{id}\times\mathsf{fgt}} & & \downarrow{\scriptstyle \mathsf{fgt}} \\ \mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_{\Bbbk} & \xrightarrow[\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)]{} & \mathsf{Ch}_{\Bbbk} \end{array},$$

and the fact that the forgetful functor both preserves and detects homotopy kernel sequences.

[63] Here we use the word "respect" instead of "preserve" due to the contravariance of $\mathsf{hom}_{\mathsf{Mod}_R}(-, M)$: recognizing that $\mathsf{Mod}_R^{\mathsf{op}}$ is also an abelian category, one might hope that this would carry kernels to cokernels and cokernels to kernels.

(a) Show that the functor

$$\mathsf{Mod}_{\mathbb{k}} \times \mathsf{Mod}_R \xrightarrow{(-)\otimes_{\mathbb{k}}(-)} \mathsf{Mod}_R$$

does not generally preserve exact sequences in either variable.

(b) Show that the functor

$$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\hom_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_R$$

does not generally preserve exact sequences in either variable.

Namely, applying either bifunctor appearing in Exercise 5.1 to an exact sequence in one of its slots, one obtains a "half-exact" sequence: $(-)\otimes_{\mathbb{k}}(-)$ preserves cokernels separately in each variable, $\hom_{\mathsf{Mod}_R}(M,-)$ preserves kernels, and $\hom_{\mathsf{Mod}_R}(-,M)$ carries cokernels to kernels. It was originally desired for these half-exact sequences to extend to long exact sequences, whose additional terms would quantify these various failures of exactness.

As illustrated by Exercises 4.7 and 4.8, towards resolving these issues it is fruitful to pass from ordinary modules and ordinary co/kernels to complexes of modules and homotopy co/kernels; the desired long exact sequences would then be those on homology discussed in §3.4, although of course we will take the perspective that the homotopy co/kernel sequences themselves are the more fundamental objects. However, given that we would like to consider quasi-isomorphisms as isomorphisms, the following results show that this maneuver does not suffice on its own.

Exercise 5.2 (6 points)**.**

(a) Show that the functor

$$\mathsf{Ch}_{\mathbb{k}} \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_{\mathbb{k}}(-)} \mathsf{Ch}_R$$

does not generally preserve acyclic objects in either variable.

(b) Show that the functor

$$\mathsf{Ch}_R^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\underline{\hom}_{\mathsf{Ch}_R}(-,-)} \mathsf{Ch}_{\mathbb{k}}$$

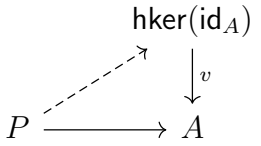
does not generally preserve acyclic objects in either variable.

Clearly, preservation of acyclics is necessary for the preservation of quasi-isomorphisms. But in fact, the converse is guaranteed by Exercise 3.10(a) (combined with Exercises 4.7 and 4.8). This motivates the notions that we introduce now.

5.2. **Projective and injective complexes.**

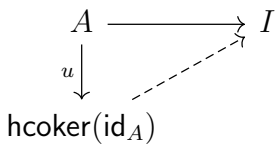
5.2.1. We write $\mathbf{A}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the acyclic complexes.

We say that a complex $P \in \mathbf{K}_R$ is ***projective*** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(P, A) \in \mathsf{Ch}_{\Bbbk}$ is acyclic. Because acyclic complexes are preserved under de/suspensions and $\Sigma^k \mathsf{hom}_{\mathbf{K}_R}(P, A) \cong \mathsf{hom}_{\mathbf{K}_R}(P, \Sigma^{-k} A)$, we may equivalently demand simply that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(P, A)) = 0$ for every acyclic complex $A \in \mathbf{A}_R$. In other words, P is projective iff every morphism $P \to A$ to an acyclic complex admits a nullhomotopy. In turn, this is equivalent to the condition that for every acyclic complex $A \in \mathbf{A}_R$ and every solid diagram



there exists a lift making the diagram commute. We write $\mathbf{P}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the projective complexes.[64]

Dually, we say that a complex $I \in \mathbf{K}_R$ is ***injective*** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(A, I) \in \mathsf{Ch}_{\Bbbk}$ is acyclic. Likewise, we may equivalently demand for all acyclic complexes $A \in \mathbf{A}_R$ that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(A, I)) = 0$, or that every morphism $A \to I$ admits a nullhomotopy, or that for every solid diagram



there exists an extension making the diagram commute. We write $\mathbf{I}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the injective complexes.

Given that our primary interest is in quasi-isomorphism classes of complexes, we would like to consider the acyclic complexes as "negligible". So, recalling that we only consider hom-complexes "up to quasi-isomorphism", one may view the condition of a complex being projective (resp. injective) as being a necessary condition in order for it to be "good" for mapping out of (resp. into). We will eventually see that these necessary conditions are also sufficient, in a precise sense.

It is easy to deduce the following facts.

Exercise 5.3 (8 points)**.**

[64]We observe for future reference that v is a levelwise surjective quasi-isomorphism: indeed, $\mathsf{hker}(\mathsf{id}_A)$ is acyclic by the long exact sequence in homology, and v is surjective by construction (or by its defining universal property). So, in order for a complex to be projective, it suffices for it to have the analogous lifting property with respect to *all* levelwise surjective quasi-isomorphisms.

(a) Show that all quasi-isomorphisms between projective (resp. injective) complexes are homotopy equivalences.[65]

(b) Show that the property of projectivity (resp. injectivity) is stable under homotopy equivalence.

(c) Show that projective complexes are preserved under tensor product: if $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$ then $P \otimes Q \in \mathbf{P}_R$.

(d) Show that for any projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P,-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(Q,-)} \mathbf{K}_{\Bbbk}$$

preserve quasi-isomorphisms.

(e) Show that for any injective complex $I \in \mathbf{I}_R$, the functors

$$\mathbf{K}_{\Bbbk}^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(-,I)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(-,I)} \mathbf{K}_{\Bbbk}$$

preserve quasi-isomorphisms.

Clearly the zero complex is both projective and injective. Hence, (a) implies that if an acyclic complex is either projective or injective then it must be contractible, while (b) implies that every contractible complex is both projective and injective. In other words, the intersections

$$\mathbf{P}_R \cap \mathbf{A}_R \qquad \text{and} \qquad \mathbf{I}_R \cap \mathbf{A}_R$$

coincide and consist of the contractible complexes.

5.2.2. Of course, the definitions of projective and injective complexes on their own are not so useful. What gives them their power is that every complex $M \in \mathbf{K}_R$ admits both a ***projective resolution*** $P \xrightarrow{\approx} M$ and an ***injective resolution*** $M \xrightarrow{\approx} I$ (i.e. quasi-isomorphisms as indicated). These are the promised representatives of the quasi-isomorphism class of M that are "well-adapted" for certain purposes. Specifically, we will respectively view the functors

$$(-) \otimes_{\Bbbk} P\ , \qquad \mathsf{hom}_{\mathbf{K}_R}(P,-) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-,I)$$

as "corrected" (a.k.a. "derived") versions of the functors

$$(-) \otimes_{\Bbbk} M\ , \qquad \mathsf{hom}_{\mathbf{K}_R}(M,-) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-,M)$$

(and similarly for projective resolutions of complexes of $\Bbbk$-modules).

As we will see, it is relatively straightforward to construct projective resolutions of *bounded-below* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \ll 0$) and to construct injective resolutions of *bounded-above* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \gg 0$).

[65] The statement for projective complexes is formally analogous to *Whitehead's theorem*, which states that a weak homotopy equivalence between cell complexes (or retracts thereof) is necessarily a homotopy equivalence.

Note in particular that these constructions apply to ordinary R-modules via the inclusion $\mathsf{Mod}_R \subseteq \mathsf{Ch}_R$.

As for the unbounded situation, it turns out to be much easier (and more conceptually satisfying) to construct projective resolutions than injective resolutions. On the other hand, we will not need to know anything specific about injective resolutions of unbounded complexes, beyond their existence. Therefore, we will discuss only projective resolutions in the unbounded case, and refer the reader to [Spa88] (or to [Hov99, §2.3]) for the construction of injective resolutions in the unbounded case.

It is easy to deduce the following facts directly from the existence of projective and injective resolutions.

Exercise 5.4 (6 points)**.**

(a) Show that if the complex $M \in \mathbf{K}_R$ has that $\mathsf{hom}_{\mathbf{K}_R}(P, M)$ is acyclic for every projective complex $P \in \mathbf{P}_R$, then M is acyclic.[66]

(b) Show that if the complex $M \in \mathbf{K}_R$ has that $\mathsf{hom}_{\mathbf{K}_R}(M, I)$ is acyclic for every injective complex $I \in \mathbf{I}_R$, then M is acyclic.

(c) Show that for any projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors

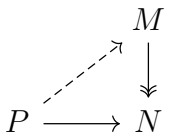
$$\mathbf{K}_R \xrightarrow{P\otimes_{\Bbbk}(-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_{\Bbbk} \xrightarrow{(-)\otimes_{\Bbbk} Q} \mathbf{K}_R$$

preserve quasi-isomorphisms.

In particular, the tensor product of a projective complex and an acyclic complex is acyclic.

We will eventually organize many of the facts enumerated in Exercises 5.3 and 5.4 in a more systematic way.

5.3. **Projective resolutions in the bounded-below case.** Recall that an R-module $P \in \mathsf{Mod}_R$ is called ***projective*** if mapping out of it preserves surjections, i.e. if for any surjection $M \twoheadrightarrow N$ in Mod_R the induced map $\mathsf{hom}_{\mathsf{Mod}_R}(P, M) \to \mathsf{hom}_{\mathsf{Mod}_R}(P, N)$ is a surjection. Said differently, given any solid diagram



there exists a dashed lift making the diagram commute.

Exercise 5.5 (2 points)**.** Show that an R-module $P \in \mathsf{Mod}_R$ is projective iff it is a summand of a free module (i.e. there exists an R-module $Q \in \mathsf{Mod}_R$ and an isomorphism $P \oplus Q \cong R^{\oplus S}$ with the free R-module on some set S).

Exercise 5.6 (4 points)**.** Show that every projective $\mathbb{Z}$-module is free.

[66] Evidently, it is already sufficient just to take $P = R$ (which is projective by Exercises 5.5 and 5.9).

Exercise 5.7 (4 points)**.** Give necessary and sufficient conditions on $n \in \mathbb{N}$ such that every projective $\mathbb{Z}/n$-module is free.

Exercise 5.8 (4 points)**.** Show that an R-module $M \in \mathsf{Mod}_R$ is projective iff the corresponding complex $\underset{\sim}{M} \in \mathsf{Ch}_R$ is projective.

We now turn from projective modules back to projective complexes.

Exercise 5.9 (6 points)**.** Show that if $P \in \mathbf{K}_R$ is a bounded-below complex such that each $P_n \in \mathsf{Mod}_R$ is projective, then $P \in \mathbf{P}_R$ is projective.

The bounded-below hypothesis in Exercise 5.9 is necessary: both parts of Exercise 3.6 give examples of unbounded complexes which are levelwise free (hence levelwise projective by Exercise 5.5) that cannot be projective. Indeed, an acyclic projective complex must be contractible, as its identity map must admit a nullhomotopy.

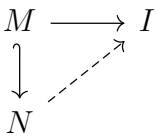
However, there is no converse statement.

Exercise 5.10 (2 points)**.** Find a projective complex that is not levelwise projective.[67]

Having studied the notions of projectivity for modules and complexes, we now turn to the existence of projective resolutions in the bounded-below case. For this, let us note that every R-module admits a surjection from a projective (in fact free) R-module; one says that the abelian category Mod_R ***has enough projectives***.

Exercise 5.11 (6 points)**.** Using the fact that the abelian category Mod_R has enough projectives (along with Exercise 5.9), construct a projective resolution $P \xrightarrow{\approx} M$ of any bounded-below complex $M \in \mathbf{K}_R$.

5.4. **Injective resolutions in the bounded-above case.** Recall that an R-module $I \in \mathsf{Mod}_R$ is called ***injective*** if mapping into it carries injections to surjections, i.e. if for any injection $M \hookrightarrow N$ in Mod_R the induced map $\mathsf{hom}_{\mathsf{Mod}_R}(N, I) \to \mathsf{hom}_{\mathsf{Mod}_R}(M, I)$ is surjective. Said differently, given any solid diagram



there exists a dashed extension making the diagram commute.

Injective modules are much more bizarre than projective modules.

Exercise 5.12 (8 points)**.**

(a) Show that if an R-module $M \in \mathsf{Mod}_R$ is injective then it is *divisible*, i.e. for every nonzero element $r \in R$ the map $M \xrightarrow{r} M$ is surjective.

(b) Show that if R is a PID then the converse holds, i.e. divisible R-modules are injective.

[67] This can be taken to be bounded-below.

So for instance, $\mathbb{Q} \in \mathsf{Mod}_{\mathbb{Z}}$ is injective while $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}$ is not.

It is easy to dualize the argument of Exercise 5.9: bounded-above levelwise injective complexes are injective. Using this, we would also like to dualize the argument of Exercise 5.11 to show that any bounded-above complex $M \in \mathbf{K}_R$ admits an injective resolution $M \xrightarrow{\approx} I$. This too is easy, once one knows that Mod_R also ***has enough injectives***, i.e. that any R-module admits an injection into an injective R-module. This relatively subtle assertion can be established as follows.

Exercise 5.13 (12 points)**.** Fix a ring homomorphism $S \to R$.

(a) Verify that the functors

$$\mathsf{Mod}_S \underset{\mathsf{hom}_{\mathsf{Mod}_S}(R,-)}{\overset{(-)\otimes_S R}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot \quad \mathsf{fgt} \quad \bot}}}} \mathsf{Mod}_R$$

participate in adjunctions as indicated.[68]

(b) Given modules $M \in \mathsf{Mod}_R$ and $N \in \mathsf{Mod}_S$ and an injection

$$\mathsf{fgt}(M) \hookrightarrow N$$

in Mod_S, show that the corresponding morphism

$$M \longrightarrow \mathsf{hom}_{\mathsf{Mod}_S}(R, N)$$

in Mod_R is also an injection.

(c) Deduce from the fact that fgt preserves injections that the functor $\mathsf{hom}_{\mathsf{Mod}_S}(R, -)$ preserves injective objects.[69]

(d) Show that injective objects are preserved under products.[70]

(e) Fix an abelian group $A \in \mathsf{Ab}$. Show that for every $a \in A$, there exists a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ carrying a to a nonzero element.[71] Deduce that there exists an injection

$$A \hookrightarrow \prod_A \mathbb{Q}/\mathbb{Z} \ .^{72}$$

Namely, suppose we are given an R-module $M \in \mathsf{Mod}_R$. By (e) we have an injection

$$\mathsf{fgt}(M) \hookrightarrow \prod_M \mathbb{Q}/\mathbb{Z} =: I$$

[68] In defining the functor $(-) \otimes_S R$ (resp. $\mathsf{hom}_{\mathsf{Mod}_S}(R, -)$), we use that R is an (S, R)-bimodule (resp. an (R, S)-bimodule).

[69] Dually, the functor $(-) \otimes_S R$ preserves projective objects because the functor fgt also preserves surjections.

[70] Dually, projective objects are preserved under coproducts.

[71] This uses Exercise 5.12.

[72] One says that $\mathbb{Q}/\mathbb{Z} \in \mathsf{Ab}$ is an ***injective cogenerator***. Dually, $R \in \mathsf{Mod}_R$ is a ***projective generator***: it is projective, and moreover for any $M \in \mathsf{Mod}_R$ there exists a surjection $R^{\oplus S} \twoheadrightarrow M$ for some set S.

in Ab, and by (d) and Exercise 5.12 we see that $I \in \mathsf{Ab}$ is injective. Thereafter, by (b) we obtain an injection

$$M \hookrightarrow \mathsf{hom}_{\mathsf{Ab}}(R, I) =: J$$

in Mod_R, and by (c) we see that $J \in \mathsf{Mod}_R$ is injective. So indeed, Mod_R has enough injectives.

5.5. **Cell complexes and lifting criteria.** For the purpose of constructing projective resolutions, we first introduce some auxiliary ideas.

For any $n \in \mathbb{Z}$, we define the objects and morphism

$$\begin{array}{ccc} S^n & & \\ {\scriptstyle i_n}\downarrow & := & \\ D^{n+1} & & \end{array} \qquad := \qquad \begin{array}{ccc} 0 & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle \mathsf{id}_R} \\ R & \xrightarrow[\mathsf{id}_R]{} & R \end{array}$$

in Ch_R, where the columns are in dimensions $n+1$ and n.[73] For any complex $M \in \mathsf{Ch}_R$, we have natural isomorphisms

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \cong \mathsf{Z}_n(M) \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \cong M_{n+1}\ ;$$

precomposition with i_n induces the morphism

$$\mathsf{Z}_n(M) \xleftarrow{d_{n+1}} M_{n+1}\ .$$[74]

We say that a morphism $M \xrightarrow{f} N$ has the ***right lifting property*** with respect to i_n (or equivalently that i_n has the ***left lifting property*** with respect to f) if for every solid commutative diagram

$$\begin{array}{ccc} S^n & \longrightarrow & M \\ {\scriptstyle i_n}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ D^{n+1} & \longrightarrow & N \end{array}$$

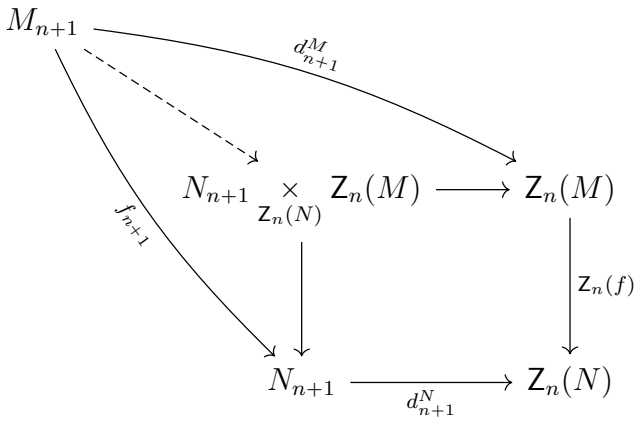
[73] Of course, this morphism will function as "the inclusion of the n-sphere into the $(n+1)$-disk": indeed, for any $n \geqslant 0$ it is the reduced simplicial chains on the inclusion $\Delta^{\{0,\dots,n\}}/\partial\Delta^{\{0,\dots,n\}} \hookrightarrow \Delta^n/\Lambda^n_0$ (which is a particularly small simplicial model for this morphism). Note too that $S^n \cong \Sigma^n S^0$ and $D^n \cong \Sigma^n D^0$. (While the former isomorphism is always an equality (so that we may take $\Sigma^n S^0$ to be the definition of S^n), the latter introduces an inconvenient sign when n is odd (which is why we do not take $\Sigma^n D^0$ to be the definition of D^n).)

[74] Categorically speaking, the reason that the morphism

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, -) \xleftarrow{(-)\circ i_n} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, -)$$

in $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_{\Bbbk})$ lifts to $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_R)$ is that the morphism i_n is in fact a morphism of complexes of (R,R)-bimodules.

there exists a dashed lift that makes the diagram commute. This is equivalent to saying that the canonical dashed morphism in the pullback



is surjective. We may abbreviate this by writing $f \in \mathsf{rlp}(i_n)$ (or by writing $i_n \in \mathsf{llp}(f)$). We also introduce the notation $I := \{i_n\}_{n\in\mathbb{Z}}$ and $\mathsf{rlp}(I) := \bigcap_{n\in\mathbb{Z}} \mathsf{rlp}(i_n)$.

Exercise 5.14 (4 points)**.** Prove that if $f \in \mathsf{rlp}(i_n)$ then $\mathsf{H}_n(f)$ is injective and $\mathsf{Z}_{n+1}(f)$ is surjective.

Now, suppose that $f \in \mathsf{rlp}(I)$. We claim this implies that f is a levelwise surjective quasi-isomorphism. First of all, Exercise 5.14 immediately implies that f is a quasi-isomorphism. Moreover, Exercise 5.14 implies that $\mathsf{Z}_n(f)$ is surjective (using that $f \in \mathsf{rlp}(i_{n-1})$). From here, the above reformulation of $\mathsf{rlp}(i_n)$ implies that f_{n+1} is surjective. So altogether, it follows that f is levelwise surjective.

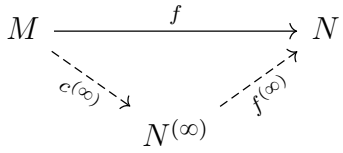
In fact, the converse is also true.

Exercise 5.15 (4 points)**.** Prove that if $M \xrightarrow{f} N$ is a levelwise surjective quasi-isomorphism, then $f \in \mathsf{rlp}(I)$.

In other words, $\mathsf{rlp}(I)$ consists precisely of the levelwise surjective quasi-isomorphisms.

5.6. **Projective resolutions as cellular approximations.**

5.6.1. We now construct projective resolutions via a general procedure known as the ***small object argument***.[75] In fact, given a morphism $M \xrightarrow{f} N$ in Ch_R, we will construct a factorization



satisfying the following conditions:

[75]This is formally analogous to the construction of a cell complex that is weak homotopy equivalent to a given topological space. The name arises from its crucial use of a certain "smallness" property of the sources of the elements of the set I. In the present situation, the relevant fact is given as Exercise 5.18.

(1) the morphism $c^{(\infty)}$ has the left lifting property with respect to levelwise surjective quasi-isomorphisms (i.e. $c^{(\infty)} \in \mathsf{llp}(\mathsf{rlp}(I))$), and
(2) the morphism $f^{(\infty)}$ is a quasi-isomorphism.

Taking $M \xrightarrow{f} N$ to be the map $0 \to N$ then yields a projective resolution of N: the projectivity of $N^{(\infty)}$ in this case follows from Footnote 64.[76]

5.6.2. We begin some auxiliary notation. First of all, we write

$$I(f)_n := \mathsf{hom}_{\mathsf{Ar}(\mathsf{Ch}_R)}(i_n, f) \cong \mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \underset{\mathsf{hom}_{\mathsf{Ch}_R}(S^n,N)}{\times} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, N)$$

for the set of morphisms

$$\begin{array}{ccc} S^n & \dashrightarrow & M \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f} \\ D^{n+1} & \dashrightarrow & N \end{array}$$

in the arrow category $\mathsf{Ar}(\mathsf{Ch}_R) := \mathsf{Fun}([1], \mathsf{Ch}_R)$ of Ch_R (i.e. pairs of morphisms $S^n \to M$ and $D^{n+1} \to N$ making the square commute). Moreover, we write

$$I(f) := \coprod_{n \in \mathbb{Z}} I(f)_n$$

for the disjoint union of these sets indexed over all $n \in \mathbb{Z}$. And finally, given an element $\alpha \in I(f)$, we write $n_\alpha := n$ if $\alpha \in I(f)_n$. (That is, we write $I(f) \xrightarrow{n_{(-)}} \mathbb{Z}$ for the evident function from the disjoint union.)

5.6.3. We now construct the factorization $f = f^{(\infty)} \circ c^{(\infty)}$.

First of all, observe that we have a canonical commutative square

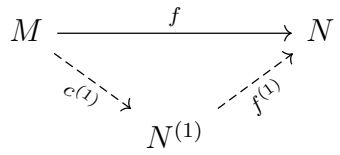
$$\begin{array}{ccc} \bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \longrightarrow & M \\ {\scriptstyle \oplus_{\alpha \in I(f)} i_{n_\alpha}}\downarrow & & \downarrow{\scriptstyle f} \\ \bigoplus_{\alpha \in I(f)} D^{n_\alpha + 1} & \longrightarrow & N \end{array} .$$

We write

$$N^{(1)} := \mathsf{colim} \left(\begin{array}{ccc} \bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \longrightarrow & M \\ {\scriptstyle \oplus_{\alpha \in I(f)} i_{n_\alpha}}\downarrow & & \\ \bigoplus_{\alpha \in I(f)} D^{n_\alpha + 1} & & \end{array} \right)$$

[76] One may interpret the general case as providing a "relatively projective" resolution of N as an object under M. (Whereas a projective complex is like a cell complex (or a retract thereof), a relatively projective complex is like a relative cell complex (or a retract thereof).)

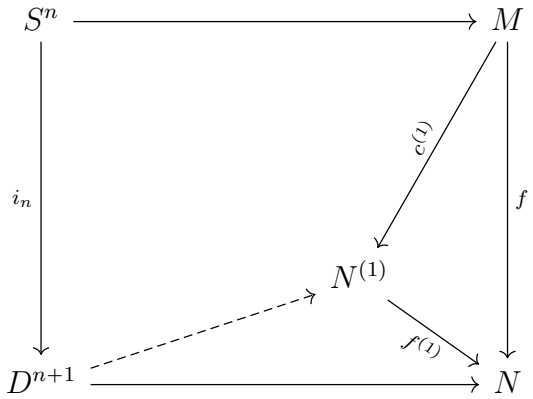
for the pushout, and we write



for the morphisms in the induced factorization.

Exercise 5.16 (2 points)**.** Show that the morphism $c^{(1)}$ has the left lifting property with respect to any levelwise surjective quasi-isomorphism (i.e. $c^{(1)} \in \mathsf{llp}(\mathsf{rlp}(I))$).

Exercise 5.17 (2 points)**.** Show that for every solid commutative diagram



there exists a dashed factorization making the diagram commute.

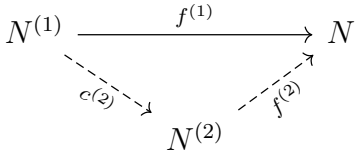
We now apply the above construction again, but to $f^{(1)}$ instead of to f: namely, we observe the canonical commutative square

$$\begin{array}{ccc} \bigoplus_{\alpha\in I(f^{(1)})} S^{n_\alpha} & \longrightarrow & N^{(1)} \\ \downarrow{\scriptstyle \oplus_{\alpha\in I(f^{(1)})} i_{n_\alpha}} & & \downarrow{\scriptstyle f^{(1)}} \\ \bigoplus_{\alpha\in I(f^{(1)})} D^{n_\alpha+1} & \longrightarrow & N \end{array},$$

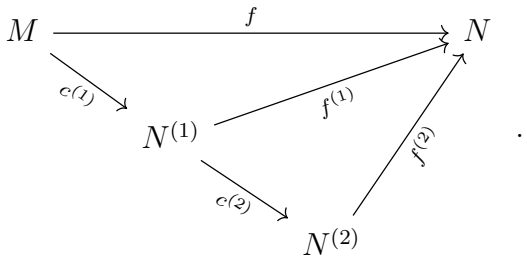
we write

$$N^{(2)} := \mathsf{colim}\left(\begin{array}{ccc} & \bigoplus_{\alpha\in I(f^{(1)})} S^{n_\alpha} & \longrightarrow N^{(1)} \\ & \downarrow{\scriptstyle \oplus_{\alpha\in I(f^{(1)})} i_{n_\alpha}} & \\ & \bigoplus_{\alpha\in I(f^{(1)})} D^{n_\alpha+1} & \end{array} \right)$$

for the pushout, and we write



for the morphisms in the induced factorization. This gives us a commutative diagram



.

Of course, we continue to iterate this construction in the obvious way. To conclude, we define

$$N^{(\infty)} := \mathsf{colim}\left(M \xrightarrow{c^{(1)}} N^{(1)} \xrightarrow{c^{(2)}} N^{(2)} \xrightarrow{c^{(3)}} \cdots\right) ,$$

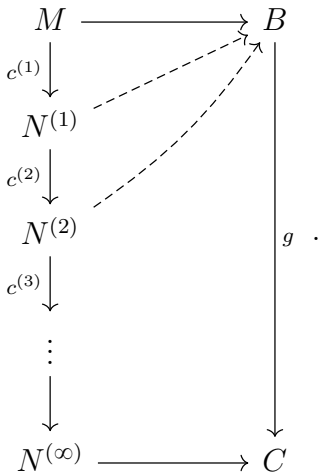
we define the map $M \xrightarrow{c^{(\infty)}} N^{(\infty)}$ to be the canonical map to the colimit, and we define the map $N^{(\infty)} \xrightarrow{f^{(\infty)}} N$ to be that induced from the maps $f^{(n)}$ by the universal property of the colimit.

5.6.4. We now verify the two conditions given in §5.6.1.

(1) Let $B \xrightarrow{g} C$ be a levelwise surjective quasi-isomorphism (i.e. an element of $\mathsf{rlp}(I)$). Given any commutative square

$$\begin{array}{ccc} M & \longrightarrow & B \\ {\scriptstyle c^{(\infty)}}\downarrow & & \downarrow{\scriptstyle g} \\ N^{(\infty)} & \longrightarrow & C \end{array} ,$$

we enlarge it to the solid commutative diagram



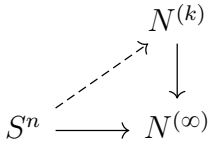
.

Here, applying Exercise 5.16 repeatedly we may inductively construct dashed lifts as indicated that make the diagram commute. By the universal property of the colimit, together these yield the desired lift in the original diagram.[77] In other words, we have shown that $c^{(\infty)} \in \mathsf{llp}(g)$. However, since $g \in \mathsf{rlp}(I)$ was arbitrary, altogether we have shown that $c^{(\infty)} \in \mathsf{llp}(\mathsf{rlp}(I))$, as desired.

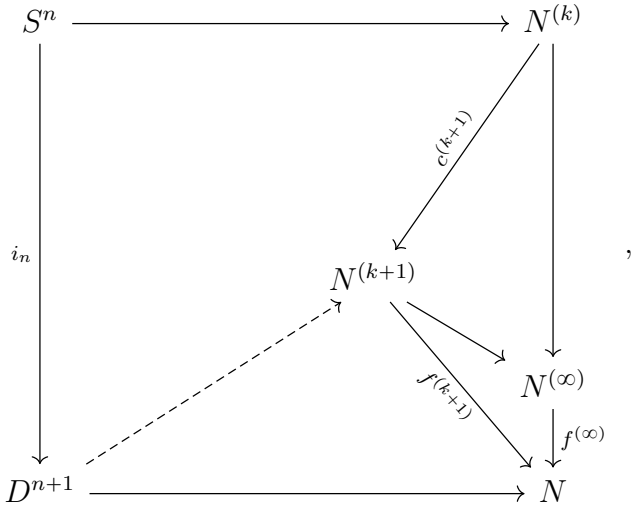
(2) In fact, we show that $f^{(\infty)} \in \mathsf{rlp}(I)$. For this, suppose we are given any commutative diagram

$$\begin{array}{ccc} S^n & \longrightarrow & N^{(\infty)} \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f^{(\infty)}} \\ D^{n+1} & \longrightarrow & N \end{array} .$$

By Exercise 5.18 below, there exists some $k \in \mathbb{N}$ and a lift



of the given map. Now, just as in Exercise 5.17, we obtain a lift



which proves the claim.

Exercise 5.18 (2 points)**.** Show that the object $S^n \in \mathsf{Ch}_R$ is ***compact***, i.e. that the functor

$$\mathsf{Ch}_R \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(S^n,-)} \mathsf{Set}$$

commutes with filtered colimits.[78]

[77] This may be phrased as choosing an element of a codirected limit of *surjective* functions between sets.

[78] Note that a functor commutes with filtered colimits if and only if it commutes with directed colimits. (This is one instance of the useful general principle of "coordinatization" of a class of colimits. As another example, given a functor between categories that admit finite colimits, it preserves them if and only if it preserves initial objects and pushouts. Another related fact is that given a functor between cocomplete

6. The derived ∞-category

6.1. The homotopy theory of the dg-category of complexes.

6.1.1. We have by now obtained a diagram

$$\begin{array}{ccccc} \mathbf{P}_R & & & & \\ & \searrow & & & \\ & & \mathbf{K}_R & \hookleftarrow & \mathbf{A}_R \\ & \nearrow & & & \\ \mathbf{I}_R & & & & \end{array} \tag{4}$$

of fully faithful inclusions among dg-categories: we began with the subcategory $\mathbf{A}_R \subseteq \mathbf{K}_R$, and then we defined $\mathbf{P}_R \subseteq \mathbf{K}_R$ (resp. $\mathbf{I}_R \subseteq \mathbf{K}_R$) to be what might be called its *homotopical left* (resp. *right*) *orthogonal* subcategory, namely

$$\mathbf{P}_R := \{M \in \mathbf{K}_R \text{ such that } \mathsf{hom}_{\mathbf{K}_R}(M, A) \approx 0 \text{ for all } A \in \mathbf{A}_R\}$$

and

$$\mathbf{I}_R := \{M \in \mathbf{K}_R \text{ such that } \mathsf{hom}_{\mathbf{K}_R}(A, M) \approx 0 \text{ for all } A \in \mathbf{A}_R\} \ .^{79}$$

Recall that for any complex $M \in \mathbf{K}_R$ we have constructed a projective resolution $M' \xrightarrow{\approx} M$, i.e. a quasi-isomorphism from a projective object. Consider the resulting homotopy kernel sequence

$$A \longrightarrow M' \longrightarrow M$$

in Ch_R, where we write $A := \mathsf{hker}(M' \to M)$ because this complex is acyclic (by the long exact sequence in homology). Now, for any projective complex $P \in \mathbf{P}_R \subseteq \mathbf{K}_R$, by Exercise 4.7 we obtain a homotopy kernel sequence

$$\underline{\mathsf{hom}}(P, A) \longrightarrow \underline{\mathsf{hom}}(P, M') \longrightarrow \underline{\mathsf{hom}}(P, M)$$

in $\mathsf{Ch}_{\Bbbk}$. Because P is projective, the complex $\underline{\mathsf{hom}}(P, A)$ is acyclic, and hence we find that the morphism

$$\underline{\mathsf{hom}}(P, M') \xrightarrow{\approx} \underline{\mathsf{hom}}(P, M)$$

is a quasi-isomorphism (again by the long exact sequence). Heuristically, we have found that "the morphism $M' \to M$ appears to be an isomorphism when mapping from projective complexes (and considering hom-complexes up to quasi-isomorphism)".

---

categories (i.e. categories admitting all (small) colimits), it preserves all (small) colimits if and only if it preserves finite colimits and filtered colimits.)

[79] This categorifies the notion of orthogonality in linear algebra: given a vector space V equipped with an inner product $\langle -, - \rangle$, the orthogonal of a subspace $W \subseteq V$ is defined to be

$$\{v \in V \text{ such that } \langle v, w \rangle = 0 \text{ for all } w \in W\} \ .$$

The categorified notion comes equipped with a handedness because hom-objects aren't symmetric (in contrast with inner products), and by "homotopical" we mean that we only demand that the hom-complexes are quasi-isomorphic (rather than isomorphic) to 0.

6.1.2. This sort of phenomenon is in fact quite ubiquitous. Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between ordinary categories. Given an object $D \in \mathcal{D}$, a ***pointwise right adjoint*** to F at D consists of an object $C \in \mathcal{C}$ and a morphism $F(C) \to D$ such that the resulting composite morphism

$$\mathsf{hom}_{\mathcal{C}}(-, C) \xrightarrow{F} \mathsf{hom}_{\mathcal{D}}(F(-), F(C)) \longrightarrow \mathsf{hom}_{\mathcal{D}}(F(-), D)$$

in $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set})$ is a natural isomorphism. In this situation, we may refer to the morphism $F(C) \to D$ as the ***pointwise counit*** and denote it by ε_D (or simply by ε).

Exercise 6.1 (6 points)**.**

(a) Show that if it exists, a pointwise right adjoint to F at D is unique up to unique isomorphism.[80]

(b) Show that the datum of a right adjoint to F is equivalent data to a choice of pointwise right adjoint to F at every object $D \in \mathcal{D}$.[81]

So, the above discussion may be summarized as saying that in a homotopical sense – that is, if we only consider hom-complexes up to quasi-isomorphism – the fully faithful inclusion $\mathbf{P}_R \hookrightarrow \mathbf{K}_R$ "should" admit a right adjoint whose pointwise counits are projective resolutions. On the other hand, because these morphisms on hom-complexes are only quasi-isomorphisms and not isomorphisms, we should not expect this structure to exist at the level of dg-categories.

6.1.3. Of course, this discussion immediately dualizes: any complex $M \in \mathbf{K}_R$ has an injective resolution $M \xrightarrow{\approx} M''$ (i.e. a quasi-isomorphism to an injective object), and this has the property that for any injective complex $I \in \mathbf{I}_R \subseteq \mathbf{K}_R$ the morphism

$$\underline{\mathsf{hom}}(M'', I) \xrightarrow{\approx} \underline{\mathsf{hom}}(M, I)$$

is a quasi-isomorphism. Heuristically, "the morphism $M \to M''$ appears to be an isomorphism when mapping to an injective complex (and considering hom-complexes up to quasi-isomorphism)". So, in a homotopical sense the fully faithful inclusion $\mathbf{I}_R \hookrightarrow \mathbf{K}_R$ "should" admit a left adjoint whose pointwise units are injective resolutions.

6.2. **A brief introduction to $\Bbbk$-linear ∞-categories.**

6.2.1. These adjoints that "should" exist in a homotopical sense actually *do* exist at the level of underlying $\Bbbk$-linear ∞-categories. So, we work within an ∞-categorical context.

However, our present usage of ∞-category theory will be extremely "soft": it will not require any real familiarity with the foundations of the theory. So in the interest of maintaining the narrative thread, here we give an extremely brief summary of the immediately relevant features of ∞-category theory, and pursue a more systematic discussion later.

[80] Said differently, the category of pointwise right adjoints to F at D is always either an empty or contractible groupoid.

[81] It follows that the category of right adjoints to F is always either an empty or contractible groupoid (because these are preserved under products).

6.2.2. Here is the most expedient definition of a $\Bbbk$-linear ∞-category.[82] We write cat for the category of ordinary categories, and we write $\mathsf{cat}^{\mathsf{dg}} \xrightarrow{\mathsf{H}_0} \mathsf{cat}$ for the functor given by applying H_0 to each hom-complex. Then, a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between dg-categories is called a ***weak equivalence*** if the following two conditions hold:

(1) it is ***homotopically fully faithful***, i.e. for all $C, C' \in \mathcal{C}$ the induced morphism $\mathsf{hom}_{\mathcal{C}}(C, C') \to \mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$ in $\mathsf{Ch}_{\Bbbk}$ is a quasi-isomorphism; and

(2) it is ***homotopically essentially surjective***, i.e. the induced functor $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ is essentially surjective.[83]

(Of course, if $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a weak equivalence of dg-categories then $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ is in fact an equivalence of categories.)

We localize the category $\mathsf{cat}^{\mathsf{dg}}_{\Bbbk}$ of $\Bbbk$-linear dg-categories at the weak equivalences. This yields a category that (for reasons that will become clear later) we will denote by $\mathsf{ho}(\mathsf{Cat}_{\Bbbk})$ and refer to as ***the homotopy category of $\Bbbk$-linear ∞-categories***; its objects are called ***$\Bbbk$-linear ∞-categories***. So by definition, there is a canonical functor

$$\mathsf{cat}^{\mathsf{dg}} \longrightarrow \mathsf{ho}(\mathsf{Cat}_{\Bbbk})$$

that carries all weak equivalences to isomorphisms. In particular, this functor is essentially surjective: every $\Bbbk$-linear ∞-category can be represented by a $\Bbbk$-linear dg-category.[84]

Given a dg-category $\mathcal{C}$, for the remainder of this section we will write $\mathcal{C}^{\infty}$ for its underlying $\Bbbk$-linear ∞-category. However, we will thereafter omit this superscript, as we will have finished drawing distinctions between dg-categories and their underlying $\Bbbk$-linear ∞-categories. We will generally refer to data in $\mathcal{C}$ as "point-set" and data in $\mathcal{C}^{\infty}$ as "∞-categorical".

Essentially by construction, given two dg-categories $\mathcal{C}, \mathcal{D} \in \mathsf{cat}^{\mathsf{dg}}$, morphisms $\mathcal{C}^{\infty} \to \mathcal{D}^{\infty}$ in $\mathsf{ho}(\mathsf{Cat}_{\Bbbk})$ are given by equivalence classes of zigzags

$$\mathcal{C} \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \cdots \longrightarrow \bullet \xleftarrow{\approx} \mathcal{D}$$

in $\mathsf{cat}^{\mathsf{dg}}$ (in which all backwards maps are weak equivalences). However, it turns out that every equivalence class contains a representative of the form

$$\mathcal{C} \xleftarrow{\approx} \mathcal{C}' \longrightarrow \mathcal{D} \ .$$

In fact, dg-categories admit "projective resolutions": every equivalence class contains a representative of this form for *any* fixed weak equivalence $\mathcal{C}' \xrightarrow{\approx} \mathcal{C}$ from a "projective" dg-category.

[82] This should be compared with the discussion of §2.4.

[83] It is possible to phrase homotopical essential surjectivity at the level of dg-categories, but this requires a bit of enriched category theory.

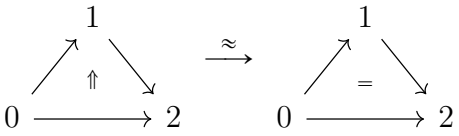
[84] This is analogous to the presentation of ordinary ∞-categories as categories enriched in simplicial sets; in particular, in both cases the presentations admit strict compositions (in contrast with the perspective on composition in ∞-categories explained in §6.2.3). A presentation of $\Bbbk$-linear ∞-categories that adheres more closely to e.g. quasicategories or complete Segal spaces is given by ($\Bbbk$-linear) $\mathbb{A}_{\infty}$-categories.

Once one is only considering hom-complexes up to quasi-isomorphism, homotopy equivalences (e.g. in $\mathbf{K}_R$ and its various subcategories) become indistinguishable from isomorphisms. In general, one simply uses the term ***equivalence*** to refer to the ∞-categorical notion of isomorphism. We will denote equivalences in an ∞-category by $\xrightarrow{\sim}$.

For brevity, for the time being we will often simply refer to $\Bbbk$-linear ∞-categories as "∞-categories".

6.2.3. To a first approximation, a $\Bbbk$-linear ∞-category is simply a category enriched in $\mathsf{H}_0(\mathbf{D}_\Bbbk)$, the derived category of $\Bbbk$ introduced in §2.4. In particular, passing from dg-categories to their underlying $\Bbbk$-linear ∞-categories does indeed "only remember their hom-complexes up to quasi-isomorphism". The distinction lies in the higher homotopies (e.g. those recording the homotopy-coherent associativity of composition) which are present in the underlying ∞-category of a dg-category but are lost in passing to its underlying $\mathsf{H}_0(\mathbf{D}_\Bbbk)$-enriched category.

Namely, as we will see when we study ∞-categories more generally, given a dg-category $\mathcal{C}$, commutative diagrams in its underlying $\Bbbk$-linear ∞-category $\mathcal{C}^\infty$ are represented by *homotopy-coherently* commutative diagrams in $\mathcal{C}$ itself. For instance, we will see the projective resolution

(5)


of the free strictly-commutative triangle $[2] = \{0 < 1 < 2\}$ by the free homotopy-coherently commutative triangle, and this persists upon passing to free dg-categories.[85]

Said differently, given morphisms $C_0 \xrightarrow{\varphi_{01}} C_1$, $C_1 \xrightarrow{\varphi_{12}} C_2$, and $C_0 \xrightarrow{\varphi_{02}} C_2$ in the ∞-category $\mathcal{C}^\infty$, we can no longer ask whether φ_{02} is *equal* to the composition $\varphi_{12} \circ \varphi_{01}$.[86] Rather, we can only ask for a homotopy that "witnesses φ_{02} as being a homotopy-coherent composite of φ_{12} and φ_{01}".[87]

6.2.4. As a particular case, it turns out that homotopy co/kernel squares in dg-categories – which fit into *homotopy-coherently* commutative squares – define co/limit diagrams – and particular, *commutative* squares – in their underlying ∞-categories.[88] For the moment, we will write coker^∞ and ker^∞ to denote ∞-categorical co/kernels.

[85]To be precise, this is a sort of projective resolution (actually a "cofibrant" resolution) among categories enriched in simplicial sets, and by "the free $\Bbbk$-linear dg-category" on a category enriched in simplicial sets we mean the dg-category obtained by taking $\Bbbk$-linear simplicial chains on its hom-objects.

[86]Indeed, e.g. in $\mathbf{K}_R$ we have that chain maps $M \to N$ correspond to 0-cycles in $\mathsf{hom}_{\mathbf{K}_R}(M, N)$, but it is not well-defined to discuss the "0-cycles" of a derived $\Bbbk$-module (because these are not generally preserved by quasi-isomorphisms).

[87]Of course, any pair of composable morphisms does admit *some* "homotopy-coherent composite", and in fact these assemble into a contractible complex.

[88]This should not be surprising, based on our discussion of the former.

It will also be convenient to discuss more general ∞-categorical pushout/pullback squares in a $\Bbbk$-linear ∞-category. As an expedient definition, we can declare that a commutative square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[i]{} & Z \end{array}$$

in a $\Bbbk$-linear ∞-category is a ***pushout*** (resp. ***pullback***) if the resulting commutative square

$$\begin{array}{ccc} W & \xrightarrow{(f,g)} & X \oplus Y \\ \downarrow & & \downarrow{\scriptstyle (h,-i)} \\ 0 & \longrightarrow & Z \end{array}$$

is a cokernel (resp. kernel) square.

It turns out that pushout and pullback squares (and in particular, cokernel and kernel squares) coincide in *any* $\Bbbk$-linear ∞-category.[89] In order to refer to them in an unbiased way, we may refer to them as ***exact squares***.

6.3. **The derived ∞-category.**

6.3.1. Having introduced $\Bbbk$-linear ∞-categories, we can now finally define the ***derived ∞-category of R-modules***: it is the ∞-categorical localization

$$\mathbf{K}_R^\infty \xrightarrow{\pi} \mathbf{D}_R$$

at the quasi-isomorphisms. In other words, for any $\Bbbk$-linear ∞-category $\mathcal{C}$, precomposition with π determines an inclusion

$$\mathsf{hom}_{\mathsf{ho}(\mathsf{Cat}_\Bbbk)}(\mathbf{K}_R^\infty, \mathcal{C}) \longhookrightarrow \mathsf{hom}_{\mathsf{ho}(\mathsf{Cat}_\Bbbk)}(\mathbf{D}_R, \mathcal{C}) \ ,$$

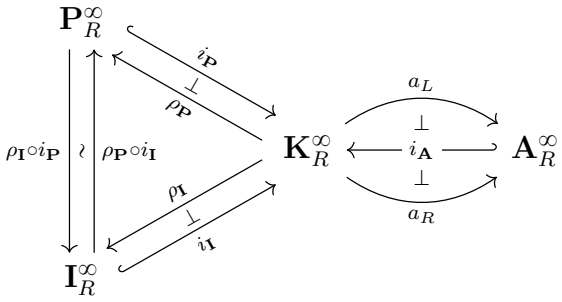
whose image consists of those functors $\mathbf{K}_R^\infty \to \mathcal{C}$ that carry all quasi-isomorphisms to equivalences.[90]

Of course, this definition of the derived ∞-category $\mathbf{D}_R$ is rather abstract. However, it turns out that it is canonically equivalent to *both* underlying ∞-categories $\mathbf{P}_R^\infty$ and $\mathbf{I}_R^\infty$ (which in particular are canonically equivalent to one another). More precisely, as we will explain,

[89] This makes crucial use of the $\Bbbk$-linearity: it is certainly not the case that pushout and pullback squares coincide in a general ∞-category.

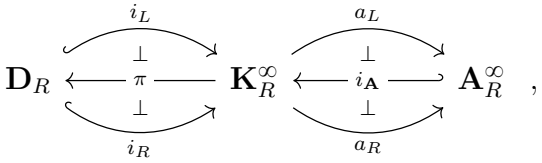
[90] We have stated this universal property in the category $\mathsf{ho}(\mathsf{Cat}_\Bbbk)$, but in fact it holds in the ∞-category $\mathsf{Cat}_\Bbbk$ as well.

the diagram (4) of dg-categories extends to a diagram

(6)


on underlying ∞-categories, in which *both* functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ satisfy the universal property of the functor π. This implies that there are canonical equivalences between $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ as functors out of $\mathbf{K}_R^\infty$ (i.e. there are canonical equivalences between their targets that commute with them), and we will see that these are precisely the equivalences indicated in the diagram.

Taking these assertions for granted, we can express diagram (6) more concisely by collapsing the vertical equivalences and using the unbiased notation $\mathbf{D}_R$ to obtain a diagram

(7)


in which i_L (resp. i_R) denotes "the inclusion of $\mathbf{D}_R$ into $\mathbf{K}_R^\infty$ as the full subcategory of projective (resp. injective) complexes", which is the left (resp. right) adjoint to the projection functor π. This latter diagram forms what is called a ***recollement***, a sort of categorified extension sequence, which notion we will discuss more later.

6.3.2. Before explaining the diagram (6), we pause to slightly sharpen the perspective on the passage from a dg-category to its underlying ∞-category given in §6.2.3, in a way that will be helpful shortly. Namely, it is possible to *define* a $\Bbbk$-linear ∞-category to be a category enriched in the derived ∞-category $\mathbf{D}_\Bbbk$ of $\Bbbk$.[91] Then, the passage from a $\Bbbk$-linear dg-category to its underlying $\Bbbk$-linear ∞-category amounts to applying the functor

$$\mathbf{K}_\Bbbk^\infty \xrightarrow{\pi} \mathbf{D}_\Bbbk$$

to each of its hom-complexes.

6.3.3. It is important to note that there are many commutative squares in $\mathbf{K}_R^\infty$ that may not be co/kernel squares themselves but nevertheless become so in $\mathbf{D}_R$, e.g. any commutative

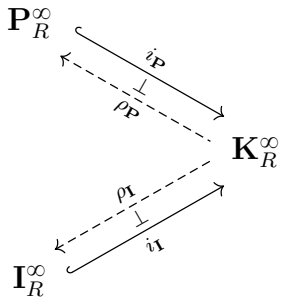
[91] It would certainly be circular to do so here. (In particular, we do not discuss the symmetric monoidal structure on $\mathbf{D}_\Bbbk$ until §6.3.7.) The formalism of enriched ∞-categories is developed in [GH15].

square of the form

$$\begin{array}{ccc} X & \xrightarrow{\approx} & Y \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[\approx]{} & Z \end{array} .$$

In general, it is only after projective resolution or injective resolution that such a square becomes a co/kernel square.

6.3.4. We begin our explanation of diagram (6) by obtaining the adjunctions in its left side. Of course, these were already hinted at in §6.1: the existence of the adjoints in the diagram



follows immediately from the ∞-categorical analog of Exercise 6.1 and its dual. We refer to $\rho_{\mathbf{P}}$ (resp. $\rho_{\mathbf{I}}$) as the ***projective*** (resp. ***injective***) ***resolution*** functor.

Recall that projective resolutions are far from unique at the "point-set" (a.k.a. "chain") level. For instance, applying the the construction of §5.6 to a projective complex $P \in \mathbf{P}_R$ will yield a quasi-isomorphism $P' \xrightarrow{\approx} P$ from another projective complex, which is generally be very far from being an isomorphism. On the other hand, by Exercise 5.3(a) this must be a homotopy equivalence, and hence it defines an equivalence in the underlying ∞-category $\mathbf{P}_R^\infty$. More broadly, the ∞-categorical analog of Exercise 6.1(a) tells us that projective resolutions of *any* complex $M \in \mathbf{K}_R^\infty$ are "unique in the ∞-categorical sense". Of course, dual comments apply to injective resolutions.

Using merely the definitions of the adjoint functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$, we can obtain the inverse equivalences on the left in diagram (6).

Exercise 6.2 (6 points)**.** Prove that the composite adjunction

$$\mathbf{P}_R^\infty \underset{\rho_{\mathbf{P}}}{\overset{i_{\mathbf{P}}}{\rightleftarrows}} \mathbf{K}_R^\infty \underset{i_{\mathbf{I}}}{\overset{\rho_{\mathbf{I}}}{\rightleftarrows}} \mathbf{I}_R^\infty$$

is an adjoint equivalence.

6.3.5. We now simultaneously address the commutativity of the left triangle in diagram (6) and explain why both functors $\mathbf{K}_R^\infty \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_R^\infty$ and $\mathbf{K}_R^\infty \xrightarrow{\rho_{\mathbf{I}}} \mathbf{I}_R^\infty$ satisfy the universal property defining $\mathbf{K}_R^\infty \xrightarrow{\pi} \mathbf{D}_R$ (namely they are both the localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms).

We say that an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

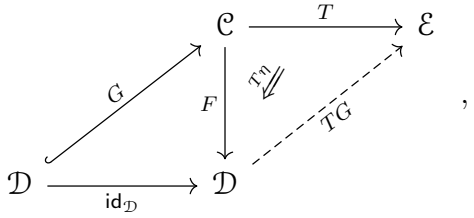
is a ***reflective localization adjunction*** if the right adjoint G is fully faithful. In this situation, we refer to the functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ as a ***reflective localization***. For concreteness, we discuss these in the context of ordinary categories, but then we will apply the theory in the context of $\Bbbk$-linear ∞-categories (in which it also holds).

Exercise 6.3 (6 points)**.**

(a) Show that G is fully faithful if and only if the counit $FG \xrightarrow{\varepsilon} \mathsf{id}_{\mathcal{D}}$ is a natural isomorphism.

(b) Show that for a reflective localization adjunction $F \dashv G$, the morphism $F \xrightarrow{F\eta} FGF$ is a natural isomorphism.

Given such a reflective localization adjunction, let us write $\mathbf{W} \subseteq \mathcal{C}$ for the subcategory consisting of those morphisms that are carried by F to isomorphisms. Then, we claim that the reflective localization $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is precisely the localization of $\mathcal{C}$ at $\mathbf{W}$.[92] To see this, fix any functor $\mathcal{C} \xrightarrow{T} \mathcal{E}$, and consider the diagram



in which the left triangle commutes via the natural isomorphism ε of Exercise 6.3(a). Considering just the solid diagram, we see that there is at most one choice of factorization of T through F, namely TG. On the other hand, we *always* have the indicated natural transformation $T \xRightarrow{T\eta} TGF$. So, it remains to check that $T\eta$ is a natural isomorphism if and only if T carries the morphisms in $\mathbf{W}$ to isomorphisms. By Exercise 6.3(b), the components of η lie in $\mathbf{W} \subseteq \mathcal{C}$, which implies the "if" direction.

Exercise 6.4 (4 points)**.** Prove the "only if" direction: that if $T\eta$ is a natural isomorphism then T carries all morphisms in $\mathbf{W}$ to isomorphisms.

A class of examples of reflective localizations is given by the following.

Exercise 6.5 (3 points)**.** Given a ring homomorphism $R \to S$, show that the corresponding adjunction

$$\mathsf{Mod}_R \underset{\mathsf{fgt}}{\overset{(-)\otimes_R S}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Mod}_S$$

is a reflective localization if and only if S is an *idempotent* R-algebra, i.e. the multiplication map $S \otimes_R S \to S$ is an isomorphism.

[92] To explain the terminology further, considering $\mathcal{D} \subseteq \mathcal{C}$ as a full subcategory via G, the functor F is also called the *reflector* of $\mathcal{C}$ into $\mathcal{D}$.

For example, the ring homomorphisms $\mathbb{Z} \to \mathbb{Z}[\frac{1}{n}]$ and $\mathbb{Z} \to \mathbb{Q}$ determine reflective localizations. So for instance, the forgetful functor

$$\mathsf{Mod}_{\mathbb{Z}} \xleftarrow{\mathsf{fgt}} \mathsf{Mod}_{\mathbb{Q}}$$

is fully faithful: a homomorphism between $\mathbb{Q}$-vector spaces is nothing more or less than a homomorphism between their underlying abelian groups.[93] Moreover, the functor

$$\mathsf{Mod}_{\mathbb{Z}} \xrightarrow{(-)\otimes_{\mathbb{Z}}\mathbb{Q}} \mathsf{Mod}_{\mathbb{Q}}$$

is the localization of $\mathsf{Mod}_{\mathbb{Z}}$ at the *rational isomorphisms* between abelian groups (i.e. those homomorphisms that rationalize to isomorphisms (e.g. $\mathbb{Z} \xrightarrow{(9,2)} \mathbb{Z} \oplus \mathbb{Z}/5$, which rationalizes to $\mathbb{Q} \xrightarrow{9} \mathbb{Q}$)). Explicitly, given a functor $\mathsf{Mod}_{\mathbb{Z}} \xrightarrow{T} \mathcal{E}$ that carries rational isomorphisms to isomorphisms, the canonical factorization is given by

$$\begin{array}{ccc} \mathsf{Mod}_{\mathbb{Z}} & \xrightarrow{T} & \mathcal{E} \\ {\scriptstyle (-)\otimes_{\mathbb{Z}}\mathbb{Q}}\downarrow & \nearrow_{T\circ\mathsf{fgt}} & \\ \mathsf{Mod}_{\mathbb{Q}} & & \end{array} .$$

By contrast, the ring homomorphisms $\mathbb{Q} \to \mathbb{Q}[\sqrt{2}]$ and $\mathbb{R} \to \mathbb{C}$ do not determine reflective localizations. So for instance, the forgetful functor

$$\mathsf{Mod}_{\mathbb{R}} \xleftarrow{\mathsf{fgt}} \mathsf{Mod}_{\mathbb{C}}$$

is *not* fully faithful: a homomorphism between $\mathbb{C}$-vector spaces is not the same thing as a homomorphism between their underlying $\mathbb{R}$-vector spaces. Rather, one must additionally demand that it respects multiplication by $i \in \mathbb{C}$.[94]

Now, by definition the adjunction $\rho_{\mathbf{I}} \dashv i_{\mathbf{I}}$ is a reflective localization adjunction.

Exercise 6.6 (2 points)**.** Prove that a morphism in $\mathbf{K}_R^\infty$ is carried by $\rho_{\mathbf{I}}$ to an equivalence if and only if it is a quasi-isomorphism.

By what we have just seen, this implies that the functor $\mathbf{K}_R^\infty \xrightarrow{\rho_{\mathbf{I}}} \mathbf{I}_R^\infty$ is indeed the localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms.

Of course, dually the functor $\mathbf{K}_R^\infty \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_R^\infty$ is a *coreflective* localization, and so it is *also* the localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms.

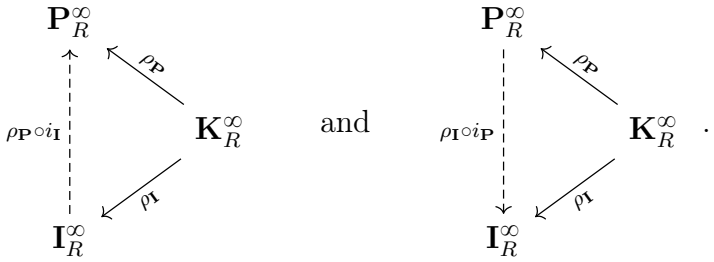
[93] And indeed, we have a further reflective localization adjunction

$$\mathsf{Grp} \underset{\mathsf{fgt}}{\overset{\perp}{\rightleftarrows}} \mathsf{Ab}$$

(in which the left adjoint is given by abelianization), so that a homomorphism between $\mathbb{Q}$-vector spaces is nothing more or less than a homomorphism between their underlying groups (which happen to be abelian).

[94] These examples illustrate a general principle, namely the dichotomy between "properties" and "structure". For instance, being a $\mathbb{Q}$-module turns out to be merely a *property* that can be satisfied by a $\mathbb{Z}$-module (in the sense that we can consider $\mathsf{Mod}_{\mathbb{Q}} \subset \mathsf{Mod}_{\mathbb{Z}}$ as a full subcategory), whereas given an $\mathbb{R}$-module we must provide additional *structure* to make it into a $\mathbb{C}$-module.

So indeed, *both* functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ are the localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms. In other words, they share the same universal property. Therefore, they must be canonically equivalent. Moreover, tracing through our proof that a reflective localization is a localization (and dualizing), we find that the canonical inverse equivalences between $\mathbf{P}_R^\infty$ and $\mathbf{I}_R^\infty$ that are induced by their shared universal property (as $\Bbbk$-linear ∞-categories under $\mathbf{K}_R^\infty$) are precisely the composites



In particular, these triangles commute.

It is also not hard to see that these triangles commute using the Yoneda lemma. For instance, given any $P \in \mathbf{P}_R^\infty$ and any $M \in \mathbf{K}_R^\infty$, we have the composite equivalence

$$\mathsf{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}} i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \simeq \mathsf{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \xrightarrow{\sim} \mathsf{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), M) \simeq \mathsf{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}}(M))$$

in $\mathbf{D}_{\Bbbk}$, using the fact that $\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(P), -)$ carries quasi-isomorphisms to quasi-isomorphisms. Since $P \in \mathbf{P}_R^\infty$ was arbitrary, it follows that the morphism

$$\rho_{\mathbf{P}} i_{\mathbf{I}} \rho_{\mathbf{I}}(M) \xrightarrow{\rho_{\mathbf{P}} \varepsilon_M} \rho_{\mathbf{P}}(M)$$

in $\mathbf{P}_R^\infty$ is an equivalence.

6.3.6. To conclude our discussion of diagram (6), we now construct the adjoint a_L; the adjoint a_R will arises from dual considerations. We refer to a_L (resp. a_R) as the ***left*** (resp. ***right***) ***acyclification*** functor.

For this, let us observe that for any complex $M \in \mathbf{K}_R$, the complex

$$a_L(M) := \mathsf{coker}^\infty \left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M)) \xrightarrow{\varepsilon_M} M \right)$$

is acyclic by Exercise 3.10(a).[95] In other words, the formula

$$a_L := \mathsf{coker}^\infty \left(i_{\mathbf{P}} \circ \rho_{\mathbf{P}} \xrightarrow{\varepsilon} \mathsf{id}_{\mathbf{K}_R} \right)$$

defines a functor $\mathbf{K}_R^\infty \xrightarrow{a_L} \mathbf{A}_R^\infty$.

We now verify that a_L defines a left adjoint to the inclusion $\mathbf{K}_R^\infty \xleftarrow{i_{\mathbf{A}}} \mathbf{A}_R^\infty$. At the ∞-categorical level, this is a straightforward computation. However, it is also a good opportunity to highlight the distinction between working in a dg-category and working in its underlying ∞-category, which makes the computation somewhat more subtle. Namely, the functor a_L is only well-defined at the ∞-categorical level, which implies that it is not literally

[95] In view of the adjunction $i_{\mathbf{P}} \dashv \rho_{\mathbf{P}}$, one may think of $a_L(M)$ as being obtained by "killing off the part of M that can be seen by projective complexes".

meaningful to compute with it at the point-set level. On the other hand, by the dual of Exercise 6.1, it suffices to verify that *any* point-set representative of the morphism $M \to i_{\mathbf{A}} a_L(M)$ in $\mathbf{K}_R^\infty$ is a pointwise unit on underlying ∞-categories. Therefore, let us write

$$i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon'_M} M$$

for a specific but arbitrary point-set representative of the morphism ε_M (i.e. an arbitrary projective resolution of M), and let us write $a_L(M)' := \mathsf{hcoker}(\varepsilon'_M)$; recall that homotopy cokernels in $\mathbf{K}_R$ compute ∞-categorical cokernels in $\mathbf{K}_R^\infty$, so the morphism $M \to i_{\mathbf{A}} a_L(M)'$ in $\mathbf{K}_R$ is indeed a point-set model for the morphism $M \to i_{\mathbf{A}} a_L(M)$ in $\mathbf{K}_R^\infty$. Now, for any acyclic complex $A \in \mathbf{A}_R^\infty$, omitting the functor $i_{\mathbf{A}}$ to simplify notation, we compute that

$$\begin{aligned}
\mathsf{hom}_{\mathbf{A}_R^\infty}(a_L(M), A) &:= \pi\left(\mathsf{hom}_{\mathbf{A}_R}(a_L(M)', A)\right) \\
&:= \pi\left(\mathsf{hom}_{\mathbf{K}_R}(a_L(M)', A)\right) \\
&:= \pi\left(\mathsf{hom}_{\mathbf{K}_R}\left(\mathsf{hcoker}\left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon'_M} M\right), A\right)\right) \\
&\simeq \pi\left(\mathsf{hker}\left(\mathsf{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)} \mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right) \quad (8) \\
&\simeq \pi\left(\mathsf{ker}^\infty\left(\mathsf{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)} \mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right) \quad (9) \\
&\simeq \mathsf{ker}^\infty\left(\pi\left(\mathsf{hom}_{\mathbf{K}_R}(M, A)\right) \xrightarrow{\pi(\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M, A))} \pi\left(\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right) \quad (10) \\
&\simeq \mathsf{ker}^\infty\left(\pi\left(\mathsf{hom}_{\mathbf{K}_R}(M, A)\right) \longrightarrow 0\right) \quad (11) \\
&\simeq \pi(\mathsf{hom}_{\mathbf{K}_R}(M, A)) \\
&=: \mathsf{hom}_{\mathbf{K}_R^\infty}(M, A)
\end{aligned}$$

as desired, by the following reasoning.

- Equivalence (8) follows from the fact that hom-complexes respect homotopy co/kernels (Exercise 4.7).
- Equivalence (9) follows from the fact that homotopy kernels in $\mathbf{K}_R$ compute ∞-categorical kernels in $\mathbf{K}_R^\infty$. (From here onwards, we implicitly consider the morphism $\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)$ as lying not in the dg-category $\mathbf{K}_{\Bbbk}$ but in its underlying ∞-category $\mathbf{K}_{\Bbbk}^\infty$.)
- As we have just seen, $\mathbf{K}_{\Bbbk}^\infty \xrightarrow{\pi} \mathbf{D}_{\Bbbk}$ is a right adjoint. It therefore preserves ∞-categorical limits, and in particular ∞-categorical kernels. This gives equivalence (10).

- By construction, the complex $i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \in \mathbf{K}_R$ is projective.[96] Hence, the complex $\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A) \in \mathbf{K}_{\Bbbk}$ is acyclic. It follows that we have an equivalence

$$\pi\left(\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right) \simeq 0$$

  in $\mathbf{D}_{\Bbbk}$, which explains equivalence (11).

So indeed, the left acyclification functor a_L is left adjoint to the inclusion $i_{\mathbf{A}}$.

Because cokernel sequences and kernel sequences in $\mathbf{K}_R^{\infty}$ agree, note that we can also view the projective (resp. injective) resolution functor as the kernel (resp. cokernel) of the left (resp. right) acyclification functor. Indeed, we have ∞-categorical co/kernel sequences

$$\begin{array}{ccc} i_{\mathbf{P}}\rho_{\mathbf{P}} & \longrightarrow & \mathsf{id}_{\mathbf{K}_R^{\infty}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{\mathbf{A}}a_L \end{array} \qquad \text{and} \qquad \begin{array}{ccc} i_{\mathbf{A}}a_R & \longrightarrow & \mathsf{id}_{\mathbf{K}_R^{\infty}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{\mathbf{I}}\rho_{\mathbf{I}} \end{array}$$

in $\mathsf{Fun}(\mathbf{K}_R^{\infty}, \mathbf{K}_R^{\infty})$.

6.3.7. The ∞-category $\mathbf{K}_{\Bbbk}^{\infty}$ of complexes over $\Bbbk$ is symmetric monoidal via tensor product, and the ∞-category $\mathbf{K}_R^{\infty}$ of complexes over R is a left module over it. It follows from Exercise 5.3(c) that the fully faithful inclusions

$$\mathbf{D}_{\Bbbk} \xhookrightarrow{i_L} \mathbf{K}_{\Bbbk}^{\infty} \qquad \text{and} \qquad \mathbf{D}_R \xhookrightarrow{i_L} \mathbf{K}_R^{\infty}$$

endow the derived ∞-categories with the same structures: $\mathbf{D}_{\Bbbk}$ is symmetric monoidal and $\mathbf{D}_R$ is a left module over it.

Note that here we must use the inclusions i_L (not i_R). This is due to the handedness of the tensor-hom adjunction: tensor products satisfying a universal *mapping out* property (instead of a universal *mapping in* property). Said differently, it is the tensor product of *projective* complexes (and not injective complexes) that computes the tensor product of derived modules.

Relatedly, the localization $\mathbf{K}_{\Bbbk}^{\infty} \xrightarrow{\pi} \mathbf{D}_{\Bbbk}$ is *not* a symmetric monoidal functor. Rather, given complexes $M, N \in \mathbf{K}_{\Bbbk}^{\infty}$, tensoring the counit maps $i_L\pi(M) := i_{\mathbf{P}}\rho_{\mathbf{P}}(M) \to M$ and $i_L\pi(N) := i_{\mathbf{P}}\rho_{\mathbf{P}}(N) \to N$ together gives a map

$$i_L(\pi(M) \otimes \pi(N)) \simeq i_L\pi(M) \otimes i_L\pi(N) \longrightarrow M \otimes N$$

in $\mathbf{K}_{\Bbbk}^{\infty}$ that is adjoint to a map

$$\pi(M) \otimes \pi(N) \longrightarrow \pi(M \otimes N) \tag{12}$$

[96] Recall from Exercise 5.3(b) that the property of projectivity is stable under homotopy equivalence: any complex that is homotopy equivalent to a projective complex is itself projective.

in $\mathbf{D}_{\mathbb{k}}$, which is not generally an equivalence.[97] Indeed, the functor π is *right-laxly* symmetric monoidal: it comes equipped with natural comparison morphisms (such as the morphism (12)) from the (iterated) tensor product in $\mathbf{D}_{\mathbb{k}}$ of its values to its value on the corresponding tensor product in $\mathbf{K}_{\mathbb{k}}^{\infty}$.[98,99]

Inasmuch as the derived tensor product behaves better than the ordinary tensor product, we view the fact that π is only right-laxly symmetric monoidal as a feature rather than a bug. Moreover, right-laxly symmetric monoidal functors still interact nicely with algebraic structures.[100] For instance, in the present situation, we have a canonical lift

$$\begin{array}{ccc} \mathsf{CAlg}(\mathbf{K}_{\mathbb{k}}^{\infty}) & \overset{\pi}{\dashrightarrow} & \mathsf{CAlg}(\mathbf{D}_{\mathbb{k}}) \\ {\scriptstyle \mathsf{fgt}}\downarrow & & \downarrow{\scriptstyle \mathsf{fgt}} \\ \mathbf{K}_{\mathbb{k}}^{\infty} & \xrightarrow[\pi]{} & \mathbf{D}_{\mathbb{k}} \end{array} ,$$

so that commutative algebra objects in $\mathbf{K}_{\mathbb{k}}^{\infty}$ define commutative algebra objects in $\mathbf{D}_{\mathbb{k}}$, and likewise for associative algebra objects.[101]

6.3.8. We briefly indicate how the derived ∞-category construction generalizes from Mod_R to an arbitrary abelian category $\mathcal{A}$, and give a preview of the broader context into which this fits.[102]

Consider the category $\mathsf{Ch}(\mathcal{A})$ of complexes in $\mathcal{A}$. This admits an enrichment in $\mathsf{Ch}_{\mathbb{Z}}$, i.e. it defines a $\mathbb{Z}$-linear dg-category, whose underlying $\mathbb{Z}$-linear ∞-category we refer to as the ***∞-category of complexes*** in $\mathcal{A}$ and denote by $\mathbf{K}(\mathcal{A})$. Then, the ***derived ∞-category*** of $\mathcal{A}$ is the localization

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\pi} \mathbf{D}(\mathcal{A})$$

[97] Beginning with ordinary $\mathbb{k}$-modules $M, N \in \mathsf{Mod}_{\mathbb{k}} \subseteq \mathbf{K}_{\mathbb{k}}^{\infty}$, the failure of this morphism to be an equivalence is measured by the Tor functor, as explained in §7.2. (Note that the inclusion $\mathsf{Mod}_{\mathbb{k}} \subseteq \mathbf{K}_{\mathbb{k}}^{\infty}$ is symmetric monoidal.)

[98] Indeed, the right adjoint of *any* symmetric monoidal functor is automatically right-laxly symmetric monoidal. (Note that it may also simply be symmetric monoidal: this structure can be seen as a *condition* on a right-laxly symmetric monoidal functor, namely the invertibility of the natural comparison morphisms.)

[99] The notion of a right-laxly symmetric monoidal functor is defined precisely in **??**.

[100] Dually, left-laxly symmetric monoidal functors interact nicely with *coalgebraic* structures.

[101] It turns out that associative algebra objects in $\mathbf{D}_{\mathbb{k}}$ always lift to associative algebra objects in $\mathbf{K}_{\mathbb{k}}$ (i.e. dgas). When $\mathbb{k}$ has characteristic zero, commutative algebra objects in $\mathbf{D}_{\mathbb{k}}$ also always lift to commutative algebra objects in $\mathbf{K}_{\mathbb{k}}$ (i.e. cdgas). However, in general there are commutative algebra objects in $\mathbf{D}_{\mathbb{k}}$ that do not lift to cdgas; the obstructions effectively arise from the cohomology of symmetric groups with coefficients in $\mathbb{k}$ (which vanish when $\mathbb{k}$ has characteristic zero).

[102] In essence, an abelian category is one that "behaves like" the category Mod_R for some ring R. (In particular, abelian categories are canonically enriched in Ab.) Indeed, the *Freyd–Mitchell embedding theorem* guarantees that every abelian category can be fully faithfully and exactly (i.e. kernel- and cokernel-preservingly) embedded into some Mod_R [Fre64, Mit64]. (More precisely, this holds under the set-theoretic assumption that the abelian category is *small*, i.e. that it only has a *set* (as opposed to a proper class) of objects.)

obtained by freely inverting the quasi-isomorphisms.

There are also the three variants $\mathbf{D}^-(\mathcal{A})$, $\mathbf{D}^+(\mathcal{A})$, and $\mathbf{D}^b(\mathcal{A})$: the bounded-above, bounded-below, and bounded derived ∞-categories.[103] All of these $\mathbb{Z}$-linear ∞-categories are ***stable***: they have zero objects and admit all co/kernels.[104]

Moreover, at least these three variants may admit various universal properties. The simplest is that of $\mathbf{D}^b(\mathcal{A})$, which is the initial stable ∞-category receiving a functor from $\mathcal{A}$ – namely, the evident composite functor

$$\mathcal{A} \longrightarrow \mathbf{K}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

(factoring through the bounded variants) – that carries all short exact sequences to co/kernel sequences [BCKW, Corollary 7.59].[105,106] The universal properties of $\mathbf{D}^{\pm}(\mathcal{A})$ are somewhat more subtle: they require hypotheses on $\mathcal{A}$ (namely the existence of enough projectives/injectives and/or the property of being a *Grothendieck* abelian category), and make reference to "analytic" notions (in the sense of convergence) involving t-structures [§A.1.3].

7. Derived functors

A substantial vein of current mathematical research takes place entirely within the derived realm, e.g. working only with the derived ∞-category $\mathbf{D}_R$ instead of with the ordinary category Mod_R. We will take this point of view when we study sheaf theory. However, it is nevertheless worthwhile to connect back with the original approach to homological algebra. We explain the general principles in §7.1, and then illustrate how passing to classical derived functors amounts to working in the derived ∞-category in §§7.2-7.3. We then briefly discuss the example of group co/homology in **??**. We refer the interested reader to [Wei94] for a more in-depth treatment of group co/homology, as well as for treatments of two other fundamental examples: Lie algebra co/homology and Hochschild co/homology.[107] We conclude with a brief discussion of decategorification in **??**, which applies to the derived approach to the counting problems described in §1.4.

7.1. **Derived functors.**

[103] In the notations $\mathbf{D}^{\pm}(\mathcal{A})$, the superscript indicates the direction in which infinitude is permitted.

[104] We discuss (not necessarily $\mathbb{Z}$-linear) stable ∞-categories in **??**. (The general definition of stability is rather different from the one indicated here in the linear context.)

[105] As illustrated by Exercise 3.9, the functor $\mathcal{A} \to \mathbf{K}^b(\mathcal{A})$ does *not* generally send all short exact sequences to co/kernel sequences.

[106] Although $\mathbf{D}^b(\mathcal{A})$ is naturally $\mathbb{Z}$-linear, this universal property refers to *all* stable ∞-categories. (Moreover, the corresponding assertion for $\mathbb{Z}$-linear ∞-categories is false, as $\mathbb{Z}$-linearity is a *structure* rather than a *property* of a stable ∞-category: the forgetful functor $\mathsf{St}_{\mathbb{Z}} \to \mathsf{St}$ is not fully faithful.)

[107] However, Hochschild homology is discussed as an instance of factorization homology (the topic of **??**) in **??**.

7.1.1. As indicated in §5.1, the original motivation for homological algebra was to repair certain failures of exactness. This led us to study projective and injective resolutions through the remainder of §5, which in turn led us to study the derived ∞-category $\mathbf{D}_R$ in §6. We defined $\mathbf{D}_R$ by a universal property (namely as the localization of $\mathbf{K}_R$ at the quasi-isomorphisms), but we immediately saw that it is canonically equivalent to *both* of the ∞-categories $\mathbf{P}_R$ and $\mathbf{I}_R$ of projective and injective complexes. We now explain the manner in which the derived ∞-category – as well as its dual incarnations – repairs these failures of exactness.

For this, it is appropriate to work in slightly more generality. Thus, let us consider an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories.[108] We say that the functor F is

- ***left exact*** if it commutes with finite limits,
- ***right exact*** if it commutes with finite colimits, and
- ***exact*** if it is both left exact and right exact.

Exactness can be very useful. For instance, suppose that an object $A \in \mathcal{A}$ is given to us as a finite colimit. If F is right exact, then we can deduce an analogous presentation of its image $F(A) \in \mathcal{B}$ as a finite colimit. Dually, if F is left exact, then it commutes with presentations of objects as finite limits.

As we will see in §7.1.3, one can contemplate the ***right*** (resp. ***left***) ***derived functor*** of *any* additive functor F, denoted $\mathbb{R}F$ (resp. $\mathbb{L}F$), although this is most salient in the case that F is *left* (resp. *right*) exact.[109] In any case, these derived functors will be "universal exact approximations" to F, running

$$\mathbf{D}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B}) \ ,$$

which will in particular commute with *both* finite colimits and finite limits – though these must now be taken in the ∞-categorical sense. For example, given an arbitrary exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{13}$$

in $\mathcal{A}$, we can consider it as a co/kernel sequence in $\mathbf{D}(\mathcal{A})$; writing $\mathbf{D}(\mathcal{A}) \xrightarrow{\mathbb{D}F} \mathbf{D}(\mathcal{A})$ to refer simultaneously to either derived functor of F, we obtain a co/kernel sequence

$$\mathbb{D}F(L) \longrightarrow \mathbb{D}F(M) \longrightarrow \mathbb{D}F(N) \tag{14}$$

in $\mathbf{D}(\mathcal{B})$. Note that we can read this either as a presentation

$$\mathbb{D}F(L) \simeq \mathsf{ker}(\mathbb{D}F(M) \longrightarrow \mathbb{D}F(N))$$

(so that $\mathbb{D}F$ commutes with the presentation $L \simeq \mathsf{ker}(M \to N)$ in $\mathbf{D}(\mathcal{A})$) or as a presentation

$$\mathbb{D}F(N) \simeq \mathsf{coker}(\mathbb{D}F(L) \longrightarrow \mathbb{D}F(M))$$

(so that $\mathbb{D}F$ commutes with the presentation $N \simeq \mathsf{coker}(L \to M)$ in $\mathbf{D}(\mathcal{A})$).

[108] To say that F is *additive* means that it preserves zero objects and finite direct sums (which exist in any abelian category).

[109] The bizarreness of taking the "wrong" derived functor is illustrated in Exercises 7.5 and 7.8.

Of course, these derived functors will not generally agree with F (when precomposed with the functor $\mathcal{A} \to \mathbf{D}(\mathcal{A})$), but as it turns out they will nevertheless "remember" F by taking 0^{th} homology – at least in the aforementioned salient cases. Moreover, it is very often the case that the derived functor of F is "better behaved" than F itself (e.g. in the context of intersection theory, as discussed in §1). From this point of view, the functor F is merely a shadow of its derived functor.

7.1.2. Before we proceed to discuss derived functors, we briefly describe an alternative (but closely related) perspective that also leads to them, which also explains the names "left exact" and "right exact" (and gives a mnemonic for remembering them).

Exercise 7.1 (6 points)**.**

(a) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is left exact if and only if it commutes with kernels, i.e. for every exact sequence (13) in $\mathcal{A}$ the sequence

$$0 \cong F(0) \longrightarrow F(L) \longrightarrow F(M) \longrightarrow F(N) \tag{15}$$

in $\mathcal{B}$ is also exact.

(b) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is right exact if and only if it commutes with cokernels, i.e. for every exact sequence (13) in $\mathcal{A}$ the sequence

$$F(L) \longrightarrow F(M) \longrightarrow F(N) \longrightarrow F(0) \cong 0 \tag{16}$$

in $\mathcal{B}$ is also exact.

So, if F is only left exact, one is inclined to extend the sequence (15) to a long exact sequence

$$\begin{array}{l} 0 \longrightarrow (\mathbb{R}^0F)(L) \longrightarrow (\mathbb{R}^0F)(M) \longrightarrow (\mathbb{R}^0F)(N) \longrightarrow \\ \quad (\mathbb{R}^1F)(L) \longrightarrow (\mathbb{R}^1F)(M) \longrightarrow (\mathbb{R}^1F)(N) \longrightarrow \\ \quad (\mathbb{R}^2F)(L) \longrightarrow \cdots \end{array}$$

in $\mathcal{B}$, where by definition $\mathbb{R}^0F := F$ and we refer to $\mathbb{R}^nF$ as the $n^{\textbf{\textit{th}}}$ ***right derived functor*** of F. Dually, if F is only right exact, one is inclined to the extend the sequence (16) to a

long exact sequence

$$\cdots \longrightarrow (\mathbb{L}_2F)(N) \longrightarrow (\mathbb{L}_1F)(L) \longrightarrow (\mathbb{L}_1F)(M) \longrightarrow (\mathbb{L}_1F)(N) \longrightarrow (\mathbb{L}_0F)(L) \longrightarrow (\mathbb{L}_0F)(M) \longrightarrow (\mathbb{L}_0F)(N) \longrightarrow 0$$

in $\mathcal{B}$, where by definition $\mathbb{L}_0F := F$ and we refer to $\mathbb{L}_nF$ as the n^{th} ***left derived functor*** of F.[110]

As we saw in §3.4, long exact sequences in algebra arise from homotopy co/kernel sequences in homological algebra. Thus, we should expect to obtain these long exact sequences from homotopy co/kernel sequences, i.e. (∞-categorical) co/kernel sequences in $\mathbf{D}(\mathcal{B})$. And indeed, as we will see in Exercise 7.3, they will arise from the co/kernel sequences (14) by defining

$$\mathbb{R}^nF := \mathsf{H}_{-n} \circ \mathbb{R}F \qquad \text{and} \qquad \mathbb{L}_nF := \mathsf{H}_n \circ \mathbb{L}F \ .$$

In order to emphasize the distinction, one may refer to $\mathbb{R}F$ and $\mathbb{L}F$ (as opposed to $\mathbb{R}^nF$ and $\mathbb{L}_nF$) as *total* derived functors of F.

7.1.3. We now introduce (total) derived functors. We actually introduce them in a slightly more general context than that described in §7.1.1. Namely, given an arbitrary functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{T} \mathcal{C}$$

to a $\mathbb{Z}$-linear ∞-category, we define its ***right derived functor*** and its ***left derived functor*** to respectively be the initial and terminal extensions

(17)
$$\begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{T} & \mathcal{C} \\ {\scriptstyle\pi}\downarrow & \swarrow & \nearrow {\scriptstyle \mathbb{R}T} \\ \mathbf{D}(\mathcal{A}) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{T} & \mathcal{C} \\ {\scriptstyle\pi}\downarrow & \nearrow & \nearrow {\scriptstyle \mathbb{L}T} \\ \mathbf{D}(\mathcal{A}) & & \end{array}$$

along π (which may not exist in general). Said differently, these derived functors participate in comparison maps

$$\pi \circ \mathbb{L}T \longrightarrow T \longrightarrow \pi \circ \mathbb{R}T$$

in $\mathsf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C})$, and by definition they are the universal objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C})$ equipped with such comparison maps. Hence, the composites $\pi \circ \mathbb{L}T$ and $\pi \circ \mathbb{R}T$ should be thought

[110] A priori, it is not clear that these n^{th} derived functors are well-defined or uniquely characterized by these conditions, although they do turn out to be so.

of two dual ways of universally forcing the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ to be sent to equivalences in $\mathcal{C}$. To apply these notions to our additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories, we take T to be the composite functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(F)} \mathbf{K}(\mathcal{B}) \xrightarrow{\pi} \mathbf{D}(\mathcal{B}) \ ,$$

and we denote the resulting derived functors (which are objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathbf{D}(\mathcal{B}))$) simply by

$$\mathbb{R}F := \mathbb{R}(\pi \circ \mathbf{K}(F)) \qquad \text{and} \qquad \mathbb{L}F := \mathbb{L}(\pi \circ \mathbf{K}(F)) \ .$$

Examining the universal extensions (17), we see that these assignments $T \mapsto \mathbb{R}T$ and $T \mapsto \mathbb{L}T$ are nothing other than adjoints

$$\mathsf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C}) \underset{\mathbb{L}(-)}{\overset{\mathbb{R}(-)}{\leftarrow \mathsf{Fun}(\pi,\mathcal{C}) -}} \mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C}) \tag{18}$$

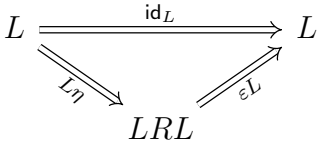
to the restriction functor.[111] (Note the unfortunate conventional clash that taking the *right* derived functor is a *left* adjoint, and reversely.) There are formulas for computing such adjoints in general, but they are extremely easy to compute in certain special cases, using the following.

Exercise 7.2 (10 points)**.** Fix two categories $\mathcal{D}$ and $\mathcal{E}$ and two functors $\mathcal{D} \xrightarrow{L} \mathcal{E}$ and $\mathcal{D} \xleftarrow{R} \mathcal{E}$.

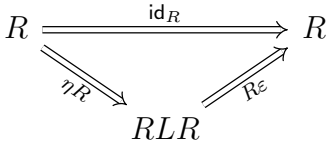
(a) Prove that to define an adjunction

$$\mathcal{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{E} \ ,$$

it is necessary and sufficient to define unit and counit natural transformations $\mathsf{id}_{\mathcal{D}} \overset{\eta}{\Rightarrow} RL$ and $LR \overset{\varepsilon}{\Rightarrow} \mathsf{id}_{\mathcal{E}}$ that satisfy the *triangle identities*, which assert that the triangles



and



commute.

(b) Using (a), show that for any category $\mathcal{F}$, applying the functors $\mathsf{Fun}(\mathcal{F}, -)$ and $\mathsf{Fun}(-, \mathcal{F})$ to the above adjunction yields adjunctions

$$\mathsf{Fun}(\mathcal{F}, \mathcal{D}) \underset{\mathsf{Fun}(\mathcal{F},R)}{\overset{\mathsf{Fun}(\mathcal{F},L)}{\rightleftarrows}} \mathsf{Fun}(\mathcal{F}, \mathcal{E}) \qquad \text{and} \qquad \mathsf{Fun}(\mathcal{D}, \mathcal{F}) \underset{\mathsf{Fun}(L,\mathcal{F})}{\overset{\mathsf{Fun}(R,\mathcal{F})}{\rightleftarrows}} \mathsf{Fun}(\mathcal{E}, \mathcal{F}) \ .$$ [112]

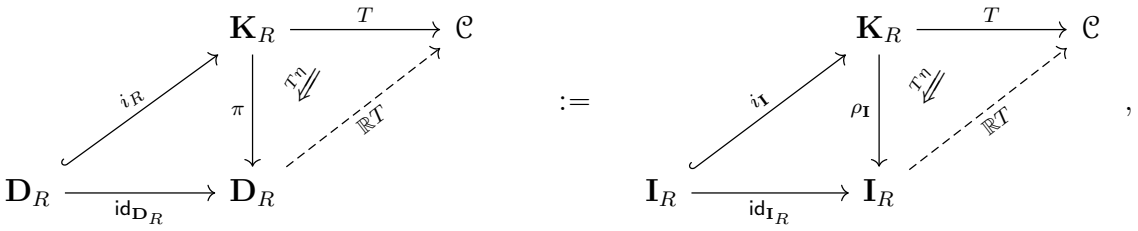
[111] In this generality, these adjoints may not exist. But when a particular derived functor exists, it is nothing other than a pointwise adjoint as indicated.

[112] It is of course also possible to establish these adjunctions directly (instead of using (a)).

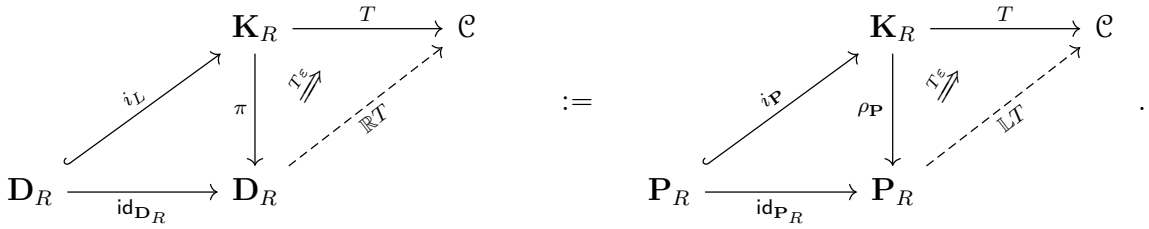
Namely, in the particular case that $\mathcal{A} = \mathsf{Mod}_R$ (so that $\mathbf{K}(\mathcal{A}) = \mathbf{K}(\mathsf{Mod}_R) =: \mathbf{K}_R$), the ∞-categorical analog of Exercise 7.2(b) identifies the adjoints (18) simply as

$$\mathsf{Fun}(\mathbf{K}_R, \mathcal{C}) \underset{\mathsf{Fun}(i_L,\mathcal{C})}{\overset{\mathsf{Fun}(i_R,\mathcal{C})}{\overset{\longleftarrow}{\underset{\longleftarrow}{\mathsf{Fun}(\pi,\mathcal{C})}}}} \mathsf{Fun}(\mathbf{D}_R, \mathcal{C}) \quad .^{113}$$

In other words, recalling the discussion of co/reflection localizations from §6.3.5, passing to right derived functors amounts to injective resolution



while passing to left derived functors amounts to projective resolution



Exercise 7.3 (12 points)**.** Verify that these formulas produce the long exact sequences sought in §7.1.2. In particular,

- in the case that F is left exact, prove that

$$\mathsf{H}_n \circ \mathbb{R}F \cong \begin{cases} F\,, & n = 0 \\ 0\,, & n > 0 \end{cases} \quad ,$$

  and
- in the case that F is right exact, prove that

$$\mathsf{H}_n \circ \mathbb{L}F \cong \begin{cases} F\,, & n = 0 \\ 0\,, & n < 0 \end{cases} \quad .$$

7.2. **Tor.**

7.2.1. We now study Tor, one of two classical examples of a derived functor. By definition, Tor is the left derived functor of relative tensor product. It will be illuminating to work relative to our base commutative ring $\Bbbk$, so we begin in that special case, and then proceed to discuss Tor over the associative $\Bbbk$-algebra R.

[113] We have passed from an arbitrary abelian category $\mathcal{A}$ to the abelian category Mod_R in order to guarantee the existence of the adjoints i_L and i_R to the projection π. In particular, in this situation the adjoints (18) exist without any hypotheses on $\mathcal{C}$.

7.2.2. The relative tensor product bifunctor

$$\mathsf{Mod}_{\mathbb{k}} \times \mathsf{Mod}_{\mathbb{k}} \xrightarrow{(-)\otimes(-)} \mathsf{Mod}_{\mathbb{k}}$$

is right exact separately in each variable, but it is not left exact in either variable. Let us fix a $\mathbb{k}$-module $M \in \mathsf{Mod}_{\mathbb{k}}$ and consider the resulting right exact functor

$$\mathsf{Mod}_{\mathbb{k}} \xrightarrow{M\otimes(-)} \mathsf{Mod}_{\mathbb{k}} \ .$$

According to the prescription of §7.1.3, we define its left derived functor $\mathbb{L}(M \otimes (-))$ as the composite

$$\mathbf{D}_{\mathbb{k}} \simeq \mathbf{P}_{\mathbb{k}} \xhookrightarrow{i_{\mathbf{P}}} \mathbf{K}_{\mathbb{k}} \xrightarrow{\mathbf{K}(M\otimes(-))} \mathbf{K}_{\mathbb{k}} \xrightarrow{\pi} \mathbf{D}_{\mathbb{k}} \ .$$

In particular, its value on ordinary $\mathbb{k}$-modules is obtained as the upper composite in the diagram

$$\mathsf{Mod}_{\mathbb{k}} \hookrightarrow \mathbf{K}_{\mathbb{k}} \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_{\mathbb{k}} \xhookrightarrow{i_{\mathbf{P}}} \mathbf{K}_{\mathbb{k}} \xrightarrow{\mathbf{K}(M\otimes(-))} \mathbf{K}_{\mathbb{k}} \xrightarrow{\pi} \mathbf{D}_{\mathbb{k}} \quad .$$

$$\varepsilon \Downarrow \quad \mathsf{id}_{\mathbf{K}_{\mathbb{k}}}$$

Of course, we can equally well perform the same operation in the second slot. We are therefore led to consider *two* possible notions of "the left derived tensor product of M and N". Luckily, they agree: by Exercise 5.4(c), we have natural quasi-isomorphisms fitting into a commutative square

$$\begin{array}{ccc}
 & & \mathbb{L}(M \otimes (-))(N) \\
 & & \vdots\wr \\
\rho_{\mathbf{P}}(M) \otimes \rho_{\mathbf{P}}(N) & \xrightarrow{\approx} & M \otimes \rho_{\mathbf{P}}(N) \\
\wr\downarrow & & \downarrow \\
\rho_{\mathbf{P}}(M) \otimes N & \longrightarrow & M \otimes N \\
\wr\vdots & & \\
\mathbb{L}((-) \otimes N)(M) & &
\end{array}$$

in $\mathbf{K}_{\mathbb{k}}$ (omitting the functor $i_{\mathbf{P}}$ for simplicity). It is customary to simply write

$$M \overset{\mathbb{L}}{\otimes} N \in \mathbf{D}_{\mathbb{k}}$$

for the common value of these derived functors and refer to it as the ***derived tensor product***. Note that this is nothing more than the tensor product in the derived ∞-category $\mathbf{D}_{\mathbb{k}}$, as defined in §6.3.7.

For any $n \geqslant 0$, we define

$$\mathsf{Tor}_n(M, N) := \mathsf{Tor}_n^{\mathbb{k}}(M, N) := \mathsf{H}_n(M \otimes^{\mathbb{L}} N) \in \mathsf{Mod}_{\mathbb{k}} \ .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{L}_n(M \otimes (-))(N) \cong \mathsf{Tor}_n(M,N) \cong \mathbb{L}_n((-)\otimes N)(M) \ .$$

The source of the name "Tor" is the following fundamental example.

Exercise 7.4 (6 points)**.** Taking $\Bbbk = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute $\mathsf{Tor}_i(M,N)$ for all $i \geqslant 0$.

As each functor $\mathsf{Tor}_i(-,-)$ preserves finite sums separately in each variable, Exercise 7.4 effectively gives a computation of Tor for all finitely generated abelian groups.

The bizarreness of taking the "wrong" derived functor can be illustrated as follows.

Exercise 7.5 (6 points)**.** Taking $\Bbbk = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute

$$\mathbb{R}^i(M \otimes (-))(N) \qquad \text{and} \qquad \mathbb{R}^i((-)\otimes N)(M)$$

for all $i \geqslant 0$.

7.2.3. We now briefly discuss Tor over the associative $\Bbbk$-algebra R. Of course, it is the left derived functor of the bifunctor

$$\mathsf{Mod}_R \times {}_R\mathsf{Mod} \xrightarrow{(-)\otimes_R(-)} \mathsf{Mod}_\Bbbk \ ;$$

the subtlety is simply that this is not a monoidal structure on a single category. Once again, it is sufficient to take a projective resolution in just one of the variables.

7.2.4. It is worth mentioning a particular small (and therefore computable) model for the derived relative tensor product of ordinary R-modules.[114] Namely, given a right R-module $M \in \mathsf{Mod}_R$ and a left R-module $N \in {}_R\mathsf{Mod}$, we define the (***two-sided***) ***bar complex*** $\mathsf{Bar}(M,R,N) \in \mathbf{K}_\Bbbk$ to be

$$\cdots \xrightarrow{d_3} M \otimes R^{\otimes 2} \otimes N \xrightarrow{d_2} M \otimes R \otimes N \xrightarrow{d_1} \underbrace{M \otimes N} \longrightarrow 0 \ ,$$

where we define $d_i := \sum_{j=0}^{i} (-1)^j d_i^j$ and we define d_i^j by multiplying the j^{th} and $(j+1)^{\text{st}}$ tensor factors (where we count starting from 0), i.e.

$$d_i^j(m \otimes r_1 \otimes \cdots \otimes r_i \otimes n) := \begin{cases} m \cdot r_1 \otimes r_2 \otimes \cdots \otimes r_i \otimes n \ , & j = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_{j-1} \otimes r_j \cdot r_{j+1} \otimes r_{j+2} \otimes \cdots \otimes r_i \otimes n \ , & 0 < j < i \\ m \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes r_i \cdot n \ , & j = i \end{cases} .$$

Then, we have a canonical equivalence

$$\mathsf{Bar}(M,R,N) \simeq M \overset{\mathbb{L}}{\underset{R}{\otimes}} N \tag{19}$$

[114] This admits a straightforward generalization to complexes of R-modules, which we do not address here.

in $\mathbf{D}_{\Bbbk}$. This is particularly easy to see if $\mathsf{fgt}(M) \in \mathsf{Mod}_{\Bbbk}$ is projective (or similarly if $\mathsf{fgt}(N) \in \mathsf{Mod}_{\Bbbk}$ is projective), e.g. if $\Bbbk$ is a field.[115] In general, if N carries a right action S-action that commutes with its left R-action, then we may consider $\mathsf{Bar}(M, R, N) \in \mathbf{K}_S$ as a complex of right S-modules. Taking $N = R = S$, we obtain a complex $\mathsf{Bar}(M, R, R) \in \mathbf{K}_R$ of right R-modules. By Exercise 5.9, this is projective: it is levelwise projective by the dual of Exercise 5.13(c), and moreover it is bounded below.

Exercise 7.6 (2 points)**.** Construct a natural quasi-isomorphism $\mathsf{Bar}(M, R, R) \xrightarrow{\approx} M$ in $\mathbf{K}_R$.

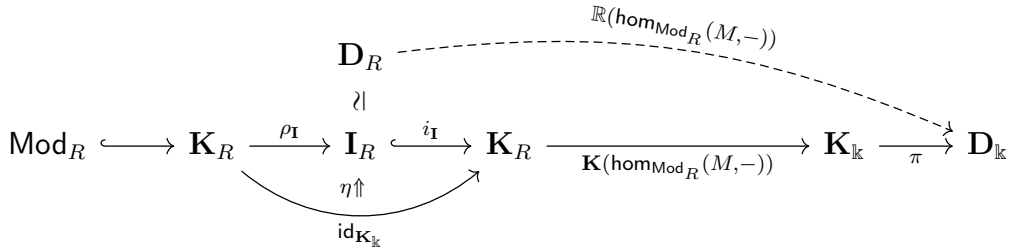
In other words, Exercise 7.6 implies that $\mathsf{Bar}(M, R, R) \xrightarrow{\approx} M$ is a projective resolution. Now, equivalence (19) follows from the evident isomorphism $\mathsf{Bar}(M, R, R) \otimes_R N \cong \mathsf{Bar}(M, R, N)$ in $\mathbf{K}_{\Bbbk}$.

7.3. **Ext.**

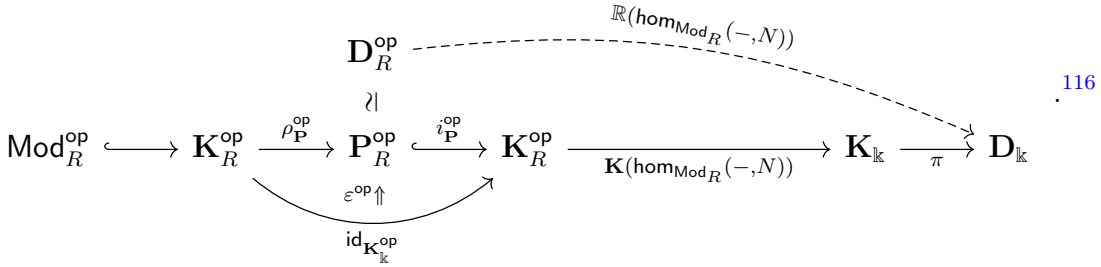
7.3.1. We now study Ext, the other classical example of a derived functor. By definition, Ext is the right derived functor of hom. The hom bifunctor

$$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\mathsf{hom}_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_{\Bbbk}$$

is left exact separately in each variable, but it is not right exact in either variable. Now, following §7.1.3 we define its two possible right derived functors as in the diagrams



and



[116]

[115]The proof in the general case requires techniques that we do not discuss here; see [Wei94, §8.6] for an explanation.

[116]Note that passing to opposites exchanges projectives and injectives.

Once again, these agree: by Exercise 5.3(d)(e), we have natural quasi-isomorphisms fitting into a commutative square

$$\begin{array}{ccc}
 & & \mathbb{R}(\mathsf{hom}_{\mathsf{Mod}_R}(M,-))(N) \\
 & & \vdots \\
 & & \| \\
\mathsf{hom}_{\mathsf{Mod}_R}(M,N) & \longrightarrow & \mathsf{hom}_{\mathbf{K}_R}(M,\rho_{\mathbf{I}}(N)) \\
\downarrow & & \downarrow \wr \\
\mathsf{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M),N) & \xrightarrow[\wr]{} & \mathsf{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M),\rho_{\mathbf{I}}(N)) \\
\| & & \\
\vdots & & \\
\mathbb{R}(\mathsf{hom}_{\mathsf{Mod}_R}(-,N))(M) & &
\end{array}$$

in $\mathbf{K}_{\Bbbk}$. It is customary to simply write

$$\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N) \in \mathbf{D}_{\Bbbk}$$

for the common value of these derived functors and refer to it as the ***derived hom***. Note that this is nothing more than the hom in $\mathbf{D}_R$:

$$\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N) \simeq \mathsf{hom}_{\mathbf{D}_R}(M,N) := \mathsf{hom}_{\mathbf{D}_R}(\pi(M),\pi(N)) \ .$$

For any $n \geqslant 0$, we define

$$\mathsf{Ext}^n(M,N) := \mathsf{Ext}^n_R(M,N) := \mathsf{H}_{-n}(\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N)) \in \mathsf{Mod}_{\Bbbk} \ .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{R}^n(\mathsf{hom}_{\mathsf{Mod}_R}(M,-))(N) \cong \mathsf{Ext}^n(M,N) \cong \mathbb{R}^n(\mathsf{hom}_{\mathsf{Mod}_R}(-,N))(M) \ .$$

Exercise 7.7 (6 points)**.** Taking $\Bbbk = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute $\mathsf{Ext}^i(M,N)$ for all $i \geqslant 0$.

The bizarreness of taking the "wrong" derived functor can be illustrated as follows.

Exercise 7.8 (6 points)**.** Taking $\Bbbk = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute

$$\mathbb{L}_i(\mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(M,-))(N) \qquad \text{and} \qquad \mathbb{L}_i(\mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(-,N))(M)$$

for all $i \geqslant 0$.

7.3.2. The name "Ext" arises from the fact that for any $n \geqslant 1$, $\mathsf{Ext}^n_R(M,N)$ classifies equivalence classes of exact sequences

$$0 \longrightarrow N \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0 \ , \tag{20}$$

called ***n-extensions*** of M by N, under the relation that there exists a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & N & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle \mathsf{id}_N}\downarrow & & \downarrow & & & & \downarrow & & {\scriptstyle \mathsf{id}_M}\downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & X'_{n-1} & \longrightarrow & \cdots & \longrightarrow & X'_0 & \longrightarrow & M & \longrightarrow & 0 \end{array} \ . \tag{21}$$

Observe that we may view an n-extension (20) as a sequence of composable morphisms

$$\begin{array}{c} \Sigma^{n-1}N \\ {\scriptstyle f}\downarrow \\ X_\bullet \\ {\scriptstyle g}\downarrow \\ M \end{array} \quad := \quad \begin{array}{ccccccccc} N & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \underset{\sim}{0} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & \underset{\sim}{X_0} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \underset{\sim}{M} \end{array}$$

in Ch_R, such that the composition equals zero. Hence, we obtain a commutative square

$$\begin{array}{ccc} \Sigma^{n-1}N & \xrightarrow{f} & X_\bullet \\ \downarrow & & {\scriptstyle g}\downarrow \\ 0 & \longrightarrow & M \end{array}$$

in the ∞-category $\mathbf{K}_R$. This may not be a co/kernel square in $\mathbf{K}_R$, but by the long exact sequence in homology it does become a co/kernel square in $\mathbf{D}_R$.[117] As such, it is completely determined by the morphism $X_\bullet \xrightarrow{g} M$, which in turn is completely specified by the morphism $M \to \mathsf{hcoker}(g)$. On the other hand, we have a canonical identification

$$\mathsf{hcoker}(g) \simeq \Sigma^n N \ :$$

both squares in the diagram

$$\begin{array}{ccccc} \Sigma^{n-1}N & \xrightarrow{f} & X_\bullet & \longrightarrow & 0 \\ \downarrow & & {\scriptstyle g}\downarrow & & \downarrow \\ 0 & \longrightarrow & M & \xrightarrow[k]{} & \mathsf{hcoker}(g) \end{array} \tag{22}$$

in $\mathbf{D}_R$ are pushouts, and so the composite rectangle is also a pushout by the ∞-categorical analog of (the first part of) the following.

[117] More precisely, there is a canonical map $\mathsf{hcoker}(f) \to M$ in $\mathbf{K}_R$, which is a quasi-isomorphism by the long exact sequence in homology for the co/kernel sequence $\Sigma^{n-1}N \xrightarrow{f} X_\bullet \to \mathsf{hcoker}(f)$.

Exercise 7.9 (6 points)**.** Consider a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

in a category $\mathcal{C}$.

(a) Show that if both squares are pushouts then so is the outer rectangle.

(b) Show that if the left square and the outer rectangle are pushouts then so is the right square.

(c) Give an example in which the right square and the outer rectangle are pushouts but the left square is not.

Of course, this morphism k is precisely the element of

$$\mathsf{Ext}^n_R(M,N) := \mathsf{H}_{-n}(\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N)) \cong \mathsf{H}_{-n}(\mathsf{hom}_{\mathbf{D}_R}(M,N)) \cong \mathsf{H}_0(\mathsf{hom}_{\mathbf{D}_R}(M,\Sigma^n N))$$

that is classified by the n-extension (20). Note too that a morphism (21) of n-extensions determines a quasi-isomorphism

$$X_\bullet \xrightarrow{\approx} X'_\bullet$$

in $(\mathsf{Ch}_R)_{\Sigma^{n-1}N//M}$, which gives a homotopy between the corresponding morphisms $M \to \Sigma^n N$ in $\mathbf{D}_R$.

Exercise 7.10 (10 points)**.**

(a) Construct an inverse function from $\mathsf{Ext}^n_R(M,N)$ to the set of equivalence classes of n-extensions of M by N.

(b) Describe the resulting $\Bbbk$-module structure (including the abelian group structure) on the set of equivalence classes of n-extensions of M by N.

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