

HIGHER ALGEBRA, CHAPTER 0

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ABSTRACT. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the spring 2023 quarter (building off of those from the winter 2021 iteration, which are available [here](#)). They are **under construction**, and will be updated at the course website at the end of each lecture.

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0. MISCELLANEA

0.1. Exercises.

0.1.1. The number of points needed to achieve various letter grades for your homework score is recorded in the syllabus.

0.1.2. Partial solutions may be submitted for partial credit.

0.1.3. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a “yes or no” question).

0.1.4. In preparing your homework, please copy down the problem statement, since it is possible that the numbering may change.

0.2. Conventions.

0.2.1. As this document is a work in progress, I will sometimes want to make changes to existing material (in addition to adding material as we cover it during lecture). Of course, it would be quite difficult for a reader to spot such changes, especially as the document grows. Therefore, any substantial such changes will be temporarily flagged by their change date (for easy searching) and additionally will be colored as follows: changes that have occurred since the most recent lecture are **red**, changes made between the past two lectures are **blue**, and changes made between two and three lectures ago are **green**. Older changes are no longer colored or dated.

0.2.2. We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and $\mathbf{Set}$ denotes the category of sets. However, notation will very often only be “local”: the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.3. The term “natural number” (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.4. We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We consider posets as categories without comment. We write e.g. $\mathbb{N}^{\leq}$ and $\mathbb{Z}^{\leq}$ for the usual poset structures on the natural numbers and the integers. Given some datum in a category (e.g. an object or morphism), we may use the superscript $(-)^{\circ}$ to denote the corresponding datum in the opposite category, although we may also omit this superscript when our meaning is sufficiently clear. We mostly ignore set-theoretic issues.¹ We refer the reader to [Shua] for a thorough discussion of the role of set theory in category theory.

0.2.5. Especially in later sections, we will frequently give references to Lurie’s books [Lur09] and [Lur]. Our exposition is nevertheless intended to be self-contained, with these references merely providing the reader with entry points for exploring those books further. For brevity, we will use the abbreviations “T” and “A” to refer to these works, and moreover we will omit environment names (except for the section symbol §). So for instance, we will refer to [Lur09, Theorem 4.1.3.1] simply as [T.4.1.3.1] and to [Lur, §1.3.3] simply as [§A.1.3.3].

0.2.6. The term “(commutative) ring” means “associative unital (resp. commutative) ring”. Likewise, modules are always unital (meaning that the unit element acts as the identity).

0.2.7. To simplify language, universal quantifiers will often be dropped. For instance, an assertion involving an integer n should generally be understood to refer to *all* integers n unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

0.2.8. For brevity, we will often use a slash to make multiple statements at once. This idiom has two possible meanings; the specific meaning should always be clear from context. On the one hand, we will write e.g. “homotopy co/kernel sequence” as a stand-in for “sequences which are simultaneously homotopy cokernel sequences and homotopy kernel sequences – and let us not forget that it suffices to check either condition in order to deduce both”. On the other hand, we will write e.g. “co/limits” as a stand in for “both colimits and limits”.

¹Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

PART I. HOMOLOGICAL ALGEBRA

1. SOME MOTIVATION FOR HOMOLOGICAL ALGEBRA

1.1. Intersection theory. A basic endeavor in geometry is to understand *intersections*. For example, given a (smooth) manifold M and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} .$$

If N_0 and N_1 intersect *transversely* (i.e. $T_p N_0 + T_p N_1 = T_p M$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

However, if the intersection of N_0 and N_1 is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either N_0 or N_1 that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.² However, this approach has a number of (related) drawbacks.

- (1) Perturbations are noncanonical.
- (2) Perturbations will generally destroy the inherent symmetries of the situation.³
- (3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally only transcendental.⁴

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. Tensor products of modules. We first recall the notion of tensor product.

Let R be a commutative ring, and let M and N be R -modules. The (*relative*) **tensor product** of M and N over R ,⁵ denoted $M \otimes_R N$, is the universal abelian group equipped with an R -balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N ,$$

i.e. a function satisfying the following axioms:

- (1) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$ and $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$;

²A good introduction to these ideas is [GP74].

³For instance, perturbations to transverse intersections need not exist in the equivariant context.

⁴It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, which is fundamental in the classical approach to intersection theory in algebraic geometry [EH16]. (In that context, “perturb” means “change to a new but rationally equivalent algebraic cycle”.)

⁵The word “relative” here is meant to emphasize that R is an arbitrary commutative ring. By contrast, the term “absolute tensor product” would emphasize that $R = \mathbb{Z}$.

$$(2) \quad \varphi(m \cdot r, n) = \varphi(m, r \cdot n).^6$$

In other words, for any abelian group A , precomposition with φ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \rightarrow A\} \xleftarrow{\cong} \{\text{abelian group homomorphisms } M \otimes_R N \rightarrow A\} .$$

In the case that R is understood (and particularly when $R = \mathbb{Z}$ or when R is a field), we may simply write $\otimes := \otimes_R$.

The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

- (1) $(m + m', n) \sim (m, n) + (m', n)$ and $(m, n + n') \sim (m, n) + (m, n')$;
- (2) $(m \cdot r, n) \sim (m, r \cdot n)$.

Exercise 1.1 (2 points). For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\gcd(m, n)$.

1.3. Basic principles of algebraic geometry. In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §I.1].

1.3.1. The polynomial functions on $\mathbb{R}^n$ are the n -variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \dots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving n implicit).

1.3.2. By definition, an **algebraic subset** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.⁷ In this case we write $Z = V(S)$, and we say that Z is the **vanishing locus** of the elements of S . If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.⁸

1.3.3. We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along Z . Then, the ring of polynomial functions on Z is

$$\mathcal{O}(Z) = R/I(Z) .$$

1.3.4. Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in J\} \subseteq \mathbb{R}^n .$$

⁶The notation here stems from the fact that more generally, we can define the relative tensor product when R is merely an associative ring, M is a right R -module, and N is a left R -module.

⁷By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

⁸Since R is noetherian, any ideal is finitely generated. In other words, we may always take S to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \xrightleftharpoons[V]{I} \{\text{ideals in } R\} \quad .^9$$

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 .$$

1.4. Intersections via tensor products. We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x - 1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x, y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \quad \text{and} \quad Z_1 = \{(x, y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \quad \text{and} \quad I_1 = I(Z_1) = (y - x) \subseteq R .$$

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned} \mathcal{O}(Z_0 \cap Z_1) &\cong R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x, y]/(y - x^2, y - x) \cong \mathbb{R}[x]/(x - x^2) \\ &= \mathbb{R}[x]/(x \cdot (1 - x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1 - x) \cong \mathbb{R} \times \mathbb{R} , \end{aligned}$$

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a principal ideal domain, in fact it is a Euclidean domain).¹⁰ The fact that this is the cartesian product of two copies of $\mathbb{R}$ corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

⁹Over an algebraically closed field $\mathbb{k}$, this construction restricts to a bijection between *closed* subsets of $\mathbb{k}^n$ (with respect to the Zariski topology) and *radical* ideals of $\mathbb{k}[x_1, \dots, x_n]$. The composite $V \circ I$ carries a subset $Y \subseteq \mathbb{k}^n$ to its closure $\overline{Y} \subseteq \mathbb{k}^n$, while the composite $I \circ V$ carries an ideal $I \subseteq \mathbb{k}[x_1, \dots, x_n]$ to its radical $\sqrt{I} = \{f \in \mathbb{k}[x_1, \dots, x_n] : \exists n > 0 \text{ s.t. } f^n \in I\} \subseteq R$. By contrast, over $\mathbb{R}$ the function V fails to be injective, e.g. $V(\mathbb{R}[x]) = V(x^2 + 1) = \emptyset$.

¹⁰An explicit inverse is given by carrying the pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b - a) \cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x \cdot (1 - x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0), g(1)}$ lies in the ideal generated by $x \cdot (x - 1)$. Observe that $g - f_{g(0), g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements, which can in good situations detect the correct multiplicity of a non-transverse intersection point.

Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in R . These correspond to the curves $y = x^2$ and $y = 0$. These intersect “twice” at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x, y]/(y - x^2, y) \cong \mathbb{R}[x]/(x^2) .$$

The 2-dimensionality of this $\mathbb{R}$ -algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect “with multiplicity two”. Namely, this $\mathbb{R}$ -algebra corresponds to “the origin along with infinitesimal fuzz in the direction of the x -axis”. This is in contrast with the previous example, where the tensor product split as a cartesian product.

These techniques are quite robust.

Exercise 1.2 (4 points). Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x - a) \subseteq R$ and $I_1 = (x - b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x - a, x - b) \cong \mathbb{R}/(a - b) \cong \begin{cases} \mathbb{R} , & a = b \\ 0 , & a \neq b \end{cases} .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$ -algebra.

Using homological algebra, namely the notion of *derived tensor products*, we will be able to obtain the expected intersection number of 0 even when $a = b$.

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

Exercise 1.3 (6 points). Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

(a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines “just barely avoid intersecting”: morally they should intersect “at infinity”.¹¹ This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3 \setminus \{0\}) / \mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have $[\lambda x : \lambda y : \lambda z] = [x : y : z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.¹²

- (b) Show that a *homogeneous* polynomial $g \in \mathbb{R}[x, y, z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.
- (c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogeneous polynomials $g_1, g_2 \in \mathbb{R}[x, y, z]$ such that $g_i([x : y : 1]) = f_i(x, y)$.
- (d) Working in an appropriate affine chart, use tensor products to compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. As we have seen in §1.4.4, given two lines in $\mathbb{R}^2$, we are more likely to get the expected number if we intersect them (or rather their closures) in $\mathbb{RP}^2$: namely, this gives the correct answer even when the lines are parallel. However, this fails to give the correct answer when the two lines are *equal*. Derived tensor products repair this failure. Namely, the derived tensor product of a (projective) line with itself in $\mathbb{RP}^2$ is “a line, but with cardinality equal to that of a single point”.

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

2. CHAIN COMPLEXES, HOMOLOGY, AND TENSOR PRODUCTS

We now proceed to introduce the basic objects of study in homological algebra.

2.1. Algebra conventions. For concreteness, we work in the context of ordinary algebra. Namely, we fix a commutative ring $\mathbb{k}$ and a $\mathbb{k}$ -algebra R . For the most part, we will work in $\mathbf{Mod}_R$, the category of (right) R -modules, and one may take $\mathbb{k}$ to be $\mathbb{Z}$. However, at times we will want to specialize to a commutative ring, and for this it is convenient for $\mathbb{k}$

¹¹A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \rightarrow 1^+$ their intersection point has $x \rightarrow -\infty$, while as $t \rightarrow 1^-$ their intersection point has $x \rightarrow +\infty$. This suggests that there should be a *single* point “at infinity” where they intersect in the case that $t = 1$.

¹²So, the “points at infinity” are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2 \setminus \{0\}) / \mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1 \setminus \mathbb{R}^1$).

to be arbitrary.¹³ Moreover, we will study some interactions between $\mathbb{k}$ -modules and R -modules. At the level of ordinary (i.e. non-homological) algebra, these are encapsulated by the following facts.

- (1) The category $\mathbf{Mod}_{\mathbb{k}}$ is symmetric monoidal via the tensor product, which we denote by $\otimes := \otimes_{\mathbb{k}}$; its unit object is $\mathbb{k}$.
- (2) The category $\mathbf{Mod}_R$ is naturally enriched in $\mathbf{Mod}_{\mathbb{k}}$. In other words, for any R -modules $M, N \in \mathbf{Mod}_R$, the set $\mathbf{hom}_{\mathbf{Mod}_R}(M, N)$ of R -linear homomorphisms carries the natural structure of a $\mathbb{k}$ -module, and moreover composition in $\mathbf{Mod}_R$ is $\mathbb{k}$ -multilinear.
- (3) Moreover, $\mathbb{k}$ -modules naturally act on R -modules in two different ways: for any $\mathbb{k}$ -module $T \in \mathbf{Mod}_{\mathbb{k}}$ and any R -modules $M, N \in \mathbf{Mod}_R$ we have R -modules

$$T \otimes_{\mathbb{k}} M \quad \text{and} \quad \mathbf{hom}_{\mathbf{Mod}_{\mathbb{k}}}(T, N) ,$$

where the (right) R -actions are induced from those on M and N , and these constructions participate in natural isomorphisms

$$(1) \quad \mathbf{hom}_{\mathbf{Mod}_R}(T \otimes_{\mathbb{k}} M, N) \cong \mathbf{hom}_{\mathbf{Mod}_{\mathbb{k}}}(T, \mathbf{hom}_{\mathbf{Mod}_R}(M, N)) \cong \mathbf{hom}_{\mathbf{Mod}_R}(M, \mathbf{hom}_{\mathbf{Mod}_{\mathbb{k}}}(T, N)) .$$

Of course, one may take $R = \mathbb{k}$ as a special case.¹⁴ As a result, the notions that we will develop relating to the interactions between $\mathbb{k}$ -modules and R -modules will all be *generalizations* of the notions that we develop relating to $\mathbb{k}$ -modules alone.

As we will see later, most of the theory works equally well for a general abelian category, although there will be some additional hiccups that do not arise when studying modules.¹⁵

Exercise 2.1 (5 points).

- (a) Given a $\mathbb{k}$ -module $T \in \mathbf{Mod}_{\mathbb{k}}$ and R -modules $M, N \in \mathbf{Mod}_R$, verify the isomorphisms (1).
- (b) Verify the naturality of the isomorphisms (1) in all three variables.

2.2. Chain complexes. A *chain complex* of R -modules is a diagram

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of R -modules such that for all $n \in \mathbb{Z}$ we have the equality

$$d_n \circ d_{n+1} = 0$$

¹³Of course, we will apply results developed for R -modules to $\mathbb{k}$ -modules without comment.

¹⁴The above facts then reduce to the assertion that $\mathbf{Mod}_{\mathbb{k}}$ is a *closed* symmetric monoidal category, i.e. that it carries a self-enrichment that is compatible with its symmetric monoidal structure. (More generally, facts 2 and 3 can be simultaneously articulated as the assertion that $\mathbf{Mod}_R$ is naturally a *closed module* over the closed symmetric monoidal category $\mathbf{Mod}_{\mathbb{k}}$.)

¹⁵On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into $\mathbf{Mod}_R$ for some ring R (although the choice of such a ring R is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

in $\mathbf{hom}_{\mathbf{Mod}_R}(M_{n+1}, M_{n-1})$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that “all indices are being referred to at once”. To emphasize the differentials, one may write $(M_\bullet, d_\bullet)$. Also, one may simply refer to a chain complex as a “complex”.¹⁶ On the other hand, we also may omit the bullet and simply write $M := M_\bullet$ for simplicity. The integer n is called the **degree** or the **dimension**. For an element $m \in M_n$, we may write $\deg(m) := n$.

The morphisms d_n are called the **differentials** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of d_n is M_n). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as d_n^M .

In this notation, a *morphism* of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is by definition a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of R -modules such that the diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{n+2}^M} & M_{n+1} & \xrightarrow{d_{n+1}^M} & M_n & \xrightarrow{d_n^M} & M_{n-1} \xrightarrow{d_{n-1}^M} \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \xrightarrow{d_{n+2}^N} & N_{n+1} & \xrightarrow{d_{n+1}^N} & N_n & \xrightarrow{d_n^N} & N_{n-1} \xrightarrow{d_{n-1}^N} \cdots \end{array}$$

commutes.¹⁷ These morphisms are often referred to as **chain maps**. We write $\mathbf{Ch}_R$ for the category of chain complexes of R -modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline whenever it appears explicitly.

Any R -module $M \in \mathbf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \underline{\underline{M}} \longrightarrow 0 \longrightarrow \cdots .$$

This construction determines a fully faithful embedding

$$\mathbf{Mod}_R \hookrightarrow \mathbf{Ch}_R .$$

As a result, we may not notationally distinguish between data in $\mathbf{Mod}_R$ and its image in $\mathbf{Ch}_R$.

When a complex only has a few nonzero terms, for brevity one may omit the terms that are zero. For instance, the above complex may also be written as $\underline{\underline{M}}$.

¹⁶The word “chain” here is historical. Namely, given a simplicial complex X , the first example of a chain complex was given in degree n by the “ n^{th} chain group” of X , i.e. the group of chains (i.e. formal linear combinations) of n -simplices of X , with differentials given by taking an n -simplex to the signed sum of its boundary $(n-1)$ -simplices (for which reason the morphisms d_n of a chain complex are also sometimes referred to as its “boundary maps”); see §8.4 (and Footnote 151) for a closely related discussion. It was only later realized that chain complexes are worth studying in their own right.

¹⁷For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

2.3. Homology. Fix a chain complex $M_\bullet$. Its n -*cycles* and n -*boundaries* are the submodules

$$Z_n(M_\bullet) := \ker(d_n) \subseteq M_n \quad \text{and} \quad B_n(M_\bullet) := \operatorname{im}(d_{n+1}) \subseteq M_n. \quad {}^{18}$$

Note that $B_n(M_\bullet) \subseteq Z_n(M_\bullet)$ because $d^2 = 0$. Then, the n^{th} *homology* of $M_\bullet$ is the quotient R -module

$$H_n(M_\bullet) := \frac{Z_n(M_\bullet)}{B_n(M_\bullet)} := \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})}.$$

Although $H_n(M_\bullet)$ is an R -module, it is common to refer to it merely as a *homology group*.

Exercise 2.2 (3 points). Verify that the constructions Z_n , B_n , and H_n define functors

$$\operatorname{Ch}_R \longrightarrow \operatorname{Mod}_R.$$

A morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R is called a **quasi-isomorphism** if the induced morphisms $H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We may indicate that a morphism is a quasi-isomorphism by decorating the arrow as $\xrightarrow{\sim}$.

We say that a complex $M_\bullet$ is **acyclic** if $H_n(M_\bullet) = 0$ for all n . So, $M_\bullet$ is acyclic if and only if either (and hence both) of the unique maps $0 \rightarrow M_\bullet$ and $M_\bullet \rightarrow 0$ is a quasi-isomorphism.

2.4. The derived category of R -modules.

2.4.1. By and large, we would like to think of quasi-isomorphic chain complexes as “essentially interchangeable”, with some representatives of a given quasi-isomorphism class (namely the *projective* and *injective* complexes introduced in §5) being “well-adapted” for certain purposes.^{19,20} In other words, one should think of quasi-isomorphisms as if they are actual isomorphisms.

This can be made literally true by **localizing** the category Ch_R at the quasi-isomorphisms, i.e. by adjoining formal inverses for them. This yields a category that (for reasons that will be explained shortly in §2.4.3) we will denote by $H_0(\mathbf{D}_R)$ and refer to as **the derived category**

¹⁸The German words for “cycle” and “boundary” respectively begin with the letters “Z” and “B”.

¹⁹This is very closely akin to how one should think of equivalent categories as “essentially interchangeable”, even when they are not isomorphic. However, in a precise sense, *all* categories are “equally well-adapted” for all purposes (in contrast with chain complexes).

²⁰While quasi-isomorphic complexes have isomorphic homology groups, the converse is generally false: the obstruction is encoded by **k-invariants**, as explained in §7.3.3. (This same name is given to the (closely analogous) obstructions to spaces with the same homotopy groups being weak homotopy equivalent.) That is, a quasi-isomorphism class of complexes is equivalent data to its homology groups along with all of its k-invariants. For this reason, we will generally consider quasi-isomorphism classes of complexes themselves as the “true” mathematical objects of lasting interest, while their homology groups are merely algebraic invariants that can be extracted therefrom.

of *R-modules*; its objects are called *derived R-modules*.²¹ So by definition, there is a canonical functor

$$\mathbf{Ch}_R \longrightarrow \mathbf{H}_0(\mathbf{D}_R)$$

that carries all quasi-isomorphisms to isomorphisms, and moreover it is universal with respect to this requirement. Indeed, for any category $\mathcal{C}$, the restriction functor

$$(2) \quad \mathbf{Fun}(\mathbf{Ch}_R, \mathcal{C}) \longleftarrow \mathbf{Fun}(\mathbf{H}_0(\mathbf{D}_R), \mathcal{C})$$

is a fully faithful inclusion, whose image consists of those functors $\mathbf{Ch}_R \rightarrow \mathcal{C}$ that carry quasi-isomorphisms to isomorphisms.²²

Note that a derived *R*-module is a “purely homotopical” object: while it can by definition be presented by a chain complex of *R*-modules, one cannot speak e.g. of its underlying *R*-module in dimension 0, as this notion is not preserved under quasi-isomorphisms.²³ On the other hand, one *can* speak e.g. of its n^{th} homology, since by definition a quasi-isomorphism induces an isomorphism on n^{th} homology. Thus, the n^{th} homology functor admits a factorization

$$\begin{array}{ccc} \mathbf{Ch}_R & \xrightarrow{H_n} & \mathbf{Mod}_R \\ \downarrow & \nearrow \text{dashed } H^n & \\ \mathbf{H}_0(\mathbf{D}_R) & & \end{array},$$

which is unique up to unique isomorphism (and which is denoted using the same notation, as is justified by the fully faithful inclusion (2)).

2.4.2. Essentially by construction, given two complexes $M, N \in \mathbf{Ch}_R$, morphisms from M to N in the derived category $\mathbf{H}_0(\mathbf{D}_R)$ are given by equivalence classes of zigzags

$$M \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \cdots \longrightarrow \bullet \xleftarrow{\approx} N$$

²¹The placement of the word “derived” in the term “the derived category of *R*-modules” is admittedly slightly unfortunate, but this term is quite common. (It arises from the fact that this term is a shortening of “the derived category of the category of *R*-modules”.)

²²This should be thought of as a “many-object” version of the localization of the $\mathbb{k}$ -algebra *R* with respect to a subset $S \subset R$: this yields a $\mathbb{k}$ -algebra $R[S^{-1}]$ equipped with a $\mathbb{k}$ -algebra homomorphism $R \rightarrow R[S^{-1}]$ such that for any $\mathbb{k}$ -algebra *T*, the restriction map $\text{hom}_{\text{Alg}_{\mathbb{k}}}(R, T) \hookrightarrow \text{hom}_{\text{Alg}_{\mathbb{k}}}(R[S^{-1}], T)$ is the inclusion of those homomorphisms $R \rightarrow T$ that carry all elements of *S* to invertible elements of *T*. (Namely, the latter is obtained by considering $\mathbb{k}$ -algebras as one-object $\mathbb{k}$ -linear categories.)

²³Likewise, one cannot speak of the underlying set of a weak homotopy equivalence class of topological spaces, nor can one speak of the set of objects of an equivalence class of categories.

(in which all backwards maps are quasi-isomorphisms).²⁴ Thankfully, it will turn out that every equivalence class contains representatives of the forms

$$M \xleftarrow{\approx} \bullet \longrightarrow N \quad \text{and} \quad M \longrightarrow \bullet \xleftarrow{\approx} N ,$$

which makes the situation substantially more manageable.²⁵

2.4.3. Although we introduce the derived category now, we will not have much use for it: it contains too little information. The richer and more primitive object is $\mathbf{D}_R$, the **derived** ∞ -**category** of R . This is a mathematical entity whose objects are still the derived R -modules, but whose hom-objects are more elaborate: namely, they are *derived $\mathbb{k}$ -modules*.²⁶ As the notation suggests, passing from $\mathbf{D}_R$ to $\mathbf{H}_0(\mathbf{D}_R)$ amounts to extracting the 0th homology groups of these hom-objects.

In fact, the relationship between the derived ∞ -categories $\mathbf{D}_R$ and $\mathbf{D}_{\mathbb{k}}$ precisely mirrors that between the categories $\mathbf{Mod}_R$ and $\mathbf{Mod}_{\mathbb{k}}$ laid out in §2.1. However, it will take us a while to see this, since the likewise relationship between the categories $\mathbf{Ch}_R$ and $\mathbf{Ch}_{\mathbb{k}}$ will have to be “homotopically corrected” in order to descend to the corresponding derived ∞ -categories. We discuss tensor products of chain complexes presently, but defer the discussion of hom-complexes to §4 in order to highlight more fundamental homotopical notions in §3.²⁷ We discuss “homotopical corrections” in §5, and finally return to the derived ∞ -category in §6.

2.5. **Tensor products of chain complexes.** Given complexes $M, N \in \mathbf{Ch}_{\mathbb{k}}$ of $\mathbb{k}$ -modules, we define their **tensor product** complex

$$(M \otimes N)_{\bullet} := (M \otimes_{\mathbb{k}} N)_{\bullet}$$

as follows. First of all, we define its k^{th} term to be

$$(M \otimes N)_k := \bigoplus_{i+j=k} (M_i \otimes N_j) := \bigoplus_{i+j=k} (M_i \otimes_{\mathbb{k}} N_j) .$$

Then, the differential is characterized by the fact that it carries a pure tensor

$$m \otimes n \in (M_i \otimes N_j) \subseteq (M \otimes N)_k$$

²⁴Returning to the context of Footnote 22, this zigzag of morphisms in $\mathbf{Ch}_R$ is analogous to a formal product $s_1^{-1} \cdot r_1 \cdot s_2^{-1} \cdot r_2 \cdot \dots \cdot s_n^{-1}$ in which the elements s_i^{-1} are formally adjoined inverses of the elements $s_i \in S$.

²⁵These reductions are guaranteed by the existence of two different *model structures* on $\mathbf{Ch}_R$, which respectively have the features that all objects are fibrant and that all objects are cofibrant; see §8.2.

²⁶So to be more precise, $\mathbf{D}_R$ is a $\mathbb{k}$ -linear ∞ -category (see §6.2).

²⁷This ordering of topics is due to the fact that the discussion of §3 makes (light) use of the tensor product of complexes.

to the sum of pure tensors

$$d(m \otimes n) := d(m) \otimes n + (-1)^i \cdot m \otimes d(n) ,^{28}$$

(an element of $((M_{i-1} \otimes N_j) \oplus (M_i \otimes N_{j-1})) \subseteq (M \otimes N)_{k-1}$). More elaborately, this may be written as

$$d_k^{M \otimes N}(m \otimes n) := d_i^M(m) \otimes n + (-1)^i \cdot m \otimes d_j^N(n) .$$

Exercise 2.3 (2 points). Verify that this formula defines a complex.

In particular, in solving Exercise 2.3 you will see why the signs are necessary in the definition of the tensor product of complexes. In fact, many sign conventions are possible (and all give equivalent symmetric monoidal categories), but it is impossible to remove all signs from the theory (unless one works over $\mathbb{F}_2$).

From here, it is straightforward to see that the above construction defines a monoidal structure

$$\mathbf{Ch}_{\mathbb{k}} \times \mathbf{Ch}_{\mathbb{k}} \xrightarrow{\otimes} \mathbf{Ch}_{\mathbb{k}} ,$$

with unit object $\mathbb{k} := \underline{\mathbb{k}} \in \mathbf{Ch}_{\mathbb{k}}$. In fact, this is a *symmetric* monoidal structure, with symmetry isomorphisms

$$M \otimes N \xrightarrow{\cong} N \otimes M$$

determined by the formula

$$m \otimes n \longmapsto (-1)^{\deg(m) \cdot \deg(n)} \cdot n \otimes m .^{29}$$

More generally, this same construction defines an action

$$\mathbf{Ch}_{\mathbb{k}} \times \mathbf{Ch}_R \xrightarrow{\otimes} \mathbf{Ch}_R$$

of the symmetric monoidal category $(\mathbf{Ch}_{\mathbb{k}}, \otimes, \mathbb{k})$ on the category $\mathbf{Ch}_R$.

Exercise 2.4 (8 points). Fix two complexes $M, N \in \mathbf{Ch}_{\mathbb{k}}$.

(a) Verify that the formula $[m] \otimes [n] \mapsto [m \otimes n]$ determines a morphism

$$H_i(M) \otimes H_j(N) \longrightarrow H_{i+j}(M \otimes N)$$

of $\mathbb{k}$ -modules.

It follows that we obtain a morphism

$$\bigoplus_{i+j=k} (H_i(M) \otimes H_j(N)) \longrightarrow H_k(M \otimes N)$$

of $\mathbb{k}$ -modules.

(b) Prove that this is an isomorphism under the assumption that $\mathbb{k}$ is a field.

²⁸The factor $(-1)^i$ is determined by the *Koszul sign rule*, which is a general principle asserting that in commuting graded quantities past one another of degrees $\alpha, \beta \in \mathbb{Z}$ one should pick up a factor of $(-1)^{\alpha \cdot \beta}$. (We consider the symbol “ d ” as an expression of degree -1 since it lowers degrees by 1.)

²⁹This formula is another instance of the Koszul sign rule.

(c) Find an example where this is not an isomorphism.

3. HOMOTOPIES, HOMOTOPY CO/KERNELS, AND EXACT SEQUENCES

3.1. Homotopies.

3.1.1. Let $M_1 \xrightarrow{f} M_0$ be a morphism of R -modules. This gives us a complex $M_\bullet := (M_1 \xrightarrow{f} \underline{M_0})$. Observe that this has a canonical morphism

$$\begin{array}{ccc} M_\bullet & & M_1 \xrightarrow{f} \underline{M_0} \\ \downarrow & := & \downarrow \quad \quad \downarrow \\ \text{coker}(f) & & 0 \longrightarrow \underline{\text{coker}(f)} \end{array}$$

to the cokernel of f (considered as a complex in degree 0). Observe further that

$$H_n(M_\bullet) \cong \begin{cases} \text{coker}(f) , & n = 0 \\ \ker(f) , & n = 1 \\ 0 , & \text{otherwise} \end{cases} .$$

Hence, the above map is a quasi-isomorphism iff f is an injection. One might think of $M_\bullet$ as a “presentation” of the underlying R -module $H_0(M_\bullet) \cong \text{coker}(f)$: the generators are M_0 , the relations are M_1 (i.e. each $m \in M_1$ gives a relation $d(m) \sim 0$), but then $H_1(M_\bullet)$ furthermore measures the “redundancy” of the relations. Said differently, $M_\bullet$ is a “homotopically correct” version of the cokernel of f , which remembers not only the literal cokernel but also the extent to which the relations are overdetermined. Indeed, it will be the *homotopy cokernel* of the morphism f .

3.1.2. Let $M_\bullet, N_\bullet \in \text{Ch}_R$ be complexes and let $f_\bullet, g_\bullet \in \text{hom}_{\text{Ch}_R}(M_\bullet, N_\bullet)$ be morphisms. A (*chain*) *homotopy* from $f_\bullet$ to $g_\bullet$ is a set of morphisms

$$M_n \xrightarrow{h_n} N_{n+1}$$

satisfying the condition that

$$g_n - f_n = d_{n+1}^N \circ h_n + h_{n-1} \circ d_n^M .^{30}$$

³⁰Chain homotopies are sometimes depicted as the dashed morphisms in the diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{n+2}^M} & M_{n+1} & \xrightarrow{d_{n+1}^M} & M_n & \xrightarrow{d_n^M} & M_{n-1} \xrightarrow{d_{n-1}^M} \cdots \\ & \swarrow h_{n+1} & \downarrow f_{n+1} & \downarrow g_{n+1} & \swarrow h_n & \downarrow f_n & \downarrow g_n & \swarrow h_{n-1} & \downarrow f_{n-1} & \downarrow g_{n-1} & \swarrow h_{n-2} & \cdots \\ \cdots & \xrightarrow{d_{n+2}^N} & N_{n+1} & \xrightarrow{d_{n+1}^N} & N_n & \xrightarrow{d_n^N} & N_{n-1} \xrightarrow{d_{n-1}^N} \cdots \end{array}$$

This may be helpful as a mnemonic device, but note that it does *not* commute in any reasonable sense.

We may write this as $f_{\bullet} \xRightarrow{h_{\bullet}} g_{\bullet}$. A **nullhomotopy** of $g_{\bullet}$ is a homotopy $0 \xRightarrow{h_{\bullet}} g_{\bullet}$ from the zero map. A **contraction** of a complex is a nullhomotopy of its identity map. If a complex admits a contraction, we say that it is **contractible**.

Exercise 3.1 (2 points). Show that the relation of homotopy on $\text{hom}_{\text{Ch}_R}(M_{\bullet}, N_{\bullet})$ is an equivalence relation.

In fact, the argument of Exercise 3.1 easily upgrades to imply that we can enhance Ch_R from an ordinary category to a category enriched in groupoids (a.k.a. a $(2, 1)$ -category):³¹ its objects are chain complexes, its 1-morphisms are chain maps, and its 2-morphisms are chain homotopies.³² Indeed, the arguments for transitivity, reflexivity, and symmetry of the relation of homotopy respectively endow these hom-categories with their composition laws, identity morphisms, and inverses (so that they are indeed hom-groupoids). It is moreover clear that homotopies may be composed appropriately, either by definition or using Exercise 3.2 below.

3.1.3. To give a more geometric perspective on chain homotopies, let us introduce the complex $\mathbb{I} \in \text{Ch}_{\mathbb{k}}$ and the morphisms $i_0, i_1 \in \text{hom}_{\text{Ch}_{\mathbb{k}}}(\mathbb{k}, \mathbb{I})$ according to the diagram

$$\begin{array}{ccc}
 \mathbb{k} & & 0 \longrightarrow \mathbb{k} \\
 \downarrow i_0 & & \downarrow \\
 \mathbb{I} & := & \mathbb{k} \xrightarrow{(-\text{id}_{\mathbb{k}}, \text{id}_{\mathbb{k}})} \mathbb{k} \oplus \mathbb{k} \\
 \uparrow i_1 & & \uparrow (0, \text{id}_{\mathbb{k}}) \\
 \mathbb{k} & & 0 \longrightarrow \mathbb{k}
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \downarrow (\text{id}_{\mathbb{k}}, 0) \\
 & & \mathbb{k} \oplus \mathbb{k} \\
 & & \uparrow (0, \text{id}_{\mathbb{k}})
 \end{array}
 .$$

This object $\mathbb{I} \in \text{Ch}_{\mathbb{k}}$ (along with the two maps i_0 and i_1) is an **interval object** for the homotopy theory of chain complexes (which explains the notation).³³ The general definition of an interval object is suggested by the discussion of §8.1.2. In the present setting, this assertion amounts to the following result.

³¹As we will see, this is the homotopy $(2, 1)$ -category of a more fundamental object, namely an ∞ -category (meaning an $(\infty, 1)$ -category).

³²This explains the notation $f_{\bullet} \Rightarrow g_{\bullet}$ just introduced.

³³This is a particularly natural choice of an interval object: as we will see, it is the (nondegenerate) simplicial chains (with coefficients in $\mathbb{k}$) on the 1-simplex Δ^1 (whose underlying topological space is a closed interval), and the two maps i_0 and i_1 are the simplicial chains on the inclusions of its two 0-simplices (i.e. the endpoints of the closed interval).

Exercise 3.2 (4 points). Given maps $f, g \in \mathbf{hom}_{\mathbf{Ch}_R}(M, N)$, prove that a homotopy $f \Rightarrow g$ is equivalent data to a map $\mathbb{I} \otimes M \xrightarrow{h} N$ that makes the diagram

$$\begin{array}{ccc}
 \mathbb{k} \otimes M \cong M & & \\
 \downarrow i_0 := i_0 \otimes \mathrm{id}_M & \searrow f & \\
 \mathbb{I} \otimes M & \xrightarrow{\quad h \quad} & N \\
 \uparrow i_1 := i_1 \otimes \mathrm{id}_M & \nearrow g & \\
 \mathbb{k} \otimes M \cong M & &
 \end{array}$$

commute.

So, a map $\mathbb{I} \otimes M \rightarrow N$ is the same thing as two maps $M \rightarrow N$ and a homotopy between them. Said differently, $\mathbb{I} \otimes M \in \mathbf{Ch}_R$ is the universal object equipped with two maps from M and a homotopy between them. Of course, the two universal maps are i_0 and i_1 .

Exercise 3.3 (2 points). Give a formula for the universal homotopy $i_0 \Rightarrow i_1$ between two maps from M .

Similarly, it follows from Exercise 3.2 that a map

$$C := \mathrm{colim} \left(\begin{array}{ccc} M & \xleftarrow{i_1} & \mathbb{I} \otimes M \\ i_0 \downarrow & & \\ \mathbb{I} \otimes M & & \end{array} \right) \rightarrow N$$

is equivalent data to a triple of morphisms $f_0, f_1, f_2 \in \mathbf{hom}_{\mathbf{Ch}_R}(M, N)$ as well as a pair of homotopies $f_0 \Rightarrow f_1 \Rightarrow f_2$.³⁴ Said differently, $C \in \mathbf{Ch}_R$ is the universal object equipped with three maps from M and two homotopies as indicated, say $j_0 \Rightarrow j_1 \Rightarrow j_2$. On the other hand, Exercise 3.1 implies that these two homotopies can be composed to a single homotopy $j_0 \Rightarrow j_2$.

Exercise 3.4 (6 points). Give a formula for the map $\mathbb{I} \otimes M \rightarrow C$ corresponding to the universal composite homotopy $j_0 \Rightarrow j_2$ of two composable homotopies between maps from M .³⁵

³⁴This colimit is an algebraic analog of the topological construction taking a space X to the space

$$(X \times [0, 1]) \coprod_{X \times \{1\}} (X \times [0, 2]) .$$

³⁵Note that co/limits in chain complexes are computed levelwise. Indeed, $\mathbf{Ch}_R$ is a full subcategory of the functor category $\mathbf{Fun}((\mathbb{Z}^{\leq})^{\mathrm{op}}, \mathbf{Mod}_R)$ (where $\mathbb{Z}^{\leq} := \{\dots \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow \dots\}$), and co/limits in any functor category are computed pointwise. So, it suffices to see that the subcategory $\mathbf{Ch}_R \subseteq \mathbf{Fun}((\mathbb{Z}^{\leq})^{\mathrm{op}}, \mathbf{Mod}_R)$ is

3.1.4. A morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$ is a **homotopy equivalence** if there exists a morphism $N \xrightarrow{g} M$ and homotopies $\mathrm{id}_M \Rightarrow g \circ f$ and $f \circ g \Rightarrow \mathrm{id}_N$.³⁶ We may indicate that a morphism is a homotopy equivalence by decorating it as $\xrightarrow{\sim}$.

As a special case, a complex $M_\bullet$ is contractible if and only if either unique map $0 \rightarrow M_\bullet$ or $M_\bullet \rightarrow 0$ is a homotopy equivalence.

Exercise 3.5 (2 points). Show that homotopic maps on complexes give equal maps on homology.

It follows immediately from Exercise 3.5 that homotopy equivalences are quasi-isomorphisms, and in particular that contractible complexes are acyclic. However, the converse is false.

Exercise 3.6 (6 points).

(a) Show that the complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots$$

of $\mathbb{Z}/4$ -modules is acyclic but not contractible.

(b) Show that the complex

$$\cdots \xrightarrow{x} \mathbb{k}[x]/x^2 \xrightarrow{x} \mathbb{k}[x]/x^2 \xrightarrow{x} \mathbb{k}[x]/x^2 \xrightarrow{x} \cdots$$

of $\mathbb{k}[x]/x^2$ -modules is acyclic. Show that it is contractible as a complex of $\mathbb{k}$ -modules, but not as a complex of $\mathbb{k}[x]/x^2$ -modules (assuming that $\mathbb{k} \neq 0$).

3.1.5. We will generally consider homotopic maps as “essentially interchangeable”. On the other hand, rather than merely positing the *existence* of a homotopy between two maps, we will always want to *keep track* of the homotopy that witnesses them as being homotopic. In fact, this will be accomplished by hom-complexes, which will also keep track of homotopies between homotopies, homotopies between those, ad infinitum (see §4.1).

The idea of keeping track of data instead of merely remembering its existence is a fundamental one in (higher) category theory, which has already appeared in §3.1.1 and will continue to feature prominently throughout this text.

3.2. Homotopy cokernels.

stable under co/limits. This follows from the more general fact that the full subcategory of the arrow category $\mathbf{Ar}(\mathbf{Mod}_R) := \mathbf{Fun}([1], \mathbf{Mod}_R)$ on those morphisms that factor through $0 \in \mathbf{Mod}_R$ is stable under co/limits; as co/limits in $\mathbf{Ar}(\mathbf{Mod}_R)$ are of course also computed pointwise, this follows from the fact that the zero object $0 \in \mathbf{Mod}_R$ is stable under co/limits (which follows from the fact that it is both initial and terminal).

³⁶The directions of these homotopies are intended to be suggestive of adjunctions (with f functioning as the left adjoint), but by Exercise 3.1 (specifically the symmetry of the relation of homotopy) they are irrelevant. (Likewise, equivalences of categories are both left adjoints and right adjoints.)

3.2.1. Recall that the cokernel of a morphism $M \xrightarrow{f} N$ in $\mathbf{Mod}_R$ is by definition an R -module $\mathrm{coker}(f) \in \mathbf{Mod}_R$ equipped with a morphism

$$N \xrightarrow{u} \mathrm{coker}(f)$$

satisfying the universal property that precomposition with u determines a natural bijection

$$\mathrm{hom}_{\mathbf{Mod}_R}(\mathrm{coker}(f), T) \xrightarrow{\cong} \left\{ \begin{array}{l} \text{morphisms } N \rightarrow T \text{ such that the} \\ \text{composite } M \xrightarrow{f} N \rightarrow T \text{ is zero} \end{array} \right\} . \quad 37$$

From here, the principles indicated in §3.1.5 lead directly to the definition of a **homotopy cokernel** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in $\mathbf{Ch}_R$: this is an object $\mathrm{hcoker}(f_\bullet) \in \mathbf{Ch}_R$ equipped with a morphism

$$N_\bullet \xrightarrow{u_\bullet} \mathrm{hcoker}(f_\bullet)$$

satisfying the universal property that precomposition with $u_\bullet$ determines a natural bijection

$$\mathrm{hom}_{\mathbf{Ch}_R}(\mathrm{hcoker}(f_\bullet), T_\bullet) \xrightarrow{\cong} \left\{ \begin{array}{l} \text{morphisms } N_\bullet \rightarrow T_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } M_\bullet \xrightarrow{f_\bullet} N_\bullet \rightarrow T_\bullet \end{array} \right\} . \quad 38$$

³⁷Said differently, the cokernel of f is the pushout

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{coker}(f) \end{array} ,$$

i.e. the initial cocone under the diagram $0 \leftarrow M \xrightarrow{f} N$.

³⁸Similarly, the homotopy cokernel of f may be characterized as the *homotopy* pushout

$$\begin{array}{ccc} M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \\ \downarrow & \nearrow & \downarrow \\ 0 & \longrightarrow & \mathrm{hcoker}(f_\bullet) \end{array} ,$$

i.e. the *initial homotopy-coherent cocone* under the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$ (though we'll refrain from defining this notion precisely here). Here, the symbol $\Rightarrow$ indicates that the square only commutes up to a specified homotopy. (This homotopy arises from the image of the identity map $\mathrm{id}_{\mathrm{hcoker}(f_\bullet)} \in \mathrm{hom}_{\mathbf{Ch}_R}(\mathrm{hcoker}(f_\bullet), \mathrm{hcoker}(f_\bullet))$ under the above natural bijection (as it must by the Yoneda lemma).) This is an algebraic analog of the topological construction taking a map $X \rightarrow Y$ of spaces to the space

$$\mathrm{hcoker}(X \rightarrow Y) := Y \coprod_X \mathrm{cone}(X) ,$$

where $\mathrm{cone}(X) := (X \times [0, 1]) / (X \times \{0\})$ denotes the *cone* on the space X (and we consider X to be embedded as the subspace $X \times \{1\}$ thereof). This topological construction is often called the (*mapping*) *cone*, and as a result it is common to refer its algebraic analog using this same term. However, we prefer the term “homotopy cokernel” as it is more directly suggestive of the object’s “role in life” (a.k.a. its *raison d’être*). (It is also common to refer to this object as the *homotopy cofiber*, but that term deviates from the more standard terminology of “cokernel” (as opposed to “cofiber”) in the context of an abelian category.)

In fact, we have already seen an example of a homotopy cokernel: namely, for any morphism $M \xrightarrow{f} N$ in $\mathbf{Mod}_R$, the complex $(M \xrightarrow{f} \underline{N}) \in \mathbf{Ch}_R$ equipped with the map

$$\begin{array}{ccc} 0 & \longrightarrow & \underline{N} \\ \downarrow & & \downarrow \text{id}_N \\ M & \xrightarrow{f} & \underline{N} \end{array}$$

is a homotopy cokernel of f (when considered in $\mathbf{Ch}_R$). More generally, given a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in $\mathbf{Ch}_R$, consider the diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\begin{pmatrix} -d_1^M & 0 \\ f_1 & d_2^N \end{pmatrix}} & \begin{matrix} M_0 \\ \oplus \\ N_1 \end{matrix} & \xrightarrow{\begin{pmatrix} -d_0^M & 0 \\ f_0 & d_1^N \end{pmatrix}} & \begin{matrix} M_{-1} \\ \oplus \\ \underline{N_0} \end{matrix} & \xrightarrow{\begin{pmatrix} -d_{-1}^M & 0 \\ f_{-1} & d_0^N \end{pmatrix}} & \begin{matrix} M_{-2} \\ \oplus \\ N_{-1} \end{matrix} & \xrightarrow{\begin{pmatrix} -d_{-2}^M & 0 \\ f_{-2} & d_{-1}^N \end{pmatrix}} & \cdots \end{array}$$

of R -modules.

Exercise 3.7 (6 points). Verify that the above diagram indeed defines a complex and moreover is a homotopy cokernel of $f_\bullet$.

Exercise 3.8 (6 points). Verify that the homotopy cokernel of f is the colimit of the diagram

$$\begin{array}{ccc} & M & \longrightarrow 0 \\ & \downarrow i_0 & \\ M & \xleftarrow{i_1} \mathbb{I} \otimes M & \\ f \downarrow & & \\ N & & \end{array} .$$

In particular, we find that for a morphism $M \xrightarrow{f} N$ in $\mathbf{Mod}_R$, we have a canonical morphism $\text{hcoker}(f) \rightarrow \text{coker}(f)$, and this is a quasi-isomorphism iff f is injective. This is an instance of the general principle that homotopically sensitive constructions are suitably equivalent (in this case, quasi-isomorphic) to their ordinary variants in “simple” situations (in this case, when there is no redundancy in the relations). More generally, we have the following.

Exercise 3.9 (6 points). Fix a morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$ that is injective in each dimension.

(a) Show that the canonical morphism

$$\text{hcoker}(f) \longrightarrow \text{coker}(f)$$

to the levelwise cokernel is a quasi-isomorphism.

(b) Give an example showing that this map need not be a homotopy equivalence.

As the following exercise illustrates, homotopy cokernels give us a way of translating conditions on morphisms between chain complexes to conditions on chain complexes themselves.

Exercise 3.10 (10 points). Fix a morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$.

- (a) Prove that f is a quasi-isomorphism iff $\mathbf{hcoker}(f)$ is acyclic.³⁹
- (b) Prove that f is a homotopy equivalence iff $\mathbf{hcoker}(f)$ is contractible.

3.2.2. As a special case of a homotopy cokernel, we simply write

$$\Sigma M_\bullet := \mathbf{hcoker}(M_\bullet \longrightarrow 0) ,^{40}$$

and refer to this as the *suspension* of $M_\bullet$.⁴¹ So by definition, giving a chain map $\Sigma M_\bullet \rightarrow T_\bullet$ is equivalent to giving a nullhomotopy of the composite map $M_\bullet \rightarrow 0 \rightarrow T_\bullet$.

It is evident from the construction that suspension defines an autoequivalence

$$\mathbf{Ch}_R \xrightarrow{\Sigma} \mathbf{Ch}_R .$$

Namely,

$$\Sigma \left(\cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} \underline{\underline{M_0}} \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots \right) \cong \left(\cdots \xrightarrow{-d_1} M_0 \xrightarrow{-d_0} \underline{\underline{M_{-1}}} \xrightarrow{-d_{-1}} M_{-2} \xrightarrow{-d_{-2}} \cdots \right) :$$

the operation of suspension simply shifts all terms up by one and negates all differentials. We write Σ^{-1} for its inverse, which we refer to as *desuspension*. More generally, for any $k \in \mathbb{N}$ we write $\Sigma^k := \Sigma^{\circ k}$ and $\Sigma^{-k} := (\Sigma^{-1})^{\circ k}$. We also note for future reference the evident natural isomorphisms

$$H_n \circ \Sigma^k \cong H_{n-k}$$

for all $n, k \in \mathbb{Z}$.

Exercise 3.11 (6 points). Given a morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$, establish a canonical homotopy equivalence

$$\Sigma M \simeq \mathbf{hcoker}(N \xrightarrow{u} \mathbf{hcoker}(f)) .^{42}$$

3.3. Homotopy kernels.

³⁹The corresponding statement fails e.g. for spaces: there exist spaces X such that the morphism $X \rightarrow \mathbf{pt}$ is not a weak homotopy equivalence and yet $\Sigma X := \mathbf{hcoker}(X \rightarrow \mathbf{pt})$ is weakly contractible. (Such spaces are called *acyclic*, as they are characterized by having the integral homology of a point.)

⁴⁰Not coincidentally, this admits a natural isomorphism

$$\Sigma M_\bullet \cong M \otimes_R (R \rightarrow 0)$$

with the R -balanced tensor product of $M_\bullet$ and the R -valued reduced simplicial chains on the simplicial circle $\Delta^1/\partial\Delta^1$ (using the fact that the latter is actually a complex of R -bimodules).

⁴¹This is more often denoted $M_\bullet[1]$ and referred to as the *shift* of M , but we prefer the more blatantly topological notation and terminology.

⁴²That these are quasi-isomorphic follows from Exercise 3.9(a).

3.3.1. Dually, recall that the kernel of a morphism $M \xrightarrow{f} N$ in $\mathbf{Mod}_R$ is by definition an R -module $\ker(f) \in \mathbf{Mod}_R$ equipped with a morphism

$$\ker(f) \xrightarrow{v} M$$

satisfying the universal property that postcomposition with v determines a bijection

$$\mathrm{hom}_{\mathbf{Mod}_R}(T, \ker(f)) \xrightarrow{\cong} \left\{ \begin{array}{l} \text{morphisms } T \rightarrow M \text{ such that the} \\ \text{composite } T \rightarrow M \xrightarrow{f} N \text{ is zero} \end{array} \right\}. \quad .^{43}$$

This leads to the dual notion of a **homotopy kernel** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in $\mathbf{Ch}_R$: this is an object $\mathrm{hker}(f_\bullet) \in \mathbf{Ch}_R$ equipped with a morphism

$$\mathrm{hker}(f_\bullet) \xrightarrow{v_\bullet} M_\bullet$$

satisfying the universal property that postcomposition with $v_\bullet$ determines a bijection

$$\mathrm{hom}_{\mathbf{Ch}_R}(T_\bullet, \mathrm{hker}(f_\bullet)) \xrightarrow{\cong} \left\{ \begin{array}{l} \text{morphisms } T_\bullet \rightarrow M_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } T_\bullet \rightarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet \end{array} \right\}. \quad .^{44}$$

Exercise 3.12 (4 points). Given a morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$, prove that the evident levelwise projection map

$$\Sigma^{-1}\mathrm{hcoker}(f) \longrightarrow M$$

is a homotopy kernel of f .⁴⁵

By combining Exercises 3.9(a) and 3.12, it follows that if a morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$ is surjective in each dimension, then the canonical morphism

$$\ker(f) \longrightarrow \mathrm{hker}(f)$$

from the levelwise kernel is a quasi-isomorphism (though again it is not necessarily a homotopy equivalence).

⁴³Said differently, the kernel of f is the pullback

$$\begin{array}{ccc} \ker(f) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}.$$

⁴⁴Similarly, the homotopy kernel of f may be characterized as the *homotopy* pullback

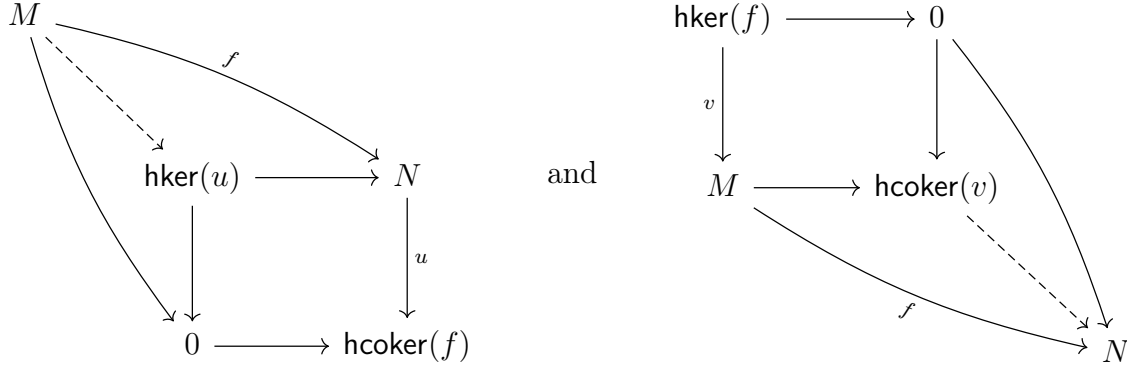
$$\begin{array}{ccc} \mathrm{hker}(f_\bullet) & \longrightarrow & 0 \\ \downarrow & \not\cong & \downarrow \\ M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \end{array},$$

i.e. the *terminal homotopy-coherent cone* over the diagram $M_\bullet \xrightarrow{f_\bullet} N_\bullet \leftarrow 0$.

⁴⁵There are unfortunately some signs that should arise here. They could be removed by tweaking the construction of hker (giving a different but isomorphic formula), but they would then arise elsewhere.

3.3.2. Although it will take some time to see why, the following feature is in some sense the *fundamental advantage* of working in the ∞ -category $\mathbf{D}_R$ instead of in the ordinary category $\mathbf{Mod}_R$.

Exercise 3.13 (6 points). Prove that for any morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$ the dashed canonical morphisms



are homotopy equivalences.⁴⁶

Namely, this implies that up to homotopy equivalence, every homotopy cokernel sequence is a homotopy kernel sequence, and conversely. This feature – along with the existence of a zero object and homotopy co/kernels in the first place – is succinctly expressed in the assertion that $\mathbf{D}_R$ is in fact a **stable** ∞ -category: its fundamental advantage over $\mathbf{Mod}_R$. By contrast, the corresponding assertion in $\mathbf{Mod}_R$ fails in a rather dramatic way: a cokernel sequence is a kernel sequence iff the original map is injective, and a kernel sequence is a cokernel sequence iff the original map is surjective.^{47,48}

3.4. Exact sequences.

3.4.1. A complex $M_\bullet \in \mathbf{Ch}_R$ is said to be **exact** at M_n if $H_n(M_\bullet) = 0$. In particular, an acyclic complex is also called an **exact sequence** in $\mathbf{Mod}_R$, or sometimes a **long exact sequence** to emphasize that it is (potentially) infinite in one or both directions.

As a special case, a **short exact sequence** in $\mathbf{Mod}_R$ is a three-term acyclic complex

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 .$$

So, a short exact sequence is such a diagram satisfying the conditions that f is injective, g is surjective, and $\ker(g) = \mathrm{im}(f)$. In this case, one may also say that M is an **extension** of

⁴⁶Note that these diagrams are only *homotopy-coherently* commutative (indeed, the canonical morphisms are induced by homotopy-coherent universal properties).

⁴⁷As $\mathbf{Mod}_R$ has no intrinsic notion of “homotopy”, the only reasonable definition of “homotopy co/kernel” therein simply reduces to the usual notion of co/kernel.

⁴⁸In fact, ordinary categories are almost never stable (when considered as ∞ -categories): an ordinary category is stable if and only if it is trivial (i.e. it has only a zero object (up to equivalence)).

N by L . For instance,

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$ -modules, which expresses $\mathbb{Z}/4$ as an extension of $\mathbb{Z}/2$ by itself.

More generally, we may refer to any (possibly finite) sequence of composable morphisms in $\mathbf{Mod}_R$ as an **exact sequence** if it is exact at all *interior* terms. So for example, one may refer to the diagram

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

as an exact sequence.⁴⁹

3.4.2. The following is the source of almost every single long exact sequence in mathematics.⁵⁰

Exercise 3.14 (2 points). For any morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$, show that the sequence

$$H_0(M) \xrightarrow{H_0(f)} H_0(N) \xrightarrow{H_0(u)} H_0(\mathrm{hcoker}(f))$$

is exact.

Namely, from Exercises 3.11, 3.12, and 3.13 we obtain an infinite sequence

$$\begin{array}{ccccccc} \dots & \xrightarrow{\Sigma^{-1}v} & \Sigma^{-1}M & \xrightarrow{\Sigma^{-1}f} & \Sigma^{-1}N & \xrightarrow{\Sigma^{-1}u} & \Sigma^{-1}\mathrm{hcoker}(f) \\ & & & & \wr & & \\ & & \mathrm{hker}(f) & \xrightarrow{v} & M & \xrightarrow{f} & N \xrightarrow{u} \mathrm{hcoker}(f) \\ & & & & & & \wr \\ & & & & & & \Sigma\mathrm{hker}(f) \xrightarrow{\Sigma v} \Sigma M \xrightarrow{\Sigma f} \Sigma N \xrightarrow{\Sigma u} \dots \end{array}$$

of morphisms in $\mathbf{Ch}_R$ in which every composable pair of morphisms is a homotopy co/kernel sequence up to homotopy equivalence.⁵¹ Thereafter, by Exercise 3.14, applying H_0 yields a long exact sequence

(3)

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_1(M) & \longrightarrow & H_1(N) & \longrightarrow & H_1(\mathrm{hcoker}(f)) \\ & & & & \wr & & \\ & & H_0(\mathrm{hker}(f)) & \longrightarrow & H_0(M) & \longrightarrow & H_0(N) \longrightarrow H_0(\mathrm{hcoker}(f)) \\ & & & & & & \wr \\ & & & & & & H_{-1}(\mathrm{hker}(f)) \longrightarrow H_{-1}(M) \longrightarrow H_{-1}(N) \longrightarrow \dots \end{array}$$

in $\mathbf{Mod}_R$.

⁴⁹Note that here we are *not* implicitly extending the sequence by zero.

⁵⁰A notable exception is the long exact sequence on (stable) homotopy groups, of which this is actually a special case.

⁵¹To be precise, here we are using the fact that homotopy co/kernels are invariant up to homotopy equivalence under homotopy equivalence (i.e. componentwise homotopy equivalence in $\mathbf{Fun}([1], \mathbf{Ch}_R)$).

and its differential

$$\prod_{i \in \mathbb{Z}} \text{hom}_{\text{Mod}_R}(M_i, N_{i+n}) \xrightarrow{d_n} \prod_{i \in \mathbb{Z}} \text{hom}_{\text{Mod}_R}(M_i, N_{i+n-1})$$

carries a tuple $\{f_i \in \text{hom}_{\text{Mod}_R}(M_i, N_{i+n})\}_{i \in \mathbb{Z}}$ to the tuple

$$\{(d_{i+n}^N \circ f_i + (-1)^{n-1} \cdot f_{i-1} \circ d_i^M) \in \text{hom}_{\text{Mod}_R}(M_i, N_{i+n-1})\}_{i \in \mathbb{Z}} .$$

Exercise 4.1 (2 points). Verify that this formula defines a complex.

Directly from the definition, we have that $Z_0(\underline{\text{hom}}_{\text{Ch}_R}(M, N)) \cong \text{hom}_{\text{Ch}_R}(M, N)$. Moreover, given any morphism $M \xrightarrow{f} N$ in Ch_R , a nullhomotopy $0 \xRightarrow{h} f$ is equivalent data to an element $h \in \underline{\text{hom}}_{\text{Ch}_R}(M, N)_1$ such that $d_1(h) = f$. So, $H_0(\underline{\text{hom}}_{\text{Ch}_R}(M, N))$ is canonically isomorphic to the abelian group of *homotopy classes* of morphisms $M \rightarrow N$. Note too that

$$\Sigma^i \underline{\text{hom}}_{\text{Ch}_R}(M, N) \cong \underline{\text{hom}}_{\text{Ch}_R}(\Sigma^{-i} M, N) \cong \underline{\text{hom}}_{\text{Ch}_R}(M, \Sigma^i N) ;$$

this gives an analogous description of all homology groups of the hom-complex, as $H_n \cong H_0 \circ \Sigma^{-n}$. On the other hand, these homology groups can also be understood in a more homotopical (although closely related) manner.

Exercise 4.2 (4 points). Given two chain maps $M \xrightarrow{f} N$ and $M \xrightarrow{g} N$ and two homotopies $f \xRightarrow{h_0} g$ and $f \xRightarrow{h_1} g$, define a notion of a homotopy $h_0 \Rightarrow h_1$ between homotopies, and prove that such a higher homotopy always exists precisely when $H_1(\underline{\text{hom}}_{\text{Ch}_R}(M, N)) = 0$.

If one views the complexes M and N as 0-cells, the maps $M \xrightarrow{0} N$ and $M \xrightarrow{f} N$ as 1-cells, and the homotopies h_i as 2-cells, then such a higher homotopy should be viewed as a 3-cell. Of course, there are analogs of this same interpretation for all the homology groups of $\underline{\text{hom}}_{\text{Ch}_R}(M, N)$.

For brevity, we may simply write

$$\underline{\text{hom}}(M, N) := \underline{\text{hom}}_{\text{Ch}_R}(M, N) .$$

4.2. The dg-category of complexes. These hom-complexes, in turn, can be naturally assembled into a single object.

Exercise 4.3 (8 points). Fix any complexes $L, M, N, O \in \text{Ch}_R$.

(a) Construct a map

$$\underline{\text{hom}}(M, N) \otimes \underline{\text{hom}}(L, M) \xrightarrow{\chi_{L, M, N}} \underline{\text{hom}}(L, N)$$

in $\text{Ch}_{\mathbb{k}}$ that encodes the composition of morphisms of complexes.⁵⁵

⁵⁵Note the ordering of the hom-complexes, which adheres to standard notation for composition. Of course, we could also consider this as a map $\underline{\text{hom}}(L, M) \otimes \underline{\text{hom}}(M, N) \rightarrow \underline{\text{hom}}(L, N)$, but note that this would introduce unnatural signs in the composition map. (Recall from §2.5 that the symmetry isomorphism of the symmetric monoidal category $\text{Ch}_{\mathbb{k}}$ follows the Koszul sign rule.)

(b) Verify that these composition morphisms are associative in the sense that the diagram

$$\begin{array}{ccc}
 \underline{\mathrm{hom}}(N, O) \otimes \underline{\mathrm{hom}}(M, N) \otimes \underline{\mathrm{hom}}(L, M) & \xrightarrow{\mathrm{id} \otimes \chi_{L, M, N}} & \underline{\mathrm{hom}}(N, O) \otimes \underline{\mathrm{hom}}(L, N) \\
 \downarrow \chi_{M, N, O} \otimes \mathrm{id} & & \downarrow \chi_{L, N, O} \\
 \underline{\mathrm{hom}}(M, O) \otimes \underline{\mathrm{hom}}(L, M) & \xrightarrow{\chi_{L, M, O}} & \underline{\mathrm{hom}}(L, O)
 \end{array}$$

commutes.⁵⁶

(c) Construct a map

$$\mathbb{k} \xrightarrow{\iota_M} \underline{\mathrm{hom}}(M, M)$$

using the identity morphism of $M \in \mathbf{Ch}_R$, and verify that it defines a two-sided identity for the above composition in the sense that the diagrams

$$\begin{array}{ccc}
 \mathbb{k} \otimes \underline{\mathrm{hom}}(L, M) & \xrightarrow{\iota_M \otimes \mathrm{id}} & \underline{\mathrm{hom}}(M, M) \otimes \underline{\mathrm{hom}}(L, M) \\
 & \searrow \cong & \downarrow \chi_{L, M, M} \\
 & & \underline{\mathrm{hom}}(L, M)
 \end{array}$$

and

$$\begin{array}{ccc}
 \underline{\mathrm{hom}}(M, N) \otimes \mathbb{k} & \xrightarrow{\mathrm{id} \otimes \iota_M} & \underline{\mathrm{hom}}(M, N) \otimes \underline{\mathrm{hom}}(M, M) \\
 & \searrow \cong & \downarrow \chi_{M, M, N} \\
 & & \underline{\mathrm{hom}}(M, N)
 \end{array}$$

commute (where the isomorphisms are the canonical ones coming from the fact that $\mathbb{k} \in \mathbf{Ch}_{\mathbb{k}}$ is the unit object).

Altogether, Exercise 4.3 yields a ($\mathbb{k}$ -*linear*) *dg-category*,⁵⁷ i.e. a category enriched in the symmetric monoidal category $(\mathbf{Ch}_{\mathbb{k}}, \otimes, \mathbb{k})$:⁵⁸ its objects are the chain complexes of R -modules and its hom-objects are the hom-complexes between them. We denote this dg-category by

⁵⁶This implicitly uses the associativity of the operation $\otimes$ in $\mathbf{Ch}_{\mathbb{k}}$.

⁵⁷Here, “dg” is short for “differential graded”. (In general, it is common to refer to a chain complex of R -modules as a *dg- R -module*.)

⁵⁸Of course, a dg-category is a particular instance of a more general notion. Namely, given a monoidal category $\mathcal{V} := (\mathcal{V}, \otimes_{\mathcal{V}}, \mathbb{1}_{\mathcal{V}})$, there is a natural notion of a *category enriched in $\mathcal{V}$* , or simply a *$\mathcal{V}$ -enriched category*: a $\mathcal{V}$ -enriched category $\mathcal{C}$ consists of a set of objects, the data of hom-objects $\underline{\mathrm{hom}}_{\mathcal{C}}(X, Y) \in \mathcal{V}$ for all $X, Y \in \mathcal{C}$, and the data of composition and identity morphisms

$$\underline{\mathrm{hom}}_{\mathcal{C}}(Y, Z) \otimes_{\mathcal{V}} \underline{\mathrm{hom}}_{\mathcal{C}}(X, Y) \xrightarrow{\chi_{X, Y, Z}} \underline{\mathrm{hom}}_{\mathcal{C}}(X, Z) \quad \text{and} \quad \mathbb{1}_{\mathcal{V}} \xrightarrow{\iota_Y} \underline{\mathrm{hom}}_{\mathcal{C}}(Y, Y)$$

in $\mathcal{V}$ for all $X, Y, Z \in \mathcal{C}$, subject to the evident associativity and unitality conditions. (As indicated, one sometimes uses an underline to emphasize that these are *enriched* hom-objects (as opposed to mere hom-sets).) On the other hand, note that the notation $\underline{\mathrm{hom}}_{\mathcal{C}}(X, Y)$ is already unambiguous, as $\mathcal{C}$ is a $\mathcal{V}$ -enriched category.

$\mathbf{K}_R$ and refer to it as *the dg-category of complexes of R -modules*.⁵⁹ In particular, we may also write $\mathrm{hom}_{\mathbf{K}_R}(M, N) := \underline{\mathrm{hom}}_{\mathrm{Ch}_R}(M, N)$; our chosen notation will depend on our desired emphasis.

The fundamental purpose of the dg-category $\mathbf{K}_R$ is to assemble the data of chain maps, homotopies, and higher (and lower) homotopies into a single object. In particular, it is the homology groups of the hom-complexes in $\mathbf{K}_R$ that are relevant. Therefore, it will be natural to view the hom-complexes in $\mathbf{K}_R$ as “only being important up to quasi-isomorphism”.

As a special case, note that for any complex $M \in \mathrm{Ch}_R$, composition in $\mathbf{K}_R$ makes the complex $\mathrm{hom}_{\mathbf{K}_R}(M, M) \in \mathrm{Ch}_{\mathbb{k}}$ into an *associative algebra object*, i.e. it is a *dg-algebra* (or *dga* for short) over $\mathbb{k}$.⁶⁰

Exercise 4.4 (4 points). Show that the following conditions are equivalent.

- (i) The complex $M \in \mathbf{K}_R$ is contractible.
- (ii) The complex $\mathrm{hom}_{\mathbf{K}_R}(M, M) \in \mathrm{Ch}_{\mathbb{k}}$ is acyclic.
- (iii) The complex $\mathrm{hom}_{\mathbf{K}_R}(M, M) \in \mathrm{Ch}_{\mathbb{k}}$ has that $H_0(\mathrm{hom}_{\mathbf{K}_R}(M, M)) = 0$.
- (iv) The element $[\mathrm{id}_M] \in H_0(\mathrm{hom}_{\mathbf{K}_R}(M, M))$ has that $[\mathrm{id}_M] = 0$.

Among the morphisms in the dg-category $\mathbf{K}_R$, the homotopy equivalences play a distinguished role.

Given a $\mathcal{V}$ -enriched category $\mathcal{C}$, we can obtain an ordinary category $\mathrm{fgt}(\mathcal{C})$ – its “underlying unenriched category” – by defining $\mathrm{fgt}(\mathcal{C})$ to have the same objects as $\mathcal{C}$ and defining

$$\mathrm{hom}_{\mathrm{fgt}(\mathcal{C})}(X, Y) := \mathrm{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \mathrm{hom}_{\mathcal{C}}(X, Y)) .$$

(Then, e.g. the identity element of $Y \in \mathrm{fgt}(\mathcal{C})$ is simply the element $\iota_Y \in \mathrm{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \mathrm{hom}_{\mathcal{C}}(Y, Y))$. More broadly and more formally, this construction yields a $(\mathrm{Set}, \times)$ -enriched category (e.g. with the expected composition maps as well as their associativity and unitality) because $\mathrm{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -)$ defines a *right-laxly monoidal* functor $(\mathcal{V}, \otimes_{\mathcal{V}}) \rightarrow (\mathrm{Set}, \times)$ (see e.g. ??).) So, a “ $\mathcal{V}$ -enrichment” of an ordinary category $\mathcal{D}$ may be defined to be a $\mathcal{V}$ -enriched category $\tilde{\mathcal{D}}$ as well as an identification $\mathcal{D} \cong \mathrm{fgt}(\tilde{\mathcal{D}})$.

It is worth acknowledging that the notion of a $\mathrm{Ch}_{\mathbb{k}}$ -enrichment of an ordinary category is somewhat more exotic than that of e.g. a $\mathrm{Mod}_{\mathbb{k}}$ - or Top -enrichment. On the one hand, the functors $\mathrm{hom}_{\mathrm{Mod}_{\mathbb{k}}}(\mathbb{k}, -)$ and $\mathrm{hom}_{\mathrm{Top}}(\mathrm{pt}, -)$ are simply the respective “underlying set” functors, so a $\mathrm{Mod}_{\mathbb{k}}$ - or Top -enrichment of an ordinary category amounts to endowing its hom-sets with the indicated structures (satisfying appropriate conditions, e.g. multilinearity or continuity respectively). By contrast, the functor $\mathrm{hom}_{\mathrm{Ch}_{\mathbb{k}}}(\mathbb{k}, -)$ can be identified as the functor $\mathrm{Ch}_{\mathbb{k}} \xrightarrow{Z_0} \mathrm{Mod}_{\mathbb{k}} \xrightarrow{\mathrm{fgt}} \mathrm{Set}$ of 0-cycles, so obtaining a $\mathrm{Ch}_{\mathbb{k}}$ -enrichment of an ordinary category involves defining hom-complexes whose 0-cycles are the original hom-sets. (Recall our trivial observation that $Z_0(\underline{\mathrm{hom}}_{\mathrm{Ch}_R}(M, N)) \cong \mathrm{hom}_{\mathrm{Ch}_R}(M, N)$.)

⁵⁹The German word for “complex” begins with the letter “K”.

⁶⁰Of course, this construct can be reversed. Namely, given a dg-algebra $A \in \mathrm{Alg}(\mathrm{Ch}_{\mathbb{k}})$, we can form a dg-category $\mathfrak{B}A \in \mathrm{cat}^{\mathrm{dg}}$ as follows: it has a single object $*$, and we declare that $\mathrm{hom}_{\mathfrak{B}A}(*, *) := A$ (with composition and identity respectively defined as the multiplication and unit in A).

Exercise 4.5 (4 points). Given a homotopy equivalence $M \rightarrow N$ in $\mathbf{K}_R$, show that the natural transformations

$$\mathrm{hom}_{\mathbf{K}_R}(-, M) \longrightarrow \mathrm{hom}_{\mathbf{K}_R}(-, N) \quad \text{and} \quad \mathrm{hom}_{\mathbf{K}_R}(M, -) \longleftarrow \mathrm{hom}_{\mathbf{K}_R}(N, -)$$

are natural homotopy equivalences.

On the other hand, the quasi-isomorphisms cannot play any such distinguished role in $\mathbf{K}_R$. For instance, given any acyclic but noncontractible complex $A \in \mathbf{K}_R$ (recall Exercise 3.6), by Exercise 4.4 the morphism

$$0 \cong \mathrm{hom}_{\mathbf{K}_R}(A, 0) \longrightarrow \mathrm{hom}_{\mathbf{K}_R}(A, A)$$

cannot be a quasi-isomorphism (let alone a homotopy equivalence).

4.3. Key features of the dg-category of complexes.

4.3.1. As we will discuss in §5.1, the basic operations of hom and tensor product among ordinary modules do *not* generally respect co/kernels. We will repair this failure in two steps. First, we will pass to the world of complexes, and from ordinary co/kernels to *homotopy* co/kernels; the latter are respected by hom and tensor product, as we will see in §4.3.3. However, these constructions fail to preserve quasi-isomorphisms. So, we will further correct them by studying *resolutions*, which are the topic of §5.

4.3.2. In order to prove that hom and tensor product among complexes preserve homotopy co/kernels, it will be convenient to use tensor-hom adjunctions among complexes, which we now establish. For any complex $T \in \mathbf{Ch}_{\mathbb{k}}$ of $\mathbb{k}$ -modules and any complexes $M, N \in \mathbf{Ch}_R$ of R -modules, we may form the complexes

$$T \otimes_{\mathbb{k}} M \quad \text{and} \quad \underline{\mathrm{hom}}_{\mathbf{Ch}_{\mathbb{k}}}(T, N)$$

of R -modules, where the (right) R -actions are induced from those on M and N . These satisfy the universal properties that

$$\mathrm{hom}_{\mathbf{Ch}_R}(T \otimes_{\mathbb{k}} M, N) \cong \mathrm{hom}_{\mathbf{Ch}_{\mathbb{k}}}(T, \underline{\mathrm{hom}}_{\mathbf{Ch}_R}(M, N)) \cong \mathrm{hom}_{\mathbf{Ch}_R}(M, \underline{\mathrm{hom}}_{\mathbf{Ch}_{\mathbb{k}}}(T, N)) .^{61}$$

In fact, these satisfy *enriched* universal properties.

Exercise 4.6 (6 points). Prove that for any $T \in \mathbf{Ch}_{\mathbb{k}}$ and any $M, N \in \mathbf{Ch}_R$ we have natural isomorphisms

$$\underline{\mathrm{hom}}_{\mathbf{Ch}_R}(T \otimes_{\mathbb{k}} M, N) \cong \underline{\mathrm{hom}}_{\mathbf{Ch}_{\mathbb{k}}}(T, \underline{\mathrm{hom}}_{\mathbf{Ch}_R}(M, N)) \cong \underline{\mathrm{hom}}_{\mathbf{Ch}_R}(M, \underline{\mathrm{hom}}_{\mathbf{Ch}_{\mathbb{k}}}(T, N))$$

in $\mathbf{Ch}_{\mathbb{k}}$.

Of course, these generalize the natural isomorphisms (1) in the context of ordinary modules (instead of chain complexes of modules).

⁶¹In the general context of enriched category theory, one says that $T \otimes M$ is the *tensoring* of M by T and that $\underline{\mathrm{hom}}_{\mathbf{Ch}_{\mathbb{k}}}(T, N)$ is the *cotensoring* of N by T .

4.3.3. We can now show that hom and tensor product among complexes preserve homotopy co/kernels.

Exercise 4.7 (8 points). Choose any complexes $M, N, T \in \mathbf{Ch}_R$ and any morphism $M \xrightarrow{f} N$. Using Exercise 4.6 and the universal property of the homotopy co/kernel, construct natural isomorphisms

$$\underline{\mathrm{hom}}_{\mathbf{Ch}_R} \left(T, \mathrm{hker} \left(M \xrightarrow{f} N \right) \right) \cong \mathrm{hker} \left(\underline{\mathrm{hom}}_{\mathbf{Ch}_R}(T, M) \xrightarrow{\underline{\mathrm{hom}}_{\mathbf{Ch}_R}(T, f)} \underline{\mathrm{hom}}_{\mathbf{Ch}_R}(T, N) \right)$$

and

$$\underline{\mathrm{hom}}_{\mathbf{Ch}_R} \left(\mathrm{hcoker} \left(M \xrightarrow{f} N \right), T \right) \cong \mathrm{hker} \left(\underline{\mathrm{hom}}_{\mathbf{Ch}_R}(N, T) \xrightarrow{\underline{\mathrm{hom}}_{\mathbf{Ch}_R}(f, T)} \underline{\mathrm{hom}}_{\mathbf{Ch}_R}(M, T) \right)$$

in $\mathbf{Ch}_k$.

Of course, essentially identical reasoning shows that the hom-complex bifunctor

$$\mathbf{Ch}_k^{\mathrm{op}} \times \mathbf{Ch}_R \xrightarrow{\underline{\mathrm{hom}}_{\mathbf{Ch}_k}(-, -)} \mathbf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.⁶²

Exercise 4.8 (4 points). Show that the tensor product bifunctor

$$\mathbf{Ch}_k \times \mathbf{Ch}_R \xrightarrow{(-) \otimes_k (-)} \mathbf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.

5. PROJECTIVE AND INJECTIVE RESOLUTIONS

5.1. **Motivation for resolutions.** The original motivation for homological algebra is the fact that many natural functors on ordinary modules do not preserve exact sequences. Equivalently but more fundamentally, they do not preserve short exact sequences, i.e. they do not respect both kernels and cokernels.⁶³

Exercise 5.1 (6 points).

⁶²Alternatively, this follows from Exercise 4.7, the commutative square

$$\begin{array}{ccc} \mathbf{Ch}_k^{\mathrm{op}} \times \mathbf{Ch}_R & \xrightarrow{\underline{\mathrm{hom}}_{\mathbf{Ch}_k}(-, -)} & \mathbf{Ch}_R \\ \mathrm{id} \times \mathrm{fgt} \downarrow & & \downarrow \mathrm{fgt} \\ \mathbf{Ch}_k^{\mathrm{op}} \times \mathbf{Ch}_k & \xrightarrow{\underline{\mathrm{hom}}_{\mathbf{Ch}_k}(-, -)} & \mathbf{Ch}_k \end{array} ,$$

and the fact that the forgetful functor both preserves and detects homotopy kernel sequences.

⁶³Here we use the word “respect” instead of “preserve” due to the contravariance of $\underline{\mathrm{hom}}_{\mathbf{Mod}_R}(-, M)$: recognizing that $\mathbf{Mod}_R^{\mathrm{op}}$ is also an abelian category, one might hope that this would carry kernels to cokernels and cokernels to kernels.

(a) Show that the functor

$$\mathbf{Mod}_{\mathbb{k}} \times \mathbf{Mod}_R \xrightarrow{(-) \otimes_{\mathbb{k}} (-)} \mathbf{Mod}_R$$

does not generally preserve exact sequences in either variable.

(b) Show that the functor

$$\mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Mod}_R \xrightarrow{\mathrm{hom}_{\mathbf{Mod}_R}(-, -)} \mathbf{Mod}_R$$

does not generally preserve exact sequences in either variable.

Namely, applying either bifunctor appearing in Exercise 5.1 to an exact sequence in one of its slots, one obtains a “half-exact” sequence: $(-) \otimes_{\mathbb{k}} (-)$ preserves cokernels separately in each variable, $\mathrm{hom}_{\mathbf{Mod}_R}(M, -)$ preserves kernels, and $\mathrm{hom}_{\mathbf{Mod}_R}(-, M)$ carries cokernels to kernels. It was originally desired for these half-exact sequences to extend to long exact sequences, whose additional terms would quantify these various failures of exactness.

As illustrated by Exercises 4.7 and 4.8, towards resolving these issues it is fruitful to pass from ordinary modules and ordinary co/kernels to complexes of modules and homotopy co/kernels; the desired long exact sequences would then be those on homology discussed in §3.4, although of course we will take the perspective that the homotopy co/kernel sequences themselves are the more fundamental objects. However, given that we would like to consider quasi-isomorphisms as isomorphisms, the following results show that this maneuver does not suffice on its own.

Exercise 5.2 (6 points).

(a) Show that the functor

$$\mathbf{Ch}_{\mathbb{k}} \times \mathbf{Ch}_R \xrightarrow{(-) \otimes_{\mathbb{k}} (-)} \mathbf{Ch}_R$$

does not generally preserve acyclic objects in either variable.

(b) Show that the functor

$$\mathbf{Ch}_R^{\mathrm{op}} \times \mathbf{Ch}_R \xrightarrow{\mathrm{hom}_{\mathbf{Ch}_R}(-, -)} \mathbf{Ch}_{\mathbb{k}}$$

does not generally preserve acyclic objects in either variable.

Clearly, preservation of acyclics is necessary for the preservation of quasi-isomorphisms. But in fact, the converse is guaranteed by Exercise 3.10(a) (combined with Exercises 4.7 and 4.8). This motivates the notions that we introduce now.

5.2. Projective and injective complexes.

5.2.1. We write $\mathbf{A}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the acyclic complexes.

We say that a complex $P \in \mathbf{K}_R$ is **projective** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathrm{hom}_{\mathbf{K}_R}(P, A) \in \mathbf{Ch}_k$ is acyclic. Because acyclic complexes are preserved under de/suspensions and $\Sigma^k \mathrm{hom}_{\mathbf{K}_R}(P, A) \cong \mathrm{hom}_{\mathbf{K}_R}(P, \Sigma^{-k} A)$, we may equivalently demand simply that $H_0(\mathrm{hom}_{\mathbf{K}_R}(P, A)) = 0$ for every acyclic complex $A \in \mathbf{A}_R$. In other words, P is projective iff every morphism $P \rightarrow A$ to an acyclic complex admits a nullhomotopy. In turn, this is equivalent to the condition that for every acyclic complex $A \in \mathbf{A}_R$ and every solid diagram

$$\begin{array}{ccc} & & \mathrm{hker}(\mathrm{id}_A) \\ & \nearrow \text{dashed} & \downarrow v \\ P & \longrightarrow & A \end{array}$$

there exists a lift making the diagram commute. We write $\mathbf{P}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the projective complexes.⁶⁴

Dually, we say that a complex $I \in \mathbf{K}_R$ is **injective** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathrm{hom}_{\mathbf{K}_R}(A, I) \in \mathbf{Ch}_k$ is acyclic. Likewise, we may equivalently demand for all acyclic complexes $A \in \mathbf{A}_R$ that $H_0(\mathrm{hom}_{\mathbf{K}_R}(A, I)) = 0$, or that every morphism $A \rightarrow I$ admits a nullhomotopy, or that for every solid diagram

$$\begin{array}{ccc} A & \longrightarrow & I \\ \downarrow u & \searrow \text{dashed} & \\ \mathrm{hcoker}(\mathrm{id}_A) & & \end{array}$$

there exists an extension making the diagram commute. We write $\mathbf{I}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the injective complexes.

Given that our primary interest is in quasi-isomorphism classes of complexes, we would like to consider the acyclic complexes as “negligible”. So, recalling that we only consider hom-complexes “up to quasi-isomorphism”, one may view the condition of a complex being projective (resp. injective) as being a necessary condition in order for it to be “good” for mapping out of (resp. into). We will eventually see that these necessary conditions are also sufficient, in a precise sense.

It is easy to deduce the following facts.

Exercise 5.3 (8 points).

⁶⁴We observe for future reference that v is a levelwise surjective quasi-isomorphism: indeed, $\mathrm{hker}(\mathrm{id}_A)$ is acyclic by the long exact sequence in homology, and v is surjective by construction (or by its defining universal property). So, in order for a complex to be projective, it suffices for it to have the analogous lifting property with respect to *all* levelwise surjective quasi-isomorphisms.

- (a) Show that all quasi-isomorphisms between projective (resp. injective) complexes are homotopy equivalences.⁶⁵
- (b) Show that the property of projectivity (resp. injectivity) is stable under homotopy equivalence.
- (c) Show that projective complexes are preserved under tensor product: if $P \in \mathbf{P}_k$ and $Q \in \mathbf{P}_R$ then $P \otimes Q \in \mathbf{P}_R$.
- (d) Show that for any projective complexes $P \in \mathbf{P}_k$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{\text{hom}_{\mathbf{K}_k}(P, -)} \mathbf{K}_R \quad \text{and} \quad \mathbf{K}_R \xrightarrow{\text{hom}_{\mathbf{K}_R}(Q, -)} \mathbf{K}_k$$

preserve quasi-isomorphisms.

- (e) Show that for any injective complex $I \in \mathbf{I}_R$, the functors

$$\mathbf{K}_k^{\text{op}} \xrightarrow{\text{hom}_{\mathbf{K}_k}(-, I)} \mathbf{K}_R \quad \text{and} \quad \mathbf{K}_R^{\text{op}} \xrightarrow{\text{hom}_{\mathbf{K}_R}(-, I)} \mathbf{K}_k$$

preserve quasi-isomorphisms.

Clearly the zero complex is both projective and injective. Hence, (a) implies that if an acyclic complex is either projective or injective then it must be contractible, while (b) implies that every contractible complex is both projective and injective. In other words, the intersections

$$\mathbf{P}_R \cap \mathbf{A}_R \quad \text{and} \quad \mathbf{I}_R \cap \mathbf{A}_R$$

coincide and consist of the contractible complexes.

5.2.2. Of course, the definitions of projective and injective complexes on their own are not so useful. What gives them their power is that every complex $M \in \mathbf{K}_R$ admits both a **projective resolution** $P \xrightarrow{\sim} M$ and an **injective resolution** $M \xrightarrow{\sim} I$ (i.e. quasi-isomorphisms as indicated). These are the promised representatives of the quasi-isomorphism class of M that are “well-adapted” for certain purposes. Specifically, we will respectively view the functors

$$(-) \otimes_k P, \quad \text{hom}_{\mathbf{K}_R}(P, -) \quad \text{and} \quad \text{hom}_{\mathbf{K}_R}(-, I)$$

as “corrected” (a.k.a. “derived”) versions of the functors

$$(-) \otimes_k M, \quad \text{hom}_{\mathbf{K}_R}(M, -) \quad \text{and} \quad \text{hom}_{\mathbf{K}_R}(-, M)$$

(and similarly for projective resolutions of complexes of k -modules).

As we will see, it is relatively straightforward to construct projective resolutions of *bounded-below* complexes (i.e. $M \in \mathbf{Ch}_R$ such that $M_n = 0$ for all $n \ll 0$) and to construct injective resolutions of *bounded-above* complexes (i.e. $M \in \mathbf{Ch}_R$ such that $M_n = 0$ for all $n \gg 0$).

⁶⁵The statement for projective complexes is formally analogous to *Whitehead’s theorem*, which states that a weak homotopy equivalence between cell complexes (or retracts thereof) is necessarily a homotopy equivalence.

Note in particular that these constructions apply to ordinary R -modules via the inclusion $\text{Mod}_R \subseteq \text{Ch}_R$.

As for the unbounded situation, it turns out to be much easier (and more conceptually satisfying) to construct projective resolutions than injective resolutions. On the other hand, we will not need to know anything specific about injective resolutions of unbounded complexes, beyond their existence. Therefore, we will discuss only projective resolutions in the unbounded case, and refer the reader to [Spa88] (or to [Hov99, §2.3]) for the construction of injective resolutions in the unbounded case.

It is easy to deduce the following facts directly from the existence of projective and injective resolutions.

Exercise 5.4 (6 points).

- (a) Show that if the complex $M \in \mathbf{K}_R$ has that $\text{hom}_{\mathbf{K}_R}(P, M)$ is acyclic for every projective complex $P \in \mathbf{P}_R$, then M is acyclic.⁶⁶
- (b) Show that if the complex $M \in \mathbf{K}_R$ has that $\text{hom}_{\mathbf{K}_R}(M, I)$ is acyclic for every injective complex $I \in \mathbf{I}_R$, then M is acyclic.
- (c) Show that for any projective complexes $P \in \mathbf{P}_k$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{P \otimes_k (-)} \mathbf{K}_R \quad \text{and} \quad \mathbf{K}_k \xrightarrow{(-) \otimes_k Q} \mathbf{K}_R$$

preserve quasi-isomorphisms.

In particular, the tensor product of a projective complex and an acyclic complex is acyclic.

We will eventually organize many of the facts enumerated in Exercises 5.3 and 5.4 in a more systematic way.

5.3. Projective resolutions in the bounded-below case. Recall that an R -module $P \in \text{Mod}_R$ is called **projective** if mapping out of it preserves surjections, i.e. if for any surjection $M \twoheadrightarrow N$ in Mod_R the induced map $\text{hom}_{\text{Mod}_R}(P, M) \rightarrow \text{hom}_{\text{Mod}_R}(P, N)$ is a surjection. Said differently, given any solid diagram

$$\begin{array}{ccc} & & M \\ & \nearrow \text{dashed} & \downarrow \\ P & \longrightarrow & N \end{array}$$

there exists a dashed lift making the diagram commute.

Exercise 5.5 (2 points). Show that an R -module $P \in \text{Mod}_R$ is projective iff it is a summand of a free module (i.e. there exists an R -module $Q \in \text{Mod}_R$ and an isomorphism $P \oplus Q \cong R^{\oplus S}$ with the free R -module on some set S).

Exercise 5.6 (4 points). Show that every projective $\mathbb{Z}$ -module is free.

⁶⁶Evidently, it is already sufficient just to take $P = R$ (which is projective by Exercises 5.5 and 5.9).

Exercise 5.7 (4 points). Give necessary and sufficient conditions on $n \in \mathbb{N}$ such that every projective $\mathbb{Z}/n$ -module is free.

Exercise 5.8 (4 points). Show that an R -module $M \in \mathbf{Mod}_R$ is projective iff the corresponding complex $\underline{M} \in \mathbf{Ch}_R$ is projective.

We now turn from projective modules back to projective complexes.

Exercise 5.9 (6 points). Show that if $P \in \mathbf{K}_R$ is a bounded-below complex such that each $P_n \in \mathbf{Mod}_R$ is projective, then $P \in \mathbf{P}_R$ is projective.

The bounded-below hypothesis in Exercise 5.9 is necessary: both parts of Exercise 3.6 give examples of unbounded complexes which are levelwise free (hence levelwise projective by Exercise 5.5) that cannot be projective. Indeed, an acyclic projective complex must be contractible, as its identity map must admit a nullhomotopy.

However, there is no converse statement.

Exercise 5.10 (2 points). Find a projective complex that is not levelwise projective.⁶⁷

Having studied the notions of projectivity for modules and complexes, we now turn to the existence of projective resolutions in the bounded-below case. For this, let us note that every R -module admits a surjection from a projective (in fact free) R -module; one says that the abelian category $\mathbf{Mod}_R$ *has enough projectives*.

Exercise 5.11 (6 points). Using the fact that the abelian category $\mathbf{Mod}_R$ has enough projectives (along with Exercise 5.9), construct a projective resolution $P \xrightarrow{\sim} M$ of any bounded-below complex $M \in \mathbf{K}_R$.

5.4. Injective resolutions in the bounded-above case. Recall that an R -module $I \in \mathbf{Mod}_R$ is called *injective* if mapping into it carries injections to surjections, i.e. if for any injection $M \hookrightarrow N$ in $\mathbf{Mod}_R$ the induced map $\mathrm{hom}_{\mathbf{Mod}_R}(N, I) \rightarrow \mathrm{hom}_{\mathbf{Mod}_R}(M, I)$ is surjective. Said differently, given any solid diagram

$$\begin{array}{ccc} M & \longrightarrow & I \\ \downarrow & \nearrow \text{dashed} & \\ N & & \end{array}$$

there exists a dashed extension making the diagram commute.

Injective modules are much more bizarre than projective modules.

Exercise 5.12 (8 points). Recall that an R -module $M \in \mathbf{Mod}_R$ is *divisible* if for every nonzero element $r \in R$ the map $M \xrightarrow{r} M$ is surjective.

(a) Show that if R is an integral domain, then all injective R -modules are divisible.

⁶⁷This can be taken to be bounded-below.

- (b) Show that if R is in fact a principal ideal domain, then all divisible R -modules are injective.

So for instance, $\mathbb{Q} \in \mathbf{Mod}_{\mathbb{Z}}$ is injective while $\mathbb{Z} \in \mathbf{Mod}_{\mathbb{Z}}$ is not.

It is easy to dualize the argument of Exercise 5.9: bounded-above levelwise injective complexes are injective. Using this, we would also like to dualize the argument of Exercise 5.11 to show that any bounded-above complex $M \in \mathbf{K}_R$ admits an injective resolution $M \xrightarrow{\sim} I$. This too is easy, once one knows that $\mathbf{Mod}_R$ also *has enough injectives*, i.e. that any R -module admits an injection into an injective R -module. This relatively subtle assertion can be established as follows.

Exercise 5.13 (12 points). Fix a ring homomorphism $S \rightarrow R$.

- (a) Verify that the functors

$$\begin{array}{ccc} & (-) \otimes_S R & \\ \text{Mod}_S & \begin{array}{c} \xleftarrow{\quad} \text{fgt} \xrightarrow{\quad} \\ \perp \quad \perp \end{array} & \text{Mod}_R \\ & \text{hom}_{\text{Mod}_S}(R, -) & \end{array}$$

participate in adjunctions as indicated.⁶⁸

- (b) Given modules $M \in \mathbf{Mod}_R$ and $N \in \mathbf{Mod}_S$ and an injection

$$\text{fgt}(M) \hookrightarrow N$$

in $\mathbf{Mod}_S$, show that the corresponding morphism

$$M \longrightarrow \text{hom}_{\text{Mod}_S}(R, N)$$

in $\mathbf{Mod}_R$ is also an injection.

- (c) Deduce from the fact that fgt preserves injections that the functor $\text{hom}_{\text{Mod}_S}(R, -)$ preserves injective objects.⁶⁹
- (d) Show that injective objects are preserved under products.⁷⁰
- (e) Fix an abelian group $A \in \mathbf{Ab}$. Show that for every $a \in A$, there exists a homomorphism $A \rightarrow \mathbb{Q}/\mathbb{Z}$ carrying a to a nonzero element.⁷¹ Deduce that there exists an injection

$$A \hookrightarrow \prod_A \mathbb{Q}/\mathbb{Z} .^{72}$$

⁶⁸In defining the functor $(-) \otimes_S R$ (resp. $\text{hom}_{\text{Mod}_S}(R, -)$), we use that R is an (S, R) -bimodule (resp. an (R, S) -bimodule).

⁶⁹Dually, the functor $(-) \otimes_S R$ preserves projective objects because the functor fgt also preserves surjections.

⁷⁰Dually, projective objects are preserved under coproducts.

⁷¹This uses Exercise 5.12.

⁷²One says that $\mathbb{Q}/\mathbb{Z} \in \mathbf{Ab}$ is an *injective cogenerator*. Dually, $R \in \mathbf{Mod}_R$ is a *projective generator*: it is projective, and moreover for any $M \in \mathbf{Mod}_R$ there exists a surjection $R^{\oplus S} \twoheadrightarrow M$ for some set S .

Namely, suppose we are given an R -module $M \in \mathbf{Mod}_R$. By (e) we have an injection

$$\mathrm{fgt}(M) \hookrightarrow \prod_M \mathbb{Q}/\mathbb{Z} =: I$$

in $\mathbf{Ab}$, and by (d) and Exercise 5.12 we see that $I \in \mathbf{Ab}$ is injective. Thereafter, by (b) we obtain an injection

$$M \hookrightarrow \mathrm{hom}_{\mathbf{Ab}}(R, I) =: J$$

in $\mathbf{Mod}_R$, and by (c) we see that $J \in \mathbf{Mod}_R$ is injective. So indeed, $\mathbf{Mod}_R$ has enough injectives.

5.5. Cell complexes and lifting criteria. For the purpose of constructing projective resolutions, we first introduce some auxiliary ideas.

For any $n \in \mathbb{Z}$, we define the objects and morphism

$$\begin{array}{ccc} S^n & & 0 \longrightarrow R \\ i_n \downarrow & := & \downarrow \quad \quad \downarrow \mathrm{id}_R \\ D^{n+1} & & R \xrightarrow{\mathrm{id}_R} R \end{array}$$

in $\mathbf{Ch}_R$, where the columns are in dimensions $n+1$ and n .⁷³ For any complex $M \in \mathbf{Ch}_R$, we have natural isomorphisms

$$\mathrm{hom}_{\mathbf{Ch}_R}(S^n, M) \cong Z_n(M) \quad \text{and} \quad \mathrm{hom}_{\mathbf{Ch}_R}(D^{n+1}, M) \cong M_{n+1} ;$$

precomposition with i_n induces the morphism

$$Z_n(M) \xleftarrow{d_{n+1}} M_{n+1} . \quad ^{74}$$

We say that a morphism $M \xrightarrow{f} N$ has the **right lifting property** with respect to i_n (or equivalently that i_n has the **left lifting property** with respect to f) if for every solid commutative diagram

$$\begin{array}{ccc} S^n & \longrightarrow & M \\ i_n \downarrow & \nearrow & \downarrow f \\ D^{n+1} & \longrightarrow & N \end{array}$$

⁷³Of course, this morphism will function as “the inclusion of the n -sphere into the $(n+1)$ -disk”: indeed, for any $n \geq 0$ it is the reduced simplicial chains on the inclusion $\Delta^{\{0, \dots, n\}} / \partial \Delta^{\{0, \dots, n\}} \hookrightarrow \Delta^n / \Lambda_0^n$ (which is a particularly small simplicial model for this morphism). Note too that $S^n \cong \Sigma^n S^0$ and $D^n \cong \Sigma^n D^0$. (While the former isomorphism is always an equality (so that we may take $\Sigma^n S^0$ to be the definition of S^n), the latter introduces an inconvenient sign when n is odd (which is why we do not take $\Sigma^n D^0$ to be the definition of D^n).)

⁷⁴Categorically speaking, the reason that the morphism

$$\mathrm{hom}_{\mathbf{Ch}_R}(S^n, -) \xleftarrow{(-) \circ i_n} \mathrm{hom}_{\mathbf{Ch}_R}(D^{n+1}, -)$$

in $\mathbf{Fun}(\mathbf{Ch}_R, \mathbf{Mod}_{\mathbb{k}})$ lifts to $\mathbf{Fun}(\mathbf{Ch}_R, \mathbf{Mod}_R)$ is that the morphism i_n is in fact a morphism of complexes of (R, R) -bimodules.

there exists a dashed lift that makes the diagram commute. This is equivalent to saying that the canonical dashed morphism in the pullback

$$\begin{array}{ccc}
 M_{n+1} & \xrightarrow{d_{n+1}^M} & Z_n(M) \\
 \searrow \scriptstyle f_{n+1} & \dashrightarrow & \downarrow \scriptstyle Z_n(f) \\
 & N_{n+1} \times_{Z_n(N)} Z_n(M) & \longrightarrow Z_n(M) \\
 & \downarrow & \\
 & N_{n+1} & \xrightarrow{d_{n+1}^N} Z_n(N)
 \end{array}$$

is surjective. We may abbreviate this by writing $f \in \mathbf{rlp}(i_n)$ (or by writing $i_n \in \mathbf{lp}(f)$). We also introduce the notation $I := \{i_n\}_{n \in \mathbb{Z}}$ and $\mathbf{rlp}(I) := \bigcap_{n \in \mathbb{Z}} \mathbf{rlp}(i_n)$.

Exercise 5.14 (4 points). Prove that if $f \in \mathbf{rlp}(i_n)$ then $H_n(f)$ is injective and $Z_{n+1}(f)$ is surjective.

Now, suppose that $f \in \mathbf{rlp}(I)$. We claim this implies that f is a levelwise surjective quasi-isomorphism. First of all, Exercise 5.14 immediately implies that f is a quasi-isomorphism. Moreover, Exercise 5.14 implies that $Z_n(f)$ is surjective (using that $f \in \mathbf{rlp}(i_{n-1})$). From here, the above reformulation of $\mathbf{rlp}(i_n)$ implies that f_{n+1} is surjective. So altogether, it follows that f is levelwise surjective.

In fact, the converse is also true.

Exercise 5.15 (4 points). Prove that if $M \xrightarrow{f} N$ is a levelwise surjective quasi-isomorphism, then $f \in \mathbf{rlp}(I)$.

In other words, $\mathbf{rlp}(I)$ consists precisely of the levelwise surjective quasi-isomorphisms.

5.6. Projective resolutions as cellular approximations.

5.6.1. We now construct projective resolutions via a general procedure known as the *small object argument*.⁷⁵ In fact, given a morphism $M \xrightarrow{f} N$ in $\mathbf{Ch}_R$, we will construct a factorization

$$\begin{array}{ccc}
 M & \xrightarrow{f} & N \\
 \searrow \scriptstyle c^{(\infty)} & & \nearrow \scriptstyle f^{(\infty)} \\
 & N^{(\infty)} &
 \end{array}$$

satisfying the following conditions:

⁷⁵This is formally analogous to the construction of a cell complex that is weak homotopy equivalent to a given topological space. The name arises from its crucial use of a certain “smallness” property of the sources of the elements of the set I . In the present situation, the relevant fact is given as Exercise 5.18.

- (1) the morphism $c^{(\infty)}$ has the left lifting property with respect to levelwise surjective quasi-isomorphisms (i.e. $c^{(\infty)} \in \mathbf{llp}(\mathbf{rlp}(I))$), and
- (2) the morphism $f^{(\infty)}$ is a quasi-isomorphism.

Taking $M \xrightarrow{f} N$ to be the map $0 \rightarrow N$ then yields a projective resolution of N : the projectivity of $N^{(\infty)}$ in this case follows from Footnote 64.⁷⁶

5.6.2. We begin some auxiliary notation. First of all, we write

$$I(f)_n := \mathbf{hom}_{\mathbf{Ar}(\mathbf{Ch}_R)}(i_n, f) \cong \mathbf{hom}_{\mathbf{Ch}_R}(S^n, M) \times_{\mathbf{hom}_{\mathbf{Ch}_R}(S^n, N)} \mathbf{hom}_{\mathbf{Ch}_R}(D^{n+1}, N)$$

for the set of morphisms

$$\begin{array}{ccc} S^n & \dashrightarrow & M \\ i_n \downarrow & & \downarrow f \\ D^{n+1} & \dashrightarrow & N \end{array}$$

in the arrow category $\mathbf{Ar}(\mathbf{Ch}_R) := \mathbf{Fun}([1], \mathbf{Ch}_R)$ of $\mathbf{Ch}_R$ (i.e. pairs of morphisms $S^n \rightarrow M$ and $D^{n+1} \rightarrow N$ making the square commute). Moreover, we write

$$I(f) := \bigsqcup_{n \in \mathbb{Z}} I(f)_n$$

for the disjoint union of these sets indexed over all $n \in \mathbb{Z}$. And finally, given an element $\alpha \in I(f)$, we write $n_\alpha := n$ if $\alpha \in I(f)_n$. (That is, we write $I(f) \xrightarrow{n(-)} \mathbb{Z}$ for the evident function from the disjoint union.)

5.6.3. We now construct the factorization $f = f^{(\infty)} \circ c^{(\infty)}$.

First of all, observe that we have a canonical commutative square

$$\begin{array}{ccc} \bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \longrightarrow & M \\ \bigoplus_{\alpha \in I(f)} i_{n_\alpha} \downarrow & & \downarrow f \\ \bigoplus_{\alpha \in I(f)} D^{n_\alpha+1} & \longrightarrow & N \end{array} .$$

We write

$$N^{(1)} := \mathbf{colim} \left(\begin{array}{ccc} \bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \longrightarrow & M \\ \bigoplus_{\alpha \in I(f)} i_{n_\alpha} \downarrow & & \downarrow f \\ \bigoplus_{\alpha \in I(f)} D^{n_\alpha+1} & \longrightarrow & N \end{array} \right)$$

⁷⁶One may interpret the general case as providing a “relatively projective” resolution of N as an object under M . (Whereas a projective complex is like a cell complex (or a retract thereof), a relatively projective complex is like a relative cell complex (or a retract thereof).)

for the pushout, and we write

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow c^{(I)} & \nearrow f^{(1)} \\ & N^{(1)} & \end{array}$$

for the morphisms in the induced factorization.

Exercise 5.16 (2 points). Show that the morphism $c^{(1)}$ has the left lifting property with respect to any levelwise surjective quasi-isomorphism (i.e. $c^{(1)} \in \mathbf{llp}(\mathbf{rlp}(I))$).

Exercise 5.17 (2 points). Show that for every solid commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\quad} & M \\ \downarrow i_n & & \searrow c^{(1)} \\ & & N^{(1)} \\ & \nearrow f^{(1)} & \downarrow f \\ D^{n+1} & \xrightarrow{\quad} & N \end{array}$$

there exists a dashed factorization making the diagram commute.

We now apply the above construction again, but to $f^{(1)}$ instead of to f : namely, we observe the canonical commutative square

$$\begin{array}{ccc} \bigoplus_{\alpha \in I(f^{(1)})} S^{n_\alpha} & \longrightarrow & N^{(1)} \\ \bigoplus_{\alpha \in I(f^{(1)})} i_{n_\alpha} \downarrow & & \downarrow f^{(1)} \\ \bigoplus_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} & \longrightarrow & N \end{array},$$

we write

$$N^{(2)} := \operatorname{colim} \left(\begin{array}{ccc} \bigoplus_{\alpha \in I(f^{(1)})} S^{n_\alpha} & \longrightarrow & N^{(1)} \\ \bigoplus_{\alpha \in I(f^{(1)})} i_{n_\alpha} \downarrow & & \downarrow f^{(1)} \\ \bigoplus_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} & \longrightarrow & N \end{array} \right)$$

for the pushout, and we write

$$\begin{array}{ccc} N^{(1)} & \xrightarrow{f^{(1)}} & N \\ & \searrow c^{(2)} \quad \nearrow f^{(2)} & \\ & N^{(2)} & \end{array}$$

for the morphisms in the induced factorization. This gives us a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow c^{(1)} & \nearrow f^{(1)} \\ & N^{(1)} & \\ & \searrow c^{(2)} & \nearrow f^{(2)} \\ & N^{(2)} & \end{array} .$$

Of course, we continue to iterate this construction in the obvious way. To conclude, we define

$$N^{(\infty)} := \operatorname{colim} \left(M \xrightarrow{c^{(1)}} N^{(1)} \xrightarrow{c^{(2)}} N^{(2)} \xrightarrow{c^{(3)}} \dots \right) ,$$

we define the map $M \xrightarrow{c^{(\infty)}} N^{(\infty)}$ to be the canonical map to the colimit, and we define the map $N^{(\infty)} \xrightarrow{f^{(\infty)}} N$ to be that induced from the maps $f^{(n)}$ by the universal property of the colimit.

5.6.4. We now verify the two conditions given in §5.6.1.

- (1) Let $B \xrightarrow{g} C$ be a levelwise surjective quasi-isomorphism (i.e. an element of $\mathbf{rlp}(I)$). Given any commutative square

$$\begin{array}{ccc} M & \longrightarrow & B \\ c^{(\infty)} \downarrow & & \downarrow g \\ N^{(\infty)} & \longrightarrow & C \end{array} ,$$

we enlarge it to the solid commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & B \\ c^{(1)} \downarrow & \nearrow & \downarrow \\ N^{(1)} & & \\ c^{(2)} \downarrow & \nearrow & \downarrow \\ N^{(2)} & & \\ c^{(3)} \downarrow & & \downarrow \\ \vdots & & \downarrow \\ N^{(\infty)} & \longrightarrow & C \end{array} .$$

Here, applying Exercise 5.16 repeatedly we may inductively construct dashed lifts as indicated that make the diagram commute. By the universal property of the colimit, together these yield the desired lift in the original diagram.⁷⁷ In other words, we have shown that $c^{(\infty)} \in \text{llp}(g)$. However, since $g \in \text{rlp}(I)$ was arbitrary, altogether we have shown that $c^{(\infty)} \in \text{llp}(\text{rlp}(I))$, as desired.

- (2) In fact, we show that $f^{(\infty)} \in \text{rlp}(I)$. For this, suppose we are given any commutative diagram

$$\begin{array}{ccc} S^n & \longrightarrow & N^{(\infty)} \\ i_n \downarrow & & \downarrow f^{(\infty)} \\ D^{n+1} & \longrightarrow & N \end{array} .$$

By Exercise 5.18 below, there exists some $k \in \mathbb{N}$ and a lift

$$\begin{array}{ccc} & N^{(k)} & \\ & \downarrow & \\ S^n & \xrightarrow{\quad} & N^{(\infty)} \end{array}$$

of the given map. Now, just as in Exercise 5.17, we obtain a lift

$$\begin{array}{ccccc} S^n & \xrightarrow{\quad} & N^{(k)} & & \\ \downarrow i_n & & \searrow c^{(k+1)} & & \downarrow \\ & & N^{(k+1)} & \xrightarrow{\quad} & N^{(\infty)} \\ & \nearrow \text{dashed} & \searrow f^{(k+1)} & & \downarrow f^{(\infty)} \\ D^{n+1} & \xrightarrow{\quad} & N & & \end{array} ,$$

which proves the claim.

Exercise 5.18 (2 points). Show that the object $S^n \in \text{Ch}_R$ is **compact**, i.e. that the functor

$$\text{Ch}_R \xrightarrow{\text{hom}_{\text{Ch}_R}(S^n, -)} \text{Set}$$

commutes with filtered colimits.⁷⁸

⁷⁷This may be phrased as choosing an element of a codirected limit of *surjective* functions between sets.

⁷⁸Note that a functor commutes with filtered colimits if and only if it commutes with directed colimits. (This is one instance of the useful general principle of “coordinatization” of a class of colimits. As another example, given a functor between categories that admit finite colimits, it preserves them if and only if it preserves initial objects and pushouts. Another related fact is that given a functor between cocomplete

6. THE DERIVED ∞ -CATEGORY

6.1. The homotopy theory of the dg-category of complexes.

6.1.1. We have by now obtained a diagram

$$(4) \quad \begin{array}{ccc} \mathbf{P}_R & \searrow & \\ & \mathbf{K}_R \longleftrightarrow \mathbf{A}_R & \\ \mathbf{I}_R & \nearrow & \end{array}$$

of fully faithful inclusions among dg-categories: we began with the subcategory $\mathbf{A}_R \subseteq \mathbf{K}_R$, and then we defined $\mathbf{P}_R \subseteq \mathbf{K}_R$ (resp. $\mathbf{I}_R \subseteq \mathbf{K}_R$) to be what might be called its *homotopical left* (resp. *right*) *orthogonal* subcategory, namely

$$\mathbf{P}_R := \{M \in \mathbf{K}_R \text{ such that } \mathrm{hom}_{\mathbf{K}_R}(M, A) \approx 0 \text{ for all } A \in \mathbf{A}_R\}$$

and

$$\mathbf{I}_R := \{M \in \mathbf{K}_R \text{ such that } \mathrm{hom}_{\mathbf{K}_R}(A, M) \approx 0 \text{ for all } A \in \mathbf{A}_R\} .^{79}$$

Recall that for any complex $M \in \mathbf{K}_R$ we have constructed a projective resolution $M' \xrightarrow{\sim} M$, i.e. a quasi-isomorphism from a projective object. Consider the resulting homotopy kernel sequence

$$A \longrightarrow M' \longrightarrow M$$

in $\mathbf{Ch}_R$, where we write $A := \mathbf{hker}(M' \rightarrow M)$ because this complex is acyclic (by the long exact sequence in homology). Now, for any projective complex $P \in \mathbf{P}_R \subseteq \mathbf{K}_R$, by Exercise 4.7 we obtain a homotopy kernel sequence

$$\underline{\mathrm{hom}}(P, A) \longrightarrow \underline{\mathrm{hom}}(P, M') \longrightarrow \underline{\mathrm{hom}}(P, M)$$

in $\mathbf{Ch}_{\mathbb{K}}$. Because P is projective, the complex $\underline{\mathrm{hom}}(P, A)$ is acyclic, and hence we find that the morphism

$$\underline{\mathrm{hom}}(P, M') \xrightarrow{\sim} \underline{\mathrm{hom}}(P, M)$$

is a quasi-isomorphism (again by the long exact sequence). Heuristically, we have found that “the morphism $M' \rightarrow M$ appears to be an isomorphism when mapping from projective complexes (and considering hom-complexes up to quasi-isomorphism)”.

categories (i.e. categories admitting all (small) colimits), it preserves all (small) colimits if and only if it preserves finite colimits and filtered colimits.)

⁷⁹This categorifies the notion of orthogonality in linear algebra: given a vector space V equipped with an inner product $\langle -, - \rangle$, the orthogonal of a subspace $W \subseteq V$ is defined to be

$$\{v \in V \text{ such that } \langle v, w \rangle = 0 \text{ for all } w \in W\} .$$

The categorified notion comes equipped with a handedness because hom-objects aren’t symmetric (in contrast with inner products), and by “homotopical” we mean that we only demand that the hom-complexes are quasi-isomorphic (rather than isomorphic) to 0.

6.1.2. This sort of phenomenon is in fact quite ubiquitous. Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between ordinary categories. Given an object $D \in \mathcal{D}$, a **pointwise right adjoint** to F at D consists of an object $C \in \mathcal{C}$ and a morphism $F(C) \rightarrow D$ such that the resulting composite morphism

$$\mathrm{hom}_{\mathcal{C}}(-, C) \xrightarrow{F} \mathrm{hom}_{\mathcal{D}}(F(-), F(C)) \longrightarrow \mathrm{hom}_{\mathcal{D}}(F(-), D)$$

in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ is a natural isomorphism. In this situation, we may refer to the morphism $F(C) \rightarrow D$ as the **pointwise counit** and denote it by ε_D (or simply by ε).

Exercise 6.1 (6 points).

- (a) Show that if it exists, a pointwise right adjoint to F at D is unique up to unique isomorphism.⁸⁰
- (b) Show that the datum of a right adjoint to F is equivalent data to a choice of pointwise right adjoint to F at every object $D \in \mathcal{D}$.⁸¹

So, the above discussion may be summarized as saying that in a homotopical sense – that is, if we only consider hom-complexes up to quasi-isomorphism – the fully faithful inclusion $\mathbf{P}_R \hookrightarrow \mathbf{K}_R$ “should” admit a right adjoint whose pointwise counits are projective resolutions. On the other hand, because these morphisms on hom-complexes are only quasi-isomorphisms and not isomorphisms, we should not expect this structure to exist at the level of dg-categories.

6.1.3. Of course, this discussion immediately dualizes: any complex $M \in \mathbf{K}_R$ has an injective resolution $M \xrightarrow{\sim} M''$ (i.e. a quasi-isomorphism to an injective object), and this has the property that for any injective complex $I \in \mathbf{I}_R \subseteq \mathbf{K}_R$ the morphism

$$\underline{\mathrm{hom}}(M'', I) \xrightarrow{\sim} \underline{\mathrm{hom}}(M, I)$$

is a quasi-isomorphism. Heuristically, “the morphism $M \rightarrow M''$ appears to be an isomorphism when mapping to an injective complex (and considering hom-complexes up to quasi-isomorphism)”. So, in a homotopical sense the fully faithful inclusion $\mathbf{I}_R \hookrightarrow \mathbf{K}_R$ “should” admit a left adjoint whose pointwise units are injective resolutions.

6.2. A brief introduction to $\mathbb{k}$ -linear ∞ -categories.

6.2.1. These adjoints that “should” exist in a homotopical sense actually *do* exist at the level of underlying $\mathbb{k}$ -linear ∞ -categories. So, we work within an ∞ -categorical context.

However, our present usage of ∞ -category theory will be extremely “soft”: it will not require any real familiarity with the foundations of the theory. So in the interest of maintaining the narrative thread, here we give an extremely brief summary of the immediately relevant features of ∞ -category theory, and pursue a more systematic discussion later.

⁸⁰Said differently, the category of pointwise right adjoints to F at D is always either an empty or contractible groupoid.

⁸¹It follows that the category of right adjoints to F is always either an empty or contractible groupoid (because these are preserved under products).

6.2.2. Here is the most expedient definition of a $\mathbb{k}$ -linear ∞ -category.⁸² We write $\mathbf{cat}$ for the category of ordinary categories, and we write $\mathbf{cat}^{\mathrm{dg}} \xrightarrow{H_0} \mathbf{cat}$ for the functor given by applying H_0 to each hom-complex. Then, a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between dg-categories is called a **weak equivalence** if the following two conditions hold:

- (1) it is **homotopically fully faithful**, i.e. for all $C, C' \in \mathcal{C}$ the induced morphism $\mathrm{hom}_{\mathcal{C}}(C, C') \rightarrow \mathrm{hom}_{\mathcal{D}}(F(C), F(C'))$ in $\mathbf{Ch}_{\mathbb{k}}$ is a quasi-isomorphism; and
- (2) it is **homotopically essentially surjective**, i.e. the induced functor $H_0(\mathcal{C}) \xrightarrow{H_0(F)} H_0(\mathcal{D})$ is essentially surjective.⁸³

(Of course, if $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a weak equivalence of dg-categories then $H_0(\mathcal{C}) \xrightarrow{H_0(F)} H_0(\mathcal{D})$ is in fact an equivalence of categories.)

We localize the category $\mathbf{cat}_{\mathbb{k}}^{\mathrm{dg}}$ of $\mathbb{k}$ -linear dg-categories at the weak equivalences. This yields a category that (for reasons that will become clear later) we will denote by $\mathbf{ho}(\mathbf{Cat}_{\mathbb{k}})$ and refer to as **the homotopy category of $\mathbb{k}$ -linear ∞ -categories**; its objects are called **$\mathbb{k}$ -linear ∞ -categories**. So by definition, there is a canonical functor

$$\mathbf{cat}^{\mathrm{dg}} \longrightarrow \mathbf{ho}(\mathbf{Cat}_{\mathbb{k}})$$

that carries all weak equivalences to isomorphisms. In particular, this functor is essentially surjective: every $\mathbb{k}$ -linear ∞ -category can be represented by a $\mathbb{k}$ -linear dg-category.⁸⁴

Given a dg-category $\mathcal{C}$, for the remainder of this section we will write $\mathcal{C}^{\infty}$ for its underlying $\mathbb{k}$ -linear ∞ -category. However, we will thereafter omit this superscript, as we will have finished drawing distinctions between dg-categories and their underlying $\mathbb{k}$ -linear ∞ -categories. We will generally refer to data in $\mathcal{C}$ as “point-set” and data in $\mathcal{C}^{\infty}$ as “ ∞ -categorical”.

Essentially by construction, given two dg-categories $\mathcal{C}, \mathcal{D} \in \mathbf{cat}^{\mathrm{dg}}$, morphisms $\mathcal{C}^{\infty} \rightarrow \mathcal{D}^{\infty}$ in $\mathbf{ho}(\mathbf{Cat}_{\mathbb{k}})$ are given by equivalence classes of zigzags

$$\mathcal{C} \xleftarrow{\sim} \bullet \longrightarrow \bullet \xleftarrow{\sim} \dots \longrightarrow \bullet \xleftarrow{\sim} \mathcal{D}$$

in $\mathbf{cat}^{\mathrm{dg}}$ (in which all backwards maps are weak equivalences). However, it turns out that every equivalence class contains a representative of the form

$$\mathcal{C} \xleftarrow{\sim} \mathcal{C}' \longrightarrow \mathcal{D} .$$

In fact, dg-categories admit “projective resolutions”: every equivalence class contains a representative of this form for *any* fixed weak equivalence $\mathcal{C}' \xrightarrow{\sim} \mathcal{C}$ from a “projective” dg-category.

⁸²This should be compared with the discussion of §2.4.

⁸³It is possible to phrase homotopical essential surjectivity at the level of dg-categories, but this requires a bit of enriched category theory.

⁸⁴This is analogous to the presentation of ordinary ∞ -categories as categories enriched in simplicial sets; in particular, in both cases the presentations admit strict compositions (in contrast with the perspective on composition in ∞ -categories explained in §6.2.3). A presentation of $\mathbb{k}$ -linear ∞ -categories that adheres more closely to e.g. quasicategories or complete Segal spaces is given by ($\mathbb{k}$ -linear) $\mathbb{A}_{\infty}$ -categories.

Once one is only considering hom-complexes up to quasi-isomorphism, homotopy equivalences (e.g. in $\mathbf{K}_R$ and its various subcategories) become indistinguishable from isomorphisms. In general, one simply uses the term *equivalence* to refer to the ∞ -categorical notion of isomorphism. We will denote equivalences in an ∞ -category by $\xrightarrow{\sim}$.

For brevity, for the time being we will often simply refer to $\mathbb{k}$ -linear ∞ -categories as “ ∞ -categories”.

6.2.3. To a first approximation, a $\mathbb{k}$ -linear ∞ -category is simply a category enriched in $H_0(\mathbf{D}_{\mathbb{k}})$, the derived category of $\mathbb{k}$ introduced in §2.4. In particular, passing from dg-categories to their underlying $\mathbb{k}$ -linear ∞ -categories does indeed “only remember their hom-complexes up to quasi-isomorphism”. The distinction lies in the higher homotopies (e.g. those recording the homotopy-coherent associativity of composition) which are present in the underlying ∞ -category of a dg-category but are lost in passing to its underlying $H_0(\mathbf{D}_{\mathbb{k}})$ -enriched category.

Namely, as we will see when we study ∞ -categories more generally, given a dg-category $\mathcal{C}$, commutative diagrams in its underlying $\mathbb{k}$ -linear ∞ -category $\mathcal{C}^\infty$ are represented by *homotopy-coherently* commutative diagrams in $\mathcal{C}$ itself. For instance, we will see the projective resolution

$$(5) \quad \begin{array}{ccc} & 1 & \\ \nearrow & \uparrow & \searrow \\ 0 & \longrightarrow & 2 \end{array} \xrightarrow{\approx} \begin{array}{ccc} & 1 & \\ \nearrow & = & \searrow \\ 0 & \longrightarrow & 2 \end{array}$$

of the free strictly-commutative triangle $[2] = \{0 < 1 < 2\}$ by the free homotopy-coherently commutative triangle, and this persists upon passing to free dg-categories.⁸⁵

Said differently, given morphisms $C_0 \xrightarrow{\varphi_{01}} C_1$, $C_1 \xrightarrow{\varphi_{12}} C_2$, and $C_0 \xrightarrow{\varphi_{02}} C_2$ in the ∞ -category $\mathcal{C}^\infty$, we can no longer ask whether φ_{02} is *equal* to the composition $\varphi_{12} \circ \varphi_{01}$.⁸⁶ Rather, we can only ask for a homotopy that “witnesses φ_{02} as being a homotopy-coherent composite of φ_{12} and φ_{01} ”.⁸⁷

6.2.4. As a particular case, it turns out that homotopy co/kernel squares in dg-categories – which fit into *homotopy-coherently* commutative squares – define co/limit diagrams – and particular, *commutative* squares – in their underlying ∞ -categories.⁸⁸ For the moment, we will write $\mathbf{coker}^\infty$ and $\mathbf{ker}^\infty$ to denote ∞ -categorical co/kernels.

⁸⁵To be precise, this is a sort of projective resolution (actually a “cofibrant” resolution) among categories enriched in simplicial sets, and by “the free $\mathbb{k}$ -linear dg-category” on a category enriched in simplicial sets we mean the dg-category obtained by taking $\mathbb{k}$ -linear simplicial chains on its hom-objects.

⁸⁶Indeed, e.g. in $\mathbf{K}_R$ we have that chain maps $M \rightarrow N$ correspond to 0-cycles in $\mathbf{hom}_{\mathbf{K}_R}(M, N)$, but it is not well-defined to discuss the “0-cycles” of a derived $\mathbb{k}$ -module (because these are not generally preserved by quasi-isomorphisms).

⁸⁷Of course, any pair of composable morphisms does admit *some* “homotopy-coherent composite”, and in fact these assemble into a contractible complex.

⁸⁸This should not be surprising, based on our discussion of the former.

It will also be convenient to discuss more general ∞ -categorical pushout/pullback squares in a $\mathbb{k}$ -linear ∞ -category. As an expedient definition, we can declare that a commutative square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \downarrow h \\ Y & \xrightarrow{i} & Z \end{array}$$

in a $\mathbb{k}$ -linear ∞ -category is a **pushout** (resp. **pullback**) if the resulting commutative square

$$\begin{array}{ccc} W & \xrightarrow{(f,g)} & X \oplus Y \\ \downarrow & & \downarrow (h,-i) \\ 0 & \longrightarrow & Z \end{array}$$

is a cokernel (resp. kernel) square.

It turns out that pushout and pullback squares (and in particular, cokernel and kernel squares) coincide in *any* $\mathbb{k}$ -linear ∞ -category.⁸⁹ In order to refer to them in an unbiased way, we may refer to them as **exact squares**.

6.3. The derived ∞ -category.

6.3.1. Having introduced $\mathbb{k}$ -linear ∞ -categories, we can now finally define the **derived ∞ -category of R -modules**: it is the ∞ -categorical localization

$$\mathbf{K}_R^\infty \xrightarrow{\pi} \mathbf{D}_R$$

at the quasi-isomorphisms. In other words, for any $\mathbb{k}$ -linear ∞ -category $\mathcal{C}$, precomposition with π determines an inclusion

$$\mathrm{hom}_{\mathrm{ho}(\mathrm{Cat}_{\mathbb{k}})}(\mathbf{K}_R^\infty, \mathcal{C}) \hookrightarrow \mathrm{hom}_{\mathrm{ho}(\mathrm{Cat}_{\mathbb{k}})}(\mathbf{D}_R, \mathcal{C}) ,$$

whose image consists of those functors $\mathbf{K}_R^\infty \rightarrow \mathcal{C}$ that carry all quasi-isomorphisms to equivalences.⁹⁰

Of course, this definition of the derived ∞ -category $\mathbf{D}_R$ is rather abstract. However, it turns out that it is canonically equivalent to *both* underlying ∞ -categories $\mathbf{P}_R^\infty$ and $\mathbf{I}_R^\infty$ (which in particular are canonically equivalent to one another). More precisely, as we will explain,

⁸⁹This makes crucial use of the $\mathbb{k}$ -linearity: it is certainly not the case that pushout and pullback squares coincide in a general ∞ -category.

⁹⁰We have stated this universal property in the category $\mathrm{ho}(\mathrm{Cat}_{\mathbb{k}})$, but in fact it holds in the ∞ -category $\mathrm{Cat}_{\mathbb{k}}$ as well.

the diagram (4) of dg-categories extends to a diagram

$$(6) \quad \begin{array}{ccccc} & & \mathbf{P}_R^\infty & & \\ & & \uparrow & \swarrow i_{\mathbf{P}} & \\ & & \downarrow \rho_{\mathbf{P}} & \searrow \perp & \\ & & \mathbf{K}_R^\infty & & \\ & & \uparrow \rho_{\mathbf{I}} & \swarrow i_{\mathbf{I}} & \\ & & \downarrow \rho_{\mathbf{I}} \circ i_{\mathbf{P}} & \searrow \perp & \\ & & \mathbf{I}_R^\infty & & \end{array} \quad \begin{array}{ccc} & a_L & \\ \perp & \swarrow & \searrow \\ \mathbf{K}_R^\infty & \xleftarrow{i_{\mathbf{A}}} & \mathbf{A}_R^\infty \\ \perp & \swarrow & \searrow \\ & a_R & \end{array}$$

on underlying ∞ -categories, in which *both* functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ satisfy the universal property of the functor π . This implies that there are canonical equivalences between $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ as functors out of $\mathbf{K}_R^\infty$ (i.e. there are canonical equivalences between their targets that commute with them), and we will see that these are precisely the equivalences indicated in the diagram.

Taking these assertions for granted, we can express diagram (6) more concisely by collapsing the vertical equivalences and using the unbiased notation $\mathbf{D}_R$ to obtain a diagram

$$(7) \quad \begin{array}{ccccc} & i_L & & a_L & \\ \swarrow & \perp & \searrow & \swarrow & \searrow \\ \mathbf{D}_R & \xleftarrow{\pi} & \mathbf{K}_R^\infty & \xleftarrow{i_{\mathbf{A}}} & \mathbf{A}_R^\infty \\ \swarrow & \perp & \searrow & \swarrow & \searrow \\ & i_R & & a_R & \end{array} ,$$

in which i_L (resp. i_R) denotes “the inclusion of $\mathbf{D}_R$ into $\mathbf{K}_R^\infty$ as the full subcategory of projective (resp. injective) complexes”, which is the left (resp. right) adjoint to the projection functor π . This latter diagram forms what is called a **recollement**, a sort of categorified extension sequence, which notion we will discuss more later.

6.3.2. Before explaining the diagram (6), we pause to slightly sharpen the perspective on the passage from a dg-category to its underlying ∞ -category given in §6.2.3, in a way that will be helpful shortly. Namely, it is possible to *define* a $\mathbb{k}$ -linear ∞ -category to be a category enriched in the derived ∞ -category $\mathbf{D}_{\mathbb{k}}$ of $\mathbb{k}$.⁹¹ Then, the passage from a $\mathbb{k}$ -linear dg-category to its underlying $\mathbb{k}$ -linear ∞ -category amounts to applying the functor

$$\mathbf{K}_{\mathbb{k}}^\infty \xrightarrow{\pi} \mathbf{D}_{\mathbb{k}}$$

to each of its hom-complexes.

6.3.3. It is important to note that there are many commutative squares in $\mathbf{K}_R^\infty$ that may not be co/kernel squares themselves but nevertheless become so in $\mathbf{D}_R$, e.g. any commutative

⁹¹It would certainly be circular to do so here. (In particular, we do not discuss the symmetric monoidal structure on $\mathbf{D}_{\mathbb{k}}$ until §6.3.7.) The formalism of enriched ∞ -categories is developed in [GH15].

square of the form

$$\begin{array}{ccc} X & \xrightarrow{\approx} & Y \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\approx} & Z \end{array} .$$

In general, it is only after projective resolution or injective resolution that such a square becomes a co/kernel square.

6.3.4. We begin our explanation of diagram (6) by obtaining the adjunctions in its left side. Of course, these were already hinted at in §6.1: the existence of the adjoints in the diagram

$$\begin{array}{ccc} \mathbf{P}_R^\infty & & \\ & \swarrow i_{\mathbf{P}} & \\ & \perp & \\ & \searrow \rho_{\mathbf{P}} & \\ & & \mathbf{K}_R^\infty \\ & \swarrow \rho_{\mathbf{I}} & \\ & \perp & \\ & \searrow i_{\mathbf{I}} & \\ \mathbf{I}_R^\infty & & \end{array}$$

follows immediately from the ∞ -categorical analog of Exercise 6.1 and its dual. We refer to $\rho_{\mathbf{P}}$ (resp. $\rho_{\mathbf{I}}$) as the **projective** (resp. **injective**) **resolution** functor.

Recall that projective resolutions are far from unique at the “point-set” (a.k.a. “chain”) level. For instance, applying the the construction of §5.6 to a projective complex $P \in \mathbf{P}_R$ will yield a quasi-isomorphism $P' \xrightarrow{\sim} P$ from another projective complex, which is generally be very far from being an isomorphism. On the other hand, by Exercise 5.3(a) this must be a homotopy equivalence, and hence it defines an equivalence in the underlying ∞ -category $\mathbf{P}_R^\infty$. More broadly, the ∞ -categorical analog of Exercise 6.1(a) tells us that projective resolutions of *any* complex $M \in \mathbf{K}_R^\infty$ are “unique in the ∞ -categorical sense”. Of course, dual comments apply to injective resolutions.

Using merely the definitions of the adjoint functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$, we can obtain the inverse equivalences on the left in diagram (6).

Exercise 6.2 (6 points). Prove that the composite adjunction

$$\mathbf{P}_R^\infty \begin{array}{c} \xleftarrow{i_{\mathbf{P}}} \\ \perp \\ \xleftarrow{\rho_{\mathbf{P}}} \end{array} \mathbf{K}_R^\infty \begin{array}{c} \xleftarrow{\rho_{\mathbf{I}}} \\ \perp \\ \xleftarrow{i_{\mathbf{I}}} \end{array} \mathbf{I}_R^\infty$$

is an adjoint equivalence.

6.3.5. We now simultaneously address the commutativity of the left triangle in diagram (6) and explain why both functors $\mathbf{K}_R^\infty \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_R^\infty$ and $\mathbf{K}_R^\infty \xrightarrow{\rho_{\mathbf{I}}} \mathbf{I}_R^\infty$ satisfy the universal property defining $\mathbf{K}_R^\infty \xrightarrow{\pi} \mathbf{D}_R$ (namely they are both the localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms).

We say that an adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{D}$$

is a **reflective localization adjunction** if the right adjoint G is fully faithful. In this situation, we refer to the functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ as a **reflective localization**. For concreteness, we discuss these in the context of ordinary categories, but then we will apply the theory in the context of $\mathbb{k}$ -linear ∞ -categories (in which it also holds).

Exercise 6.3 (6 points).

- (a) Show that G is fully faithful if and only if the counit $FG \xrightarrow{\varepsilon} \text{id}_{\mathcal{D}}$ is a natural isomorphism.
- (b) Show that for a reflective localization adjunction $F \dashv G$, the morphism $F \xrightarrow{F\eta} FGF$ is a natural isomorphism.

Given such a reflective localization adjunction, let us write $\mathbf{W} \subseteq \mathcal{C}$ for the subcategory consisting of those morphisms that are carried by F to isomorphisms. Then, we claim that the reflective localization $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is precisely the localization of $\mathcal{C}$ at $\mathbf{W}$.⁹² To see this, fix any functor $\mathcal{C} \xrightarrow{T} \mathcal{E}$, and consider the diagram

$$\begin{array}{ccc}
 & \mathcal{C} & \xrightarrow{T} \mathcal{E} \\
 G \nearrow & \downarrow F & \swarrow T\eta \\
 \mathcal{D} & \xrightarrow{\text{id}_{\mathcal{D}}} \mathcal{D} & \xrightarrow{TG} \mathcal{E}
 \end{array}
 ,$$

in which the left triangle commutes via the natural isomorphism ε of Exercise 6.3(a). Considering just the solid diagram, we see that there is at most one choice of factorization of T through F , namely TG . On the other hand, we *always* have the indicated natural transformation $T \xrightarrow{T\eta} TGF$. So, it remains to check that $T\eta$ is a natural isomorphism if and only if T carries the morphisms in $\mathbf{W}$ to isomorphisms. By Exercise 6.3(b), the components of η lie in $\mathbf{W} \subseteq \mathcal{C}$, which implies the “if” direction.

Exercise 6.4 (4 points). Prove the “only if” direction: that if $T\eta$ is a natural isomorphism then T carries all morphisms in $\mathbf{W}$ to isomorphisms.

A class of examples of reflective localizations is given by the following.

Exercise 6.5 (3 points). Given a ring homomorphism $R \rightarrow S$, show that the corresponding adjunction

$$\text{Mod}_R \begin{array}{c} \xrightarrow{(-) \otimes_R S} \\ \xleftarrow[\text{fgt}]{\perp} \end{array} \text{Mod}_S$$

is a reflective localization if and only if S is an *idempotent* R -algebra, i.e. the multiplication map $S \otimes_R S \rightarrow S$ is an isomorphism.

⁹²To explain the terminology further, considering $\mathcal{D} \subseteq \mathcal{C}$ as a full subcategory via G , the functor F is also called the *reflector* of $\mathcal{C}$ into $\mathcal{D}$.

For example, the ring homomorphisms $\mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{n}]$ and $\mathbb{Z} \rightarrow \mathbb{Q}$ determine reflective localizations. So for instance, the forgetful functor

$$\mathbf{Mod}_{\mathbb{Z}} \xleftarrow{\text{fgt}} \mathbf{Mod}_{\mathbb{Q}}$$

is fully faithful: a homomorphism between $\mathbb{Q}$ -vector spaces is nothing more or less than a homomorphism between their underlying abelian groups.⁹³ Moreover, the functor

$$\mathbf{Mod}_{\mathbb{Z}} \xrightarrow{(-) \otimes_{\mathbb{Z}} \mathbb{Q}} \mathbf{Mod}_{\mathbb{Q}}$$

is the localization of $\mathbf{Mod}_{\mathbb{Z}}$ at the *rational isomorphisms* between abelian groups (i.e. those homomorphisms that rationalize to isomorphisms (e.g. $\mathbb{Z} \xrightarrow{(9,2)} \mathbb{Z} \oplus \mathbb{Z}/5$, which rationalizes to $\mathbb{Q} \xrightarrow{9} \mathbb{Q}$)). Explicitly, given a functor $\mathbf{Mod}_{\mathbb{Z}} \xrightarrow{T} \mathcal{E}$ that carries rational isomorphisms to isomorphisms, the canonical factorization is given by

$$\begin{array}{ccc} \mathbf{Mod}_{\mathbb{Z}} & \xrightarrow{T} & \mathcal{E} \\ (-) \otimes_{\mathbb{Z}} \mathbb{Q} \downarrow & \nearrow T \circ \text{fgt} & \\ \mathbf{Mod}_{\mathbb{Q}} & & \end{array} .$$

By contrast, the ring homomorphisms $\mathbb{Q} \rightarrow \mathbb{Q}[\sqrt{2}]$ and $\mathbb{R} \rightarrow \mathbb{C}$ do not determine reflective localizations. So for instance, the forgetful functor

$$\mathbf{Mod}_{\mathbb{R}} \xleftarrow{\text{fgt}} \mathbf{Mod}_{\mathbb{C}}$$

is *not* fully faithful: a homomorphism between $\mathbb{C}$ -vector spaces is not the same thing as a homomorphism between their underlying $\mathbb{R}$ -vector spaces. Rather, one must additionally demand that it respects multiplication by $i \in \mathbb{C}$.⁹⁴

Now, by definition the adjunction $\rho_{\mathbf{I}} \dashv i_{\mathbf{I}}$ is a reflective localization adjunction.

Exercise 6.6 (2 points). Prove that a morphism in $\mathbf{K}_R^{\infty}$ is carried by $\rho_{\mathbf{I}}$ to an equivalence if and only if it is a quasi-isomorphism.

By what we have just seen, this implies that the functor $\mathbf{K}_R^{\infty} \xrightarrow{\rho_{\mathbf{I}}} \mathbf{I}_R^{\infty}$ is indeed the localization of $\mathbf{K}_R^{\infty}$ at the quasi-isomorphisms.

Of course, dually the functor $\mathbf{K}_R^{\infty} \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_R^{\infty}$ is a *coreflective* localization, and so it is *also* the localization of $\mathbf{K}_R^{\infty}$ at the quasi-isomorphisms.

⁹³And indeed, we have a further reflective localization adjunction

$$\mathbf{Grp} \xrightleftharpoons[\text{fgt}]{\perp} \mathbf{Ab}$$

(in which the left adjoint is given by abelianization), so that a homomorphism between $\mathbb{Q}$ -vector spaces is nothing more or less than a homomorphism between their underlying groups (which happen to be abelian).

⁹⁴These examples illustrate a general principle, namely the dichotomy between “properties” and “structure”. For instance, being a $\mathbb{Q}$ -module turns out to be merely a *property* that can be satisfied by a $\mathbb{Z}$ -module (in the sense that we can consider $\mathbf{Mod}_{\mathbb{Q}} \subset \mathbf{Mod}_{\mathbb{Z}}$ as a full subcategory), whereas given an $\mathbb{R}$ -module we must provide additional *structure* to make it into a $\mathbb{C}$ -module.

So indeed, *both* functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ are the localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms. In other words, they share the same universal property. Therefore, they must be canonically equivalent. Moreover, tracing through our proof that a reflective localization is a localization (and dualizing), we find that the canonical inverse equivalences between $\mathbf{P}_R^\infty$ and $\mathbf{I}_R^\infty$ that are induced by their shared universal property (as $\mathbb{k}$ -linear ∞ -categories under $\mathbf{K}_R^\infty$) are precisely the composites

$$\begin{array}{ccc} \mathbf{P}_R^\infty & & \mathbf{P}_R^\infty \\ \uparrow \scriptstyle \rho_{\mathbf{P}} \circ i_{\mathbf{I}} & \swarrow \scriptstyle \rho_{\mathbf{P}} & \swarrow \scriptstyle \rho_{\mathbf{P}} \\ & \mathbf{K}_R^\infty & \mathbf{K}_R^\infty \\ & \searrow \scriptstyle \rho_{\mathbf{I}} & \searrow \scriptstyle \rho_{\mathbf{I}} \\ \mathbf{I}_R^\infty & & \mathbf{I}_R^\infty \\ & \downarrow \scriptstyle \rho_{\mathbf{I}} \circ i_{\mathbf{P}} & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{P}_R^\infty & & \mathbf{P}_R^\infty \\ \downarrow \scriptstyle \rho_{\mathbf{I}} \circ i_{\mathbf{P}} & \swarrow \scriptstyle \rho_{\mathbf{P}} & \swarrow \scriptstyle \rho_{\mathbf{P}} \\ & \mathbf{K}_R^\infty & \mathbf{K}_R^\infty \\ & \searrow \scriptstyle \rho_{\mathbf{I}} & \searrow \scriptstyle \rho_{\mathbf{I}} \\ \mathbf{I}_R^\infty & & \mathbf{I}_R^\infty \end{array} .$$

In particular, these triangles commute.

It is also not hard to see that these triangles commute using the Yoneda lemma. For instance, given any $P \in \mathbf{P}_R^\infty$ and any $M \in \mathbf{K}_R^\infty$, we have the composite equivalence

$$\mathrm{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}} i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \simeq \mathrm{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \xrightarrow{\sim} \mathrm{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), M) \simeq \mathrm{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}}(M))$$

in $\mathbf{D}_{\mathbb{k}}$, using the fact that $\mathrm{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), -)$ carries quasi-isomorphisms to quasi-isomorphisms. Since $P \in \mathbf{P}_R^\infty$ was arbitrary, it follows that the morphism

$$\rho_{\mathbf{P}} i_{\mathbf{I}} \rho_{\mathbf{I}}(M) \xrightarrow{\rho_{\mathbf{P}} \varepsilon_M} \rho_{\mathbf{P}}(M)$$

in $\mathbf{P}_R^\infty$ is an equivalence.

6.3.6. To conclude our discussion of diagram (6), we now construct the adjoint a_L ; the adjoint a_R will arise from dual considerations. We refer to a_L (resp. a_R) as the *left* (resp. *right*) **acyclification** functor.

For this, let us observe that for any complex $M \in \mathbf{K}_R$, the complex

$$a_L(M) := \mathrm{coker}^\infty \left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M)) \xrightarrow{\varepsilon_M} M \right)$$

is acyclic by Exercise 3.10(a).⁹⁵ In other words, the formula

$$a_L := \mathrm{coker}^\infty \left(i_{\mathbf{P}} \circ \rho_{\mathbf{P}} \xrightarrow{\varepsilon} \mathrm{id}_{\mathbf{K}_R} \right)$$

defines a functor $\mathbf{K}_R^\infty \xrightarrow{a_L} \mathbf{A}_R^\infty$.

We now verify that a_L defines a left adjoint to the inclusion $\mathbf{K}_R^\infty \xleftarrow{i_{\mathbf{A}}} \mathbf{A}_R^\infty$. At the ∞ -categorical level, this is a straightforward computation. However, it is also a good opportunity to highlight the distinction between working in a dg-category and working in its underlying ∞ -category, which makes the computation somewhat more subtle. Namely, the functor a_L is only well-defined at the ∞ -categorical level, which implies that it is not literally

⁹⁵In view of the adjunction $i_{\mathbf{P}} \dashv \rho_{\mathbf{P}}$, one may think of $a_L(M)$ as being obtained by “killing off the part of M that can be seen by projective complexes”.

meaningful to compute with it at the point-set level. On the other hand, by the dual of Exercise 6.1, it suffices to verify that *any* point-set representative of the morphism $M \rightarrow i_{\mathbf{A}} a_L(M)$ in $\mathbf{K}_R^\infty$ is a pointwise unit on underlying ∞ -categories. Therefore, let us write

$$i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon'_M} M$$

for a specific but arbitrary point-set representative of the morphism ε_M (i.e. an arbitrary projective resolution of M), and let us write $a_L(M)' := \mathbf{hcoker}(\varepsilon'_M)$; recall that homotopy cokernels in $\mathbf{K}_R$ compute ∞ -categorical cokernels in $\mathbf{K}_R^\infty$, so the morphism $M \rightarrow i_{\mathbf{A}} a_L(M)'$ in $\mathbf{K}_R$ is indeed a point-set model for the morphism $M \rightarrow i_{\mathbf{A}} a_L(M)$ in $\mathbf{K}_R^\infty$. Now, for any acyclic complex $A \in \mathbf{A}_R^\infty$, omitting the functor $i_{\mathbf{A}}$ to simplify notation, we compute that

$$\begin{aligned}
\text{hom}_{\mathbf{A}_R^\infty}(a_L(M), A) &:= \pi(\text{hom}_{\mathbf{A}_R}(a_L(M)', A)) \\
&:= \pi(\text{hom}_{\mathbf{K}_R}(a_L(M)', A)) \\
&:= \pi\left(\text{hom}_{\mathbf{K}_R}\left(\mathbf{hcoker}\left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon'_M} M\right), A\right)\right) \\
(8) \quad &\simeq \pi\left(\mathbf{hker}\left(\text{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\text{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)} \text{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right) \\
(9) \quad &\simeq \pi\left(\mathbf{ker}^\infty\left(\text{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\text{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)} \text{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right) \\
(10) \quad &\simeq \mathbf{ker}^\infty\left(\pi(\text{hom}_{\mathbf{K}_R}(M, A)) \xrightarrow{\pi(\text{hom}_{\mathbf{K}_R}(\varepsilon'_M, A))} \pi(\text{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A))\right) \\
(11) \quad &\simeq \mathbf{ker}^\infty(\pi(\text{hom}_{\mathbf{K}_R}(M, A)) \longrightarrow 0) \\
&\simeq \pi(\text{hom}_{\mathbf{K}_R}(M, A)) \\
&=: \text{hom}_{\mathbf{K}_R^\infty}(M, A)
\end{aligned}$$

as desired, by the following reasoning.

- Equivalence (8) follows from the fact that hom-complexes respect homotopy co/kernels (Exercise 4.7).
- Equivalence (9) follows from the fact that homotopy kernels in $\mathbf{K}_R$ compute ∞ -categorical kernels in $\mathbf{K}_R^\infty$. (From here onwards, we implicitly consider the morphism $\text{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)$ as lying not in the dg-category $\mathbf{K}_k$ but in its underlying ∞ -category $\mathbf{K}_k^\infty$.)
- As we have just seen, $\mathbf{K}_k^\infty \xrightarrow{\pi} \mathbf{D}_k$ is a right adjoint. It therefore preserves ∞ -categorical limits, and in particular ∞ -categorical kernels. This gives equivalence (10).

- By construction, the complex $i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \in \mathbf{K}_R$ is projective.⁹⁶ Hence, the complex $\mathrm{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A) \in \mathbf{K}_{\mathbb{k}}$ is acyclic. It follows that we have an equivalence

$$\pi(\mathrm{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)) \simeq 0$$

in $\mathbf{D}_{\mathbb{k}}$, which explains equivalence (11).

So indeed, the left acyclification functor a_L is left adjoint to the inclusion $i_{\mathbf{A}}$.

Because cokernel sequences and kernel sequences in $\mathbf{K}_R^{\infty}$ agree, note that we can also view the projective (resp. injective) resolution functor as the kernel (resp. cokernel) of the left (resp. right) acyclification functor. Indeed, we have ∞ -categorical co/kernel sequences

$$\begin{array}{ccc} i_{\mathbf{P}}\rho_{\mathbf{P}} & \longrightarrow & \mathrm{id}_{\mathbf{K}_R^{\infty}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{\mathbf{A}}a_L \end{array} \quad \text{and} \quad \begin{array}{ccc} i_{\mathbf{A}}a_R & \longrightarrow & \mathrm{id}_{\mathbf{K}_R^{\infty}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{\mathbf{I}}\rho_{\mathbf{I}} \end{array}$$

in $\mathrm{Fun}(\mathbf{K}_R^{\infty}, \mathbf{K}_R^{\infty})$.

6.3.7. The ∞ -category $\mathbf{K}_{\mathbb{k}}^{\infty}$ of complexes over $\mathbb{k}$ is symmetric monoidal via tensor product, and the ∞ -category $\mathbf{K}_R^{\infty}$ of complexes over R is a left module over it. It follows from Exercise 5.3(c) that the fully faithful inclusions

$$\mathbf{D}_{\mathbb{k}} \xhookrightarrow{i_L} \mathbf{K}_{\mathbb{k}}^{\infty} \quad \text{and} \quad \mathbf{D}_R \xhookrightarrow{i_L} \mathbf{K}_R^{\infty}$$

endow the derived ∞ -categories with the same structures: $\mathbf{D}_{\mathbb{k}}$ is symmetric monoidal and $\mathbf{D}_R$ is a left module over it.

Note that here we must use the inclusions i_L (not i_R). This is due to the handedness of the tensor-hom adjunction: tensor products satisfying a universal *mapping out* property (instead of a universal *mapping in* property). Said differently, it is the tensor product of *projective* complexes (and not injective complexes) that computes the tensor product of derived modules.

Relatedly, the localization $\mathbf{K}_{\mathbb{k}}^{\infty} \xrightarrow{\pi} \mathbf{D}_{\mathbb{k}}$ is *not* a symmetric monoidal functor. Rather, given complexes $M, N \in \mathbf{K}_{\mathbb{k}}^{\infty}$, tensoring the counit maps $i_L\pi(M) := i_{\mathbf{P}}\rho_{\mathbf{P}}(M) \rightarrow M$ and $i_L\pi(N) := i_{\mathbf{P}}\rho_{\mathbf{P}}(N) \rightarrow N$ together gives a map

$$i_L(\pi(M) \otimes \pi(N)) \simeq i_L\pi(M) \otimes i_L\pi(N) \longrightarrow M \otimes N$$

in $\mathbf{K}_{\mathbb{k}}^{\infty}$ that is adjoint to a map

$$(12) \quad \pi(M) \otimes \pi(N) \longrightarrow \pi(M \otimes N)$$

⁹⁶Recall from Exercise 5.3(b) that the property of projectivity is stable under homotopy equivalence: any complex that is homotopy equivalent to a projective complex is itself projective.

in $\mathbf{D}_{\mathbb{k}}$, which is not generally an equivalence.⁹⁷ Indeed, the functor π is *right-laxly* symmetric monoidal: it comes equipped with natural comparison morphisms (such as the morphism (12)) from the (iterated) tensor product in $\mathbf{D}_{\mathbb{k}}$ of its values to its value on the corresponding tensor product in $\mathbf{K}_{\mathbb{k}}^{\infty}$.^{98,99}

Inasmuch as the derived tensor product behaves better than the ordinary tensor product, we view the fact that π is only right-laxly symmetric monoidal as a feature rather than a bug. Moreover, right-laxly symmetric monoidal functors still interact nicely with algebraic structures.¹⁰⁰ For instance, in the present situation, we have a canonical lift

$$\begin{array}{ccc} \mathbf{CAlg}(\mathbf{K}_{\mathbb{k}}^{\infty}) & \xrightarrow{\pi} & \mathbf{CAlg}(\mathbf{D}_{\mathbb{k}}) \\ \text{fgt} \downarrow & & \downarrow \text{fgt} \\ \mathbf{K}_{\mathbb{k}}^{\infty} & \xrightarrow{\pi} & \mathbf{D}_{\mathbb{k}} \end{array} ,$$

so that commutative algebra objects in $\mathbf{K}_{\mathbb{k}}^{\infty}$ define commutative algebra objects in $\mathbf{D}_{\mathbb{k}}$, and likewise for associative algebra objects.¹⁰¹

6.3.8. We briefly indicate how the derived ∞ -category construction generalizes from $\mathbf{Mod}_R$ to an arbitrary abelian category $\mathcal{A}$, and give a preview of the broader context into which this fits.¹⁰²

Consider the category $\mathbf{Ch}(\mathcal{A})$ of complexes in $\mathcal{A}$. This admits an enrichment in $\mathbf{Ch}_{\mathbb{Z}}$, i.e. it defines a $\mathbb{Z}$ -linear dg-category, whose underlying $\mathbb{Z}$ -linear ∞ -category we refer to as the *∞ -category of complexes* in $\mathcal{A}$ and denote by $\mathbf{K}(\mathcal{A})$. Then, the *derived ∞ -category* of $\mathcal{A}$ is the localization

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\pi} \mathbf{D}(\mathcal{A})$$

⁹⁷Beginning with ordinary $\mathbb{k}$ -modules $M, N \in \mathbf{Mod}_{\mathbb{k}} \subseteq \mathbf{K}_{\mathbb{k}}^{\infty}$, the failure of this morphism to be an equivalence is measured by the Tor functor, as explained in §7.2. (Note that the inclusion $\mathbf{Mod}_{\mathbb{k}} \subseteq \mathbf{K}_{\mathbb{k}}^{\infty}$ is symmetric monoidal.)

⁹⁸Indeed, the right adjoint of *any* symmetric monoidal functor is automatically right-laxly symmetric monoidal. (Note that it may also simply be symmetric monoidal: this structure can be seen as a *condition* on a right-laxly symmetric monoidal functor, namely the invertibility of the natural comparison morphisms.)

⁹⁹The notion of a right-laxly symmetric monoidal functor is defined precisely in ??.

¹⁰⁰Dually, left-laxly symmetric monoidal functors interact nicely with *coalgebraic* structures.

¹⁰¹It turns out that associative algebra objects in $\mathbf{D}_{\mathbb{k}}$ always lift to associative algebra objects in $\mathbf{K}_{\mathbb{k}}$ (i.e. dgas). When $\mathbb{k}$ has characteristic zero, commutative algebra objects in $\mathbf{D}_{\mathbb{k}}$ also always lift to commutative algebra objects in $\mathbf{K}_{\mathbb{k}}$ (i.e. cdgas). However, in general there are commutative algebra objects in $\mathbf{D}_{\mathbb{k}}$ that do not lift to cdgas; the obstructions effectively arise from the cohomology of symmetric groups with coefficients in $\mathbb{k}$ (which vanish when $\mathbb{k}$ has characteristic zero).

¹⁰²In essence, an abelian category is one that “behaves like” the category $\mathbf{Mod}_R$ for some ring R . (In particular, abelian categories are canonically enriched in $\mathbf{Ab}$.) Indeed, the *Freyd–Mitchell embedding theorem* guarantees that every abelian category can be fully faithfully and exactly (i.e. kernel- and cokernel-preservingly) embedded into some $\mathbf{Mod}_R$ [Fre64, Mit64]. (More precisely, this holds under the set-theoretic assumption that the abelian category is *small*, i.e. that it only has a *set* (as opposed to a proper class) of objects.)

obtained by freely inverting the quasi-isomorphisms.

There are also the three variants $\mathbf{D}^-(\mathcal{A})$, $\mathbf{D}^+(\mathcal{A})$, and $\mathbf{D}^b(\mathcal{A})$: the bounded-above, bounded-below, and bounded derived ∞ -categories.¹⁰³ All of these $\mathbb{Z}$ -linear ∞ -categories are *stable*: they have zero objects and admit all co/kernels.¹⁰⁴

Moreover, at least these three variants may admit various universal properties. The simplest is that of $\mathbf{D}^b(\mathcal{A})$, which is the initial stable ∞ -category receiving a functor from $\mathcal{A}$ – namely, the evident composite functor

$$\mathcal{A} \longrightarrow \mathbf{K}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

(factoring through the bounded variants) – that carries all short exact sequences to co/kernel sequences [BCKW, Corollary 7.59].^{105,106} The universal properties of $\mathbf{D}^\pm(\mathcal{A})$ are somewhat more subtle: they require hypotheses on $\mathcal{A}$ (namely the existence of enough projectives/injectives and/or the property of being a *Grothendieck* abelian category), and make reference to “analytic” notions (in the sense of convergence) involving t-structures [§A.1.3].

7. DERIVED FUNCTORS

A substantial vein of current mathematical research takes place entirely within the derived realm, e.g. working only with the derived ∞ -category $\mathbf{D}_R$ instead of with the ordinary category $\mathbf{Mod}_R$. We will take this point of view when we study sheaf theory. However, it is nevertheless worthwhile to connect back with the original approach to homological algebra. We explain the general principles in §7.1, and then illustrate how passing to classical derived functors amounts to working in the derived ∞ -category in §§7.2–7.3. We then briefly discuss the example of group co/homology in §7.4. We refer the interested reader to [Wei94] for a more in-depth treatment of group co/homology, as well as for treatments of two other fundamental examples: Lie algebra co/homology and Hochschild co/homology.¹⁰⁷ We conclude with a brief discussion of decategorification in §7.5, which applies to the derived approach to the counting problems described in §1.4.

7.1. Derived functors.

¹⁰³In the notations $\mathbf{D}^\pm(\mathcal{A})$, the superscript indicates the direction in which infinitude is permitted.

¹⁰⁴We discuss (not necessarily $\mathbb{Z}$ -linear) stable ∞ -categories in ???. (The general definition of stability is rather different from the one indicated here in the linear context.)

¹⁰⁵As illustrated by Exercise 3.9, the functor $\mathcal{A} \rightarrow \mathbf{K}^b(\mathcal{A})$ does *not* generally send all short exact sequences to co/kernel sequences.

¹⁰⁶Although $\mathbf{D}^b(\mathcal{A})$ is naturally $\mathbb{Z}$ -linear, this universal property refers to *all* stable ∞ -categories. (Moreover, the corresponding assertion for $\mathbb{Z}$ -linear ∞ -categories is false, as $\mathbb{Z}$ -linearity is a *structure* rather than a *property* of a stable ∞ -category: the forgetful functor $\mathbf{St}_{\mathbb{Z}} \rightarrow \mathbf{St}$ is not fully faithful.)

¹⁰⁷However, Hochschild homology is discussed as an instance of factorization homology (the topic of §10.5) in §10.5.6.

7.1.1. As indicated in §5.1, the original motivation for homological algebra was to repair certain failures of exactness. This led us to study projective and injective resolutions through the remainder of §5, which in turn led us to study the derived ∞ -category $\mathbf{D}_R$ in §6. We defined $\mathbf{D}_R$ by a universal property (namely as the localization of $\mathbf{K}_R$ at the quasi-isomorphisms), but we immediately saw that it is canonically equivalent to *both* of the ∞ -categories $\mathbf{P}_R$ and $\mathbf{I}_R$ of projective and injective complexes. We now explain the manner in which the derived ∞ -category – as well as its dual incarnations – repairs these failures of exactness.

For this, it is appropriate to work in slightly more generality. Thus, let us consider an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories.¹⁰⁸ We say that the functor F is

- **left exact** if it commutes with finite limits,
- **right exact** if it commutes with finite colimits, and
- **exact** if it is both left exact and right exact.¹⁰⁹

Exactness can be very useful. For instance, suppose that an object $A \in \mathcal{A}$ is given to us as a finite colimit. If F is right exact, then we can deduce an analogous presentation of its image $F(A) \in \mathcal{B}$ as a finite colimit. Dually, if F is left exact, then it commutes with presentations of objects as finite limits.

As we will see in §7.1.3, one can contemplate the **right** (resp. **left**) **derived functor** of *any* additive functor F , denoted $\mathbb{R}F$ (resp. $\mathbb{L}F$), although this is most salient in the case that F is *left* (resp. *right*) exact.¹¹⁰ In any case, these derived functors will be “universal exact approximations” to F , running

$$\mathbf{D}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B}) ,$$

which will in particular commute with *both* finite colimits and finite limits – though these must now be taken in the ∞ -categorical sense. For example, given an arbitrary exact sequence

$$(13) \quad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\mathcal{A}$, we can consider it as a co/kernel sequence in $\mathbf{D}(\mathcal{A})$; writing $\mathbf{D}(\mathcal{A}) \xrightarrow{\mathbb{D}F} \mathbf{D}(\mathcal{A})$ to refer simultaneously to either derived functor of F , we obtain a co/kernel sequence

$$(14) \quad \mathbb{D}F(L) \longrightarrow \mathbb{D}F(M) \longrightarrow \mathbb{D}F(N)$$

in $\mathbf{D}(\mathcal{B})$. Note that we can read this either as a presentation

$$\mathbb{D}F(L) \simeq \ker(\mathbb{D}F(M) \longrightarrow \mathbb{D}F(N))$$

(so that $\mathbb{D}F$ commutes with the presentation $L \simeq \ker(M \rightarrow N)$ in $\mathbf{D}(\mathcal{A})$) or as a presentation

$$\mathbb{D}F(N) \simeq \operatorname{coker}(\mathbb{D}F(L) \longrightarrow \mathbb{D}F(M))$$

¹⁰⁸To say that F is *additive* means that it preserves zero objects and finite direct sums (which exist in any abelian category).

¹⁰⁹For present purposes, a category is called *finite* if it has finitely many objects and finitely many morphisms. (A slightly different definition of finiteness (for ∞ -categories) is given in ??.)

¹¹⁰The bizarreness of taking the “wrong” derived functor is illustrated in Exercises 7.5 and 7.8.

Of course, these derived functors will not generally agree with F (when precomposed with the functor $\mathcal{A} \rightarrow \mathbf{D}(\mathcal{A})$), but as it turns out they will nevertheless “remember” F by taking 0th homology – at least in the aforementioned salient cases. Moreover, it is very often the case that the derived functor of F is “better behaved” than F itself (e.g. in the context of intersection theory, as discussed in §1). From this point of view, the functor F is merely a shadow of its derived functor.

7.1.2. Before we proceed to discuss derived functors, we briefly describe an alternative (but closely related) perspective that also leads to them, which also explains the names “left exact” and “right exact” (and gives a mnemonic for remembering them).

(a) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is left exact if and only if it commutes with kernels, i.e. for every exact sequence (13) in $\mathcal{A}$ the sequence

in $\mathcal{B}$ is also exact.

(b) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is right exact if and only if it commutes with cokernels, i.e. for every exact sequence (13) in $\mathcal{A}$ the sequence

in $\mathcal{B}$ is also exact.¹¹¹

So, if F is only left exact, one is inclined to extend the sequence (15) to a long exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\mathbb{R}^0 F)(L) & \longrightarrow & (\mathbb{R}^0 F)(M) & \longrightarrow & (\mathbb{R}^0 F)(N) \longrightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (\mathbb{R}^1 F)(L) & \longrightarrow & (\mathbb{R}^1 F)(M) & \longrightarrow & (\mathbb{R}^1 F)(N) \longrightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (\mathbb{R}^2 F)(L) & \longrightarrow & \cdots & &
\end{array}$$

in $\mathcal{B}$, where by definition $\mathbb{R}^0 F := F$ and we refer to $\mathbb{R}^n F$ as the n^{th} *right derived functor* of F . Dually, if F is only right exact, one is inclined to extend the sequence (16) to a

¹¹¹For this, it is helpful to prove that any limit (in any category) can be equivalently expressed as the equalizer of a morphism between products, and that an equalizer in an abelian category can be equivalently expressed as a kernel.

long exact sequence

$$\begin{array}{ccccccc}
 & & \cdots & \longrightarrow & (\mathbb{L}_2 F)(N) & & \\
 & & & & \downarrow & & \\
 & \longrightarrow & (\mathbb{L}_1 F)(L) & \longrightarrow & (\mathbb{L}_1 F)(M) & \longrightarrow & (\mathbb{L}_1 F)(N) \\
 & & & & \downarrow & & \\
 & \longrightarrow & (\mathbb{L}_0 F)(L) & \longrightarrow & (\mathbb{L}_0 F)(M) & \longrightarrow & (\mathbb{L}_0 F)(N) \longrightarrow 0
 \end{array}$$

in $\mathcal{B}$, where by definition $\mathbb{L}_0 F := F$ and we refer to $\mathbb{L}_n F$ as the n^{th} *left derived functor* of F .¹¹²

As we saw in §3.4, long exact sequences in algebra arise from homotopy co/kernel sequences in homological algebra. Thus, we should expect to obtain these long exact sequences from homotopy co/kernel sequences, i.e. (∞ -categorical) co/kernel sequences in $\mathbf{D}(\mathcal{B})$. And indeed, as we will see in Exercise 7.3, they will arise from the co/kernel sequences (14) by defining

$$\mathbb{R}^n F := \mathbf{H}_{-n} \circ \mathbb{R}F \quad \text{and} \quad \mathbb{L}_n F := \mathbf{H}_n \circ \mathbb{L}F.$$

In order to emphasize the distinction, one may refer to $\mathbb{R}F$ and $\mathbb{L}F$ (as opposed to $\mathbb{R}^n F$ and $\mathbb{L}_n F$) as *total* derived functors of F .

7.1.3. We now introduce (total) derived functors. We actually introduce them in a slightly more general context than that described in §7.1.1. Namely, given an arbitrary functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{T} \mathcal{C}$$

to a $\mathbb{Z}$ -linear ∞ -category, we define its *right derived functor* and its *left derived functor* to respectively be the initial and terminal extensions

$$(17) \quad \begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{T} & \mathcal{C} \\ \pi \downarrow & \swarrow \text{dashed } \mathbb{R}T & \nearrow \text{dashed } \mathbb{L}T \\ \mathbf{D}(\mathcal{A}) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{T} & \mathcal{C} \\ \pi \downarrow & \searrow \text{dashed } \mathbb{L}T & \nwarrow \text{dashed } \mathbb{R}T \\ \mathbf{D}(\mathcal{A}) & & \end{array}$$

along π (which may not exist in general). Said differently, these derived functors participate in comparison maps

$$\pi \circ \mathbb{L}T \longrightarrow T \longrightarrow \pi \circ \mathbb{R}T$$

in $\mathbf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C})$, and by definition they are the universal objects of $\mathbf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C})$ equipped with such comparison maps. Hence, the composites $\pi \circ \mathbb{L}T$ and $\pi \circ \mathbb{R}T$ should be thought

¹¹²A priori, it is not clear that these n^{th} derived functors are well-defined or uniquely characterized by these conditions, although they do turn out to be so.

of two dual ways of universally forcing the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ to be sent to equivalences in $\mathcal{C}$. To apply these notions to our additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories, we take T to be the composite functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(F)} \mathbf{K}(\mathcal{B}) \xrightarrow{\pi} \mathbf{D}(\mathcal{B}) ,$$

and we denote the resulting derived functors (which are objects of $\mathbf{Fun}(\mathbf{D}(\mathcal{A}), \mathbf{D}(\mathcal{B}))$) simply by

$$\mathbb{R}F := \mathbb{R}(\pi \circ \mathbf{K}(F)) \quad \text{and} \quad \mathbb{L}F := \mathbb{L}(\pi \circ \mathbf{K}(F)) .$$

Examining the universal extensions (17), we see that these assignments $T \mapsto \mathbb{R}T$ and $T \mapsto \mathbb{L}T$ are nothing other than adjoints

$$(18) \quad \begin{array}{ccc} & \mathbb{R}(-) & \\ \swarrow \text{dashed} & \downarrow \perp & \searrow \text{dashed} \\ \mathbf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C}) & \leftarrow \mathbf{Fun}(\pi, \mathcal{C}) = & \mathbf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C}) \\ \nwarrow \text{dashed} & \downarrow \perp & \nearrow \text{dashed} \\ & \mathbb{L}(-) & \end{array}$$

to the restriction functor.¹¹³ (Note the unfortunate conventional clash that taking the *right* derived functor is a *left* adjoint, and reversely.) There are formulas for computing such adjoints in general, but they are extremely easy to compute in certain special cases, using the following.

Exercise 7.2 (10 points). Fix two categories $\mathcal{D}$ and $\mathcal{E}$ and two functors $\mathcal{D} \xrightarrow{L} \mathcal{E}$ and $\mathcal{D} \xleftarrow{R} \mathcal{E}$.

(a) Prove that to define an adjunction

$$\mathcal{D} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathcal{E} ,$$

it is necessary and sufficient to define unit and counit natural transformations $\mathrm{id}_{\mathcal{D}} \xRightarrow{\eta} RL$ and $LR \xRightarrow{\varepsilon} \mathrm{id}_{\mathcal{E}}$ that satisfy the *triangle identities*, which assert that the triangles

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{id}_L} & L \\ & \searrow L\eta & \nearrow \varepsilon_L \\ & LRL & \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{\mathrm{id}_R} & R \\ & \searrow \eta_R & \nearrow R\varepsilon \\ & RLR & \end{array}$$

commute.

(b) Using (a), show that for any category $\mathcal{F}$, applying the functors $\mathbf{Fun}(\mathcal{F}, -)$ and $\mathbf{Fun}(-, \mathcal{F})$ to the above adjunction yields adjunctions

$$\mathbf{Fun}(\mathcal{F}, \mathcal{D}) \begin{array}{c} \xrightarrow{\mathbf{Fun}(\mathcal{F}, L)} \\ \perp \\ \xleftarrow{\mathbf{Fun}(\mathcal{F}, R)} \end{array} \mathbf{Fun}(\mathcal{F}, \mathcal{E}) \quad \text{and} \quad \mathbf{Fun}(\mathcal{D}, \mathcal{F}) \begin{array}{c} \xrightarrow{\mathbf{Fun}(R, \mathcal{F})} \\ \perp \\ \xleftarrow{\mathbf{Fun}(L, \mathcal{F})} \end{array} \mathbf{Fun}(\mathcal{E}, \mathcal{F}) .^{114}$$

¹¹³In this generality, these adjoints may not exist. But when a particular derived functor exists, it is nothing other than a pointwise adjoint as indicated.

¹¹⁴It is of course also possible to establish these adjunctions directly (instead of using (a)).

Namely, in the particular case that $\mathcal{A} = \mathbf{Mod}_R$ (so that $\mathbf{K}(\mathcal{A}) = \mathbf{K}(\mathbf{Mod}_R) =: \mathbf{K}_R$), the ∞ -categorical analog of Exercise 7.2(b) identifies the adjoints (18) simply as

$$\begin{array}{ccc} & \text{Fun}(i_R, \mathcal{C}) & \\ \text{Fun}(\mathbf{K}_R, \mathcal{C}) & \xleftarrow{\text{Fun}(\pi, \mathcal{C})} \text{Fun}(\mathbf{D}_R, \mathcal{C}) & \xrightarrow{\text{Fun}(i_L, \mathcal{C})} \\ & \perp & \\ & \text{Fun}(i_L, \mathcal{C}) & \end{array} \quad .^{115}$$

In other words, recalling the discussion of co/reflection localizations from §6.3.5, passing to right derived functors amounts to injective resolution

$$\begin{array}{ccc} & \mathbf{K}_R & \xrightarrow{T} \mathcal{C} \\ & \uparrow i_R & \nearrow T^{\eta} \\ \mathbf{D}_R & \xrightarrow{\text{id}_{\mathbf{D}_R}} \mathbf{D}_R & \searrow \mathbb{R}T \\ & \downarrow \pi & \\ & \mathbf{I}_R & \end{array} \quad := \quad \begin{array}{ccc} & \mathbf{K}_R & \xrightarrow{T} \mathcal{C} \\ & \uparrow i_I & \nearrow T^{\eta} \\ \mathbf{I}_R & \xrightarrow{\text{id}_{\mathbf{I}_R}} \mathbf{I}_R & \searrow \mathbb{R}T \\ & \downarrow \rho_I & \\ & \mathbf{I}_R & \end{array} ,$$

while passing to left derived functors amounts to projective resolution

$$\begin{array}{ccc} & \mathbf{K}_R & \xrightarrow{T} \mathcal{C} \\ & \uparrow i_L & \nearrow T^{\epsilon} \\ \mathbf{D}_R & \xrightarrow{\text{id}_{\mathbf{D}_R}} \mathbf{D}_R & \searrow \mathbb{R}T \\ & \downarrow \pi & \\ & \mathbf{P}_R & \end{array} \quad := \quad \begin{array}{ccc} & \mathbf{K}_R & \xrightarrow{T} \mathcal{C} \\ & \uparrow i_P & \nearrow T^{\epsilon} \\ \mathbf{P}_R & \xrightarrow{\text{id}_{\mathbf{P}_R}} \mathbf{P}_R & \searrow \mathbb{L}T \\ & \downarrow \rho_P & \\ & \mathbf{P}_R & \end{array} .$$

Exercise 7.3 (12 points). Verify that these formulas produce the long exact sequences sought in §7.1.2. In particular,

- in the case that F is left exact, prove that

$$\mathbf{H}_n \circ \mathbb{R}F \cong \begin{cases} F, & n = 0 \\ 0, & n > 0 \end{cases} ,$$

and

- in the case that F is right exact, prove that

$$\mathbf{H}_n \circ \mathbb{L}F \cong \begin{cases} F, & n = 0 \\ 0, & n < 0 \end{cases} .$$

7.2. Tor.

7.2.1. We now study Tor, one of two classical examples of a derived functor. By definition, Tor is the left derived functor of relative tensor product. It will be illuminating to work relative to our base commutative ring $\mathbb{k}$, so we begin in that special case, and then proceed to discuss Tor over the associative $\mathbb{k}$ -algebra R .

¹¹⁵We have passed from an arbitrary abelian category $\mathcal{A}$ to the abelian category $\mathbf{Mod}_R$ in order to guarantee the existence of the adjoints i_L and i_R to the projection π . In particular, in this situation the adjoints (18) exist without any hypotheses on $\mathcal{C}$.

7.2.2. The relative tensor product bifunctor

$$\mathbf{Mod}_{\mathbb{k}} \times \mathbf{Mod}_{\mathbb{k}} \xrightarrow{(-) \otimes (-)} \mathbf{Mod}_{\mathbb{k}}$$

is right exact separately in each variable, but it is not left exact in either variable. Let us fix a $\mathbb{k}$ -module $M \in \mathbf{Mod}_{\mathbb{k}}$ and consider the resulting right exact functor

$$\mathbf{Mod}_{\mathbb{k}} \xrightarrow{M \otimes (-)} \mathbf{Mod}_{\mathbb{k}} .$$

According to the prescription of §7.1.3, we define its left derived functor $\mathbb{L}(M \otimes (-))$ as the composite

$$\mathbf{D}_{\mathbb{k}} \simeq \mathbf{P}_{\mathbb{k}} \xleftarrow{i_{\mathbf{P}}} \mathbf{K}_{\mathbb{k}} \xrightarrow{\mathbf{K}(M \otimes (-))} \mathbf{K}_{\mathbb{k}} \xrightarrow{\pi} \mathbf{D}_{\mathbb{k}} .$$

In particular, its value on ordinary $\mathbb{k}$ -modules is obtained as the upper composite in the diagram

$$\begin{array}{ccccccc} \mathbf{Mod}_{\mathbb{k}} & \hookrightarrow & \mathbf{K}_{\mathbb{k}} & \xrightarrow{\rho_{\mathbf{P}}} & \mathbf{P}_{\mathbb{k}} & \xleftarrow{i_{\mathbf{P}}} & \mathbf{K}_{\mathbb{k}} \xrightarrow{\mathbf{K}(M \otimes (-))} \mathbf{K}_{\mathbb{k}} \xrightarrow{\pi} \mathbf{D}_{\mathbb{k}} . \\ & & & \searrow \varepsilon \Downarrow & \nearrow & & \\ & & & \text{id}_{\mathbf{K}_{\mathbb{k}}} & & & \end{array}$$

Of course, we can equally well perform the same operation in the second slot. We are therefore led to consider *two* possible notions of “the left derived tensor product of M and N ”. Luckily, they agree: by Exercise 5.4(c), we have natural quasi-isomorphisms fitting into a commutative square

$$\begin{array}{ccc} & & \mathbb{L}(M \otimes (-))(N) \\ & & \Downarrow \\ \rho_{\mathbf{P}}(M) \otimes \rho_{\mathbf{P}}(N) & \xrightarrow{\approx} & M \otimes \rho_{\mathbf{P}}(N) \\ \downarrow \wr & & \downarrow \\ \rho_{\mathbf{P}}(M) \otimes N & \longrightarrow & M \otimes N \\ \Downarrow & & \\ \mathbb{L}((-) \otimes N)(M) & & \end{array}$$

in $\mathbf{K}_{\mathbb{k}}$ (omitting the functor $i_{\mathbf{P}}$ for simplicity). It is customary to simply write

$$M \overset{\mathbb{L}}{\otimes} N \in \mathbf{D}_{\mathbb{k}}$$

for the common value of these derived functors and refer to it as the **derived tensor product**. Note that this is nothing more than the tensor product in the derived ∞ -category $\mathbf{D}_{\mathbb{k}}$, as defined in §6.3.7.

For any $n \geq 0$, we define

$$\mathrm{Tor}_n(M, N) := \mathrm{Tor}_n^{\mathbb{k}}(M, N) := H_n(M \overset{\mathbb{L}}{\otimes} N) \in \mathbf{Mod}_{\mathbb{k}} .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{L}_n(M \otimes (-))(N) \cong \mathbf{Tor}_n(M, N) \cong \mathbb{L}_n((-) \otimes N)(M) .$$

The source of the name “Tor” is the following fundamental example.

Exercise 7.4 (6 points). Taking $\mathbb{k} = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute $\mathbf{Tor}_i(M, N)$ for all $i \geq 0$.

As each functor $\mathbf{Tor}_i(-, -)$ preserves finite sums separately in each variable, Exercise 7.4 effectively gives a computation of Tor for all finitely generated abelian groups.

The bizarreness of taking the “wrong” derived functor can be illustrated as follows.

Exercise 7.5 (6 points). Taking $\mathbb{k} = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute

$$\mathbb{R}^i(M \otimes (-))(N) \quad \text{and} \quad \mathbb{R}^i((-) \otimes N)(M)$$

for all $i \in \mathbb{Z}$.

7.2.3. We now briefly discuss Tor over the associative $\mathbb{k}$ -algebra R . Of course, it is the left derived functor of the bifunctor

$$\mathbf{Mod}_R \times {}_R\mathbf{Mod} \xrightarrow{(-) \otimes_R (-)} \mathbf{Mod}_{\mathbb{k}} ;$$

the subtlety is simply that this is not a monoidal structure on a single category. Once again, it is sufficient to take a projective resolution in just one of the variables.

7.2.4. It is worth mentioning a particular small (and therefore computable) model for the derived relative tensor product of ordinary R -modules.¹¹⁶ Namely, given a right R -module $M \in \mathbf{Mod}_R$ and a left R -module $N \in {}_R\mathbf{Mod}$, we define the (*two-sided*) *bar complex* $\mathbf{Bar}(M, R, N) \in \mathbf{K}_{\mathbb{k}}$ to be

$$\cdots \xrightarrow{d_3} M \otimes R^{\otimes 2} \otimes N \xrightarrow{d_2} M \otimes R \otimes N \xrightarrow{d_1} \underbrace{M \otimes N}_{\text{}} \longrightarrow 0 ,$$

where we define $d_i := \sum_{j=0}^i (-1)^j d_i^j$ and we define d_i^j by multiplying the j^{th} and $(j+1)^{\text{st}}$ tensor factors (where we count starting from 0), i.e.

$$d_i^j(m \otimes r_1 \otimes \cdots \otimes r_i \otimes n) := \begin{cases} m \cdot r_1 \otimes r_2 \otimes \cdots \otimes r_i \otimes n , & j = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_{j-1} \otimes r_j \cdot r_{j+1} \otimes r_{j+2} \otimes \cdots \otimes r_i \otimes n , & 0 < j < i \\ m \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes r_i \cdot n , & j = i \end{cases} .$$

Then, we have a canonical equivalence

$$(19) \quad \mathbf{Bar}(M, R, N) \simeq M \overset{\mathbb{L}}{\underset{R}{\otimes}} N$$

¹¹⁶This admits a straightforward generalization to complexes of R -modules, which we do not address here.

in $\mathbf{D}_{\mathbb{k}}$. This is particularly easy to see if $\mathbf{fgt}(M) \in \mathbf{Mod}_{\mathbb{k}}$ is projective (or similarly if $\mathbf{fgt}(N) \in \mathbf{Mod}_{\mathbb{k}}$ is projective), e.g. if $\mathbb{k}$ is a field.¹¹⁷ In general, if N carries a right action S -action that commutes with its left R -action, then we may consider $\mathbf{Bar}(M, R, N) \in \mathbf{K}_S$ as a complex of right S -modules. Taking $N = R = S$, we obtain a complex $\mathbf{Bar}(M, R, R) \in \mathbf{K}_R$ of right R -modules. By Exercise 5.9, this is projective: it is levelwise projective by the dual of Exercise 5.13(c), and moreover it is bounded below.

Exercise 7.6 (2 points). Construct a natural quasi-isomorphism $\mathbf{Bar}(M, R, R) \xrightarrow{\sim} M$ in $\mathbf{K}_R$.

In other words, Exercise 7.6 implies that $\mathbf{Bar}(M, R, R) \xrightarrow{\sim} M$ is a projective resolution. Now, equivalence (19) follows from the evident isomorphism $\mathbf{Bar}(M, R, R) \otimes_R N \cong \mathbf{Bar}(M, R, N)$ in $\mathbf{K}_{\mathbb{k}}$.

7.3. Ext.

7.3.1. We now study $\mathbf{Ext}$, the other classical example of a derived functor. By definition, $\mathbf{Ext}$ is the right derived functor of $\mathbf{hom}$. The $\mathbf{hom}$ bifunctor

$$\mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Mod}_R \xrightarrow{\mathbf{hom}_{\mathbf{Mod}_R}(-, -)} \mathbf{Mod}_{\mathbb{k}}$$

is left exact separately in each variable, but it is not right exact in either variable. Now, following §7.1.3 we define its two possible right derived functors as in the diagrams

$$\begin{array}{ccccccc} & & & \mathbf{D}_R & & \xrightarrow{\mathbb{R}(\mathbf{hom}_{\mathbf{Mod}_R}(M, -))} & \\ & & \wr & \downarrow & & & \\ \mathbf{Mod}_R & \hookrightarrow & \mathbf{K}_R & \xrightarrow{\rho_{\mathbf{I}}} & \mathbf{I}_R & \xrightarrow{i_{\mathbf{I}}} & \mathbf{K}_R & \xrightarrow{\mathbf{K}(\mathbf{hom}_{\mathbf{Mod}_R}(M, -))} & \mathbf{K}_{\mathbb{k}} & \xrightarrow{\pi} & \mathbf{D}_{\mathbb{k}} \\ & & \eta \uparrow & & \uparrow & & & & & & \\ & & \mathrm{id}_{\mathbf{K}_{\mathbb{k}}} & & & & & & & & \end{array}$$

and

$$\begin{array}{ccccccc} & & & \mathbf{D}_R^{\mathrm{op}} & & \xrightarrow{\mathbb{R}(\mathbf{hom}_{\mathbf{Mod}_R}(-, N))} & \\ & & \wr & \downarrow & & & \\ \mathbf{Mod}_R^{\mathrm{op}} & \hookrightarrow & \mathbf{K}_R^{\mathrm{op}} & \xrightarrow{\rho_{\mathbf{P}}^{\mathrm{op}}} & \mathbf{P}_R^{\mathrm{op}} & \xrightarrow{i_{\mathbf{P}}^{\mathrm{op}}} & \mathbf{K}_R^{\mathrm{op}} & \xrightarrow{\mathbf{K}(\mathbf{hom}_{\mathbf{Mod}_R}(-, N))} & \mathbf{K}_{\mathbb{k}} & \xrightarrow{\pi} & \mathbf{D}_{\mathbb{k}} \\ & & \varepsilon^{\mathrm{op}} \uparrow & & \uparrow & & & & & & \\ & & \mathrm{id}_{\mathbf{K}_{\mathbb{k}}^{\mathrm{op}}} & & & & & & & & \end{array} \quad .^{118}$$

¹¹⁷The proof in the general case requires techniques that we do not discuss here; see [Wei94, §8.6] for an explanation.

¹¹⁸Note that passing to opposites exchanges projectives and injectives.

Once again, these agree: by Exercise 5.3(d)(e), we have natural quasi-isomorphisms fitting into a commutative square

$$\begin{array}{ccc}
 & & \mathbb{R}(\mathrm{hom}_{\mathrm{Mod}_R}(M, -))(N) \\
 & & \Downarrow \\
 \mathrm{hom}_{\mathrm{Mod}_R}(M, N) & \longrightarrow & \mathrm{hom}_{\mathbf{K}_R}(M, \rho_{\mathbf{I}}(N)) \\
 \downarrow & & \downarrow \wr \\
 \mathrm{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M), N) & \xrightarrow{\approx} & \mathrm{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M), \rho_{\mathbf{I}}(N)) \\
 \Downarrow & & \\
 \mathbb{R}(\mathrm{hom}_{\mathrm{Mod}_R}(-, N))(M) & &
 \end{array}$$

in $\mathbf{K}_{\mathbb{k}}$. It is customary to simply write

$$\mathbb{R}\mathrm{hom}_{\mathrm{Mod}_R}(M, N) \in \mathbf{D}_{\mathbb{k}}$$

for the common value of these derived functors and refer to it as the *derived hom*. Note that this is nothing more than the hom in $\mathbf{D}_R$:

$$\mathbb{R}\mathrm{hom}_{\mathrm{Mod}_R}(M, N) \simeq \mathrm{hom}_{\mathbf{D}_R}(M, N) := \mathrm{hom}_{\mathbf{D}_R}(\pi(M), \pi(N)) .$$

For any $n \geq 0$, we define

$$\mathrm{Ext}^n(M, N) := \mathrm{Ext}_R^n(M, N) := H_{-n}(\mathbb{R}\mathrm{hom}_{\mathrm{Mod}_R}(M, N)) \in \mathrm{Mod}_{\mathbb{k}} .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{R}^n(\mathrm{hom}_{\mathrm{Mod}_R}(M, -))(N) \cong \mathrm{Ext}^n(M, N) \cong \mathbb{R}^n(\mathrm{hom}_{\mathrm{Mod}_R}(-, N))(M) .$$

Exercise 7.7 (6 points). Taking $\mathbb{k} = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute $\mathrm{Ext}^i(M, N)$ for all $i \geq 0$.

The bizarreness of taking the “wrong” derived functor can be illustrated as follows.

Exercise 7.8 (6 points). Taking $\mathbb{k} = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute

$$\mathbb{L}_i(\mathrm{hom}_{\mathrm{Mod}_{\mathbb{Z}}}(M, -))(N) \quad \text{and} \quad \mathbb{L}_i(\mathrm{hom}_{\mathrm{Mod}_{\mathbb{Z}}}(-, N))(M)$$

for all $i \in \mathbb{Z}$.

7.3.2. The name “Ext” arises from the fact that for any $n \geq 1$, $\text{Ext}_R^n(M, N)$ classifies equivalence classes of exact sequences

$$(20) \quad 0 \longrightarrow N \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0,$$

called *n-extensions* of M by N , under the relation that there exists a commutative diagram

$$(21) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & N & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow \text{id}_N & & \downarrow & & & & \downarrow & & \downarrow \text{id}_M & & \\ 0 & \longrightarrow & N & \longrightarrow & X'_{n-1} & \longrightarrow & \cdots & \longrightarrow & X'_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}.$$

Observe that we may view an n -extension (20) as a sequence of composable morphisms

$$\begin{array}{c} \Sigma^{n-1}N \\ \downarrow f \\ X_\bullet \\ \downarrow g \\ M \end{array} \quad := \quad \begin{array}{ccccccc} N & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & X_0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \widetilde{M} \end{array}$$

in $\mathbf{Ch}_R$, such that the composition equals zero. Hence, we obtain a commutative square

$$\begin{array}{ccc} \Sigma^{n-1}N & \xrightarrow{f} & X_\bullet \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & M \end{array}$$

in the ∞ -category $\mathbf{K}_R$. This may not be a co/kernel square in $\mathbf{K}_R$, but by the long exact sequence in homology it does become a co/kernel square in $\mathbf{D}_R$.¹¹⁹ As such, it is completely determined by the morphism $X_\bullet \xrightarrow{g} M$, which in turn is completely specified by the morphism $M \rightarrow \text{hcoker}(g)$. On the other hand, we have a canonical identification

$$\text{hcoker}(g) \simeq \Sigma^n N :$$

both squares in the diagram

$$(22) \quad \begin{array}{ccccc} \Sigma^{n-1}N & \xrightarrow{f} & X_\bullet & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \longrightarrow & M & \xrightarrow{k} & \text{hcoker}(g) \end{array}$$

in $\mathbf{D}_R$ are pushouts, and so the composite rectangle is also a pushout by the ∞ -categorical analog of (the first part of) the following.

¹¹⁹More precisely, there is a canonical map $\text{hcoker}(f) \rightarrow M$ in $\mathbf{K}_R$, which is a quasi-isomorphism by the long exact sequence in homology for the co/kernel sequence $\Sigma^{n-1}N \xrightarrow{f} X_\bullet \rightarrow \text{hcoker}(f)$.

Exercise 7.9 (6 points). Consider a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

in a category $\mathcal{C}$.

- (a) Show that if both squares are pushouts then so is the outer rectangle.
- (b) Show that if the left square and the outer rectangle are pushouts then so is the right square.
- (c) Give an example in which the right square and the outer rectangle are pushouts but the left square is not.

Of course, this morphism k is precisely the element of

$$\mathrm{Ext}_R^n(M, N) := \mathrm{H}_{-n}(\mathbb{R}\mathrm{hom}_{\mathrm{Mod}_R}(M, N)) \cong \mathrm{H}_{-n}(\mathrm{hom}_{\mathbf{D}_R}(M, N)) \cong \mathrm{H}_0(\mathrm{hom}_{\mathbf{D}_R}(M, \Sigma^n N))$$

that is classified by the n -extension (20). Note too that a morphism (21) of n -extensions determines a quasi-isomorphism

$$X_\bullet \xrightarrow{\sim} X'_\bullet$$

in $(\mathbf{Ch}_R)_{\Sigma^{n-1}N//M}$, which gives a homotopy between the corresponding morphisms $M \rightarrow \Sigma^n N$ in $\mathbf{D}_R$.

Exercise 7.10 (10 points).

- (a) Construct an inverse function from $\mathrm{Ext}_R^n(M, N)$ to the set of equivalence classes of n -extensions of M by N .
- (b) Describe the resulting $\mathbb{k}$ -module structure (including the abelian group structure) on the set of equivalence classes of n -extensions of M by N .

7.3.3. To simplify our discussion, let us assume that $n \geq 2$. Then, by the long exact sequence in homology, the complex $X \in \mathbf{Ch}_R$ determined by an n -extension (20) has homology groups

$$\mathrm{H}_i(X) \cong \begin{cases} M, & i = 0 \\ N, & i = n - 1 \\ 0, & \text{otherwise} \end{cases}.$$

Said differently, the derived R -module $X \in (\mathbf{D}_R)_{\Sigma^{n-1}N//M}$ has the property that its structure maps $\Sigma^{n-1}N \xrightarrow{f} X \xrightarrow{g} M$ respectively induce isomorphisms on H_{n-1} and H_0 . On the other hand, as we have seen, this does *not* characterize X as an object of $(\mathbf{D}_R)_{\Sigma^{n-1}N//M}$: equivalences $X \xrightarrow{\sim} X'$ therein correspond to homotopies $k \Rightarrow k'$ in $\mathrm{hom}_{\mathbf{D}_R}(M, \Sigma^n N)$, which are obstructed by $\mathrm{H}_0(\mathrm{hom}_{\mathbf{D}_R}(M, \Sigma^n N)) \cong \mathrm{Ext}_R^n(M, N)$. Indeed, the equivalence class of the object $X \in (\mathbf{D}_R)_{\Sigma^{n-1}N//M}$ is *equivalent data* to the morphism $M \xrightarrow{k} \Sigma^n N$, which is called the $(n-1)^{\mathrm{st}}$ ***k*-invariant** of X .

More generally, we can now make more precise the assertion of Footnote 20 that a derived R -module is specified by the data of its homology groups and its k -invariants. For this, let us write $\mathbf{D}_R^{\leq n} \subseteq \mathbf{D}_R$ for the full subcategory of n -truncated derived R -modules, i.e. those with vanishing homology above dimension n . Its inclusion admits a left adjoint

$$\mathbf{D}_R \begin{array}{c} \xrightarrow{\tau_{\leq n}} \\ \xleftarrow{\perp} \end{array} \mathbf{D}_R^{\leq n} \quad , \quad ^{120}$$

and from here it should be plausible that we have an equivalence

$$\mathrm{id}_{\mathbf{D}_R} \xrightarrow{\sim} \lim \left(\cdots \longrightarrow \tau_{\leq(n+1)} \longrightarrow \tau_{\leq n} \longrightarrow \tau_{\leq(n-1)} \longrightarrow \cdots \right)$$

in $\mathrm{Fun}(\mathbf{D}_R, \mathbf{D}_R)$:¹²¹ every derived R -module $Y \in \mathbf{D}_R$ is canonically equivalent to the limit of its *Postnikov tower*

$$\cdots \longrightarrow \tau_{\leq(n+1)} Y \longrightarrow \tau_{\leq n} Y \longrightarrow \tau_{\leq(n-1)} Y \longrightarrow \cdots .$$

Moreover, we see from the iterated co/kernel diagram

$$\begin{array}{ccccc} \Sigma^n H_n(Y) & \longrightarrow & \tau_{\leq n} Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tau_{\leq(n-1)} Y & \xrightarrow{k_n(Y)} & \Sigma^{n+1} H_n(Y) \end{array}$$

in $\mathbf{D}_R$ (akin to diagram (22)) that given $\tau_{\leq(n-1)} Y$, the data of $\tau_{\leq n} Y$ (equipped with its canonical n -truncation map to $\tau_{\leq(n-1)} Y$) is equivalent to the data of the n^{th} ***k-invariant*** of Y , namely the morphism

$$k_n(Y) \in \mathrm{hom}_{\mathbf{D}_R}(\tau_{\leq(n-1)} Y, \Sigma^{n+1} H_n(Y)) .$$

7.4. Group co/homology.

7.4.1. Let G be a discrete group. Then, a ($\mathbb{k}$ -linear) ***G-module*** is a $\mathbb{k}$ -module equipped with a (right) G -action. It is easy to see that these are equivalent to modules over the group algebra $\mathbb{k}[G]$.

Given any G -module $M \in \mathrm{Mod}_{\mathbb{k}[G]}$, its ***G-invariants*** and ***G-coinvariants*** are the $\mathbb{k}$ -modules

$$M^G := \{m \in M : m \cdot g = m \text{ for all } g \in G\} \quad \text{and} \quad M_G := M / \{m - m \cdot g\}_{m \in M, g \in G} .$$

These are respectively the maximal (i.e. terminal) $\mathbb{k}$ -submodule and minimal (i.e. initial) quotient $\mathbb{k}$ -module on which G act trivially.

¹²⁰A dual adjunction (from which this one may be immediately deduced) is essentially established in Exercise 8.7.

¹²¹This assertion is slightly subtle, because homology does *not* commute with (∞ -categorical) inverse limits in $\mathbf{D}_R$ in general. However, it *does* commute with the limits of inverse systems whose structure maps are surjective on homology.

Let us equip $\mathbb{k}$ itself with the trivial G -bimodule structure: $g \cdot \lambda := \lambda =: \lambda \cdot g$ for all $g \in G$ and for all $\lambda \in \mathbb{k}$. Then, it is clear that we have isomorphisms

$$M^G \cong \text{hom}_{\text{Mod}_{\mathbb{k}[G]}}(\mathbb{k}, M) \quad \text{and} \quad M_G \cong M \otimes_{\mathbb{k}[G]} \mathbb{k}.^{122}$$

This makes it clear how to derive these functors, and we define the **homotopy G -invariants** and **homotopy G -coinvariants** of M to be the derived $\mathbb{k}$ -modules

$$M^{\text{h}G} := \mathbb{R}\text{hom}_{\text{Mod}_{\mathbb{k}[G]}}(\mathbb{k}, M) \quad \text{and} \quad M_{\text{h}G} := M \overset{\mathbb{L}}{\otimes}_{\mathbb{k}[G]} \mathbb{k}.$$

Said differently, we have an evident *augmentation* homomorphism $\mathbb{k}[G] \rightarrow \mathbb{k}$ given by the formula $g \mapsto 1$ for all $g \in G$, and these derived functors can then be interpreted as defining adjoints

$$(23) \quad \begin{array}{ccc} & \begin{array}{c} \xrightarrow{(-)_{\text{h}G}} \\ \perp \\ \xleftarrow{\text{fgt}} \\ \perp \\ \xrightarrow{(-)^{\text{h}G}} \end{array} & \\ \mathbf{D}_{\mathbb{k}[G]} & & \mathbf{D}_{\mathbb{k}} \end{array}$$

(as in Exercise 5.13(a)). Thereafter, we recover the **cohomology** and **homology** of G with coefficients in M as

$$\mathrm{H}^n(G; M) := \mathrm{H}_{-n}(M^{\text{h}G}) =: \text{Ext}_{\mathbb{k}[G]}^n(\mathbb{k}, M) \quad \text{and} \quad \mathrm{H}_n(G; M) := \mathrm{H}_n(M_{\text{h}G}) =: \text{Tor}_n^{\mathbb{k}[G]}(M, \mathbb{k}).^{123}$$

7.4.2. In view of Exercise 5.3(c)(d), to compute the co/homology of G it suffices to choose a projective resolution of $\mathbb{k}$ as a left or right $\mathbb{k}[G]$ -module; this is certainly easier than injectively resolving M in order to compute cohomology, and if we do it once and for all then we need not projectively resolve M in order to compute homology. For this, it is standard to use the one-sidedly projective (in fact free) resolutions

$$\text{Bar}(\mathbb{k}[G], \mathbb{k}[G], \mathbb{k}) \in \left(\mathbb{k}[G] \mathbf{P} \overset{\times}{\underset{\mathbf{K}_{\mathbb{k}[G]}}{\mathbb{k}[G]}} \right) \quad \text{and} \quad \text{Bar}(\mathbb{k}, \mathbb{k}[G], \mathbb{k}[G]) \in \left(\mathbf{P}_{\mathbb{k}[G]} \overset{\times}{\underset{\mathbf{K}_{\mathbb{k}[G]}}{\mathbb{k}[G]}} \mathbf{K}_{\mathbb{k}[G]} \right).$$

Exercise 7.11 (8 points). Use bar resolutions to compute the homology and cohomology of the cyclic group $C_n := \mathbb{Z}/n$ with coefficients in the trivial module $\mathbb{Z} \in \text{Mod}_{\mathbb{Z}[C_n]}$.

In solving Exercise 7.11, one finds that group homology and group cohomology exhibit “mostly periodic” behavior. The corresponding complexes can be combined into a single periodic complex that computes *Tate cohomology*, a fascinating construction originating in the study of class field theory.

¹²²The first uses the right G -action on $\mathbb{k}$, while the second uses the left G -action on $\mathbb{k}$. (So these both carry residual right G -actions, but of course these actions are trivial.)

¹²³As we will see, the G -module M defines a local system on the space BG , and these are precisely its co/homology groups.

7.4.3. Homotopy co/invariants compose nicely. For instance, for any subgroup $H \subseteq G$, taking (homotopy) H -coinvariants is implemented by taking the (resp. derived) tensor product with $\mathbb{k}[G/H] \in \mathbb{k}[G]\mathbf{Mod}$. When H is normal this carries a commuting right action of G/H ,¹²⁴ and we find that

$$(-)_{h(G/H)} \circ (-)_{hH} := (-) \otimes_{\mathbb{k}[G]}^{\mathbb{L}} \mathbb{k}[G/H] \otimes_{\mathbb{k}[G/H]}^{\mathbb{L}} \mathbb{k} \simeq (-) \otimes_{\mathbb{k}[G]}^{\mathbb{L}} \mathbb{k} =: (-)_{hG} .^{125}$$

Of course, an analogous relation holds for homotopy invariants. From the perspective of the diagram (23) describing homotopy co/invariants as adjoint functors, we can also view these relations as arising from a diagram

$$\begin{array}{ccccc}
 & & (-)_{hG} & & \\
 & \swarrow & & \searrow & \\
 & (-)_{hH} & & (-)_{h(G/H)} & \\
 & \swarrow & & \searrow & \\
 \mathbf{D}_{\mathbb{k}[G]} & \xleftarrow{\text{fgt}} & \mathbf{D}_{\mathbb{k}[G/H]} & \xleftarrow{\text{fgt}} & \mathbf{D}_{\mathbb{k}} \\
 & \nwarrow & & \nearrow & \\
 & (-)_{hH} & & (-)_{h(G/H)} & \\
 & \swarrow & & \searrow & \\
 & & (-)_{hG} & &
 \end{array}$$

(Note: The diagram shows two hyperbolic triangles. The left triangle has vertices $\mathbf{D}_{\mathbb{k}[G]}$, $\mathbf{D}_{\mathbb{k}[G/H]}$, and $\mathbf{D}_{\mathbb{k}}$ with arrows $\mathbf{D}_{\mathbb{k}[G]} \xleftarrow{\text{fgt}} \mathbf{D}_{\mathbb{k}[G/H]$, $\mathbf{D}_{\mathbb{k}[G/H]} \xleftarrow{\text{fgt}} \mathbf{D}_{\mathbb{k}}$, and $\mathbf{D}_{\mathbb{k}[G]} \xleftarrow{(-)_{hH}} \mathbf{D}_{\mathbb{k}}$. The right triangle has vertices $\mathbf{D}_{\mathbb{k}[G/H]}$, $\mathbf{D}_{\mathbb{k}}$, and $\mathbf{D}_{\mathbb{k}}$ with arrows $\mathbf{D}_{\mathbb{k}[G/H]} \xleftarrow{\text{fgt}} \mathbf{D}_{\mathbb{k}}$, $\mathbf{D}_{\mathbb{k}} \xleftarrow{(-)_{h(G/H)}} \mathbf{D}_{\mathbb{k}}$, and $\mathbf{D}_{\mathbb{k}[G/H]} \xleftarrow{(-)_{hH}} \mathbf{D}_{\mathbb{k}}$. The outer arrows are $\mathbf{D}_{\mathbb{k}[G]} \xleftarrow{(-)_{hG}} \mathbf{D}_{\mathbb{k}}$ and $\mathbf{D}_{\mathbb{k}[G]} \xleftarrow{(-)_{hG}} \mathbf{D}_{\mathbb{k}}$.)

in which the upper and lower (hyperbolic) triangles commute by the uniqueness of adjoints.

Homotopy co/invariants admit other subtler relations.

Exercise 7.12 (6 points). Let $G * H$ denote the coproduct (a.k.a. free product) of groups G and H . For any derived $(G * H)$ -module $M \in \mathbf{D}_{\mathbb{k}[G * H]}$, construct exact squares

$$\begin{array}{ccc}
 M & \longrightarrow & M_{hG} \\
 \downarrow & & \downarrow \\
 M_{hH} & \longrightarrow & M_{h(G * H)}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 M & \longleftarrow & M^{hG} \\
 \uparrow & & \uparrow \\
 M^{hH} & \longleftarrow & M^{h(G * H)}
 \end{array}$$

in $\mathbf{D}_{\mathbb{k}}$.¹²⁶

7.5. Decategorification and algebraic K-theory.

¹²⁴More generally, this carries an action of the *Weyl group* of H in G , the quotient $W_G(H) := N_G(H)/H$ by it of its normalizer. (In fact, $W_G(H)$ is precisely the monoid of endomorphisms of G/H considered as a left G -set.)

¹²⁵Here we use the facts that the derived relative tensor product is associative and the equivalence $\mathbb{k}[G/H] \otimes_{\mathbb{k}[G/H]}^{\mathbb{L}} \mathbb{k} \xrightarrow{\sim} \mathbb{k}$ of derived left $\mathbb{k}[G]$ -modules.

¹²⁶This may be seen as a Mayer–Vietoris decomposition for (twisted) co/homology via the wedge sum decomposition $B(G * H) \simeq BG \vee BH$, as we will see below.

7.5.1. The term **decategorification** refers informally to the process of taking a mathematical object and extracting an invariant of it that lies at a lower categorical level – e.g. extracting a vector space from a (possibly linear) category, or extracting a number from a vector space. Indeed, a first example of decategorification – which appeared in §1 – is given by taking the dimension of a finite-dimensional vector space over a field $\mathbb{k}$.

Continuing to assume that $\mathbb{k}$ is a field, this decategorification admits a straightforward generalization to the derived setting. Namely, given a derived $\mathbb{k}$ -module $M \in \mathbf{D}_{\mathbb{k}}$, we can attempt to define its **Euler characteristic** to be the alternating sum

$$\chi(M) := \sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim_{\mathbb{k}}(H_n(M)) .^{127}$$

Of course, this is not always well-defined; we say that M is **perfect** when this is in fact a finite sum of integers (i.e. all homology groups of M are finite-dimensional $\mathbb{k}$ -vector spaces and only finitely many of them are nonzero). Thus, perfectness is a derived generalization of finite-dimensionality, and the Euler characteristic is well-defined for perfect derived $\mathbb{k}$ -modules. We write $\mathbf{D}_{\mathbb{k}}^{\text{perf}} \subseteq \mathbf{D}_{\mathbb{k}}$ for the full subcategory on the perfect derived $\mathbb{k}$ -modules.

Exercise 7.13 (2 points). Suppose that $M \in \mathbf{Ch}_{\mathbb{k}}$ is a bounded complex over a field $\mathbb{k}$ whose terms are all finite-dimensional vector spaces. Show that its Euler characteristic (i.e. that of its underlying derived $\mathbb{k}$ -module) is given by the formula

$$\chi(M) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim_{\mathbb{k}}(M_n) .$$

We will discuss Euler characteristics over more general base rings in §7.5.5; this is more subtle, and leads to the topic of **algebraic K-theory** (which is *itself* a decategorification, but one level up). Before that, we illustrate how Euler characteristics can be used to compute “corrected” intersection numbers in the two examples given in §1.4 (both of which take place over the field $\mathbb{R}$).

7.5.2. Let us briefly recall the setup of §1. We wished to compute algebraic intersection numbers of subvarieties of complementary codimensions. That is, we wished to compute the “expected” number of intersection points that would be literally obtained by perturbing the intersection to be transverse, without actually making such a perturbation. For a transverse intersection (and in somewhat more generality (recall §1.4.2)), we saw that computing the dimension of a corresponding tensor product gave the correct answer. It was asserted that derived tensor products would give the correct answer unconditionally.

7.5.3. As we saw in §1.4.3, the first instance where ordinary tensor products fail to give the correct answer is for the self-intersection of a single point $a \in \mathbb{R}^1$. In this case, the

¹²⁷The signs in this definition are justified by Exercise 7.18.

corresponding **derived intersection** is computed by the derived tensor product

$$(24) \quad \mathbb{R}[x]/(x-a) \underset{\mathbb{R}[x]}{\overset{\mathbb{L}}{\otimes}} \mathbb{R}[x]/(x-a) .$$

That is, we can enhance the ordinary intersection by passing to the world of **derived algebraic geometry**: the derived intersection has an underlying variety given by the ordinary intersection (which is just the point $a \in \mathbb{R}^1$), but then it has a *derived* commutative $\mathbb{R}$ -algebra of functions, namely this derived relative tensor product.¹²⁸

In order to compute this derived relative tensor product, we form the projective resolution

$$P := \left(\mathbb{R}[x] \xrightarrow{(x-a)} \underset{\sim}{\mathbb{R}[x]} \right) \xrightarrow{\sim} \mathbb{R}[x]/(x-a)$$

(as a complex of $\mathbb{R}[x]$ -modules). Then, the ordinary relative tensor product

$$P \underset{\mathbb{R}[x]}{\otimes} \mathbb{R}[x]/(x-a) \cong \left(\mathbb{R} \xrightarrow{0} \mathbb{R} \right)$$

in $\mathbf{K}_{\mathbb{R}}$ represents the derived relative tensor product (24) in $\mathbf{D}_{\mathbb{R}}$.¹²⁹ Hence, we find that

$$\chi \left(\mathbb{R}[x]/(x-a) \underset{\mathbb{R}[x]}{\overset{\mathbb{L}}{\otimes}} \mathbb{R}[x]/(x-a) \right) = \dim_{\mathbb{R}}(\mathbb{R}) - \dim_{\mathbb{R}}(\mathbb{R}) = 0 ,$$

as desired: generically, the intersection of two points in the line is empty.

7.5.4. We now explain how to compute the derived self-intersection of a projective curve in $\mathbb{RP}^2$ (as asserted in §1.4.5), using some basic facts about the cohomology of line bundles on projective space (see e.g. [Har77, §III.5]).

Given a curve $Z \subseteq X := \mathbb{RP}^2$, let us write $\mathcal{O}_Z$ for the corresponding ideal sheaf on X . Then, the **derived self-intersection** of Z with itself in X is the derived scheme

$$Y := Z \underset{X}{\overset{\mathbb{R}}{\times}} Z := \left(|Z|, \mathcal{O}_Z \underset{\mathcal{O}_X}{\overset{\mathbb{L}}{\otimes}} \mathcal{O}_Z \right)$$

(expressed as a topological space equipped with a sheaf of derived commutative $\mathbb{R}$ -algebras, and omitting the pullback functor from the notation). Now, because the underlying variety of Y is no longer just a point, in order to extract a numerical invariant we must take the Euler characteristic of Y : that is, the Euler characteristic of the derived global sections (a.k.a. (hyper)cohomology) of its structure sheaf.¹³⁰

Exercise 7.14 (8 points). For definiteness, let us take $Z \subseteq X := \mathbb{RP}^2$ to be the closure of the x -axis in $\mathbb{R}^2$, i.e. the locus $\{[x : y : z] \in \mathbb{RP}^2 : y = 0\}$.

(a) Find a line bundle $\mathcal{L}$ over X and a section s of $\mathcal{L}$ such that $Z = s^{-1}(0)$.

¹²⁸Recall from §6.3.7 that projective resolution carries commutative algebras to commutative algebras.

¹²⁹It is not hard to see that the commutative algebra structure is the evident one that exists at the point-set level on this particular chain complex representative.

¹³⁰The fact that Y arose as the intersection of two subvarieties of complementary codimension is reflected in its *virtual dimension*, which can be read off from its cotangent complex (which is introduced in §8.6.4).

This gives a short exact sequence $0 \rightarrow \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$, which we may view as defining a quasi-isomorphism $\mathcal{M} := (\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X) \xrightarrow{\sim} \mathcal{O}_Z$. Thereafter, we may compute the derived tensor product as $\mathcal{O}_Z \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{O}_Z \simeq \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$.¹³¹

- (b) Compute the Euler characteristic of $Y := Z \times_X^{\mathbb{R}} Z$, i.e. the Euler characteristic of the derived global sections (a.k.a. hypercohomology) of its structure sheaf.¹³²

These same techniques can be applied more generally.

Exercise 7.15 (8 points). For definiteness, let us take $Z \subseteq X := \mathbb{RP}^2$ to be the closure of the degree- d curve $y = x^d$ in $\mathbb{R}^2$, i.e. the locus $\{[x : y : z] \in \mathbb{RP}^2 : yz^{d-1} - x^d = 0\}$. As in Exercise 7.14, compute the Euler characteristic of $Y := Z \times_X^{\mathbb{R}} Z$.

7.5.5. The Euler characteristic of derived modules over more general rings is not a priori well-defined. This is rectified by a universal construction known as **algebraic K-theory**, as we now explain.

We begin with the following motivating observation.

Exercise 7.16 (4 points). Assume that $\mathbb{k}$ is a field.

- (a) Show that for any co/kernel sequence $L \rightarrow M \rightarrow N$ in $\mathbf{D}_{\mathbb{k}}$, if any two of L , M , and N are perfect then so is the third.
- (b) Show that perfect derived $\mathbb{k}$ -modules are closed under retracts: given any commutative triangle

$$\begin{array}{ccc} & M & \\ \nearrow \eta & & \searrow \gamma \\ N & \xrightarrow[\text{id}_N]{\sim} & N \end{array}$$

in $\mathbf{D}_{\mathbb{k}}$, if M is perfect then so is N .¹³³

Namely, we define the full subcategory $\mathbf{D}_R^{\text{perf}} \subseteq \mathbf{D}_R$ of **perfect** derived R -modules to be the smallest subcategory containing R and closed under co/kernels and retracts. Because

¹³¹That this indeed computes the derived tensor product follows from the fact that this is a *flat* resolution of $\mathcal{O}_X$ -modules. (An $\mathcal{O}_X$ -module is flat iff it is locally flat.) Note that there are no nonzero projective objects in the abelian category of $\mathcal{O}_X$ -modules, so our previous considerations do not immediately apply here.

¹³²Equivalently (by an enhancement of Exercise 7.13), one can compute the alternating sum of the Euler characteristics of [the individual terms in the specific chain complex representative](#)

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \cong \left(\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \xrightarrow{s \otimes_{\mathcal{O}_X} \text{id}_{\mathcal{O}_Z}} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Z \right) \cong \left(\mathcal{L}^{-1}|_Z \xrightarrow{s} \mathcal{O}_Z \right)$$

of its structure sheaf (which is only well-defined in the derived sense).

¹³³Recall from §6.2.3 that a commutative triangle in the $\mathbb{k}$ -linear ∞ -category $\mathbf{D}_{\mathbb{k}}$ may only be represented by a homotopy-commutative triangle in a dg-category representing it (e.g. that of projective or injective complexes); likewise, an identity morphism may only be represented by a chain map that is homotopic to the identity.

perfectness is an intrinsically derived concept, it is common to refer to perfect derived R -modules simply as “perfect R -modules”.

Clearly the zero object $0 \in \mathbf{D}_R$ is perfect, since $0 \simeq \mathbf{coker}(R \xrightarrow{\text{id}_R} R)$ (or alternatively since it is a retract of R). It follows that $\mathbf{D}_R^{\text{perf}}$ is closed under de/suspension, since we have

$$\Sigma(-) := \mathbf{coker}((-) \longrightarrow 0) \quad \text{and} \quad \Sigma^{-1}(-) \simeq \mathbf{ker}(0 \longrightarrow (-)) .$$

Thereafter, we see that $\mathbf{D}_R^{\text{perf}}$ is closed under extensions: for any co/kernel sequence $L \rightarrow M \rightarrow N$ in $\mathbf{D}_R$ we have $M \simeq \mathbf{coker}(\Sigma^{-1}N \rightarrow L)$, and so if L and N are perfect then so is M .

More interesting is the fact that $\mathbf{D}_R^{\text{perf}}$ is closed under retracts.¹³⁴ Recalling Exercise 5.5, we see that finitely generated projective R -modules are perfect when considered as derived R -modules. In fact, we have the following characterization of perfect R -modules.

Exercise 7.17 (8 points). Show that a derived R -module is perfect iff it can be presented by a bounded complex of finitely generated projective R -modules.

Having generalized perfectness from fields to arbitrary rings, we proceed to do the same for Euler characteristics. We begin with the following motivating observation.

Exercise 7.18 (2 points). Assume that $\mathbb{k}$ is a field.

- (a) Show that Euler characteristics are **additive**, i.e. for any co/kernel sequence $L \rightarrow M \rightarrow N$ in $\mathbf{D}_{\mathbb{k}}^{\text{perf}}$ we have $\chi(M) = \chi(L) + \chi(N)$.^{135,136}
- (b) Show that the Euler characteristic function

$$\{\text{objects of } \mathbf{D}_{\mathbb{k}}^{\text{perf}}\} \xrightarrow{\chi} \mathbb{Z}$$

is uniquely characterized by its additivity (in the sense of (a)) and the fact that $\chi(\mathbb{k}) = 1$.

Namely, we define the 0^{th} **algebraic K -group** of $\mathbf{D}_R^{\text{perf}}$ to be the abelian group

$$K_0(\mathbf{D}_R^{\text{perf}})$$

defined as follows: it has a generator $[M]$ for every perfect R -module $M \in \mathbf{D}_R^{\text{perf}}$, and for every co/kernel sequence $L \rightarrow M \rightarrow N$ it has a relation $[M] = [L] + [N]$.¹³⁷ (In particular, if $M \simeq M'$ then $[M] = [M']$, because the co/kernel of an equivalence is zero.) By construction, we have a “generalized Euler characteristic” function

$$\{\text{objects of } \mathbf{D}_R^{\text{perf}}\} \xrightarrow{M \mapsto [M]} K_0(\mathbf{D}_R^{\text{perf}}) ,$$

¹³⁴In the ∞ -categorical setting (unlike the 1-categorical setting), retracts *cannot* generally be expressed as finite colimits (or finite limits); see ???. (Nevertheless, it is reasonable to imagine that passing to a retract only makes an object “smaller”, which justifies the closure under retracts in the definition of $\mathbf{D}_R^{\text{perf}}$.)

¹³⁵The Euler characteristic for (suitably finite) spaces is likewise additive for cokernel (but not kernel!) sequences.

¹³⁶This fact can be used to solve Exercise 7.13.

¹³⁷The German word for “class” (as in “equivalence class”) begins with the letter “K”.

which is additive for co/kernel sequences.¹³⁸

In fact, it is not so hard to compute $K_0(\mathbf{D}_R^{\text{perf}})$. Let us define an abelian group $K_0(\text{Mod}_R^{\text{f.g.,proj}})$ as follows: it has a generator $[M]$ for each finitely generated projective R -module $M \in \text{Mod}_R^{\text{f.g.,proj}}$, and for every short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ among such it has a relation $[M] = [L] + [N]$. Observe that there is an evident homomorphism

$$(25) \quad K_0(\text{Mod}_R^{\text{f.g.,proj}}) \longrightarrow K_0(\mathbf{D}_R^{\text{perf}})$$

arising from the functor $\text{Mod}_R^{\text{f.g.,proj}} \rightarrow \mathbf{D}_R$ (which carries short exact sequences to co/kernel sequences).¹³⁹

Exercise 7.19 (10 points). Prove that there is a well-defined homomorphism

$$\begin{array}{ccc} K_0(\mathbf{D}_R^{\text{perf}}) & \longrightarrow & K_0(\text{Mod}_R^{\text{f.g.,proj}}) \\ \psi & & \psi \\ [M] & \longmapsto & \sum_{n \in \mathbb{Z}} (-1)^n \cdot [H_n(M)] \end{array}$$

that defines an inverse to the homomorphism (25).

In particular, we see that K-theory classes are insensitive to k-invariants.

As the notation indicates, it is possible to define higher (and in fact also lower) K-groups. In parallel with the discussion of §7.1.1, these constructions are motivated by a desire to repair certain failures of exactness of the functor K_0 , although now the notion of “exactness” becomes substantially more subtle. To a first approximation, this is accomplished by defining a functor K taking values in derived $\mathbb{Z}$ -modules, and we recover the n^{th} algebraic K-group as $K_n := H_n \circ K$. We refer the reader to [Wei13] for a comprehensive introduction to algebraic K-theory.

PART II. HIGHER CATEGORY THEORY

In this part, we give a rapid introduction to ∞ -category theory. We take the point of view that it is easiest to learn this theory through examples; those that we choose to highlight are provided by sheaf theory, which is the subject of ???. So, the present part will contain relatively few examples, in the interest of reaching sheaf theory as quickly as possible. In

¹³⁸Of course we could have made an analogous definition with all of $\mathbf{D}_R$, but this would be trivial as a result of the so-called “Eilenberg swindle”: for any $M \in \mathbf{D}_R$, the split co/kernel sequence

$$M \longrightarrow M \oplus M^{\oplus \mathbb{N}} \longrightarrow M^{\oplus \mathbb{N}}$$

in $\mathbf{D}_R$ yields the relation that $[M] = 0$ (since $M \oplus M^{\oplus \mathbb{N}} \simeq M^{\oplus \mathbb{N}}$).

¹³⁹In fact, this functor is a fully faithful inclusion because the fully faithful inclusion $\text{Mod}_R^{\text{f.g.,proj}} \hookrightarrow \mathbf{K}_R$ factors through the full subcategory $\mathbf{P}_R \subseteq \mathbf{K}_R$ (recall Exercise 5.9).

particular, the present account of ∞ -category theory is very far from exhaustive. Rather, it is intended to simultaneously explain

- (1) how ∞ -categories are actually used in practice, and
- (2) how this usage connects with the rigorous definitions.

We address these goals in reverse order: the foundations of ∞ -category theory are based in the theory of model categories, and so we begin with a brief summary of the latter topic.¹⁴⁰

The character of our exposition will change fairly dramatically here: we will introduce many results here without any semblance of proof, especially after §8 (although we will give references where relevant). Based on these black boxes, we will return to proving most of our assertions in ??.

8. MODEL CATEGORIES

8.1. Model categories, Quillen adjunctions, and Quillen equivalences.

8.1.1. A **relative category** is a pair $(\mathcal{C}, \mathbf{W})$ consisting of a category $\mathcal{C}$ equipped with a subcategory $\mathbf{W} \subseteq \mathcal{C}$, which is called the subcategory of **weak equivalences**. We generally denote weak equivalences by $\xrightarrow{\sim}$. Given a relative category $(\mathcal{C}, \mathbf{W})$, its **localization** (also sometimes called its **homotopy category**) is the category $\mathcal{C}[\mathbf{W}^{-1}]$ obtained from $\mathcal{C}$ by freely inverting the morphisms in $\mathbf{W}$.¹⁴¹

A **model structure** on a relative category $(\mathcal{C}, \mathbf{W})$ consists of additional structure which makes it feasible to perform computations in the localization $\mathcal{C}[\mathbf{W}^{-1}]$. In particular, this gives a means of obtaining adjunctions and equivalences among localizations of relative categories.

We first give the relevant definitions, and then illustrate them through some examples. We note here that the examples will be much more important for us than the specific details of the definitions.

8.1.2. A **model structure** on a relative category $(\mathcal{C}, \mathbf{W})$ consists of subcategories $\mathbf{C}, \mathbf{F} \subseteq \mathcal{C}$, whose morphisms are respectively called **cofibrations** and **fibrations**, which are respectively denoted by $\rightarrowtail$ and $\twoheadrightarrow$; morphisms in $\mathbf{W} \cap \mathbf{C}$ and $\mathbf{W} \cap \mathbf{F}$ are respectively called **acyclic cofibrations** and **acyclic fibrations**. These data are required to satisfy the following axioms.

- (1) The category $\mathcal{C}$ has all finite limits and colimits.
- (2) The subcategory $\mathbf{W} \subseteq \mathcal{C}$ satisfies the two-out-of-three property: given any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, if any two of the morphisms f , g , and gf lie in $\mathbf{W}$ then so does the third.

¹⁴⁰Also, some indication of how ∞ -categories are used in practice already appeared in Part I.

¹⁴¹Beware that we previously wrote $\mathcal{C}[\mathbf{W}^{-1}]$ to denote ∞ -categorical localizations.

(3) Every morphism $X \rightarrow Y$ admits factorizations

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow \wr & \nearrow \wr \\ & X' & \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow \wr & \nearrow \wr \\ & Y' & \end{array} .$$

(4) We have equalities $(\mathbf{W} \cap \mathbf{C}) = \text{llp}(\mathbf{F})$ (or equivalently $\mathbf{F} = \text{rlp}(\mathbf{W} \cap \mathbf{C})$) and $\mathbf{C} = \text{llp}(\mathbf{W} \cap \mathbf{F})$ (or equivalently $(\mathbf{W} \cap \mathbf{F}) = \text{rlp}(\mathbf{C})$).¹⁴²

In particular, axiom 4 implies that given solid commutative squares

$$\begin{array}{ccc} W & \xrightarrow{\quad} & X \\ \wr \downarrow & \nearrow & \downarrow \wr \\ Y & \xrightarrow{\quad} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \xrightarrow{\quad} & X \\ \downarrow & \nearrow & \downarrow \wr \\ Y & \xrightarrow{\quad} & Z \end{array}$$

there always exist dashed lifts making the diagrams commute. It additionally requires that these various classes of morphisms are *characterized* by these lifting conditions.¹⁴³

In general, we will notate a model category and its various attendant data using a subscript that indicates the name of the model structure. However, we will notate the weak equivalences with a subscript that indicates the more standard term for them, when one exists; for instance, we will always write $\mathbf{W}_{\text{q.i.}} \subseteq \mathbf{Ch}_R$ for the subcategory of quasi-isomorphisms.

Fix a model category $\mathcal{C}$. We say that an object $X \in \mathcal{C}$ is **cofibrant** if the unique map $\emptyset_{\mathcal{C}} \rightarrow X$ from the initial object is a cofibration, **fibrant** if the unique map $X \rightarrow \text{pt}_{\mathcal{C}}$ to the terminal object is a fibration, and **bifibrant** if it is both cofibrant and fibrant. We respectively write $\mathcal{C}^c, \mathcal{C}^f, \mathcal{C}^{cf} \subseteq \mathcal{C}$ for the full subcategories on the cofibrant, fibrant, and bifibrant objects.

Note that every object is weakly equivalent to both a cofibrant object and a fibrant object. Indeed, we define a **cofibrant resolution** and a **fibrant resolution** of an object $X \in \mathcal{C}$ to respectively be factorizations

$$\begin{array}{ccc} \emptyset_{\mathcal{C}} & \xrightarrow{\quad} & X \\ & \searrow \wr & \nearrow \wr \\ & X^c & \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\quad} & \text{pt}_{\mathcal{C}} \\ \wr \searrow & & \nearrow \wr \\ & X^f & \end{array} ;$$

these are not unique, but they are guaranteed to exist by axiom 3. Even further, axiom 3 can be used to construct a bifibrant object that is weakly equivalent to X (albeit generally only through a zigzag of weak equivalences).

¹⁴²Conditions 3 and 4 are sometimes expressed by saying that the pairs $(\mathbf{W} \cap \mathbf{C}, \mathbf{F})$ and $(\mathbf{C}, \mathbf{W} \cap \mathbf{F})$ form *weak factorization systems* on $\mathcal{C}$.

¹⁴³There are a number of slight variations on the axioms: notably, some authors strengthen axiom 1 to require that $\mathcal{C}$ have all (not necessarily finite) limits and colimits, and some authors strengthen axiom 3 to require that the factorizations can be made to be functorial. These stronger axioms hold in essentially all examples of interest.

8.1.3. The first purpose of co/fibrant objects is that they are “good for mapping from/to”, as we now explain.¹⁴⁴

Fix a model category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$. We define a **cylinder object** for X and a **path object** for Y to respectively be factorizations

$$\begin{array}{ccc} X \amalg X & \xrightarrow{(id_X, id_X)} & X \\ \swarrow (i_0, i_1) & \searrow \simeq & \uparrow \\ & cyl(X) & \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{(id_Y, id_Y)} & Y \times Y \\ \searrow \simeq & \swarrow (p_0, p_1) & \uparrow \\ & path(Y) & \end{array} ;$$

these are not unique, but they are guaranteed to exist by axiom 3. Then, for any pair of morphisms $f, g \in \text{hom}_{\mathcal{C}}(X, Y)$, we define a **left homotopy** and a **right homotopy** from f to g (with respect to these choices) to respectively be dashed morphisms

$$\begin{array}{ccc} X & & Y \\ \downarrow i_0 & \searrow f & \uparrow p_0 \\ cyl(X) & \dashrightarrow & path(Y) \\ \uparrow i_1 & \nearrow g & \downarrow p_1 \\ X & & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & & Y \\ & \nearrow f & \uparrow p_0 \\ & \dashrightarrow & path(Y) \\ & \searrow g & \downarrow p_1 \\ X & & Y \end{array}$$

making the diagrams commute. We respectively denote the existence of such homotopies by $f \stackrel{l}{\sim} g$ and $f \stackrel{r}{\sim} g$. Straightforward considerations then lead to the **fundamental theorem of model categories**: if X is cofibrant and Y is fibrant, then the map

$$\text{hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{hom}_{\mathcal{C}[W^{-1}]}(X, Y)$$

is surjective, with the equivalence relation implementing it given by either left homotopy or right homotopy (i.e. these relations are both equivalence relations and moreover they coincide). We refer the reader to [Hov99, §1.2] for a proof; we include the following to give a representative sample of the arguments involved.

Exercise 8.1 (8 points). Prove the following statements.

(a) Given a diagram

$$W \xrightarrow{e} X \xrightarrow[f]{g} Y \xrightarrow{h} Z$$

in $\mathcal{C}$, $f \stackrel{l}{\sim} g$ implies $hf \stackrel{l}{\sim} hg$, and if Y is fibrant then $f \stackrel{l}{\sim} g$ implies $fe \stackrel{l}{\sim} ge$.

(b) If X is cofibrant then $\stackrel{l}{\sim}$ defines an equivalence relation on $\text{hom}_{\mathcal{C}}(X, Y)$.

¹⁴⁴More generally, a cofibration $W \rightarrowtail X$ makes X “good for mapping from in $\mathcal{C}_{W/}$ ”, while a fibration $Y \twoheadrightarrow Z$ makes Y “good for mapping to in $\mathcal{C}_{/Z}$ ”. Indeed, a model structure on $\mathcal{C}$ determines a model structure on $\mathcal{C}_{W//Z}$ in an evident way.

- (c) If X is cofibrant and moreover $Y \xrightarrow{h} Z$ is either an acyclic fibration or a weak equivalence between fibrant objects, then postcomposition with h induces an isomorphism

$$\begin{array}{ccc} \mathrm{hom}_{\mathcal{C}}(X, Y) & \xrightarrow{h \circ (-)} & \mathrm{hom}_{\mathcal{C}}(X, Z) \\ \downarrow & & \downarrow \\ \mathrm{hom}_{\mathcal{C}}(X, Y) / \sim & \xrightarrow{\cong} & \mathrm{hom}_{\mathcal{C}}(X, Z) / \sim \end{array} .$$

- (d) Suppose X is cofibrant. Then, for any $f, g \in \mathrm{hom}_{\mathcal{C}}(X, Y)$, $f \stackrel{l}{\sim} g$ implies $f \stackrel{r}{\sim} g$. Moreover, the relation of right homotopy is realized by any fixed path object for Y .

We emphasize that the fundamental theorem of model categories should be seen as quite striking: a priori, morphisms in $\mathcal{C}[\mathbf{W}^{-1}]$ are given by zigzags (or arbitrary length) in $\mathcal{C}$ in which the backwards maps are weak equivalences, whereas it implies that for any objects $X, Y \in \mathcal{C}$, every morphism in $\mathrm{hom}_{\mathcal{C}[\mathbf{W}^{-1}]}(X, Y)$ is represented by a zigzag

$$X \xleftarrow{\sim} X^c \longrightarrow Y^f \xleftarrow{\sim} Y$$

involving arbitrary but fixed co/fibrant resolutions of X and Y .

8.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A **Quillen adjunction** is an adjunction

$$(26) \quad \mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{D}$$

such that F preserves cofibrations and acyclic cofibrations; by axiom 4, these conditions are respectively equivalent to requiring that G preserves acyclic fibrations and fibrations.

As we now explain, a Quillen adjunction (26) determines a diagram

$$\begin{array}{ccccc} & & \curvearrowright & & \\ \mathcal{C}^c[\mathbf{W}^{-1}] & \xrightarrow{\sim} & \mathcal{C}[\mathbf{W}^{-1}] & \xrightarrow{\mathbb{L}F} & \mathcal{D}[\mathbf{W}^{-1}] \\ \uparrow & & \uparrow & \Downarrow & \uparrow \\ \mathcal{C}^c & \xleftarrow{\quad} & \mathcal{C} & \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} & \mathcal{D} \xleftarrow{\quad} \mathcal{D}^f \\ & & \downarrow & \Downarrow & \downarrow \\ & & \mathcal{C}[\mathbf{W}^{-1}] & \xleftarrow{\mathbb{R}G} & \mathcal{D}[\mathbf{W}^{-1}] \xleftarrow{\sim} \mathcal{D}^f[\mathbf{W}^{-1}] \\ & & & & \curvearrowleft \end{array} ,$$

where we simply write $\mathbf{W}$ for all relevant subcategories of weak equivalences. First of all, the curved arrows arise from the easy consequence that F (resp. G) preserves weak equivalences between cofibrant (resp. fibrant) objects [Hov99, Lemma 1.1.12]. Moreover, the horizontal

equivalences follow easily from the fundamental theorem of model categories. Hence, we may define the dashed arrows so that the curved regions commute; we respectively call $\mathbb{L}F$ and $\mathbb{R}G$ the **left derived functor** of F and the **right derived functor** of G .¹⁴⁵ So by definition, for any objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we respectively have isomorphisms

$$(\mathbb{L}F)(X) \cong F(X^c) \quad \text{and} \quad (\mathbb{R}G)(Y) \cong G(Y^f)$$

in $\mathcal{D}[\mathbf{W}^{-1}]$ and $\mathcal{C}[\mathbf{W}^{-1}]$ for any co/fibrant resolutions $X^c \xrightarrow{\sim} X$ and $Y \xrightarrow{\sim} Y^f$. The natural transformations arise from the resolution maps: their components are

$$(\mathbb{L}F)(X) \cong F(X^c) \longrightarrow F(X) \quad \text{and} \quad G(Y) \longrightarrow G(Y^f) \cong (\mathbb{R}G)(Y) .^{146}$$

Of course, implicit is the assertion that these are independent of the choices of co/fibrant resolutions.

The main point of a Quillen adjunction (26) is the following result.

Exercise 8.2 (8 points). Prove that the derived functors participate in a canonical **derived adjunction**

$$\mathcal{C}[\mathbf{W}^{-1}] \begin{array}{c} \xrightarrow{\mathbb{L}F} \\ \xleftarrow{\mathbb{R}G} \end{array} \mathcal{D}[\mathbf{W}^{-1}] .$$

The Quillen adjunction is called a **Quillen equivalence** if for every $X \in \mathcal{C}^c$ and every $Y \in \mathcal{D}^f$, a morphism $X \rightarrow G(Y)$ in $\mathcal{C}$ is a weak equivalence if and only if the adjunct morphism $F(X) \rightarrow Y$ in $\mathcal{D}$ is a weak equivalence. It is clear from (the solution of) Exercise 8.2 that the derived adjunction is an equivalence if and only if the Quillen adjunction is a Quillen equivalence.

In practice, most functors between model categories do *not* preserve weak equivalences. Indeed, if a Quillen functor preserves all weak equivalences, then its point-set values compute its derived values, obviating the need for co/fibrant resolutions. Given a functor between model categories, we will say that it is **automatically derived** if it preserves weak equivalences. For instance, a left (resp. right) Quillen functor is automatically derived if in its source all objects are cofibrant (resp. fibrant). We simply write $F := \mathbb{L}F$ for the left derived functor of a left Quillen functor F that is automatically derived, and similarly we simply write $G := \mathbb{R}G$ for the right derived functor of a right Quillen functor G that is automatically derived.

8.2. Model categories of derived modules.

8.2.1. The category $\mathbf{Ch}_R$ is a relative category via the subcategory $\mathbf{W}_{\text{q.i.}} \subseteq \mathbf{Ch}_R$ of quasi-isomorphisms. Its localization is the derived category of R : $\mathbf{Ch}_R[\mathbf{W}^{-1}] \simeq \mathbf{H}_0(\mathbf{D}_R)$.

¹⁴⁵As the notation and terminology suggest, these are closely related to the derived functors introduced in §7.1.3.

¹⁴⁶Note that F and G do *not* generally preserve weak equivalences between arbitrary objects.

8.2.2. The relative category $(\mathbf{Ch}_R, \mathbf{W}_{\text{q.i.}})$ admits a **projective model structure**, which is characterized by the fact that the fibrations are the levelwise surjective chain maps. So, all objects are fibrant, and the cofibrant objects are precisely the projective complexes. Hence, all weak equivalences are fibrant resolutions, and cofibrant resolutions are simply projective resolutions.

We now discuss axiom 3. For this, consider the sets

$$I := \{S^n \hookrightarrow D^{n+1}\}_{n \in \mathbb{Z}} \quad \text{and} \quad J := \{0 \xrightarrow{\sim} D^n\}_{n \in \mathbb{Z}}$$

of morphisms in $\mathbf{Ch}_R$, where I is the set introduced in §5.5. Now, the factorizations $\xrightarrow{\sim} \twoheadrightarrow$ were constructed in §5.6 using the small object argument applied to the set I . It is easy to see that the small object argument applied to the set J yields the factorizations $\twoheadrightarrow \xrightarrow{\sim}$.¹⁴⁷ We summarize this situation by saying that the projective model structure is **cofibrantly generated** by the sets I and J , which are respectively called the set of *generating cofibrations* and the set of *generating acyclic cofibrations*.

Let us say that a morphism is a **relative I -cell complex** if it can be expressed as a transfinite composition of pushouts of elements of I (such as the morphism $c^{(\infty)}$ constructed in §5.6).¹⁴⁸ Then, the relative I -cell complexes are cofibrations, and in fact every cofibration is a retract of a relative I -cell complex. Similarly, every acyclic cofibration is a retract of a relative J -cell complex. We also have $\mathbf{F} = \mathbf{rlp}(J)$ and $(\mathbf{W} \cap \mathbf{F}) = \mathbf{rlp}(I)$: to detect (resp. acyclic) fibrations, it suffices to check the right lifting property merely against the set J (resp. the set I). Of course, these statements hold in any cofibrantly generated model category.

8.2.3. The relative category $(\mathbf{Ch}_R, \mathbf{W}_{\text{q.i.}})$ also admits an **injective model structure**, which is characterized by the fact that the cofibrations are the levelwise injective chain maps. So, all objects are cofibrant, and the fibrant objects are precisely the injective complexes. Hence, all weak equivalences are cofibrant resolutions, and fibrant resolutions are simply injective resolutions.¹⁴⁹

8.2.4. The identity adjunction defines a Quillen equivalence

$$(\mathbf{Ch}_R)_{\text{proj}} \begin{array}{c} \xrightarrow{\text{id}_{\mathbf{Ch}_R}} \\ \perp \\ \xleftarrow{\text{id}_{\mathbf{Ch}_R}} \end{array} (\mathbf{Ch}_R)_{\text{inj}}$$

in which both adjoints are automatically derived, whose derived equivalence is the identity functor on the derived category $\mathbf{H}_0(\mathbf{D}_R) := \mathbf{Ch}_R[\mathbf{W}_{\text{q.i.}}^{-1}]$.

¹⁴⁷To apply the small object argument to the set I , we needed to know that the source objects $S^n \in \mathbf{Ch}_R$ were compact (Exercise 5.18). It is trivial to verify that the object $0 \in \mathbf{Ch}_R$ is compact, so that we can indeed apply the small object argument to the set J .

¹⁴⁸To be precise, in §5.6 we constructed transfinite compositions of pushouts of *coproducts* of elements of I , but it is not hard to see that these are also I -cell complexes.

¹⁴⁹This model structure is also cofibrantly generated, but the generating sets are quite inexplicit; see [Hov99, Theorem 2.3.13].

8.2.5. The model category $(\mathbf{Ch}_{\mathbb{k}})_{\text{proj}}$ is a *symmetric monoidal model category*. In particular, this means that the bifunctor

$$(\mathbf{Ch}_{\mathbb{k}})_{\text{proj}} \times (\mathbf{Ch}_{\mathbb{k}})_{\text{proj}} \xrightarrow{(-) \otimes_{\mathbb{k}} (-)} (\mathbf{Ch}_{\mathbb{k}})_{\text{proj}}$$

is a *left Quillen bifunctor*, which in particular means that for any cofibrant object $P \in (\mathbf{Ch}_{\mathbb{k}})_{\text{proj}}^c = \mathbf{P}_{\mathbb{k}}$, the functors $P \otimes_{\mathbb{k}} (-)$ and $(-) \otimes_{\mathbb{k}} P$ are left Quillen functors (i.e. they preserve cofibrations and acyclic cofibrations). This also means that the internal hom bifunctor

$$(\mathbf{Ch}_{\mathbb{k}})_{\text{proj}}^{\text{op}} \times (\mathbf{Ch}_{\mathbb{k}})_{\text{proj}} \xrightarrow{\text{hom}_{\mathbf{Ch}_{\mathbb{k}}}(-, -)} (\mathbf{Ch}_{\mathbb{k}})_{\text{proj}}$$

is also suitably compatible with the projective model structure.

8.2.6. Similarly, the bifunctor

$$(\mathbf{Ch}_{\mathbb{k}})_{\text{proj}} \times (\mathbf{Ch}_R)_{\text{proj}} \xrightarrow{(-) \otimes_{\mathbb{k}} (-)} (\mathbf{Ch}_R)_{\text{proj}}$$

is a left Quillen bifunctor. This also means that its two-variable adjoints

$$(\mathbf{Ch}_R)_{\text{proj}}^{\text{op}} \times (\mathbf{Ch}_R)_{\text{proj}} \xrightarrow{\text{hom}_{\mathbf{Ch}_R}(-, -)} (\mathbf{Ch}_{\mathbb{k}})_{\text{proj}} \quad \text{and} \quad (\mathbf{Ch}_{\mathbb{k}})_{\text{proj}}^{\text{op}} \times (\mathbf{Ch}_R)_{\text{proj}} \xrightarrow{\text{hom}_{\mathbf{Ch}_{\mathbb{k}}}(-, -)} (\mathbf{Ch}_R)_{\text{proj}}$$

are also suitably compatible with the projective model structures.

8.3. Model categories of spaces.

8.3.1. The category $\mathbf{Top}$ is a relative category via the subcategory $\mathbf{W}_{\text{w.h.e.}} \subset \mathbf{Top}$ of weak homotopy equivalences. We refer to objects of the localization $\mathbf{Top}[\mathbf{W}_{\text{w.h.e.}}^{-1}]$ as *spaces*.¹⁵⁰ So by definition, a space is a weak homotopy equivalence class of topological spaces. For reasons that will become clear later, we write $\text{ho}(\mathcal{S}) := \mathbf{Top}[\mathbf{W}_{\text{w.h.e.}}^{-1}]$ and refer to this as *the homotopy category of spaces*.

8.3.2. The relative category $(\mathbf{Top}, \mathbf{W}_{\text{w.h.e.}})$ admits a model structure known as the *Quillen–Serre model structure*, which we denote by $\mathbf{Top}_{\text{QS}}$. This model structure is cofibrantly generated by the sets

$$I_{\text{QS}} := \{S^{n-1} \hookrightarrow D^n\}_{n \geq 0} \quad \text{and} \quad J_{\text{QS}} := \{D^n \cong D^n \times \{0\} \xrightarrow{\approx} D^n \times [0, 1]\}_{n \geq 0}.$$

So, the cofibrations are the retracts of relative cell complexes (in the standard sense), and the fibrations are precisely the Serre fibrations. In particular, the cofibrant objects are the cell complexes and their retracts, and all objects are fibrant.

In any model category $\mathcal{C}$, if X is fibrant and Y is cofibrant, then any weak equivalence $X \xrightarrow{\approx} Y$ admits a “weak inverse”, i.e. a weak equivalence $Y \xrightarrow{\approx} X$ that becomes an inverse in $\mathcal{C}[\mathbf{W}^{-1}]$. Applied to $\mathbf{Top}_{\text{QS}}$, this recovers (a slight strengthening of) *Whitehead’s theorem*.

¹⁵⁰We will later also refer to spaces as *∞ -groupoids*.

8.3.3. Recall the category Δ of finite nonempty totally ordered sets and order-preserving functions, and recall the object $[n] := \{0 < 1 < \cdots < n\} \in \Delta$ for every $n \geq 0$. Evidently, every object of Δ is isomorphic to $[n]$ for some $n \geq 0$.

The category of *cosimplicial objects* in a category $\mathcal{C}$ is the category $\mathbf{cC} := \mathbf{Fun}(\Delta, \mathcal{C})$. In general, given a cosimplicial object $X \in \mathbf{cC}$, we generally write $X^n := X([n]) \in \mathcal{C}$ for its value at the object $[n] \in \Delta$; correspondingly, we may write $X^\bullet := X$ for emphasis.

Dually, the category of *simplicial objects* in a category $\mathcal{C}$ is the category $\mathbf{sC} := \mathbf{Fun}(\Delta^{\text{op}}, \mathcal{C})$. Given a simplicial object $X \in \mathbf{sC}$, we generally write $X_n := X([n]^\circ) \in \mathcal{C}$ for its value at the object $[n]^\circ \in \Delta^{\text{op}}$; correspondingly, we may write $X_\bullet := X$ for emphasis.

Simplicial sets can be used as models for spaces. That is, the category $\mathbf{sSet}$ admits a model structure known as the *Kan–Quillen model structure*, which we denote by $\mathbf{sSet}_{\text{KQ}}$, which participates in a Quillen equivalence

$$(27) \quad \mathbf{sSet}_{\text{KQ}} \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow[\text{Sing}]{\perp} \end{array} \mathbf{Top}_{\text{QS}} \quad ,$$

as we explain shortly. Of course, it will follow that we have an equivalence $\mathbf{sSet}[\mathbf{W}_{\text{KQ}}^{-1}] \simeq \mathbf{ho}(\mathcal{S})$, and in particular that a space can *also* be defined as a weak equivalence class of simplicial set.

8.3.4. We begin with some background on simplicial sets.

We write $\Delta^n := \mathbf{hom}_\Delta(-, [n])$ for the (*combinatorial*) *n-simplex*. These assemble into a functor $\Delta \xrightarrow{\Delta^\bullet} \mathbf{sSet}$, i.e. a cosimplicial object in $\mathbf{sSet}$, which is simply the Yoneda embedding. Hence, given a finite nonempty totally ordered set $S \in \Delta$ (e.g. a nonempty subset of $[n]$ with the induced ordering), we may write $\Delta^S := \mathbf{hom}_\Delta(-, S)$ for the functor that it represents.

Given a simplicial set $X \in \mathbf{sSet}$, we refer to the set $X_n := X([n]) \cong \mathbf{hom}_{\mathbf{sSet}}(\Delta^n, X)$ as its set of *n-simplices*. We also refer to 0-simplices as *vertices* and to 1-simplices as *edges*. An *n-simplex* of X is called *degenerate* if it arises as a composition $\Delta^n \rightarrow \Delta^i \rightarrow X$ for

some $i < n$, and *nondegenerate* otherwise.¹⁵¹ The simplices of X assemble into a category

$$\Delta_{/X} := \Delta \times_{\mathbf{sSet}} \mathbf{sSet}_{/X},$$

which is called the *category of simplices* of X .

As in any presheaf category, limits and colimits in $\mathbf{sSet}$ are computed pointwise.¹⁵²

Exercise 8.3 (4 points). For every $n \geq 0$, determine the number of degenerate and nondegenerate n -simplices of $\Delta^1 \times \Delta^1$.

Exercise 8.4 (6 points). Prove that any simplicial set $X \in \mathbf{sSet}$ is the colimit of its simplices, i.e. that the canonical morphism

$$\operatorname{colim}_{(\Delta^n \downarrow X) \in \Delta_{/X}} \Delta^n \longrightarrow X$$

is an isomorphism.

We can rephrase Exercise 8.4 as saying that the left Kan extension

$$\begin{array}{ccc} \Delta & \xleftarrow{\Delta^\bullet} & \mathbf{sSet} \\ \Delta^\bullet \downarrow & \nearrow & \\ \mathbf{sSet} & & \end{array}$$

is the identity functor.¹⁵³ Indeed, in general the left Kan extension

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{F} & \mathcal{C} \\ \varphi \downarrow & \nearrow \varphi_* F & \\ \mathcal{J} & & \end{array}$$

¹⁵¹Simplicial sets have come to largely replace certain more pedestrian combinatorial objects called *simplicial complexes*, in which one only keeps track of *nondegenerate* simplices. At first glance, simplicial complexes may seem more simple and intuitive than simplicial sets, but they have significant drawbacks.

Crucially, it is not so simple to define the desired notion of a *morphism* between simplicial complexes: one would like to allow higher-dimensional simplices to collapse onto lower-dimensional ones, and this is not easily afforded within the context of simplicial complexes. Indeed, given a simplicial complex Y , one can associate a simplicial set whose set of n -simplices is precisely the set of such desired morphisms $\Delta^n \rightarrow Y$ (i.e. a choice of an i -simplex of Y and a morphism $\Delta^n \rightarrow \Delta^i$, modulo an appropriate equivalence relation). In this sense (and as amplified by Exercise 8.4), simplicial sets are the “minimal extension” of simplicial complexes that accommodate the desired notion of morphism.

More broadly, it turns out that the category of simplicial sets is remarkably (and perhaps surprisingly) well-adapted to our geometric intuition, e.g. with respect to quotients and products (see Exercise 8.3).

¹⁵²More generally, in any functor category $\operatorname{Fun}(\mathcal{J}, \mathcal{C})$, if $\mathcal{C}$ admits co/limits indexed over $\mathcal{J}$, then so does $\operatorname{Fun}(\mathcal{J}, \mathcal{C})$ and these co/limits are computed pointwise. However, beware that there may exist co/limits in $\operatorname{Fun}(\mathcal{J}, \mathcal{C})$ that are *not* computed pointwise (necessarily arising in the case that $\mathcal{C}$ does *not* admit such co/limits).

¹⁵³Note that Kan extensions along a fully faithful functor do not change the values on the full subcategory, so the natural transformation that should appear here is a natural isomorphism.

is given by the formula

$$j \longmapsto (\varphi_! F)(j) \cong \operatorname{colim} \left(\mathcal{J} \times_{\mathcal{J}} \mathcal{J}_{/j} \xrightarrow{\operatorname{fgt}} \mathcal{J} \xrightarrow{F} \mathcal{C} \right) .$$

(whenever these colimits in $\mathcal{C}$ exist).¹⁵⁴

Exercise 8.5 (6 points). Prove that any simplicial set $X \in \mathbf{sSet}$ is also the colimit of its *nondegenerate* simplices. That is, writing $\Delta_{\operatorname{nondeg}/X} \subseteq \Delta_{/X}$ for the full subcategory on the nondegenerate simplices, prove that the canonical morphism

$$\operatorname{colim}_{(\Delta^n \downarrow X) \in \Delta_{\operatorname{nondeg}/X}} \Delta^n \longrightarrow X$$

is an isomorphism.

In fact, Exercise 8.5 follows from the more general claim that *any* colimit over $\Delta_{/X}$ is isomorphic to the colimit of the restriction to $\Delta_{\operatorname{nondeg}/X}$, as we will see below.

8.3.5. We now return to the Quillen equivalence (27).

For any $n \geq 0$, the **topological n -simplex** is the topological space

$$\Delta_{\operatorname{top}}^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1 \text{ and } x_i \geq 0 \right\} \in \mathbf{Top} .$$

These assemble into a cosimplicial topological space $\Delta \xrightarrow{\Delta_{\operatorname{top}}^\bullet} \mathbf{Top}$.¹⁵⁵ We can now define the right adjoint **Sing**: it is simply the restricted Yoneda functor. That is, for any topological space $Y \in \mathbf{Top}$, its **singular simplicial set** is the functor

$$\operatorname{Sing}(Y)_\bullet : \Delta^{\operatorname{op}} \xrightarrow{\operatorname{hom}_{\mathbf{Top}}(\Delta_{\operatorname{top}}^\bullet, Y)} \mathbf{Set} .$$

We refer to the set $\operatorname{Sing}(Y)_n := \operatorname{hom}_{\mathbf{Top}}(\Delta_{\operatorname{top}}^n, Y)$ as the set of (**singular**) **n -simplices** of Y . Its left adjoint functor $|-|$ is called **geometric realization**. The most efficient definition

¹⁵⁴Heuristically, this operation can be seen as replacing each object $j \in \mathcal{J}$ with the formal colimit of the diagram $\mathcal{J} \times_{\mathcal{J}} \mathcal{J}_{/j} \xrightarrow{\operatorname{fgt}} \mathcal{J}$, which is then carried to $(\varphi_! F)(j) \in \mathcal{C}$ by (an appropriate extension of) the functor $\mathcal{J} \xrightarrow{F} \mathcal{C}$.

¹⁵⁵The functoriality of $\Delta_{\operatorname{top}}^\bullet$ is uniquely specified by the following requirements: all maps in its image are linear (in the evident sense), and the function $\operatorname{fgt}([n]) \rightarrow \operatorname{fgt}(\Delta_{\operatorname{top}}^n)$ on underlying sets taking $i \in [n]$ to the i^{th} unit basis vector assembles into a natural transformation

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta_{\operatorname{top}}^\bullet} & \mathbf{Top} \\ & \searrow \operatorname{fgt} & \downarrow \operatorname{fgt} \\ & & \mathbf{Set} \end{array} .$$

is that it is the left Kan extension

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta^\bullet_{\text{top}}} & \mathbf{Top} \\ \Delta^\bullet \downarrow & \nearrow \text{dashed arrow} & \\ \mathbf{sSet} & & \end{array}$$

of $\Delta^\bullet_{\text{top}}$ along the Yoneda embedding $\Delta^\bullet$. So by definition, for any simplicial set $X \in \mathbf{sSet}$, we have

$$|X| := \text{colim} \left(\Delta_{/X} \xrightarrow{\text{fgt}} \Delta \xrightarrow{\Delta^\bullet_{\text{top}}} \mathbf{Top} \right) .$$

So, the topological space $|X| \in \mathbf{Top}$ is built by gluing together topological n -simplices, one for each combinatorial n -simplex of X .

Exercise 8.6 (4 points). Verify the adjunction (27).

Now, we define the subcategory $\mathbf{W}_{\text{w.h.e.}} \subset \mathbf{sSet}$ of *weak homotopy equivalences* simply by pullback along $\mathbf{sSet} \xrightarrow{|\cdot|} \mathbf{Top}$. These are the weak equivalences of the Kan–Quillen model structure. The cofibrations are the monomorphisms (i.e. the levelwise injections), so that every object is cofibrant. To describe the fibrations, we introduce some notation. First of all, for any $n \geq 0$, we write $\partial\Delta^n \subseteq \Delta^n$ for the largest simplicial subset not containing the unique nondegenerate n -simplex. For any $0 \leq i \leq n$, the i^{th} **face** of Δ^n is the (nondegenerate) $(n-1)$ -simplex given by the morphism $[n-1] \xrightarrow{d_n^i} [n]$ in Δ defined by

$$d_n^i(j) = \begin{cases} j, & 0 \leq j < i \\ j+1, & i \leq j \leq n \end{cases} .$$

Then, the i^{th} **horn** of Δ^n is the largest simplicial subset $\Lambda_i^n \subseteq \Delta^n$ not containing the i^{th} face. So for instance, we may depict Λ_0^2 as

$$\begin{array}{ccc} & & 1 \\ & \nearrow & \\ 0 & \longrightarrow & 2 \end{array} .$$

Finally, the Kan–Quillen model structure is cofibrantly generated by the sets

$$I_{\text{KQ}} := \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \geq 0} \quad \text{and} \quad J_{\text{KQ}} := \{\Lambda_i^n \hookrightarrow \Delta^n\}_{0 \leq i \leq n \geq 1} .$$

In particular, an object is fibrant iff it has the extension property with respect to all horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$. A fibrant object of $\mathbf{sSet}_{\text{KQ}}$ is called a **Kan complex**.¹⁵⁶

¹⁵⁶It is worth noting that Kan complexes are necessarily very large. In fact, a Kan complex with any nondegenerate positive-dimensional simplices will necessarily have infinitely many simplices in all dimensions. In a strong sense, this largeness reflects the fundamental incalculability of unstable homotopy theory. This may be compared e.g. with the “small simplicial circle”, namely $\Delta^1/\partial\Delta^1 \in \mathbf{sSet}$: it only has one nondegenerate edge, whereas a fibrant replacement must have at least as many edges as there are homotopy classes of maps $S^1 \rightarrow S^1$.

Now, it is immediate from the definitions that the left adjoint $\mathbf{sSet}_{\mathbf{kQ}} \xrightarrow{|\cdot|} \mathbf{Top}_{\mathbf{QS}}$ is a left Quillen functor; indeed, it even carries the generating (resp. acyclic) cofibrations in $\mathbf{sSet}_{\mathbf{kQ}}$ to the generating (resp. acyclic) cofibrations in $\mathbf{Top}_{\mathbf{QS}}$ (up to certain straightforward homeomorphisms). To see that the Quillen adjunction (27) is in fact a Quillen equivalence, for any $X \in \mathbf{sSet}^c = \mathbf{sSet}$ and any $Y \in \mathbf{Top}^f = \mathbf{Top}$ we must have that a morphism $|X| \xrightarrow{f} Y$ lies in $\mathbf{W}_{\mathbf{QS}}$ iff its adjunct morphism $X \xrightarrow{f^\#} \mathbf{Sing}(Y)$ lies in $\mathbf{W}_{\mathbf{kQ}}$. By definition, the latter holds iff the morphism $|X| \xrightarrow{|f^\#|} |\mathbf{Sing}(Y)|$ lies in $\mathbf{W}_{\mathbf{QS}}$. Via the commutative triangle

$$\begin{array}{ccc} & & |\mathbf{Sing}(Y)| \\ & \nearrow |f^\#| & \downarrow \varepsilon \\ |X| & & Y \\ & \searrow f & \end{array}$$

and the two-out-of-three property for $\mathbf{W}_{\mathbf{QS}} \subset \mathbf{Top}$, it is equivalent to verify that the counit morphism $|\mathbf{Sing}(Y)| \xrightarrow{\varepsilon} Y$ lies in $\mathbf{W}_{\mathbf{QS}}$. This is a nontrivial fact, but it should be plausible; indeed, this morphism is a cellular approximation (i.e. a weak equivalence from a cell complex).

We note that both adjoints of the Quillen equivalence (27) are automatically derived, simply because $\mathbf{sSet}_{\mathbf{kQ}}^c = \mathbf{sSet}$ and $\mathbf{Top}_{\mathbf{QS}}^f = \mathbf{Top}$. Also, $\mathbf{sSet}_{\mathbf{kQ}}$ is a symmetric monoidal model category with respect to cartesian product, and $\mathbf{Top}_{\mathbf{QS}}$ is as well once restricting to a “convenient” subcategory (so that cartesian product is a two-variable left adjoint).

8.4. Homology.

8.4.1. As we now explain, homology of spaces essentially amounts to *derived linearization*. Beyond its intrinsic interest, this will be relevant for us in understanding the relationship between ∞ -categories and $\mathbb{k}$ -linear ∞ -categories.

8.4.2. We begin with a brief digression.

We write $\mathbf{Ch}_R^{\geq 0} \subseteq \mathbf{Ch}_R$ for the full subcategory on the *nonnegatively-graded complexes* of R -modules, i.e. those $M \in \mathbf{Ch}_R$ such that $M_n = 0$ for all $n < 0$. We write

$$\mathbf{H}_0(\mathbf{D}_R^{\geq 0}) := (\mathbf{Ch}_R^{\geq 0})[\mathbf{W}_{\text{q.i.}}^{-1}]$$

for its localization, and refer to it as the category of *nonnegatively-graded derived R -modules*,¹⁵⁷ as justified by Exercise 8.7.

Exercise 8.7 (6 points).

- (a) Prove that the projective model structure on $\mathbf{Ch}_R$ restricts to a (“projective”) model structure on $\mathbf{Ch}_R^{\geq 0} \subseteq \mathbf{Ch}_R$.

¹⁵⁷It is also common to call these *connective* derived R -modules. (This term is meant to sound similar to “connected”, but connectedness requires the vanishing of $\mathbf{H}_0$ as well.) This terminology has the advantage that it carries over without change when one uses cohomological indexing conventions.

(b) Construct a Quillen coreflective localization adjunction

$$(\mathbf{Ch}_R^{\geq 0})_{\text{proj}} \begin{array}{c} \xleftarrow{i_{\geq 0}} \\ \xleftarrow[\tau_{\geq 0}]{\perp} \end{array} (\mathbf{Ch}_R)_{\text{proj}} \quad ,$$

and verify that both adjoints are automatically derived.

(c) Prove that the resulting derived adjunction

$$\mathbf{H}_0(\mathbf{D}_R^{\geq 0}) := \mathbf{Ch}_R^{\geq 0}[\mathbf{W}_{\text{q.i.}}^{-1}] \begin{array}{c} \xleftarrow{i_{\geq 0}} \\ \xleftarrow[\tau_{\geq 0}]{\perp} \end{array} \mathbf{Ch}_R[\mathbf{W}_{\text{q.i.}}^{-1}] =: \mathbf{H}_0(\mathbf{D}_R)$$

is also a coreflective localization adjunction, and prove that the image of the derived left adjoint $i_{\geq 0} = \mathbb{L}(i_{\geq 0})$ consists of precisely those derived R -modules $M \in \mathbf{H}_0(\mathbf{D}_R)$ such that $\mathbf{H}_n(M) = 0$ for all $n < 0$.¹⁵⁸

8.4.3. We now study certain intermediate categories that are relevant in the definition of homology.

Given a category $\mathcal{C}$, we write $\mathbf{Mod}_R(\mathcal{C})$ for the category of right R -module objects in $\mathcal{C}$. Note that $\mathbf{Mod}_R(\mathbf{sSet}) \simeq \mathbf{sMod}_R(\mathbf{Set}) =: \mathbf{sMod}_R$.¹⁵⁹ As a special case, we have the category $\mathbf{Ab}(\mathcal{C}) := \mathbf{Mod}_{\mathbb{Z}}(\mathcal{C})$ of abelian group objects, and no intuition is lost by restricting to this special case.

It will be convenient to refer to certain *lifted* model structures. Namely, we have Quillen adjunctions

$$\mathbf{sSet}_{\text{KQ}} \begin{array}{c} \xrightarrow{R\{-\}} \\ \xleftarrow[\text{fgt}]{\perp} \end{array} (\mathbf{sMod}_R)_{\text{KQ}} \quad \text{and} \quad \mathbf{Top}_{\text{QS}} \begin{array}{c} \xrightarrow{R\{-\}} \\ \xleftarrow[\text{fgt}]{\perp} \end{array} \mathbf{Mod}_R(\mathbf{Top})_{\text{QS}} \quad . \quad ^{160,161}$$

The model structures on categories of R -module objects are “created by the right adjoints”, in the sense that a morphism is a (resp. acyclic) fibration iff its image under fgt is a (resp. acyclic) fibration. Because every object of $\mathbf{Top}_{\text{QS}}$ is fibrant, so is every object of $\mathbf{Mod}_R(\mathbf{Top})_{\text{QS}}$.

Exercise 8.8 (2 points). Show that every object of $(\mathbf{sMod}_R)_{\text{KQ}}$ is fibrant.

¹⁵⁸In other words, a derived R -module with nonnegatively-graded homology can be presented by a nonnegatively-graded complex of R -modules.

¹⁵⁹This equivalence follows from the fact that products in $\mathbf{sSet}$ are computed pointwise.

¹⁶⁰The free simplicial R -module on a simplicial set $X \in \mathbf{sSet}$ has $R\{X\}_n := R\{X_n\}$, i.e. it is given by applying the free R -module functor $\mathbf{Set} \xrightarrow{R\{-\}} \mathbf{Mod}_R$ levelwise. That is, this adjunction between simplicial objects is obtained by applying the functor $\mathbf{Fun}(\Delta^{\text{op}}, -)$ to the adjunction

$$\mathbf{Set} \begin{array}{c} \xrightarrow{R\{-\}} \\ \xleftarrow[\text{fgt}]{\perp} \end{array} \mathbf{Mod}_R \quad .$$

¹⁶¹The free topological R -module on a topological space is the free R -module on its underlying set equipped with a suitable topology. So, its elements are given by finite (unordered) configurations of points in X labeled by elements of R , and when points collide their labels are added; addition and right R -action are pointwise. (In particular, given a topological space $X \in \mathbf{Top}$ we have $R\{X\} \cong \mathbb{Z}\{X\} \otimes_{\mathbb{Z}} R$: the underlying topological abelian group of $R\{X\}$ only depends on the underlying abelian group of R .)

8.4.4. There is a close relationship between the categories $\mathbf{sMod}_R$ and $\mathbf{Ch}_R^{\geq 0}$, which is known as the **Dold–Kan correspondence**.

Namely, there are two functors $\bar{\mathbf{C}}_\bullet, \mathbf{C}_\bullet \in \mathbf{Fun}(\mathbf{sMod}_R, \mathbf{Ch}_R^{\geq 0})$, the *normalized* and *unnormalized chains* on a simplicial R -module. These are related by a commutative diagram

$$(28) \quad \begin{array}{ccc} \bar{\mathbf{C}}_\bullet & \xrightarrow{\approx} & \mathbf{C}_\bullet \\ & \searrow \text{id}_\bullet & \downarrow d \\ & & \bar{\mathbf{C}}_\bullet \end{array}$$

of natural quasi-isomorphisms, and moreover $\bar{\mathbf{C}}_\bullet$ is an equivalence of categories [Wei94, §8.3].

We do not discuss the diagram (28), but we at least define the functors that it involves. First of all, for a simplicial R -module $M \in \mathbf{sMod}_R$, we define $\mathbf{C}_\bullet(M) \in \mathbf{Ch}_R^{\geq 0}$ by setting $\mathbf{C}_n(M) = M_n$ and defining the differentials $\mathbf{C}_n(M) \xrightarrow{d_n} \mathbf{C}_{n-1}(M)$ to be

$$\mathbf{C}_n(M) := M_n \xrightarrow{\sum_{i=0}^n (-1)^i \cdot M(d_n^i)} M_{n-1} =: \mathbf{C}_{n-1}(M) .$$

Then, $\bar{\mathbf{C}}_\bullet(M) \subseteq \mathbf{C}_\bullet(M)$ is the subcomplex defined levelwise by

$$\bar{\mathbf{C}}_n(M) := \bigcap_{i=0}^{n-1} \ker \left(M_n \xrightarrow{M(d_n^i)} M_{n-1} \right) ,$$

so that its differential is simply $\bar{\mathbf{C}}_n(M) \xrightarrow{(-1)^n \cdot M(d_n^n)} \bar{\mathbf{C}}_{n-1}(M)$.

Exercise 8.9 (8 points). For any simplicial R -module $M \in \mathbf{sMod}_R$ and any $n \geq 0$, establish a commutative diagram

$$\begin{array}{ccc} \mathrm{hom}_{\mathbf{sSet}_*}(\Delta^n / \partial \Delta^n, \mathrm{fgt}(M)) & \xrightarrow{\cong} & Z_n(\bar{\mathbf{C}}_\bullet(M)) \\ \uparrow & & \uparrow d_{n+1} \\ \mathrm{hom}_{\mathbf{sSet}_*}(\Delta^{n+1} / \Lambda_{n+1}^{n+1}, \mathrm{fgt}(M)) & \xrightarrow{\cong} & \bar{\mathbf{C}}_{n+1}(M) \end{array}$$

in $\mathbf{Mod}_R$ (where we consider $\mathrm{fgt}(M) := \mathrm{fgt}(0 \rightarrow M) \in \mathbf{sSet}_*$), and use this to deduce an isomorphism

$$\pi_n(\mathrm{fgt}(M)) \cong H_n(\mathbf{C}_\bullet(M))$$

in $\mathbf{Mod}_R$.

It follows from Exercise 8.9 and the above discussion that the functors $\bar{\mathbf{C}}_\bullet$ and $\mathbf{C}_\bullet$ induce canonically equivalent equivalences

$$\mathbf{sMod}_R[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}] \xrightarrow{\sim} \mathbf{Ch}_R^{\geq 0}[\mathbf{W}_{\mathrm{q.i.}}^{-1}]$$

on localizations.¹⁶² So, the model category $(\mathbf{sMod}_R)_{\mathrm{KQ}}$ is a presentation of $H_0(\mathbf{D}_R^{\geq 0})$.

¹⁶²In fact, it is not hard to see that the equivalence of categories

$$(\mathbf{sMod}_R)_{\mathrm{KQ}} \xrightarrow{\bar{\mathbf{C}}_\bullet} (\mathbf{Ch}_R^{\geq 0})_{\mathrm{proj}}$$

8.4.5. Now, both singular and simplicial homology (with coefficients in R) may be located in the diagram

$$(29) \quad \begin{array}{ccc} \text{Top}_{\text{QS}} & \xrightleftharpoons[\text{fgt}]{R\{-\}} & \text{Mod}_R(\text{Top})_{\text{QS}} \\ \uparrow \downarrow \text{Sing} & & \uparrow \downarrow \text{Mod}_R(|-|) \dashv \text{Mod}_R(\text{Sing}) \\ \text{sSet}_{\text{KQ}} & \xrightleftharpoons[\text{fgt}]{R\{-\}} & (\text{sMod}_R)_{\text{KQ}} \\ & \downarrow \text{C}_\bullet & \\ & (\text{Ch}_R)_{\text{proj}} & \end{array} \quad \begin{array}{c} \nearrow \tau_* \\ \searrow H_* \end{array} \quad \text{Fun}(\mathbb{N}, \text{Mod}_R) \quad :$$

namely, *simplicial homology* is the composite functor

$$H_*^\Delta(-; R) : \text{sSet} \xrightarrow{R\{-\}} \text{sMod}_R \xrightarrow{\text{C}_\bullet} \text{Ch}_R \xrightarrow{H_*} \text{Fun}(\mathbb{N}, \text{Mod}_R) ,$$

and *singular homology* is the composite functor

$$H_*^{\text{sing}}(-; R) : \text{Top} \xrightarrow{\text{Sing}} \text{sSet} \xrightarrow{H_*^\Delta(-; R)} \text{Fun}(\mathbb{N}, \text{Mod}_R) .^{163}$$

The adjunction $\text{Mod}_R(|-| \dashv \text{Sing})$ arises from the fact that both functors in the adjunction $| - | \dashv \text{Sing}$ preserve finite products, and it is clear that the square in diagram (29) commutes after omitting all left adjoints or all right adjoints.

Exercise 8.10 (4 points). Prove that the adjunction $\text{Mod}_R(|-| \dashv \text{Sing})$ is a Quillen equivalence.

Combining Exercise 8.10 with the discussion of §8.4.4, we see that the model category $\text{Mod}_R(\text{Top})_{\text{QS}}$ is also a presentation of $\mathbf{H}_0(\mathbf{D}_R^{\geq 0})$ and that the triangle in diagram (29) commutes.

From diagram (29), we also see that homology with coefficients in R is indeed derived R -linearization; note that both composites

$$\text{C}_\bullet^\Delta(-; R) : \text{sSet} \xrightarrow{R\{-\}} \text{sMod}_R \xrightarrow{\text{C}_\bullet} \text{Ch}_R$$

and

$$\text{C}_\bullet^{\text{sing}}(-; R) : \text{Top} \xrightarrow{\text{Sing}} \text{sSet} \xrightarrow{\text{C}_\bullet^\Delta(-; R)} \text{Ch}_R$$

is both a left and right Quillen equivalence (using the notation of Exercise 8.7).

¹⁶³More generally, homology with coefficients in a (possibly derived) R -module is obtained by taking the (resp. derived) tensor product before taking homology of complexes.

are automatically derived.¹⁶⁴ Indeed, diagram (29) yields the commutative diagram

$$\begin{array}{ccc}
 \text{ho}(\mathcal{S}) & & H_0(\mathbf{D}_R^{\geq 0}) \\
 \parallel & & \parallel \\
 \text{Top}[\mathbf{W}_{\text{w.h.e.}}^{-1}] & \xrightleftharpoons[\text{fgt}]{\mathbb{L}(R\{-\})} & \text{Mod}_R(\text{Top})[\mathbf{W}_{\text{w.h.e.}}^{-1}] \\
 \uparrow \downarrow \text{Sing} & & \uparrow \downarrow \text{Mod}_R(\text{Sing}) \\
 \text{sSet}[\mathbf{W}_{\text{w.h.e.}}^{-1}] & \xrightleftharpoons[\text{fgt}]{R\{-\}} & \text{sMod}_R[\mathbf{W}_{\text{w.h.e.}}^{-1}] \\
 & & \downarrow i_{\geq 0} \\
 & & \text{Ch}_R[\mathbf{W}_{\text{q.i.}}^{-1}] \\
 & & \parallel \\
 & & H_0(\mathbf{D}_R)
 \end{array}
 \quad
 \begin{array}{c}
 \text{Fun}(\mathbb{N}, \text{Mod}_R) \\
 \nearrow \pi_* \\
 \searrow H_*
 \end{array}$$

on localizations.¹⁶⁵ From this, we immediately deduce the **Dold–Thom theorem**: for any cofibrant object $X \in \text{Top}_{\text{QS}}$ (i.e. a cell complex or retract thereof), we have a canonical isomorphism

$$\pi_*(R\{X\}) \cong H_*^{\text{sing}}(X; R) .$$

Said differently, the singular homology of X with coefficients in R is computed by the homotopy groups of the free topological R -module on X .¹⁶⁶ We also deduce a sort of free/forget adjunction

$$\text{ho}(\mathcal{S}) := \text{Top}[\mathbf{W}_{\text{w.h.e.}}^{-1}] \xrightleftharpoons[\text{fgt} \circ \tau_{\geq 0}]{\mathcal{C}^{\text{sing}}(-; R)} \text{Ch}_R[\mathbf{W}_{\text{q.i.}}^{-1}] =: H_0(\mathbf{D}_R)$$

between the homotopy category of spaces and the category of derived R -modules.¹⁶⁷

¹⁶⁴As always, we prefer to interpret these “homology” functors without actually passing to homology groups of complexes.

¹⁶⁵That is, the two adjunctions are identified via the equivalences, so that in particular the diagram commutes upon omitting both left adjoints or both right adjoints.

¹⁶⁶More generally, given an R -module $M \in \text{Mod}_R$, let us write $M\{X\} := M \otimes_R R\{X\} \cong M \otimes_{\mathbb{Z}} \mathbb{Z}\{X\}$. (The points of this space are now given by finite configurations of points in X labeled by elements of M .) Then, still assuming that $X \in \text{Top}_{\text{QS}}$ is cofibrant, we have a canonical isomorphism $\pi_*(M\{X\}) \cong H_*^{\text{sing}}(X; M)$.

¹⁶⁷Note that this adjunction restricts to the free/forget adjunction

$$\text{Set} \xrightleftharpoons[\text{fgt}]{R\{-\}} \text{Mod}_R$$

between sets and R -modules. This analogy will be amplified below.

8.5. Homotopy co/limits.

8.5.1. Given a category $\mathcal{J}$ and a model category $\mathcal{C}$, we obtain a relative category $(\mathrm{Fun}(\mathcal{J}, \mathcal{C}), \mathbf{W})$ whose weak equivalences are the componentwise weak equivalences. Under various conditions (which are satisfied in all our cases of interest) there exist *projective* and *injective* model structures on this relative category, which are defined so that the adjunctions

$$\mathrm{Fun}(\mathcal{J}, \mathcal{C})_{\mathrm{proj}} \begin{array}{c} \xrightarrow{\mathrm{colim}} \\ \perp \\ \xleftarrow{\mathrm{const}} \end{array} \mathcal{C} \quad \text{and} \quad \mathcal{C} \begin{array}{c} \xrightarrow{\mathrm{const}} \\ \perp \\ \xleftarrow{\mathrm{lim}} \end{array} \mathrm{Fun}(\mathcal{J}, \mathcal{C})_{\mathrm{inj}}$$

are Quillen adjunctions. The derived adjoints $\mathbb{L}\mathrm{colim}$ and $\mathbb{R}\mathrm{lim}$ are respectively called the *homotopy colimit* and the *homotopy limit*. In particular, homotopy co/limits preserve componentwise weak equivalences. We will discuss specific examples of homotopy co/limits below.

8.5.2. Being a derived functor, the homotopy co/limit of a diagram is computed as the ordinary co/limit of an appropriate co/fibrant resolution of the diagram. However, there are various assumptions under which the ordinary co/limit of a diagram also computes its homotopy co/limit, which will be convenient for us.

- (1) (a) To compute a homotopy pushout (i.e. the homotopy colimit of a span), it suffices to assume that all objects are cofibrant and that at least one of the morphisms is a cofibration.
- (b) Dually, to compute a homotopy pullback (i.e. the homotopy limit of a cospan), it suffices to assume that all objects are fibrant and that at least one of the morphisms is a fibration.
- (2) (a) To compute a sequential homotopy colimit (i.e. a homotopy colimit over $\mathbb{N}^{\leq}$), it suffices to assume that all objects are cofibrant and that all morphisms are cofibrations.
- (b) Dually, to compute a sequential homotopy limit (i.e. a homotopy limit over $(\mathbb{N}^{\leq})^{\mathrm{op}}$), it suffices to assume that all objects are fibrant and that all morphisms are fibrations.

8.6. A few other model categories.

8.6.1. We conclude §8 by briefly discussing three other model categories; the first will be relevant to us in §9, and the latter two are interesting (or at least curious) in their own rights.

8.6.2. The category $\mathbf{cat}$ of categories admits a *canonical model structure*, in which the weak equivalences are the equivalences of categories and the cofibrations are the functors that are injective on objects. The fibrations are the *isofibrations*, i.e. the functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ such that any isomorphism $F(C) \xrightarrow{\cong} D$ in $\mathcal{D}$ lifts to an isomorphism $C \xrightarrow{\cong} C'$ in $\mathcal{C}$. In

particular, every object is bifibrant, and morphisms in $\mathbf{cat}[\mathbf{W}^{-1}]$ are equivalence classes of functors under the relation of natural isomorphism.

8.6.3. The category $\mathbf{Top}$ is a relative category via the subcategory $\mathbf{W}_{\text{h.e.}} \subset \mathbf{Top}$ of homotopy equivalences. The relative category $(\mathbf{Top}, \mathbf{W}_{\text{h.e.}})$ admits a **Strøm model structure**, whose fibrations are the Hurewicz fibrations. Because every homotopy equivalence is a weak homotopy equivalence and every Hurewicz fibration is a Serre fibration, the identity adjunction is a Quillen adjunction

$$\mathbf{Top}_{\text{QS}} \begin{array}{c} \xrightarrow{\text{id}_{\mathbf{Top}}} \\ \perp \\ \xleftarrow{\text{id}_{\mathbf{Top}}} \end{array} \mathbf{Top}_{\text{Strøm}} .$$

The right adjoint is automatically derived, and from this it is easy to check that the derived adjunction

$$\mathbf{ho}(\mathcal{S}) := \mathbf{Top}[\mathbf{W}_{\text{w.h.e.}}^{-1}] \begin{array}{c} \xleftarrow{\mathbb{L}(\text{id}_{\mathbf{Top}})} \\ \perp \\ \xrightarrow{\mathbb{R}(\text{id}_{\mathbf{Top}})} \end{array} \mathbf{Top}[\mathbf{W}_{\text{h.e.}}^{-1}]$$

is a right localization adjunction; its counit may be thought of as cellular approximation. ¹⁶⁸

8.6.4. Let $\mathbf{CAlg}_{\mathbb{k}} := \mathbf{CAlg}(\mathbf{Mod}_{\mathbb{k}})$ denote the category of commutative $\mathbb{k}$ -algebras. For any commutative $\mathbb{k}$ -algebra $A \in \mathbf{CAlg}_{\mathbb{k}}$ and any A -module $M \in \mathbf{Mod}_A$, a **$\mathbb{k}$ -linear derivation of A into M relative to $\mathbb{k}$** is a $\mathbb{k}$ -module homomorphism $A \xrightarrow{d} M$ such that the composite $\mathbb{k} \xrightarrow{\eta} A \xrightarrow{d} M$ is zero and satisfying the Leibniz rule that $d(f \cdot g) = f \cdot d(g) + g \cdot d(f)$. It is not hard to construct a universal A -module equipped with a derivation of A , namely the A -module of **Kähler differentials**

$$\Omega_{A|\mathbb{k}}^1 := A\{da\}_{a \in A} \Bigg/ \left\{ \begin{array}{l} d(a+b) = da + db \text{ for all } a, b \in A \\ d(\eta(\lambda)) = 0 \text{ for all } \lambda \in \mathbb{k} \\ d(a \cdot b) = a \cdot db + b \cdot da \text{ for all } a, b \in A \end{array} \right\} ,$$

with universal derivation $A \rightarrow \Omega_{A|\mathbb{k}}^1$ given by the formula $a \mapsto da$.

This construction plays the role of **differential 1-forms** in algebraic geometry, so that the A -module $\Omega_{A|\mathbb{k}}^1$ is the (relative) cotangent space. Therefore, it should not be surprising that it does not behave well when A is singular. It was for this reason that Quillen invented model categories: namely, in order to take the *derived functor of derivations*.

In order to explain this, we begin by reformulating the construction of Kähler differentials.

Exercise 8.11 (6 points).

(a) Construct inverse equivalences

$$\mathbf{Ab}((\mathbf{CAlg}_{\mathbb{k}})_{/A}) \begin{array}{c} \xrightarrow{\text{kernel of map to } A} \\ \sim \\ \xleftarrow{\text{square-zero extension}} \end{array} \mathbf{Mod}_A .$$

¹⁶⁸This should be compared with the adjunction $i_{\mathbf{P}} \dashv \rho_{\mathbf{P}}$ of §6.3.4 (whose counit is given by projective resolution).

(b) Prove that the left adjoint in the adjunction

$$(30) \quad (\mathbf{CAlg}_{\mathbb{k}})_{/A} \begin{array}{c} \xrightarrow{\text{free}} \\ \perp \\ \xleftarrow{\text{forget}} \end{array} \mathbf{Ab}((\mathbf{CAlg}_{\mathbb{k}})_{/A}) \begin{array}{c} \xrightarrow{\text{kernel of map to } A} \\ \sim \\ \xleftarrow{\text{square-zero extension}} \end{array} \mathbf{Mod}_A$$

carries $A \in (\mathbf{CAlg}_{\mathbb{k}})_{/A}$ to $\Omega_{A|\mathbb{k}}^1 \in \mathbf{Mod}_A$.

Now, Quillen endowed the category $\mathbf{sCAlg}_{\mathbb{k}}$ of *simplicial* commutative $\mathbb{k}$ -algebras with a model structure whose weak equivalences and fibrations are created by the right adjoint in the adjunction

$$\mathbf{sSet}_{\mathbf{KQ}} \begin{array}{c} \xrightarrow{\mathbb{k}[-]} \\ \perp \\ \xleftarrow{\text{fgt}} \end{array} \mathbf{sCAlg}_{\mathbb{k}}$$

(as in §8.4.3). Because $\mathbf{sSet}_{\mathbf{KQ}}$ is cofibrantly generated, so is $(\mathbf{CAlg}_{\mathbb{k}})_{\text{Quillen}}$, and the cofibrant objects are the levelwise free commutative $\mathbb{k}$ -algebras such that the degeneracy maps carry generators to generators. Meanwhile, by Exercise 8.8, all objects of $(\mathbf{CAlg}_{\mathbb{k}})_{\text{Quillen}}$ are fibrant. As a result, applying $\mathbf{Fun}(\Delta^{\text{op}}, (-))$ to the adjunction (30) yields a Quillen adjunction

$$(\mathbf{s}(\mathbf{CAlg}_{\mathbb{k}})_{/A})_{\text{Quillen}} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} (\mathbf{sMod}_A)_{\mathbf{KQ}}$$

(because the right adjoint preserves fibrations and weak equivalences). The value of the derived left adjoint on the object A is called the *cotangent complex* of A , a derived A -module denoted

$$\mathbb{L}\Omega_{A|\mathbb{k}} \in (\mathbf{sMod}_A)[\mathbf{W}_{\mathbf{KQ}}^{-1}] \simeq \mathbf{H}_0(\mathbf{D}_A^{\geq 0}),$$

and its homology groups are called the *André–Quillen homology groups* of A .

Exercise 8.12 (6 points). Take $A = \mathbb{k}[x]/x^2 \in \mathbf{CAlg}_{\mathbb{k}} \subseteq \mathbf{sCAlg}_{\mathbb{k}}$.

- (a) Construct a cofibrant resolution of $A \in (\mathbf{sCAlg}_{\mathbb{k}})_{\text{Quillen}}$ (which will also be a cofibrant resolution in $((\mathbf{sCAlg}_{\mathbb{k}})_{/A})_{\text{Quillen}}$).
- (b) Compute the André–Quillen homology groups of A .

9. BASIC NOTIONS IN ∞ -CATEGORY THEORY

9.1. Model categories of ∞ -categories.

9.1.1. We now introduce ∞ -categories. Our discussion is effectively a more in-depth treatment of the nonlinear version of the introduction to $\mathbb{k}$ -linear ∞ -categories given in §6.2. For more details, we refer the reader to [§T.1].

9.1.2. We write $\mathbf{cat}(\mathcal{V})$ for the category of categories enriched in a monoidal category $\mathcal{V} := (\mathcal{V}, \boxtimes)$. If $\mathcal{V}$ is *symmetric* monoidal, then $\mathbf{cat}(\mathcal{V})$ is symmetric monoidal as well: for any $\mathcal{C}, \mathcal{D} \in \mathbf{cat}(\mathcal{V})$, an object of $\mathcal{C} \boxtimes \mathcal{D}$ is a pair of objects of $\mathcal{C}$ and $\mathcal{D}$, and we define

$$\mathrm{hom}_{\mathcal{C} \boxtimes \mathcal{D}}((C, D), (C', D')) := \mathrm{hom}_{\mathcal{C}}(C, C') \boxtimes \mathrm{hom}_{\mathcal{D}}(D, D') .$$

As a particular case, we have $\mathbf{cat}_{\mathbf{k}}^{\mathrm{dg}} := \mathbf{cat}(\mathbf{Ch}_{\mathbf{k}})$. Other than this example, $\mathcal{V}$ will always be symmetric monoidal via the cartesian product. We simply write $\mathbf{cat} := \mathbf{cat}(\mathbf{Set})$ for the category of ordinary unenriched categories. We refer to objects of $\mathbf{cat}(\mathbf{sSet})$ as *simplicial categories*. This is for brevity, but it is slightly abusive: these are *not* the same thing as simplicial objects in $\mathbf{cat}$. Similarly, we refer to objects of $\mathbf{cat}(\mathbf{Top})$ as *topological categories*.¹⁶⁹

9.1.3. Observe the adjunction

$$\mathbf{sSet} \begin{array}{c} \xrightarrow{\pi_0} \\ \perp \\ \xleftarrow{\mathrm{const}} \end{array} \mathbf{Set} ,$$

where we write $\pi_0 := \pi_0 \circ | - |$ for simplicity.¹⁷⁰ Both functors preserve products, and so we obtain an adjunction

$$(31) \quad \mathbf{cat}(\mathbf{sSet}) \begin{array}{c} \xrightarrow{\mathrm{ho}} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathbf{cat}(\mathbf{Set}) =: \mathbf{cat}$$

between simplicial categories and ordinary categories. We refer to the left adjoint as the *homotopy category* functor, and we do not give notation or terminology for the right adjoint.

A morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathbf{cat}(\mathbf{sSet})$ is called a *weak equivalence* if the following two conditions hold:

- (1) it is *homotopically fully faithful*, i.e. for all $C, C' \in \mathcal{C}$ the induced morphism $\mathrm{hom}_{\mathcal{C}}(C, C') \rightarrow \mathrm{hom}_{\mathcal{D}}(F(C), F(C'))$ in $\mathbf{sSet}$ is a weak homotopy equivalence;
- (2) it is *homotopically essentially surjective*, i.e. the induced morphism $\mathrm{ho}(\mathcal{C}) \xrightarrow{\mathrm{ho}(F)} \mathrm{ho}(\mathcal{D})$ in $\mathbf{cat}$ is essentially surjective.¹⁷¹

(Of course, if $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a weak equivalence of simplicial categories then $\mathrm{ho}(\mathcal{C}) \xrightarrow{\mathrm{ho}(F)} \mathrm{ho}(\mathcal{D})$ is an equivalence of categories.)

We localize the category $\mathbf{cat}(\mathbf{sSet})$ at the weak equivalences. This yields a category that (for reasons that will become clear presently) we denote by $\mathbf{ho}(\mathbf{Cat})$ and refer to as *the homotopy category of ∞ -categories*; its objects are called *∞ -categories*. So by definition, there is a canonical functor

$$\mathbf{cat}(\mathbf{sSet}) \longrightarrow \mathbf{ho}(\mathbf{Cat})$$

¹⁶⁹As implied in §6.2.2, we reserve the notation $\mathbf{Cat}$ for the ∞ -category of ∞ -categories.

¹⁷⁰This left adjoint can also be identified as the colimit functor.

¹⁷¹It is possible to phrase homotopical essential surjectivity at the level of simplicial categories, but this requires a bit of enriched category theory.

that carries all weak equivalences to isomorphisms. In particular, this functor is essentially surjective: every ∞ -category can be represented by a simplicial category.

Equipped with these weak equivalences, $\mathbf{cat}(\mathbf{sSet})$ admits a *Bergner model structure*. This is characterized in terms of its fibrations. First of all, a morphism in $\mathcal{D} \in \mathbf{cat}(\mathbf{sSet})$ is called an *equivalence* (or a *homotopy equivalence*) if it becomes an isomorphism in $\mathbf{ho}(\mathcal{D})$; we denote these by $\xrightarrow{\sim}$. Then, a morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}$ is a fibration iff the following conditions hold:

- (1) for every $C, C' \in \mathcal{C}$, the induced morphism $\mathbf{hom}_{\mathcal{C}}(C, C') \rightarrow \mathbf{hom}_{\mathcal{D}}(F(C), F(C'))$ lies in $\mathbf{F}_{\mathbf{KQ}} \subseteq \mathbf{sSet}$;
- (2) any equivalence $F(C) \xrightarrow{\sim} D$ in $\mathcal{D}$ lifts to an equivalence $C \xrightarrow{\sim} C'$ in $\mathcal{C}$.

Note that condition 2 is a homotopical version of the notion of an isofibration (recall §8.6.2); indeed, it is not hard to see that the adjunction (31) becomes a Quillen adjunction

$$\mathbf{cat}(\mathbf{sSet})_{\text{Bergner}} \begin{array}{c} \xrightarrow{\text{ho}} \\ \xleftarrow{\perp} \end{array} \mathbf{cat}_{\text{can}} \quad .$$

9.1.4. There is similarly a Bergner model structure on $\mathbf{cat}(\mathbf{Top})$, and applying the Quillen equivalence $|-| \dashv \dashv \mathbf{Sing}$ to hom-objects determines a Quillen equivalence

$$\mathbf{cat}(\mathbf{sSet})_{\text{Bergner}} \begin{array}{c} \xrightarrow{\mathbf{cat}(|-|)} \\ \xleftarrow[\mathbf{cat}(\mathbf{Sing})]{\perp} \end{array} \mathbf{cat}(\mathbf{Top})_{\text{Bergner}} \quad .$$

It follows that we have an equivalence $\mathbf{cat}(\mathbf{Top})[\mathbf{W}^{-1}] \simeq \mathbf{ho}(\mathbf{Cat})$, and in particular an ∞ -category can *also* be defined as a weak equivalence class of topological category.

9.1.5. The Bergner model structure on $\mathbf{cat}(\mathbf{sSet})$ is defined in terms of its fibrations. As a result, the cofibrations are relatively inexplicit. In particular, $\mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}$ has a fatal flaw: it is not a *monoidal* model category, because the cartesian product of cofibrant objects will not generally be cofibrant. This is a major issue, as it implies that the internal hom bifunctor among simplicial categories does not generally give an internal hom at the level of underlying ∞ -categories, even under co/fibrancy assumptions (recall §8.2.5).¹⁷² It is of course absolutely imperative that any flavor of “category theory” have a well-behaved internal hom, i.e. a corresponding notion of “functor category”. Of course, $\mathbf{cat}(\mathbf{Top})_{\text{Bergner}}$ suffers from precisely the same fatal flaw.

In §9.2.3 we will introduce an alternative model category of ∞ -categories that *is* a symmetric monoidal model category, and in particular has a homotopically well-behaved internal

¹⁷²Given $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathbf{cat}(\mathbf{sSet})$, at the level of underlying ∞ -categories we would like an internal hom object $\underline{\mathbf{hom}}(\mathcal{D}, \mathcal{E}) \in \mathbf{ho}(\mathbf{Cat})$ such that

$$\mathbf{hom}_{\mathbf{ho}(\mathbf{Cat})}(\mathcal{C}, \underline{\mathbf{hom}}(\mathcal{D}, \mathcal{E})) \cong \mathbf{hom}_{\mathbf{ho}(\mathbf{Cat})}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \quad .$$

Because $\mathcal{C} \times \mathcal{D} \in \mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}$ will not generally be cofibrant even if $\mathcal{C}$ and $\mathcal{D}$ are, the right side must in general be computed using a further cofibrant replacement of $\mathcal{C} \times \mathcal{D}$ (even if $\mathcal{E}$ is fibrant).

hom (as we will see in §9.5.1). In fact, this will be a model structure on the category $\mathbf{sSet}$ of simplicial sets, called the *Joyal model structure*.

9.2. The Joyal model structure.

9.2.1. As background, we first discuss the relationship between simplicial sets and ordinary categories. By definition, the category Δ is a full subcategory of $\mathbf{cat}$. We write $\Delta \xhookrightarrow{[\bullet]} \mathbf{cat}$ for the inclusion. Just as in §8.3, this determines an adjunction

$$\mathbf{sSet} \begin{array}{c} \xrightarrow{\text{ho}} \\ \xleftarrow[\text{N}]{\perp} \end{array} \mathbf{cat} ,$$

as follows. First of all, the right adjoint $\mathbf{N}$ is the restricted Yoneda functor: for any category $\mathcal{C} \in \mathbf{cat}$, its *nerve* is the functor

$$\mathbf{N}(\mathcal{C})_\bullet : \Delta^{\text{op}} \xrightarrow{\text{hom}_{\mathbf{cat}}([\bullet], \mathcal{C})} \mathbf{Set} ,$$

so that $\mathbf{N}(\mathcal{C})_n$ is the set of sequences of n composable morphisms in $\mathcal{C}$. Then, the left adjoint ho is the left Kan extension

$$\begin{array}{ccc} \Delta & \xrightarrow{[\bullet]} & \mathbf{cat} \\ \Delta^\bullet \downarrow & \nearrow \text{ho} & \\ \mathbf{sSet} & & \end{array}$$

of $[\bullet]$ along the Yoneda embedding $\Delta^\bullet$; for reasons that will become clear shortly (see Exercise 9.5), we refer to this as the *homotopy category* functor.¹⁷³ So by definition, for any simplicial set $X \in \mathbf{sSet}$, its homotopy category is

$$\text{ho}(X) := \text{colim} \left(\Delta_{/X} \xrightarrow{\text{fgt}} \Delta \xrightarrow{[\bullet]} \mathbf{cat} \right) .$$

So, the homotopy category $\text{ho}(X) \in \mathbf{cat}$ is built by gluing together copies of the categories $[n] \in \mathbf{cat}$, one for each combinatorial n -simplex of X .

Exercise 9.1 (4 points).

- Prove that the nerve functor $\mathbf{N}$ is fully faithful, and that its image consists of those simplicial sets $X \in \mathbf{sSet}$ that admit *unique* inner horn fillers.
- Given a category $\mathcal{C} \in \mathbf{cat}$, prove that its nerve $\mathbf{N}(\mathcal{C}) \in \mathbf{sSet}$ admits outer horn fillers if and only if $\mathcal{C}$ is a groupoid.

9.2.2. Given any $i, j \in [n]$, we write P_{ij} for the poset of sequences $i = k_0 < k_1 < \cdots < k_l = j$ for $l \geq 0$, ordered by inclusion. Observe that for $i \leq j$, P_{ij} is simply the power set of the set $\{k \in [n] : i < k < j\}$, ordered by inclusion, while for $i > j$, P_{ij} is empty.

We define an object $\mathfrak{C}[n] \in \mathbf{cat}(\mathbf{sSet})$ as follows: its objects are those of $[n] \in \Delta \subset \mathbf{cat}$, its hom-objects are given by the formula

$$\text{hom}_{\mathfrak{C}[n]}(i, j) := \mathbf{N}(P_{ij}) \in \mathbf{sSet} ,$$

¹⁷³Of course, this will be closely related to the functor $\mathbf{cat}(\mathbf{sSet}) \xrightarrow{\text{ho}} \mathbf{cat}$ introduced in §9.1.3.

and its composition is given by the concatenation functors $P_{ij} \times P_{jk} \rightarrow P_{ik}$ (note that $\mathbf{N}$ preserves products, being a right adjoint). Now, power sets have weakly contractible nerves (they are isomorphic to cubes), and so the natural morphism

$$\mathfrak{C}[n] \longrightarrow [n]$$

is a weak equivalence in $\mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}$. This weak equivalence depicted for $n = 2$ in diagram (5); whereas $[2] \in \mathbf{cat}(\mathbf{sSet})$ corepresents *strictly* commutative triangles, $\mathfrak{C}[2] \in \mathbf{cat}(\mathbf{sSet})$ corepresents *homotopy-coherently* commutative triangles.

These simplicial categories assemble into a cosimplicial object $\Delta \xrightarrow{\mathfrak{C}[\bullet]} \mathbf{cat}(\mathbf{sSet})$ in simplicial categories, which determines an adjunction

$$\mathbf{sSet} \xrightleftharpoons[\mathbf{N}^{\text{h.c.}}]{\mathfrak{C}} \mathbf{cat}(\mathbf{sSet}) \quad ,$$

as follows. First of all, the right adjoint $\mathbf{N}^{\text{h.c.}}$ is simply the restricted Yoneda functor, so that for any simplicial category $\mathcal{C} \in \mathbf{cat}(\mathbf{sSet})$, its *homotopy-coherent nerve* is the simplicial set

$$\mathbf{N}^{\text{h.c.}}(\mathcal{C})_{\bullet} : \Delta^{\text{op}} \xrightarrow{\text{hom}_{\mathbf{cat}(\mathbf{sSet})}(\mathfrak{C}[\bullet], \mathcal{C})} \mathbf{Set} \quad .$$

So in low dimensions, the simplicial set $\mathbf{N}^{\text{h.c.}}(\mathcal{C}) \in \mathbf{sSet}$ may be described as follows: its 0-simplices are given by objects of $\mathcal{C}$, its 1-simplices are given by morphisms in $\mathcal{C}$, and its 2-simplices are given by homotopy-coherently commutative triangles in $\mathcal{C}$. And then, the left adjoint $\mathfrak{C}$ is the left Kan extension

$$\begin{array}{ccc} \Delta & \xrightarrow{\mathfrak{C}[\bullet]} & \mathbf{cat}(\mathbf{sSet}) \\ \Delta^{\bullet} \downarrow & \nearrow \mathfrak{C} & \\ \mathbf{sSet} & & \end{array}$$

of $\mathfrak{C}[\bullet]$ along the Yoneda embedding $\Delta^{\bullet}$. So by definition, for any simplicial set $X \in \mathbf{sSet}$, we have

$$\mathfrak{C}X := \text{colim} \left(\Delta_{/X} \xrightarrow{\text{fgt}} \Delta \xrightarrow{\mathfrak{C}[\bullet]} \mathbf{cat}(\mathbf{sSet}) \right) \quad .$$

9.2.3. Recall that a simplicial set $X \in \mathbf{sSet}$ is called a *Kan complex* if it has the extension property with respect to all horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$ for all $0 \leq i \leq n \geq 1$. For short, we will say that such a simplicial set *admits horn fillers*. Note that these horn fillers (i.e. extensions) are not generally unique (recall that the prototypical example of a Kan complex is $\text{Sing}(Y) \in \mathbf{sSet}_{\text{KQ}}^f$ for any $Y \in \mathbf{Top}$).

We refine this notion by declaring that a horn inclusion $\Lambda_i^n \hookrightarrow \Delta^n$ for $0 \leq i \leq n \geq 1$ is an *inner horn inclusion* if $0 < i < n \geq 2$ and is an *outer horn inclusion* otherwise. We then define a common weakening of the notion of a Kan complex and the nerve of a category (recall Exercise 9.1): a *quasicategory* is a simplicial set that admits (not necessarily unique) inner horn fillers. So, these four notions fit into the following table.

	fillers are unique	fillers are not necessarily unique
fillers for all horns	nerves of groupoids	Kan complexes
fillers for inner horns	nerves of categories	quasicategories

Now, the *Joyal model structure* on $\mathbf{sSet}$ is characterized by the facts that its weak equivalences are created by $\mathfrak{C}$ and that its cofibrations are the monomorphisms. Its fibrant objects are the quasicategories. It turns out that the adjunction

$$\mathbf{sSet}_{\text{Joyal}} \begin{array}{c} \xrightarrow{\mathfrak{C}} \\ \xleftarrow[\mathbf{N}^{\text{h.c.}}]{\perp} \end{array} \mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}$$

is a Quillen equivalence [T.2.2.5.1]. Of course, it follows that we have an equivalence $\mathbf{sSet}[\mathbf{W}_{\text{Joyal}}^{-1}] \simeq \mathbf{ho}(\mathbf{Cat})$, and in particular that an ∞ -category can *also* be defined as a Joyal weak equivalence class of simplicial set.

A particular consequence of this result is the following.

Exercise 9.2 (8 points). Given any fibrant simplicial category $\mathcal{C} \in \mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}^f$, prove that its homotopy-coherent nerve $\mathbf{N}^{\text{h.c.}}(\mathcal{C}) \in \mathbf{sSet}$ is a quasicategory.

In general, we write e.g. $\mathbf{C} \in \mathbf{sSet}_{\text{Joyal}}^f$, etc. to denote a quasicategory and $\mathcal{C} \in \mathbf{ho}(\mathbf{Cat})$ to denote its underlying ∞ -category. Similarly, we will write e.g. $\mathbf{x} \in \mathbf{C}$ to denote an object of that represents an object $x \in \mathcal{C}$. This is intended to emphasize the difference between “underlying” objects and the objects that represent them. In fact, we will use this same font more generally to denote chosen presentations of underlying objects.

9.3. Basic notions in quasicategories.

9.3.1. Just as the prototypical example of a Kan complex is $\text{Sing}(Y)$ for $Y \in \mathbf{Top} = \mathbf{Top}_{\text{QS}}^f$, the prototypical example of a quasicategory is $\mathbf{N}^{\text{h.c.}}(\mathcal{C})$ for $\mathcal{C} \in \mathbf{Cat}(\mathbf{sSet})_{\text{Bergner}}^f$ (recall Exercise 9.2). This motivates much of the following basic discussion of how one “does category theory” in an arbitrary quasicategory $\mathbf{C} \in \mathbf{sSet}_{\text{Joyal}}^f$.

9.3.2. An *object* of $\mathbf{C}$ is a vertex. Given an object $\mathbf{x} \in \mathbf{C}_0$, we often simply write $\mathbf{x} \in \mathbf{C}$.

A *morphism* in $\mathbf{C}$ is an edge. Given a morphism $\mathbf{f} \in \mathbf{C}_1 \cong \mathbf{hom}_{\mathbf{sSet}}(\Delta^1, \mathbf{C})$, its *source* is $\sigma(\Delta^{\{0\}}) \in \mathbf{C}_0$ and its *target* is $\sigma(\Delta^{\{1\}}) \in \mathbf{C}_0$. Given an object $\mathbf{x} \in \mathbf{C}_0$, its *identity morphism* is the degenerate 1-simplex $\Delta^1 \rightarrow \Delta^0 \xrightarrow{\mathbf{x}} \mathbf{C}$, which we denote by $\text{id}_{\mathbf{x}} \in \mathbf{C}_1$.

A 2-simplex $\sigma \in \mathbf{C}_2$ is called a (*homotopy-coherently*) *commutative triangle* in $\mathbf{C}$. This may be depicted as

$$(32) \quad \begin{array}{ccc} & \mathbf{y} & \\ \nearrow \mathbf{f} & & \searrow \mathbf{g} \\ \mathbf{x} & \xrightarrow{\mathbf{h}} & \mathbf{z} \end{array} \quad .$$

We say that such a 2-simplex *witnesses h as a composite of f and g* . In this situation, we write $h \simeq g \circ f$, even though the right side does *not* have prior meaning.

The data of the pair of composable morphisms

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & & z \end{array}$$

in $\mathcal{C}$ is specified by a morphism $\Lambda_1^2 \rightarrow \mathcal{C}$. Because $\mathcal{C}$ admits inner horn fillers, there always exists *some* composite of f and g (equipped with a witnessing 2-simplex). However, in general there is no preferred choice: any composite is “just as good as” any other one. This may sound bizarre, and in particular it may sound like a major deficiency of quasicategories compared to simplicial categories; but as we will see, it is in fact *the entire point of ∞ -categories*. In particular, we will see that we gain power and flexibility by passing from $\mathcal{C} \in \mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}^f$ (in which strict compositions exist) to $\mathbf{N}^{\text{h.c.}}(\mathcal{C}) \in \mathbf{sSet}_{\text{Joyal}}^f$ (in which they do not).¹⁷⁴

9.3.3. Suppose we are given a 2-simplex (32) in $\mathcal{C}$. In the case that $y = z$ and $g = \text{id}_y = \text{id}_z$, we say that σ is a **homotopy** from h to f .¹⁷⁵ Likewise, in the case that $x = y$ and $f = \text{id}_x = \text{id}_y$, we say that σ is a **homotopy** from h to g .

Exercise 9.3 (6 points). Show that for any objects $x, y \in \mathcal{C}$, these two notions of homotopy give equivalent equivalence relations on the set

$$\lim \left(\begin{array}{ccc} & \text{hom}_{\mathbf{sSet}}(\Delta^1, \mathcal{C}) & \\ & \downarrow (s,t) & \\ \text{pt}_{\mathbf{Set}} & \xrightarrow{(x,y)} & \mathcal{C}_0 \times \mathcal{C}_0 \end{array} \right) \in \mathbf{Set}$$

of morphisms from x to y .

Exercise 9.4 (6 points). Given a pair of composable morphisms in $\mathcal{C}$, show that any two composites are homotopic.

We will see a strengthening of Exercise 9.4 in §9.6.2.

The **homotopy category** of $\mathcal{C}$ is defined as follows: its objects are those of $\mathcal{C}$ and its morphisms are homotopy classes of morphisms in $\mathcal{C}$. Its composition is well-defined by Exercise 9.4.

Exercise 9.5 (4 points). Show that the functor $\mathbf{sSet} \xrightarrow{\text{ho}} \mathbf{cat}$ carries a quasicategory $\mathcal{C}$ to its homotopy category.

¹⁷⁴We refer the reader to [MG] for a diatribe that attempts to make this point.

¹⁷⁵The directionality is motivated by the definition of $\mathcal{C}[2] \in \mathbf{cat}(\mathbf{sSet})$, but it is irrelevant by Exercise 9.3.

Exercise 9.6 (4 points). Show that the adjunction

$$(33) \quad \mathbf{sSet}_{\text{Joyal}} \begin{array}{c} \xrightarrow{\text{ho}} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathbf{cat}_{\text{can}}$$

is a Quillen adjunction.

Both functors in the Quillen adjunction (33) are automatically derived, and moreover the underlying adjunction is a reflective localization. Hence, the derived adjunction is a reflective localization adjunction

$$\text{ho}(\mathbf{Cat}) := \mathbf{sSet}[\mathbf{W}_{\text{Joyal}}^{-1}] \begin{array}{c} \xrightarrow{\text{ho}} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathbf{cat}[\mathbf{W}^{-1}] \quad ,$$

whose left adjoint we also refer to as the *homotopy category* functor.

9.3.4. Given objects $\mathbf{x}, \mathbf{y} \in \mathbf{C}$, there is not merely a set of morphisms from $\mathbf{x}$ to $\mathbf{y}$, but in fact a *simplicial set* of morphisms from $\mathbf{x}$ to $\mathbf{y}$. To define this, we use the internal hom of $\mathbf{sSet}$: given simplicial sets $X, Y \in \mathbf{sSet}$, we may define a simplicial set

$$\underline{\text{hom}}(X, Y) := \underline{\text{hom}}_{\mathbf{sSet}}(X, Y) \in \mathbf{sSet}$$

by the formula

$$\underline{\text{hom}}(X, Y)_n := \text{hom}_{\mathbf{sSet}}(\Delta^n \times X, Y) \quad .^{176}$$

So, a vertex of $\underline{\text{hom}}(X, Y)$ is a morphism $X \rightarrow Y$, an edge is a morphism $\Delta^1 \times X \rightarrow Y$, and so on.

Exercise 9.7 (4 points). Given any simplicial set $Z \in \mathbf{sSet}$, construct a natural isomorphism

$$\text{hom}_{\mathbf{sSet}}(Z, \underline{\text{hom}}(X, Y)) \cong \text{hom}_{\mathbf{sSet}}(Z \times X, Y) \quad .$$

Then, for any objects $\mathbf{x}, \mathbf{y} \in \mathbf{C}$, we define

$$\text{hom}_{\mathbf{C}}(\mathbf{x}, \mathbf{y}) := \lim \left(\begin{array}{ccc} & \underline{\text{hom}}_{\mathbf{sSet}}(\Delta^1, \mathbf{C}) & \\ & \downarrow (s,t) & \\ \mathbf{pt}_{\mathbf{sSet}} & \xrightarrow{(x,y)} & \mathbf{C} \times \mathbf{C} \end{array} \right) \in \mathbf{sSet} \quad .^{177}$$

We refer to the underlying space of $\text{hom}_{\mathbf{C}}(\mathbf{x}, \mathbf{y}) \in \mathbf{sSet}$ as the *hom-space* from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbf{C}$. It turns out that there is a canonical weak homotopy equivalence

$$\text{hom}_{\mathbf{C}}(x, y) \xrightarrow{\approx} \text{hom}_{\mathcal{C}(\mathbf{C})}(x, y)$$

in $\mathbf{sSet}_{\text{KQ}}$, which is an important consistency check. Similarly, for any $\mathcal{C} \in \mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}^f$ we have a canonical weak homotopy equivalence

$$\text{hom}_{\mathbf{N}^{\text{h.c.}}(\mathcal{C})}(x, y) \xrightarrow{\approx} \text{hom}_{\mathcal{C}}(x, y)$$

¹⁷⁶More precisely, $\underline{\text{hom}}(X, Y)_{\bullet} := \text{hom}_{\mathbf{sSet}}(\Delta^{\bullet} \times X, Y)$.

¹⁷⁷Because $\mathbf{sSet}_{\text{Joyal}}$ is compatibly self-enriched (see §9.5.4), this pullback is in fact a homotopy pullback in $\mathbf{sSet}_{\text{Joyal}}$.

in $\mathbf{sSet}_{\mathbf{KQ}}$.

By definition, a vertex of $\mathrm{hom}_{\mathbf{C}}(\mathbf{x}, \mathbf{y})$ is a morphism from $\mathbf{x}$ to $\mathbf{y}$, and more generally an n -simplex of $\mathrm{hom}_{\mathbf{C}}(\mathbf{x}, \mathbf{y})$ is an extension

$$\begin{array}{ccc}
 \Delta^{\{0\}} \times \Delta^n & & \\
 \downarrow & \searrow \text{const}_{\mathbf{x}} & \\
 \Delta^1 \times \Delta^n & \cdots \cdots & \mathbf{C} \\
 \uparrow & \nearrow \text{const}_{\mathbf{y}} & \\
 \Delta^{\{1\}} \times \Delta^n & &
 \end{array}$$

that makes the diagram commute. Although it is not immediate from the definition, morphisms from $\mathbf{x}$ to $\mathbf{y}$ are homotopic if and only if there exists a path between them (in either direction) in $\mathrm{hom}_{\mathbf{C}}(\mathbf{x}, \mathbf{y})$.

9.3.5. We say that a 2-simplex

$$\begin{array}{ccc}
 & \mathbf{y} & \\
 \mathbf{f} \nearrow & \simeq & \searrow \mathbf{g} \\
 \mathbf{x} & \xrightarrow{\mathrm{id}_{\mathbf{x}}} & \mathbf{x}
 \end{array}$$

in $\mathbf{C}$ *witnesses $\mathbf{g}$ as a left inverse of $\mathbf{f}$* , or equivalently that it *witnesses $\mathbf{f}$ as a right inverse of $\mathbf{g}$* .¹⁷⁸ We say that a morphism in $\mathbf{C}$ is an *equivalence* if it admits both a left inverse and a right inverse.

Exercise 9.8 (6 points). Show that for any equivalence $\mathbf{x} \xrightarrow{\mathbf{f}} \mathbf{y}$ in $\mathbf{C}$, there exists a morphism $\mathbf{y} \xrightarrow{\mathbf{g}} \mathbf{x}$ that is both a left inverse and a right inverse of $\mathbf{f}$.

Exercise 9.9 (4 points). Show that a morphism in $\mathbf{C} \in \mathbf{sSet}_{\mathbf{Joyal}}^f$ becomes an isomorphism in $\mathrm{ho}(\mathbf{C}) \in \mathbf{cat}$ if and only if it is an equivalence.

9.3.6. Morphisms in $\mathbf{W}_{\mathbf{Joyal}} \subset \mathbf{sSet}$ are called *categorical equivalences*. From Exercise 9.9, it is easy to see that equivalences in a quasicategory $\mathbf{C} \in \mathbf{sSet}_{\mathbf{Joyal}}^f = \mathbf{sSet}_{\mathbf{Joyal}}^{cf}$ are carried to equivalences in its corresponding simplicial category $\mathfrak{C}(\mathbf{C}) \in \mathbf{cat}(\mathbf{sSet})_{\mathbf{Bergner}}$. Hence, recalling §9.1.3, we see that a morphism $\mathbf{C} \xrightarrow{\mathbf{F}} \mathbf{D}$ between quasicategories is a categorical equivalence iff the following two conditions hold:

- (1) it is *homotopically fully faithful*, i.e. for all $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$ the induced morphism $\mathrm{hom}_{\mathbf{C}}(\mathbf{c}, \mathbf{c}') \rightarrow \mathrm{hom}_{\mathbf{D}}(\mathbf{F}(\mathbf{c}), \mathbf{F}(\mathbf{c}'))$ in $\mathbf{sSet}$ is a weak homotopy equivalence (i.e. it lies in $\mathbf{W}_{\mathbf{KQ}} := \mathbf{W}_{\mathrm{w.h.e.}}$);
- (2) it is *homotopically essentially surjective*, i.e. for every $\mathbf{d} \in \mathbf{D}$ there exists an object $\mathbf{c} \in \mathbf{C}$ and an equivalence $\mathbf{F}(\mathbf{c}) \xrightarrow{\sim} \mathbf{d}$ in $\mathbf{D}$.

¹⁷⁸These conventions align with the standard convention for function composition.

9.3.7. It is clear that if a quasicategory $\mathcal{C}$ is in fact a Kan complex, then $\mathrm{ho}(\mathcal{C})$ is a groupoid. It is an important and nontrivial fact that the converse holds: given a quasicategory $\mathcal{C}$ such that $\mathrm{ho}(\mathcal{C})$ is a groupoid, $\mathcal{C}$ is in fact a Kan complex. In view of Exercise 9.9, this implies that whereas quasicategories present ∞ -categories, Kan complexes present ∞ -**groupoids**.¹⁷⁹ Of course, this is just to say that “ ∞ -groupoids” are synonymous with “spaces”.¹⁸⁰

Now, the identity adjunction defines a Quillen adjunction

$$\mathrm{sSet}_{\mathrm{Joyal}} \begin{array}{c} \xrightarrow{\mathrm{id}_{\mathrm{sSet}}} \\ \perp \\ \xleftarrow{\mathrm{id}_{\mathrm{sSet}}} \end{array} \mathrm{sSet}_{\mathrm{KQ}} \quad .^{181}$$

As the left adjoint is automatically derived, the derived counit is a natural equivalence, and so the derived adjunction is a reflective localization adjunction

$$\mathrm{ho}(\mathrm{Cat}) \simeq \mathrm{sSet}[\mathbf{W}_{\mathrm{Joyal}}^{-1}] \begin{array}{c} \xrightarrow{(-)^{\mathrm{gpd}}} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathrm{sSet}[\mathbf{W}_{\mathrm{KQ}}^{-1}] \simeq \mathrm{ho}(\mathcal{S}) \quad .$$

We leave the derived right adjoint implicit, and we refer to the derived left adjoint as ∞ -**groupoid completion**: given an ∞ -category $\mathcal{C} \in \mathrm{ho}(\mathrm{Cat})$, the unit morphism $\mathcal{C} \rightarrow \mathcal{C}^{\mathrm{gpd}}$ is the initial morphism in $\mathrm{ho}(\mathrm{Cat})$ from $\mathcal{C}$ to an ∞ -groupoid.

9.3.8. There is another way of passing from an ∞ -category to an ∞ -groupoid: rather than freely adjoining inverses to all morphisms, we can also simply discard the morphisms that

¹⁷⁹An ∞ -groupoid may also be called an $(\infty, 0)$ -**category**: in general, for $0 \leq k \leq n \leq \infty$, an (n, k) -**category** is an n -category in which every i -morphism is invertible for $k < i \leq n$, and an n -**groupoid** is an $(n, 0)$ -category. Another way of saying this is that an (n, k) -category should be a category enriched in $(n-1, k-1)$ -categories (at least for $k > 0$ – the simplest fix is to declare here that $0-1 = 0$). Recalling §9.3.4, we see that what we have been calling “ ∞ -categories” may be more properly termed “ $(\infty, 1)$ -categories”.

The n -morphisms in an $(\infty, 1)$ -category can be spelled out more concretely as follows. In general, an n -*morphism* should be a sort of morphism between $(n-1)$ -morphisms; these $(n-1)$ -morphisms are required to be *parallel*, which means that they share the same source and target. We begin by declaring that a 0 -*morphism* is simply an object, and that any two 0 -morphisms are parallel. Then, for $n \geq 1$, an n -morphism in an $(\infty, 1)$ -category is an $(n-1)$ -disk in one of its hom-spaces. More precisely, the boundary $(n-2)$ -sphere of this $(n-1)$ -cell should be thought of as “hemispherically stratified”, with the two hemispheres giving the source and target $(n-1)$ -cells (and their agreement on their common boundary encoding the requirement that they be parallel). Note that these n -morphisms are indeed invertible for every $n \geq 2$, because $(n-1)$ -morphisms in a space are invertible for every $n \geq 2$.

¹⁸⁰This is a “definitional” assertion, which can be phrased more carefully as the assertion that “any good theory of $(\infty, 1)$ -categories should have that the $(\infty, 0)$ -groupoids are equivalent to spaces”. This idea is motivated by the equivalence between the homotopy categories of n -groupoids and homotopy n -types (i.e. topological spaces with homotopy groups vanishing above dimension n) for small values of n ; it is due to Grothendieck, and is referred to as the *homotopy hypothesis*.

¹⁸¹Of course, the underlying adjunction of this Quillen adjunction is an equivalence. However, we use the notation $\perp$ instead of $\sim$, because this Quillen adjunction should not be thought of as an equivalence (because it is not a Quillen equivalence).

are not already invertible. This is called the *maximal subgroupoid* construction, which determines a right adjoint

$$(34) \quad \mathrm{ho}(\mathcal{S}) \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad \perp \quad} \\ \xleftarrow{\quad \iota_0 \quad} \end{array} \mathrm{ho}(\mathrm{Cat})$$

to the defining inclusion.

This construction can be modeled as follows. Given a quasicategory $\mathcal{C}$, it turns out that the largest simplicial subset $\iota_0\mathcal{C} \subseteq \mathcal{C}$ containing only the equivalences is in fact a Kan complex. It follows that this construction defines a right adjoint

$$(35) \quad \mathrm{sSet}_{\mathrm{KQ}}^f \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad \perp \quad} \\ \xleftarrow{\quad \iota_0 \quad} \end{array} \mathrm{sSet}_{\mathrm{Joyal}}^f$$

to the inclusion from Kan complexes into quasicategories. We will see in Exercise 9.11 that this adjunction upgrades to an adjunction in $\mathrm{cat}(\mathrm{sSet})$, which on homotopy categories (in the sense of §9.1.3) presents the adjunction (34).

9.3.9. A *subcategory* of an ∞ -category $\mathcal{C} \in \mathrm{ho}(\mathrm{Cat})$ is equivalent data to a subcategory of its homotopy category $\mathrm{ho}(\mathcal{C}) \in \mathrm{cat}[\mathbf{W}^{-1}]$.¹⁸² Namely, we can uniquely specify a subcategory $\mathcal{D} \subseteq \mathcal{C}$ by declaring which equivalence classes of objects and which homotopy classes of morphisms among them belong to $\mathcal{D}$, subject to the condition that those homotopy classes of morphisms are closed under composition.

By definition, the morphisms in a subcategory $\mathcal{D} \subseteq \mathcal{C}$ are stable under equivalence. Note that this implies that a subcategory inclusion is necessarily *fully faithful on equivalences*. So for example, the forgetful functor $\Delta \rightarrow \mathrm{Fin}$ is *not* the inclusion of a subcategory.¹⁸³ And in particular, a subcategory of an ∞ -groupoid is an inclusion of path components (and is necessarily a *full* subcategory). We will generally refer to a subcategory of a space as a *subspace*.

Of course, given a quasicategory $\mathcal{C} \in \mathrm{sSet}_{\mathrm{Joyal}}^f$ representing $\mathcal{C} \in \mathrm{ho}(\mathrm{Cat})$, it is straightforward to construct a sub-quasicategory $\mathcal{D} \subseteq \mathcal{C}$ that represents a given subcategory $\mathcal{D} \subseteq \mathcal{C}$.

9.4. Underlying ∞ -categories of model categories.

9.4.1. A *relative ∞ -category* is a pair $(\mathcal{C}, \mathbf{W})$ consisting of an ∞ -category $\mathcal{C} \in \mathrm{ho}(\mathrm{Cat})$ equipped with a subcategory $\mathbf{W} \subseteq \mathcal{C}$. Its (*∞ -categorical*) *localization* is the initial morphism

$$\mathcal{C} \longrightarrow \mathcal{C}[\![\mathbf{W}^{-1}]\!]$$

¹⁸²We find the term “sub- ∞ -category” to be too awkward, so we omit the “ ∞ ” in this case. Likewise, we say “subgroupoid” instead of “sub- ∞ -groupoid”.

¹⁸³The utility of this definition is the fact that the inclusion of a subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ is a monomorphism: it is merely a *condition* for a functor to $\mathcal{C}$ to factor through $\mathcal{D}$.

in $\mathbf{ho}(\mathbf{Cat})$ that sends all morphisms in the subcategory $\mathbf{W} \subseteq \mathcal{C}$ to equivalences.¹⁸⁴ Although we have not defined all the relevant terms yet, it should be plausible that this can be defined as an ∞ -categorical pushout

$$\mathcal{C}[\mathbf{W}^{-1}] \simeq \operatorname{colim} \left(\begin{array}{ccc} \mathbf{W} & \longrightarrow & \mathcal{C} \\ \downarrow & & \\ \mathbf{W}_{\text{gpd}} & & \end{array} \right)$$

in $\mathbf{Cat}$ (the ∞ -category of ∞ -categories).

Following §9.3.7, we can construct the ∞ -categorical localization explicitly as follows. We first choose any quasicategory $\mathbf{C}$ presenting $\mathcal{C}$, any sub-quasicategory $\mathbf{W} \subseteq \mathbf{C}$ presenting the subcategory $\mathbf{W} \subseteq \mathcal{C}$, and any fibrant resolution $\mathbf{W} \rightarrow \mathbf{W}^{f_{\mathbf{KQ}}}$ in $\mathbf{sSet}_{\mathbf{KQ}}^f$. Then, the localization $\mathcal{C}[\mathbf{W}^{-1}] \in \mathbf{ho}(\mathbf{Cat})$ is presented in $\mathbf{sSet}_{\mathbf{Joyal}}$ by the pushout

$$\operatorname{colim} \left(\begin{array}{ccc} \mathbf{W} & \longrightarrow & \mathbf{C} \\ \downarrow & & \\ \mathbf{W}^{f_{\mathbf{KQ}}} & & \end{array} \right)$$

(which is also a homotopy pushout (recall §8.5.2)). Note that this will not generally be a quasicategory, however.

Exercise 9.10 (4 points). Given a relative ∞ -category $(\mathcal{C}, \mathbf{W})$, prove that $\mathbf{ho}(\mathcal{C}[\mathbf{W}^{-1}])$ and $\mathbf{ho}(\mathcal{C})[\mathbf{W}^{-1}]$ are canonically isomorphic in $\mathbf{cat}[\mathbf{W}^{-1}]$.

9.4.2. We now define our ∞ -categories of lasting interest.¹⁸⁵

First of all, given a model category, we may refer to its ∞ -categorical localization as its *underlying ∞ -category*: we have seen in §8 that the model structure gives a way of computing hom-sets in the localization, and in fact an elaboration of the same techniques gives a way of computing hom-spaces in the ∞ -categorical localization (so that a model category may also be seen as a convenient presentation of its underlying ∞ -category). In particular, it turns out that Quillen adjunctions give adjunctions between ∞ -categories (to be defined more carefully below), and Quillen equivalences give equivalences between ∞ -categories; we simply refer to these as the *derived adjunction* (resp. *derived equivalence*) of a Quillen adjunction (resp. Quillen equivalence).

We define *the ∞ -category of ∞ -categories* to be

$$\mathbf{Cat} := \mathbf{sSet}[\mathbf{W}_{\mathbf{Joyal}}^{-1}] \simeq \mathbf{cat}(\mathbf{sSet})[\mathbf{W}_{\mathbf{Bergner}}^{-1}] \simeq \mathbf{cat}(\mathbf{Top})[\mathbf{W}_{\mathbf{Bergner}}^{-1}].$$

¹⁸⁴This notation is meant to be suggestive of the ordinary localization of relative categories, but equipped with “higher order” structure. In particular, even when $\mathcal{C}$ is an ordinary category, the ∞ -categorical localization $\mathcal{C}[\mathbf{W}^{-1}]$ will rarely be so (even up to equivalence).

¹⁸⁵Note that these are all of an essentially (higher-)algebraic flavor. However, there are also ∞ -categories of interest that arise in differential topology, as we indicate in §10.5.

Note that as we have defined it (and ignoring set-theoretic issues), this is an object

$$\mathbf{Cat} \in \mathbf{ho}(\mathbf{Cat})$$

of the homotopy category of ∞ -categories. However, we will for the most part work inside of $\mathbf{Cat}$ itself.

We define *the ∞ -category of spaces* (or *∞ -groupoids*) to be

$$\mathcal{S} := \mathbf{Gpd} := \mathbf{sSet}[\![\mathbf{W}_{\text{w.h.e.}}^{-1}]\!] \simeq \mathbf{Top}[\![\mathbf{W}_{\text{w.h.e.}}^{-1}]\!] .$$

We write

$$\mathbf{sSet} \xrightarrow{|\!-\!|} \mathcal{S} \xleftarrow{\Pi_\infty} \mathbf{Top}$$

for the canonical functors. We refer to the former as the *geometric realization* functor, and we refer to the latter as the *fundamental ∞ -groupoid* functor.¹⁸⁶

We define *the ∞ -category of derived R -modules* to be

$$\mathbf{D}_R := \mathbf{Ch}_R[\![\mathbf{W}_{\text{q.i.}}^{-1}]\!] .$$

Below, we will upgrade this to the $\mathbb{k}$ -linear ∞ -category which we studied in §6 (using the same notation).

We write

$$\mathbf{Cat}_1 := \mathbf{cat}[\![\mathbf{W}^{-1}]\!] \quad \text{and} \quad \mathbf{Gpd}_1 := \mathbf{gpd}[\![\mathbf{W}^{-1}]\!] .$$

The hom-spaces in these two ∞ -categories lie in $\mathbf{Gpd}_1 \subseteq \mathbf{Gpd} := \mathcal{S}$ (so that they are in fact $(2,1)$ -categories). We henceforth use the terms *categories* and *groupoids* to refer to objects of these ∞ -categories; we use the terms *strict category* and *strict groupoid* to respectively refer to objects of $\mathbf{cat}$ and $\mathbf{gpd}$. So, a category is an equivalence class of strict category. For any categories $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}_1$ and any representatives $\mathbf{C}, \mathbf{D} \in \mathbf{cat}$ by strict categories, the hom-groupoid $\mathbf{hom}_{\mathbf{Cat}_1}(\mathcal{C}, \mathcal{D}) \in \mathbf{Gpd}_1$ is the underlying groupoid of the strict groupoid $\mathbf{hom}_{\mathbf{cat}}(\mathbf{C}, \mathbf{D}) \in \mathbf{gpd}$ (whose objects are functors and whose morphisms are natural equivalences).

9.4.3. Of course, all of the various derived adjunctions on 1-categorical localizations that we have seen previously upgrade to derived adjunctions on the ∞ -categorical localizations introduced in §9.4.2; for instance, we have a diagram

$$\begin{array}{ccc} & \xrightarrow{(-)^{\mathbf{gpd}}} & \\ \mathbf{Cat} & \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} & \mathcal{S} \\ & \xleftarrow{\iota_0} & \end{array} ,$$

¹⁸⁶Note that the notation $|\!-\!|$ and the terminology “geometric realization” were already given to the functor $\mathbf{sSet} \xrightarrow{|\!-\!|} \mathbf{Top}$. It should always be clear from context which notion is intended. The ∞ -categorical notions will be explained further in §9.7 (and this abuse of notation is further extended in §9.7.5).

in which the lower adjunction follows from Exercise 9.11 below. Also, we have a diagram

$$\begin{array}{ccccc}
 & & \Pi_1 & & \\
 & \swarrow & & \searrow & \\
 \text{Top} & \xrightarrow{\Pi_\infty} & \text{Gpd} & \xrightleftharpoons[\perp]{\Pi_1} & \text{Gpd}_1 \\
 & \downarrow \scriptstyle (-)^{\text{gpd}} & \downarrow \scriptstyle (-)^{\text{gpd}} & & \downarrow \scriptstyle (-)^{\text{gpd}} \\
 & \downarrow \scriptstyle \iota_0 & \downarrow \scriptstyle \iota_0 & & \downarrow \scriptstyle \iota_0 \\
 & \text{Cat} & \xrightleftharpoons[\perp]{\text{ho}} & \text{Cat}_1 &
 \end{array}$$

that commutes in the evident senses, in which we may refer to either functor Π_1 as the *fundamental groupoid* functor.¹⁸⁷

We highlight one other derived adjunction for future use. Recall the composite Quillen adjunction

$$(36) \quad \text{sSet}_{\text{KQ}} \xrightleftharpoons[\text{fgt}]{R\{-\}} (\text{sMod}_R)_{\text{KQ}} \xrightleftharpoons[\sim]{\bar{\cdot}} (\text{Ch}_R^{\geq 0})_{\text{proj}} \xrightleftharpoons[\tau_{\geq 0}]{i_{\geq 0}} (\text{Ch}_R)_{\text{proj}}$$

from §8.4. This determines a composite derived adjunction

$$(37) \quad \mathcal{S} \xrightleftharpoons[\text{fgt}]{R\{-\}} \mathbf{D}_R^{\geq 0} \xrightleftharpoons[\tau_{\geq 0}]{i_{\geq 0}} \mathbf{D}_R$$

on underlying ∞ -categories. We refer to the left adjoint $R\{-\}$ (or the composite left adjoint $i_{\geq 0} \circ R\{-\}$) as the *free derived R -module* functor.¹⁸⁸

9.5. Enriched ∞ -categories.

9.5.1. Given a monoidal ∞ -category $\mathcal{V}$, there is an ∞ -category $\text{Cat}(\mathcal{V})$ of $\mathcal{V}$ -*enriched ∞ -categories*. We do not define these notions in full generality here, but here we explore a few basic instances of lasting interest. We begin by discussing a few general notions.

In general, we use an underline to denote an enrichment.

Given a monoidal ∞ -category $\mathcal{V}$ with unit object $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$, applying $\text{Cat}(-)$ to the functor

$$\mathcal{V} \xrightarrow{\text{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -)} \mathcal{S}$$

determines a forgetful functor

$$\text{Cat}(\mathcal{V}) \xrightarrow{\text{fgt}} \text{Cat}(\mathcal{S}) \simeq \text{Cat} ,$$

which carries a $\mathcal{V}$ -enriched ∞ -category to its *underlying ∞ -category*.

Suppose that $\mathbf{V}$ is a monoidal model category that presents the monoidal ∞ -category $\mathcal{V}$. Then, there is a Bergner model structure on the category $\text{cat}(\mathbf{V})$ of ordinary $\mathbf{V}$ -enriched categories that presents the ∞ -category $\text{Cat}(\mathcal{V})$.

¹⁸⁷Of course, there is also a factorization $\text{Top} \xrightarrow{\Pi_1} \text{gpd} \rightarrow \text{Gpd}_1$ through the category of strict groupoids.

¹⁸⁸Recall from §8.4.5 that the R -homology of a space $X \in \mathcal{S}$ is by definition the homology of the derived R -module $R\{X\} \in \mathbf{D}_R$.

In turn, suppose that $\mathcal{C}$ is a V -enriched model category, i.e. it is a model category and at the same time it is enriched in V in a way that is compatible with its model structure. Then, the V -enriched category

$$\underline{\mathcal{C}^{cf}} \in \mathbf{cat}(V)_{\text{Bergner}}$$

of bifibrant objects is a presentation of a V -enriched ∞ -category

$$\underline{\mathcal{C}[\![W^{-1}]\!] } \in \mathbf{Cat}(V)$$

that enhances the underlying (unenriched) ∞ -category of $\mathcal{C}$:

$$\mathbf{fgt}(\underline{\mathcal{C}[\![W^{-1}]\!] }) \simeq \mathcal{C}[\![W^{-1}]\!] .$$

All of the examples that we discuss fall into this paradigm.

9.5.2. The model categories $\mathbf{sSet}_{\mathbf{k}Q}$ and $\mathbf{Top}_{QS}$ are both compatibly self-enriched, relative to their cartesian product symmetric monoidal structures.¹⁸⁹ Hence, the ∞ -category $\mathcal{S}$ of spaces is presented both by the simplicial category

$$\underline{\mathbf{sSet}_{\mathbf{k}Q}^{cf}} = \underline{\mathbf{sSet}_{\mathbf{k}Q}^f} \in \mathbf{cat}(\mathbf{sSet})_{\text{Bergner}}$$

of Kan complexes and by the topological category

$$\underline{\mathbf{Top}_{QS}^{cf}} = \underline{\mathbf{Top}_{QS}^c} \in \mathbf{cat}(\mathbf{Top})_{\text{Bergner}}$$

of retracts of cell complexes.

9.5.3. The model category $(\mathbf{Ch}_{\mathbf{k}})_{\text{proj}}$ is compatibly self-enriched, relative to its tensor product symmetric monoidal structure (as first discussed in §8.2.5). Therefore, $\mathbf{Ch}_{\mathbf{k}}$ -enriched categories are presentations of $\mathbf{D}_{\mathbf{k}}$ -enriched ∞ -categories. We write

$$\mathbf{Cat}_{\mathbf{k}} := \mathbf{Cat}(\mathbf{D}_{\mathbf{k}}) ,$$

and refer to $\mathbf{D}_{\mathbf{k}}$ -enriched ∞ -categories as **$\mathbf{k}$ -linear ∞ -categories** (as first introduced in §6.2.2).

Because the model category $(\mathbf{Ch}_R)_{\text{proj}}$ is compatibly enriched in $(\mathbf{Ch}_{\mathbf{k}})_{\text{proj}}$, we find that the ∞ -category $\mathbf{D}_R := \mathbf{Ch}_R[\![W_{\text{q.i.}}^{-1}]\!] \in \mathbf{Cat}$ enhances to a $\mathbf{k}$ -linear ∞ -category, which is presented by the dg-category

$$\underline{(\mathbf{Ch}_R)_{\text{proj}}^{cf}} = \underline{(\mathbf{Ch}_R)_{\text{proj}}^c} = \mathbf{P}_R \in \mathbf{cat}(\mathbf{Ch}_{\mathbf{k}})_{\text{Bergner}}$$

of projective complexes of R -modules. Of course, as we saw in §6, this same $\mathbf{k}$ -linear ∞ -category is also presented by the dg-category

$$\underline{(\mathbf{Ch}_R)_{\text{inj}}^{cf}} = \underline{(\mathbf{Ch}_R)_{\text{inj}}^f} = \mathbf{I}_R \in \mathbf{cat}(\mathbf{Ch}_{\mathbf{k}})_{\text{Bergner}}$$

of injective complexes of R -modules.

¹⁸⁹In the latter case, one should restrict to a subcategory of “convenient” topological spaces. We elide this point here.

We have a Quillen adjunction

$$\mathrm{cat}(\mathbf{sSet})_{\mathrm{Bergner}} \xrightleftharpoons[\perp]{} \mathrm{cat}(\mathbf{Ch}_{\mathbb{k}})_{\mathrm{Bergner}}$$

obtained by applying $\mathrm{cat}(-)$ to the Quillen adjunction (36). This yields a derived adjunction

$$(38) \quad \mathrm{Cat} \xrightleftharpoons[\mathrm{Cat}(\mathrm{fgt})]{\mathrm{Cat}(\mathbb{k}\{-\})} \mathrm{Cat}_{\mathbb{k}}$$

on underlying ∞ -categories (obtained by applying $\mathrm{Cat}(-)$ to the adjunction (37) in the case that $R = \mathbb{k}$), whose right adjoint is the underlying ∞ -category functor. Note that this coincides with the description given in §6.2.3.

9.5.4. The model category $\mathbf{sSet}_{\mathrm{Joyal}}$ is compatibly self-enriched, relative to its cartesian product symmetric monoidal structure. This gives a presentation of the self-enrichment of Cat , i.e. its structure as an $(\infty, 2)$ -category:

$$\underline{\mathrm{Cat}} \in \mathrm{Cat}_2 := \mathrm{Cat}(\mathrm{Cat}) .$$

Since this structure is crucially important (indeed, it is the reason that we passed from $\mathrm{cat}(\mathbf{sSet})_{\mathrm{Bergner}}$ to $\mathbf{sSet}_{\mathrm{Joyal}}$), we describe it explicitly at the point-set level.

For a simplicial set $\mathbf{C} \in \mathbf{sSet}_{\mathrm{Joyal}}^c = \mathbf{sSet}$ and a quasicategory $\mathbf{D} \in \mathbf{sSet}_{\mathrm{Joyal}}^f$, we write

$$\mathrm{Fun}(\mathbf{C}, \mathbf{D}) := \underline{\mathrm{hom}}_{\mathbf{sSet}}(\mathbf{C}, \mathbf{D}) \in \mathbf{sSet}_{\mathrm{Joyal}}^f$$

for the simplicial set of morphisms between them: this is a quasicategory, which gives a homotopically well-behaved internal hom.¹⁹⁰ We refer to it as the *quasicategory of functors* from $\mathbf{C}$ to $\mathbf{D}$. We refer to its vertices as *functors* and its edges as *natural transformations*.

In particular, the fact that this construction is homotopically well-behaved implies that it is invariant up to categorical equivalence under categorical equivalence among co/fibrant objects in the source/target. In particular, we do not need to assume that $\mathbf{C}$ is a quasicategory: if $\mathbf{C} \rightarrow \mathbf{C}^{f_{\mathrm{Joyal}}}$ is a fibrant resolution in $\mathbf{sSet}_{\mathrm{Joyal}}$, then the restriction morphism

$$\mathrm{Fun}(\mathbf{C}^{f_{\mathrm{Joyal}}}, \mathbf{D}) \longrightarrow \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

is a categorical equivalence – and it even lies in $(\mathbf{W} \cap \mathbf{F})_{\mathrm{Joyal}}$ if the morphism $\mathbf{C} \rightarrow \mathbf{C}^{f_{\mathrm{Joyal}}}$ lies in $(\mathbf{W} \cap \mathbf{C})_{\mathrm{Joyal}}$.¹⁹¹

¹⁹⁰Namely, the bifunctor

$$\mathbf{sSet}_{\mathrm{Joyal}} \times \mathbf{sSet}_{\mathrm{Joyal}} \xrightarrow{(-) \times (-)} \mathbf{sSet}_{\mathrm{Joyal}}$$

is a left Quillen bifunctor (recall §8.2.5). In particular, the product of cofibrant objects in $\mathbf{sSet}_{\mathrm{Joyal}}$ is again cofibrant. This fails in $\mathrm{cat}(\mathbf{sSet})_{\mathrm{Bergner}}$ (recall §9.1.5).

¹⁹¹This strongly illustrates is the sense in which ∞ -categories “want” to only have homotopy-coherently well-defined composition (as opposed to strictly well-defined composition), i.e. the sense in which $\mathbf{sSet}_{\mathrm{Joyal}}$ harmonizes with the true nature of ∞ -categories while $\mathrm{cat}(\mathbf{sSet})_{\mathrm{Bergner}}$ does not. Namely, if one presents ∞ -categories $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ as strictly-enriched categories $\mathbf{C}, \mathbf{D} \in \mathrm{cat}(\mathbf{sSet})_{\mathrm{Bergner}}$, then even to construct the correct hom-set $\underline{\mathrm{hom}}_{\mathrm{ho}(\mathrm{Cat})}(\mathcal{C}, \mathcal{D})$ one must in particular require that $\mathbf{C}$ be “equipped with coherent homotopies” (e.g. $\mathcal{C}[n]$ is cofibrant whereas $[n]$ is not).

Given a natural transformation $\Delta^1 \times \mathcal{C} \rightarrow \mathcal{D}$, its **component** at an object $x \in \mathcal{C}$ is the morphism in $\mathcal{D}$ selected by the composite

$$\Delta^1 \xrightarrow{(\text{id}_{\Delta^1}, \text{const}_x)} \Delta^1 \times \mathcal{C} \longrightarrow \mathcal{D}.$$

Among the natural transformations are the **natural equivalences**; these are simply the equivalences in $\text{Fun}(\mathcal{C}, \mathcal{D})$. It is a nontrivial fact that a natural transformation is a natural equivalence if and only if all of its components are equivalences.

Of course, the self-enrichment of $\mathbf{Cat}$ restricts to a self-enrichment of $\mathbf{Cat}_1$, which is presented by the self-enrichment of $\mathbf{cat}$.

The above discussion allows us to enhance the category $\mathbf{sSet}_{\text{Joyal}}^f$ of quasicategories to a $\mathbf{sSet}_{\text{KQ}}^f$ -enriched category as follows: for any $\mathcal{C}, \mathcal{D} \in \mathbf{sSet}_{\text{Joyal}}^f$ we define the hom-simplicial set

$$\text{hom}_{\mathbf{sSet}_{\text{Joyal}}^f}(\mathcal{C}, \mathcal{D}) := \iota_0 \text{Fun}(\mathcal{C}, \mathcal{D}) \in \mathbf{sSet}_{\text{KQ}}^f. \quad 192$$

Exercise 9.11 (4 points). Show that the adjunction (35) in $\mathbf{cat}$ upgrades to an adjunction

$$\mathbf{sSet}_{\text{KQ}}^f \xrightleftharpoons[\iota_0]{\perp} \mathbf{sSet}_{\text{Joyal}}^f$$

in $\mathbf{cat}(\mathbf{sSet})$.

9.6. Uniqueness in quasicategory theory.

9.6.1. In ordinary category theory, co/representing objects are *unique up to unique isomorphism* – assuming that they exist. For example, in any category, the subcategory of initial objects is an empty or contractible groupoid.¹⁹³ More generally, given a diagram in a category, the category of its colimits is an empty or contractible groupoid.¹⁹⁴

In ∞ -category theory, the notion of “uniqueness” must be slightly reenvisioned. Namely, co/representing objects form **contractible ∞ -groupoids** – assuming that they exist. More generally, one often says that a given construction is well-defined “up to a contractible space of choices”.

These notions are also captured by the technical usage of the word “essentially”: for instance, one may say that an initial object in an ∞ -category is *essentially unique*, just as one may say that a functor is *essentially surjective* – the word “essentially” essentially means “up to equivalences” (and equivalences between equivalences, etc., when appropriate). On the other hand, in situations where the meaning is clear, it is common to omit this modifier:

¹⁹²Unfortunately, this enrichment does *not* make $\mathbf{sSet}_{\text{Joyal}}$ into a $\mathbf{sSet}_{\text{KQ}}$ -enriched model category. This is one of the reasons that Lurie studies the category of *marked simplicial sets* in [§T.3]: it admits a $\mathbf{sSet}_{\text{KQ}}$ -enriched model structure that presents the ∞ -category of ∞ -categories.

¹⁹³Recall that a *contractible groupoid* is a category in which all hom-sets are singletons (which is necessarily a groupoid).

¹⁹⁴This is a special case of the previous example, as a colimit is an initial object in the category of cocones.

one may simply say that an initial object in an ∞ -category is *unique*, or that a functor is *surjective*. Similarly, one may often say “is” to mean “is canonically equivalent to”.

In this subsection, we illustrate two basic examples of this phenomenon as it is implemented within the context of quasicategories.

9.6.2. Composition in an ∞ -category is essentially unique. For example, if we work in $\mathbf{cat}(\mathbf{Top})_{\text{Bergner}}$, then “ ∞ -categorical composition” is explicitly implemented by homotopy-coherent composition, i.e. extension along the cofibration $\mathbf{cat}(|-|)(\mathfrak{C}(\Lambda_1^2 \rightarrow \Delta^2))$ (whose source is simply $[2] \in \mathbf{cat} \subset \mathbf{cat}(\mathbf{Top})$). Given a pair of composable morphisms in a topological category, a homotopy-coherent composite is therefore a morphism equipped with a path from their strict composite. So in this context, the “essential uniqueness” claim follows from the fact that for any based topological space $X \in \mathbf{Top}_*$ the based path space $\{\gamma \in \underline{\mathbf{hom}}_{\mathbf{Top}}([0, 1], X) : \gamma(0) = x\}$ is contractible.

Let us verify the essential uniqueness of composition within the context of quasicategories. Fix a quasicategory $\mathbf{C} \in \mathbf{sSet}_{\text{Joyal}}^f$ and a map $\Lambda_1^2 \xrightarrow{\varphi} \mathbf{C}$ classifying a pair of composable morphisms. Recall from §9.3.3 that a *composition* is an extension

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{\varphi} & \mathbf{C} \\ \downarrow & \nearrow & \\ \Delta^2 & & \end{array} .$$

This is guaranteed to exist (because $\mathbf{C}$ is a quasicategory), and we saw in Exercise 9.4 that any two composites are homotopic. We can strengthen this as follows.

Exercise 9.12 (2 points). Show that the morphism $\Lambda_1^2 \rightarrow \Delta^2$ is an acyclic cofibration in $\mathbf{sSet}_{\text{Joyal}}$.

Then, the compatible self-enrichment of $\mathbf{sSet}_{\text{Joyal}}$ implies that the resulting restriction morphism

$$\mathbf{Fun}(\Delta^2, \mathbf{C}) \longrightarrow \mathbf{Fun}(\Lambda_1^2, \mathbf{C})$$

is an acyclic fibration in $\mathbf{sSet}_{\text{Joyal}}$ between quasicategories (recall §9.5.4). In particular, we obtain a pullback diagram

$$\begin{array}{ccc} \mathbf{Fun}(\Delta^2, \mathbf{C})_{\varphi} & \longrightarrow & \mathbf{Fun}(\Delta^2, \mathbf{C}) \\ \Downarrow & & \Downarrow \\ \Delta^0 & \xrightarrow{\varphi} & \mathbf{Fun}(\Lambda_1^2, \mathbf{C}) \end{array}$$

which is also a homotopy pullback diagram in $\mathbf{sSet}_{\text{Joyal}}$ (recall §8.5.2). In particular, we find that the simplicial set of composites $\mathbf{Fun}(\Delta^2, \mathbf{C})_{\varphi}$ is a quasicategory (since its map to Δ^0 is

a fibration) which is categorically equivalent to the Kan complex Δ^0 . It follows that this quasicategory is in fact a contractible Kan complex.¹⁹⁵

9.6.3. Observe that the functor $[1] \rightarrow \mathbf{pt}$ is an ∞ -groupoid completion.¹⁹⁶ It follows that for any ∞ -category $\mathcal{C} \in \mathbf{Cat}$, the morphism therein classified by a functor $[1] \rightarrow \mathcal{C}$ is an equivalence if and only if there exists an extension

$$\begin{array}{ccc} [1] & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathbf{pt} & & \end{array},$$

and moreover that the space of such extensions is either empty or contractible. In this situation, it is almost tautological to extract its inverse: it is the lower composite in the diagram

$$(39) \quad \begin{array}{ccccc} S^0 & \longrightarrow & [1] & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & \nearrow & \\ [1]^{\mathrm{op}} & \longrightarrow & \mathbf{pt} & & \end{array},$$

in which the copy of $S^0 \in \mathcal{S} \subset \mathbf{Cat}$ witnesses the fact that the source and target have been exchanged.¹⁹⁷

The passage to inverses is actually somewhat clearer in the context of quasicategories, because these are closer to the familiar context of strict categories. Then, we model the left

¹⁹⁵This can also be seen as follows (without implicitly using the discussion of §9.3.6): we have $\mathbf{C}_{\mathrm{KQ}} = \mathbf{C}_{\mathrm{Joyal}}$, and so $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}} = (\mathbf{W} \cap \mathbf{F})_{\mathrm{Joyal}}$, and so the acyclic fibrant objects in $\mathbf{sSet}_{\mathrm{KQ}}$ and $\mathbf{sSet}_{\mathrm{Joyal}}$ coincide.

¹⁹⁶Recall from §9.3.7 that this is just to say that the morphism $\Delta^1 \rightarrow \Delta^0$ lies in $\mathbf{W}_{\mathrm{w.h.e.}} =: \mathbf{W}_{\mathrm{KQ}} \subset \mathbf{sSet}_{\mathrm{KQ}}$.

¹⁹⁷A potential source of confusion here is the fact that every equivalence in $\mathcal{C}$ is equivalent in $\mathbf{Ar}(\mathcal{C})$ to an identity morphism. More generally, given two objects in an ∞ -category, note that it is *additional data* to witness them as being equivalent: namely, this is the data of a factorization

$$\begin{array}{ccc} S^0 & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathbf{pt} & & \end{array},$$

which is generally *not* unique even if it exists. (Note that it is *not* invariant under categorical equivalence of quasicategories to ask whether two vertices of a quasicategory are equal. Rather, it is only invariant to ask for a specified equivalence between them. (And it is *not* invariant under categorical equivalence to ask whether this equivalence is an identity morphism, i.e. a degenerate edge.) For instance, if $\mathcal{C} \in \mathbf{Gpd} \subset \mathbf{Cat}$ is an ∞ -groupoid, then the space of such factorizations is precisely the *path space* (with fixed endpoints.) In particular, this applies to the source and target of the proposed inverse morphisms in $\mathcal{C}$. (Another illustrative example of this phenomenon is the fact that it is *additional data* to witness a morphism as an endomorphism.)

square in diagram (39) in $\mathbf{Cat}$ as the diagram

$$\left(\begin{array}{ccc} \Delta^0 \sqcup \Delta^0 & \xrightarrow{\quad} & \Delta^1 \\ \downarrow & & \downarrow \\ (\Delta^1)^{\text{op}} & \xrightarrow{\quad} & \mathbf{J} \end{array} \right) \quad := \quad \mathbf{N} \left(\begin{array}{ccc} [0] \sqcup [0] & \xrightarrow{\quad} & [1] \\ \downarrow & & \downarrow \\ [1]^{\text{op}} & \xrightarrow{\quad} & [1]^{\text{gpd}} \end{array} \right)$$

in $\mathbf{sSet}_{\text{Joyal}}^c$, where $[1]^{\text{gpd}} \in \mathbf{gpd} \subset \mathbf{cat}$ denotes the strict groupoid completion of $[1] \in \mathbf{cat}$ (which has two objects and two nonidentity morphisms (which are inverse isomorphisms)).¹⁹⁸

Now, we have the following elaboration of Exercise 9.8.

Exercise 9.13 (6 points). Show that a morphism $x \xrightarrow{f} y$ in a quasicategory $\mathbf{C}$ is an equivalence if and only if there exists an extension

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{f} & \mathbf{C} \\ \downarrow & \nearrow \text{dashed} & \\ \mathbf{J} & & \end{array} .$$

In fact, there f is an equivalence, then there is a *contractible Kan complex* of such extensions. This yields a solid diagram

$$\begin{array}{ccccc} & & \text{Fun}(\mathbf{J}, \mathbf{C}) & & \\ & \swarrow \approx & & \searrow \approx & \\ \text{Fun}(\Delta^1, \mathbf{C}) & \longleftrightarrow & \text{Fun}(\Delta^1, \mathbf{C})' & \xrightarrow{\text{dashed}} & \text{Fun}((\Delta^1)^{\text{op}}, \mathbf{C})' \longleftrightarrow \text{Fun}((\Delta^1)^{\text{op}}, \mathbf{C}) \end{array}$$

in $\mathbf{sSet}_{\text{Joyal}}$ (using its compatible self-enrichment), where $\text{Fun}(\Delta^1, \mathbf{C})' \cong \text{Fun}((\Delta^1)^{\text{op}}, \mathbf{C})'$ denotes the quasicategory of equivalences in $\mathbf{C}$, and choosing a dashed section as indicated determines a functor $\text{Fun}(\Delta^1, \mathbf{C})' \rightarrow \text{Fun}((\Delta^1)^{\text{op}}, \mathbf{C})'$ carrying an equivalence in $\mathbf{C}$ to its inverse.¹⁹⁹ Of course, such a section is not unique; rather such sections assemble into a contractible Kan complex.

9.7. Co/limits in quasicategories.

9.7.1. We now illustrate the notion of a colimit in a quasicategory; this can be easily dualized to obtain the notion of a limit in a quasicategory. Of course, this will be an instance of the discussion of §9.6 regarding uniqueness in quasicategories.

¹⁹⁸In particular, note that $\mathbf{J}$ is a contractible Kan complex. (Indeed, the geometric realization $|\mathbf{J}| \in \mathbf{Top}$ equipped with its resulting CW-complex structure is the standard “hemispherical” CW-structure on the infinite-dimensional sphere.)

¹⁹⁹This section can be constructed by working in the model category $(\mathbf{sSet}_{/\text{Fun}(\Delta^1, \mathbf{C})'})_{\text{Joyal}}$ (so that the map is literally a section and not merely a section up to homotopy).

9.7.2. We briefly summarize the situation in strict categories, which we will then mimic in the context of quasicategories. Let $\mathcal{J}, \mathcal{C} \in \mathbf{cat}$ be strict categories. The **right cone** on $\mathcal{J}$ is the strict category $\mathcal{J}^\triangleright \in \mathbf{cat}$ obtained by freely adjoining a terminal object. Then, given a functor $\mathcal{J} \xrightarrow{F} \mathcal{C}$, a **cocone under F** is an extension

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{F} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ \mathcal{J}^\triangleright & & \end{array} .$$

These assemble into a strict category

$$\mathcal{C}_{F/} := \lim \left(\begin{array}{ccc} & & \text{Fun}(\mathcal{J}^\triangleright, \mathcal{C}) \\ & \downarrow & \\ \text{pt} & \xrightarrow{F} & \text{Fun}(\mathcal{J}, \mathcal{C}) \end{array} \right) ,$$

of which a **colimit** of F is an initial object. Observe that this is precisely the data of a pointwise left adjoint to the functor

$$\text{Fun}(\mathcal{J}, \mathcal{C}) \xleftarrow{\text{const}} \mathcal{C}$$

at the object $F \in \text{Fun}(\mathcal{J}, \mathcal{C})$.

9.7.3. We fix quasicategories $\mathcal{I}, \mathcal{C} \in \mathbf{sSet}_{\text{Joyal}}^f$.

We say that an object $\mathbf{x} \in \mathcal{C}$ is **initial** if for every object $\mathbf{y} \in \mathcal{C}$, the hom-space $|\text{hom}_{\mathcal{C}}(\mathbf{x}, \mathbf{y})| \in \mathcal{S}$ is contractible (i.e. the Kan complex $\text{hom}_{\mathcal{C}}(\mathbf{x}, \mathbf{y}) \in \mathbf{sSet}_{\text{KQ}}^f$ is contractible).

Exercise 9.14 (4 points). Show that the sub-quasicategory of initial objects in a quasicategory is an empty or contractible Kan complex.

The **right cone** on $\mathcal{I}$ is defined to be the pushout

$$\mathcal{I}^\triangleright := \text{colim} \left(\begin{array}{ccc} \mathcal{I} \times \Delta^{\{1\}} & \hookrightarrow & \mathcal{I} \times \Delta^1 \\ \downarrow & & \\ \Delta^0 & & \end{array} \right) \in \mathbf{sSet}_{\text{Joyal}} .^{200}$$

Given a functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$, a **cocone under F** is an extension

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ \mathcal{I}^\triangleright & & \end{array} .$$

²⁰⁰Note that this is a homotopy pushout in $\mathbf{sSet}_{\text{Joyal}}$. Hence, (as $(-) \times \Delta^1$ preserves categorical equivalences) a categorical equivalence $\mathcal{I} \xrightarrow{\sim} \mathcal{J}$ in $\mathbf{sSet}_{\text{Joyal}}$ determines a categorical equivalence $\mathcal{I}^\triangleright \xrightarrow{\sim} \mathcal{J}^\triangleright$. In particular, our constructions go through equally well even when $\mathcal{I}$ is not a quasicategory.

These assemble into a quasicategory

$$\mathbf{C}_{\mathbf{F}/} := \lim \left(\begin{array}{ccc} & & \mathbf{Fun}(\mathbf{I}^{\triangleright}, \mathbf{C}) \\ & & \downarrow \\ \Delta^0 & \xrightarrow{\mathbf{F}} & \mathbf{Fun}(\mathbf{I}, \mathbf{C}) \end{array} \right),$$

of which a **colimit** of $\mathbf{F}$ is an initial object. Of course, (once we have defined this notion in quasicategories we will see that) this is likewise a pointwise left adjoint.

In fact (and correspondingly), colimits can be constructed in families, as follows. Let us write $\mathbf{Fun}(\mathbf{I}, \mathbf{C})' \subseteq \mathbf{Fun}(\mathbf{I}, \mathbf{C})$ for the full sub-quasicategory on those functors that admit colimits and $\mathbf{Fun}(\mathbf{I}^{\triangleright}, \mathbf{C})' \subseteq \mathbf{Fun}(\mathbf{I}^{\triangleright}, \mathbf{C})$ for the full sub-quasicategory on those functors $\mathbf{I}^{\triangleright} \rightarrow \mathbf{C}$ whose restrictions to $\mathbf{I}$ are colimits.

Exercise 9.15 (4 points). Prove that we have a diagram

$$\begin{array}{ccc} \mathbf{Fun}(\mathbf{I}^{\triangleright}, \mathbf{C})' & \hookrightarrow & \mathbf{Fun}(\mathbf{I}^{\triangleright}, \mathbf{C}) \\ \Downarrow & & \Downarrow \\ \mathbf{Fun}(\mathbf{I}, \mathbf{C})' & \hookrightarrow & \mathbf{Fun}(\mathbf{I}, \mathbf{C}) \end{array}$$

in $\mathbf{sSet}_{\mathbf{Joyal}}^f$.

It follows that there exists a section

$$\begin{array}{ccc} \mathbf{Fun}(\mathbf{I}^{\triangleright}, \mathbf{C})' & & \\ \Downarrow \wr & \nearrow & \\ \mathbf{Fun}(\mathbf{I}, \mathbf{C})' & & \end{array},$$

which may be called a **colimit functor**. Of course, such a section is not unique; rather such sections assemble into a contractible Kan complex.

We have the following important consistency check for these constructions.

Exercise 9.16 (8 points). Prove that $\mathbf{I}$ -indexed colimits in $\mathbf{C}$ – and in particular, their existence – are invariant under categorical equivalence in both variables $\mathbf{I}$ and $\mathbf{C}$.

9.7.4. It turns out that homotopy co/limits in model categories present ∞ -categorical co/limits in their underlying ∞ -categories. This is nicely illustrated using the model category $\mathbf{Top}_{\mathbf{QS}}$.

Suppose we are given a span

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \\ Z & & \end{array}$$

in $\mathbf{Top}_{\mathbf{QS}}$, and assume for simplicity that $X \in \mathbf{Top}_{\mathbf{QS}}^c$ is cofibrant. Then, to compute the homotopy pushout, it suffices to replace both maps by cofibrations. That is, for any factorizations

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \scriptstyle \sim & \nearrow \scriptstyle \sim \\ & Y' & \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{g} & Z \\ & \searrow \scriptstyle \sim & \nearrow \scriptstyle \sim \\ & Z' & \end{array} ,$$

the homotopy pushout is computed as the pushout

$$(40) \quad \operatorname{colim} \left(\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ g' \downarrow \scriptstyle \sim & & \\ X & & Z' \end{array} \right) .$$

Now, a natural construction of the above factorizations is given by the *mapping cylinder* construction: namely, we can define

$$Y' := \operatorname{colim} \left(\begin{array}{ccc} X \times \{1\} & \xrightarrow{f} & Y \\ \wr \downarrow & & \\ X \times [0, 1] & & \end{array} \right) \quad \text{and} \quad Z' := \operatorname{colim} \left(\begin{array}{ccc} X \times \{1\} & \xrightarrow{g} & Z \\ \wr \downarrow & & \\ X \times [0, 1] & & \end{array} \right) ,$$

and define both maps f' and g' to be given by the formula $x \mapsto (x, 0)$ (and both unlabeled weak homotopy equivalences to be the evident projection maps). In this case, we find that the colimit (40) in $\mathbf{Top}$ corepresents the data of a map from Y , a map from Z , and a homotopy between their pullbacks to X . Note that this is indeed a model for the ∞ -categorical colimit.²⁰¹

9.7.5. Broadly speaking, simplicial objects in ordinary categories tend to be stand-ins for corresponding ∞ -categorical colimits. For example, the functor $\mathbf{sSet} \xrightarrow{|\cdot|} \mathcal{S}$ introduced in §9.4.2 is in fact the composite

$$\mathbf{sSet} := \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set}) \hookrightarrow \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\operatorname{colim}_{\Delta^{\mathrm{op}}}} \mathcal{S} .$$

Similarly, the canonical functor

$$\mathbf{sMod}_R \longrightarrow \mathbf{sMod}_R[\![\mathbf{W}_{\mathbf{KQ}}^{-1}]\!] \simeq \mathbf{D}_R^{\geq 0}$$

is in fact the composite

$$\mathbf{sMod}_R := \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathbf{Mod}_R) \hookrightarrow \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathbf{D}_R^{\geq 0}) \xrightarrow{\operatorname{colim}_{\Delta^{\mathrm{op}}}} \mathbf{D}_R^{\geq 0} . \quad ^{202}$$

²⁰¹More precisely, (assuming Y and Z are cofibrant) this defines a homotopy-coherent commutative square in the topological category $\mathbf{Top}_{\mathbf{QS}}^c \in \operatorname{cat}(\mathbf{Top})_{\text{Bergner}} = \operatorname{cat}(\mathbf{Top})_{\text{Bergner}}^f$ (and therefore also in the corresponding quasicategory $\mathbf{N}^{\text{h.c.}}(\operatorname{cat}(\mathbf{Sing})(\mathbf{Top}_{\mathbf{QS}}^c)) \in \mathbf{sSet}_{\text{Joyal}}^f$), which is an ∞ -categorical pushout diagram in the sense we have just defined.

²⁰²Note too that the inclusion $\mathbf{D}_R^{\geq 0} \hookrightarrow \mathbf{D}_R$ preserves colimits, being a left adjoint.

In ∞ -category theory, it is common to refer to the colimit of a simplicial object as its *geometric realization*, and to denote it by $|-| := \operatorname{colim}_{\Delta^{\operatorname{op}}}(-)$.

9.8. Working model-independently.

9.8.1. Having discussed many point-set aspects of ∞ -category theory, we conclude §9 by discussing the notion of *working model-independently* with ∞ -categories. We will generally work model-independently for the remainder of this document.

9.8.2. One way of articulating the idea of working model-independently is to say that we work with quasicategories, but never make reference to *specific* quasicategories or their point-set features. Rather, we only perform operations on them that are invariant under categorical equivalence up to categorical equivalence. For example, the discussions of §9.6.2, §9.6.3, and §9.7.3 respectively imply that composition, inversion of equivalences, and formation of colimits can be discussed in a model-independent fashion, and in particular that these operations are *essentially well-defined* in this context (in the sense of §9.6.1). Indeed, every homotopically meaningful operation that one can perform on a quasicategory can be discussed in a model-independent fashion (which is exactly the point – or perhaps even a tautology). This manner of interacting with ∞ -categories is formally analogous to the manner in which most mathematicians interact with the foundations of set theory.

9.8.3. If one wishes to be more philosophically aggressive, one could alternatively articulate the idea of working model-independently as saying that one is employing a Platonic theory of ∞ -categories that exists independently of any particular model. One aim of the emerging field of homotopy type theory [Pro13] is to give mathematically rigorous foundations for such a philosophical posture.

9.8.4. The model-independent notion of ∞ -categories will be amplified in §10.2, where (for both technical and philosophical reasons) we will discuss a presentation of ∞ -categories using the ∞ -category of ∞ -groupoids. Although it is of course logically circular, we find this to be the most “conceptually correct” definition of ∞ -categories.²⁰³

9.8.5. As a practical linguistic matter, when working model-independently we will essentially always omit technical uses of the word “essentially” (as discussed in §9.6.1). For example, we will simply refer to *the* composite of a pair of composable morphisms, rather than saying “a composite, which is unique up to a contractible space of choices”. Likewise, on a notational level we may implicitly invert equivalences, e.g. we may write $A \rightarrow B \xleftarrow{\sim} C \rightarrow D$ to denote a morphism $A \rightarrow D$ obtained by choosing an inverse (if it arises most naturally in this way).

²⁰³More generally (and less circularly), we find Θ_n -spaces to be the most “conceptually correct” definition of (∞, n) -categories [Rez10].

9.8.6. In particular, in this paradigm we generally implicitly consider strict categories as categories. So for example, “the ∞ -category of simplicial objects in an ∞ -category $\mathcal{C}$ ” may be interpreted to mean “the quasicategory $\mathrm{Fun}(\mathbf{N}(\Delta^{\mathrm{op}}), \mathcal{C})$, for any particular quasicategory $\mathcal{C} \in \mathbf{sSet}_{\mathrm{Joyal}}^f$ representing the ∞ -category $\mathcal{C} \in \mathbf{Cat}$ ” (and for any strict category $\Delta^{\mathrm{op}} \in \mathbf{cat}$ representing the category $\Delta^{\mathrm{op}} \in \mathbf{Cat}_1$ (e.g. a skeleton)).

9.8.7. Let $\mathcal{C}$ be a model category with underlying ∞ -category $\mathcal{C} := \mathcal{C}[\mathbf{W}^{-1}]$. At least when $\mathcal{C}$ is sufficiently nice, for any strict category $\mathcal{J} \in \mathbf{cat}$ the functor ∞ -category $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ is the localization of $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ at the componentwise weak equivalences.²⁰⁴ For example, the ∞ -category $\mathbf{sS} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ is the localization of the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Top})$ at the componentwise weak homotopy equivalences. This fact may be helpful inasmuch as it gives a concrete way of thinking about certain functor ∞ -categories (e.g. the ∞ -category of simplicial spaces, which features heavily in §10), but of course actually choosing such rectifications is antithetical to the practice of model-independent mathematics.

10. MONOIDS, COMPLETE SEGAL SPACES, AND (SYMMETRIC) MONOIDAL ∞ -CATEGORIES

10.1. Monoids and groups.

10.1.1. Let $\mathcal{X}$ be an ∞ -category that admits finite products. A **monoid** (or **monoid object**, or even **∞ -monoid object**) in $\mathcal{X}$ is a simplicial object $M_{\bullet} \xrightarrow{M_{\bullet}} \mathcal{X}$ satisfying the following conditions:

- (1) $M_{\bullet}$ is **reduced**, i.e. the canonical morphism $M_0 \rightarrow \mathbf{pt}_{\mathcal{X}}$ is an equivalence;
- (2) $M_{\bullet}$ is **Segal**, i.e. for any $n \geq 2$ the n^{th} **Segal map**

$$M_n := M_{\{0 < 1 < \dots < n\}} \longrightarrow M_{\{0 < 1\}} \times M_{\{1 < 2\}} \times \dots \times M_{\{n-1 < n\}} =: (M_1)^{\times n}$$

is an equivalence.

One thinks of $M := \mathbf{fgt}(M_{\bullet}) := M_1 \in \mathcal{X}$ as the underlying object, and of the maps

$$\mu : M_1 \times M_1 \xleftarrow{\sim} M_2 \xrightarrow{M_{[2] \leftarrow \{0 < 2\}}} M_1 \quad \text{and} \quad \eta : \mathbf{pt} \xleftarrow{\sim} M_0 \xrightarrow{M_{[0] \leftarrow [1]}} M_1$$

respectively as the multiplication and unit maps. The higher data record the *homotopy-coherent associativity* of the multiplication. We write $\mathbf{Mon}(\mathcal{X}) \subseteq \mathbf{sX}$ for the full subcategory on the monoid objects. We simply write $\mathbf{Mon} := \mathbf{Mon}(\mathcal{S})$, and refer to its objects as **monoids** or **∞ -monoids**. A monoid object in $\mathbf{Cat}$ is called a **monoidal ∞ -category**; we will discuss these further in §10.3.

It is common to abuse terminology by simply referring to an object $M \in \mathcal{X}$ as a monoid object. To be precise, this refers to a specific monoid object $M_{\bullet} \in \mathbf{Mon}(\mathcal{X})$ equipped with an equivalence $M \simeq M_1$ in $\mathcal{X}$, i.e. a lift of M through the forgetful functor $\mathbf{Mon}(\mathcal{X}) \xrightarrow{\mathbf{fgt}} \mathcal{X}$.

²⁰⁴This is a fairly nontrivial fact, and may be referred to as *rectification of diagrams*.

Exercise 10.1 (6 points).

- (a) Prove that ∞ -monoids in **Set** are equivalent to ordinary monoids.
- (b) Prove that if $\mathcal{X}$ is a 1-category (e.g. **Set**), then the composite functor

$$\mathbf{Mon}(\mathcal{X}) \hookrightarrow \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{X}) \longrightarrow \mathbf{Fun}(\Delta_{\leq 3}^{\mathrm{op}}, \mathcal{X})$$

is fully faithful.

In fact, Exercise 10.1(b) generalizes as follows: for any $n \geq 1$, if $\mathcal{X}$ is an $(n, 1)$ -category, then the composite functor

$$\mathbf{Mon}(\mathcal{X}) \hookrightarrow \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{X}) \longrightarrow \mathbf{Fun}(\Delta_{\leq n+2}^{\mathrm{op}}, \mathcal{X})$$

is fully faithful. A particular case of interest is the $(2, 1)$ -category $\mathbf{Cat}_1$, in which situation Mac Lane’s coherence theorem effectively asserts that it suffices to verify the *pentagon axiom* for 4-fold associativity [ML63].²⁰⁵ Of course, when $\mathcal{X}$ is an arbitrary $(\infty, 1)$ -category we must use the entire indexing category Δ^{op} .

Exercise 10.2 (4 points). Given monoid objects $M, N \in \mathbf{Mon}(\mathcal{X})$, endow the product $M \times N \in \mathcal{X}$ with the structure of a monoid object.

10.1.2. Given any ∞ -category $\mathcal{C} \in \mathbf{Cat}$ equipped with a distinguished object $x \in \mathcal{C}$, the endomorphism space

$$\mathrm{end}_{\mathcal{C}}(x) := \mathrm{hom}_{\mathcal{C}}(x, x) \in \mathcal{S}$$

admits a canonical ∞ -monoid structure via composition. We simply write $\mathbf{Cat}_* \xrightarrow{\mathrm{end}} \mathbf{Mon}$ for this functor from pointed ∞ -categories to ∞ -monoids.

Exercise 10.3 (4 points). Construct a cosimplicial object in pointed ∞ -categories $E^\bullet \in \mathbf{cCat}_* := \mathbf{Fun}(\Delta, \mathbf{Cat}_*)$ such that there is a natural equivalence

$$\mathrm{hom}_{\mathbf{Cat}_*}(E^\bullet, (\mathcal{C}, x)) \simeq \mathrm{end}_{\mathcal{C}}(x).$$

in $\mathbf{Mon} \subset \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ (i.e. such that $E^\bullet$ corepresents the composite functor $\mathbf{Cat}_* \xrightarrow{\mathrm{end}} \mathbf{Mon} \hookrightarrow \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$).

As will be clear from the discussion of §10.2, there exists a fully faithful left adjoint

$$\mathbf{Mon} \xleftarrow[\mathrm{end}]{\mathfrak{B}} \mathbf{Cat}_*$$

²⁰⁵There is a subtlety here: the classical definition of a monoidal category is a presentation of an object of $\mathbf{Mon}(\mathbf{Cat}_1)$ via data in **cat**. In particular, the associator and left and right unitors are natural isomorphisms in **cat** that are intended to present paths in hom-groupoids in $\mathbf{Cat}_1$ that witness the *homotopy-coherent* associativity and unitality of the monoidal structure. Mac Lane’s coherence theorem may be read as asserting that these data in **cat** do indeed suffice to define objects of $\mathbf{Mon}(\mathbf{Cat}_1)$. Objects of $\mathbf{Mon}(\mathbf{cat})$ may be called (strictly) monoidal strict categories; these are defined by requiring these natural isomorphisms to be natural *equalities*.

from ∞ -monoids to pointed ∞ -categories, which we refer to as the (*categorical*) *deloop* functor. This carries an ∞ -monoid M to a pointed ∞ -category $(* \in \mathfrak{B}M) \in \mathbf{Cat}_*$, which is characterized by the fact that it has a single equivalence class of object and that $\mathrm{end}_{\mathfrak{B}M}(*) \simeq M$ (as ∞ -monoids).²⁰⁶

10.1.3. Note that the functor $\mathcal{S} \xrightarrow{\pi_0} \mathbf{Set}$ preserves finite products. An ∞ -monoid $M \in \mathbf{Mon}$ is called an ∞ -*group* if the ordinary monoid $\pi_0(M) \in \mathbf{Mon}(\mathbf{Set})$ is in fact a group. We write $\mathbf{Grp} \subset \mathbf{Mon}$ for the full subcategory on the ∞ -groups. Given an ∞ -group G , we write $\mathbf{B}G := \mathfrak{B}G \in \mathcal{S}_*$ for the corresponding pointed ∞ -groupoid, which we refer to as its *classifying space* or its *deloop*.²⁰⁷

Given any ∞ -groupoid $X \in \mathcal{S}$ equipped with a distinguished object $x \in X$, we have a canonical identification

$$\mathrm{end}_X(x) \simeq \Omega X := \Omega_x X := \mathrm{hom}_{\mathcal{S}_*}(S^1, (X, x))$$

with the *based loop space*, under which composition of morphisms corresponds to concatenation of paths. Hence, this ∞ -monoid is in fact an ∞ -group, because the ordinary monoid $\pi_0(\Omega_x X) =: \pi_1(X, x)$ is a group. A fundamental theorem in homotopy theory [Sta63] asserts that *all* ∞ -groups arise in this way. Indeed, the functors $\mathfrak{B}$ and $\mathbf{B}$ participate in a diagram

$$\begin{array}{ccccc} \mathbf{Mon} & \xrightleftharpoons[\mathrm{end}]{\mathfrak{B}} & \mathbf{Cat}_*^{\mathrm{one\,obj}} & \xrightleftharpoons[\perp]{} & \mathbf{Cat}_* \\ \downarrow \scriptstyle{(-)^{\mathrm{gp}}} \uparrow \scriptstyle{\mathrm{max'l\,subgp}} & & \downarrow \scriptstyle{(-)^{\mathrm{gpd}}} \uparrow \scriptstyle{\iota_0} & & \downarrow \scriptstyle{(-)^{\mathrm{gpd}}} \uparrow \scriptstyle{\iota_0} \\ \mathbf{Grp} & \xrightleftharpoons[\Omega]{\mathbf{B}} & \mathcal{S}_*^{\geq 1} & \xrightleftharpoons[\perp]{} & \mathcal{S}_* \end{array} ,$$

in which the full subcategory $\mathbf{Cat}_*^{\mathrm{one\,obj}} \subseteq \mathbf{Cat}_*$ (resp. $\mathcal{S}_*^{\geq 1} \subseteq \mathcal{S}_*$) is that on those pointed ∞ -categories (resp. ∞ -groupoids) that have exactly one equivalence class of object (and the right adjoints to their inclusions are given by restriction to the distinguished objects), and

²⁰⁶Note that the maximal subgroupoid of $\mathfrak{B}M$ will be $\mathfrak{B}$ of the maximal sub(∞ -)group of M (as implicitly defined in §10.1.3).

²⁰⁷The notation $\mathfrak{B}$ instead of $\mathbf{B}$ is meant to emphasize the categorical nature of the former. More generally, for an ∞ -monoid M , the notation $\mathbf{B}M$ is sometimes used to denote the ∞ -groupoid completion $(\mathfrak{B}M)^{\mathrm{gpd}} \in \mathcal{S}_*$. These letters “ $\mathbf{B}$ ” both stand for “bar construction”, although this can be somewhat confusing: here we generally take “bar construction” to mean “corresponding simplicial object” (so that we have *defined* ∞ -monoids in terms of their bar constructions), whereas historically the bar construction often referred to the geometric realization thereof. Namely, in terms of simplicial objects, for any ∞ -monoid $M \in \mathbf{Mon} \subset \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ we have an equivalence

$$(\mathfrak{B}M)^{\mathrm{gpd}} \simeq |M_\bullet| := \mathrm{colim} \left(\Delta^{\mathrm{op}} \xrightarrow{M_\bullet} \mathcal{S} \right)$$

of pointed spaces, with the basepoint on the right side coming from the canonical morphism $\mathbf{pt} \simeq M_0 \rightarrow |M_\bullet|$ to the colimit.

which commutes in the evident ways.²⁰⁸ In particular, this explains the term “deloop” for the functor $\mathbf{B}$: it carries an ∞ -group to the unique pointed connected space of which it is the based loop space.

10.2. Complete Segal spaces.

10.2.1. Building on the discussion of §10.1, we discuss a convenient and conceptually clarifying presentation of ∞ -categories as *complete Segal spaces*.²⁰⁹ This discussion may be thought of as an ∞ -categorical (in particular, model-independent) analog of §9.2.1.

10.2.2. Recall the functor $\Delta \xrightarrow{[\bullet]} \mathbf{cat} \rightarrow \mathbf{Cat}_1 \hookrightarrow \mathbf{Cat}$. The ∞ -*nerve* functor is the restricted Yoneda embedding, i.e. the composite

$$\mathbf{N}_\infty : \mathbf{Cat} \xrightarrow{\mathrm{hom}_{\mathbf{Cat}}(=, -)} \mathbf{Fun}(\mathbf{Cat}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{([\bullet]^{\mathrm{op}})^*} \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}) =: \mathbf{sS}.$$

In other words, given an ∞ -category $\mathcal{C} \in \mathbf{Cat}$, its ∞ -nerve is a simplicial space $\mathbf{N}_\infty(\mathcal{C})_\bullet \in \mathbf{sS}$ whose n^{th} space is the space

$$\mathbf{N}_\infty(\mathcal{C})_n := \mathrm{hom}_{\mathbf{Cat}}([n], \mathcal{C}) \in \mathcal{S}$$

of n composable morphisms in $\mathcal{C}$.

10.2.3. Let $C \in \mathbf{sS}$ be a simplicial space. We say that C is a *Segal space* if for any $n \geq 2$ the n^{th} *Segal map*

$$C_n := C_{\{0 < 1 < \dots < n\}} \longrightarrow C_{\{0 < 1\}} \times_{C_{\{1\}}} C_{\{1 < 2\}} \times_{C_{\{2\}}} \dots \times_{C_{\{n-1\}}} C_{\{n-1 < n\}} =: C_1 \times_{C_0} \dots \times_{C_0} C_1$$

is an equivalence.²¹⁰ We write

$$\mathbf{SS} \subset \mathbf{sS}$$

for the full subcategory on the Segal spaces.

²⁰⁸A remarkable theorem [McD79] asserts that the long composite in the commutative diagram

$$\begin{array}{ccccc} \mathbf{Mon}(\mathbf{Set}) & \hookrightarrow & \mathbf{Mon} & \xrightarrow[\sim]{\mathfrak{B}} & \mathbf{Cat}_*^{\mathrm{one\ obj}} \\ & & \downarrow (-)^{\mathrm{gp}} & & \downarrow (-)^{\mathrm{gpd}} \\ & & \mathbf{Grp} & \xrightarrow[\sim]{\mathbf{B}} & \mathcal{S}_*^{\geq 1} \end{array}$$

is surjective: every pointed connected space is equivalent to the classifying space of an ordinary monoid. (This result may also be read as an indication of the essential complexity of (ordinary) monoids: whereas ordinary groups present connected homotopy 1-types, ordinary monoids present all connected homotopy types.)

²⁰⁹Complete Segal spaces were originally introduced as the bifibrant objects in a model structure on the category $\mathbf{ssSet} := \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})) \simeq \mathbf{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{Set})$ of bisimplicial sets [Rez01]. This model structure is a “left Bousfield localization” of a different model structure $\mathbf{s}(\mathbf{sSet}_{\mathbf{KQ}})_{\mathrm{Reedy}}$, which presents $\mathbf{sS}$; this situation is a model-categorical counterpart of the present ∞ -categorical discussion.

²¹⁰Note that this reduces to the definition of an ∞ -monoid in the case that C is reduced (i.e. that $C_0 \xrightarrow{\sim} \mathrm{pt}$).

Exercise 10.4 (6 points). Using the Joyal model structure, prove that we have an equivalence

$$\operatorname{colim} \left(\{0 < 1\} \coprod_{\{1\}} \{1 < 2\} \coprod_{\{2\}} \cdots \coprod_{\{n-1\}} \{n-1 < n\} \right) \xrightarrow{\sim} [n]$$

in $\mathbf{Cat}$.

It follows from Exercise 10.4 that the ∞ -nerve of an ∞ -category is a Segal space. Conversely, given any Segal space $C \in \mathbf{SS}$, we can make the following heuristic definition of a corresponding ∞ -category (which we will make rigorous momentarily): its objects are the points of $C_0 \in \mathcal{S}$, for any $x, y \in C_0$ we define the hom-space

$$\operatorname{hom}_C(x, y) := \lim \left(\begin{array}{ccc} & & C_1 \\ & & \downarrow \\ \operatorname{pt} & \xrightarrow{(x, y)} & C_0 \times C_0 \end{array} \right) \in \mathcal{S},$$

and composition and identities are induced by the maps

$$\mu : C_1 \times_{C_0} C_1 \xleftarrow{\sim} C_2 \xrightarrow{C_{[2] \leftarrow \{0 < 2\}}} C_1 \quad \text{and} \quad \eta : C_0 \xrightarrow{C_{[0] \leftarrow [1]}} C_1.$$

In particular, we consider C_n as “the space of n composable morphisms in C ”, and the Segal condition guarantees that this terminology is sound.

Given this definition, it is easy to define the notion of a **homotopy equivalence** in the Segal space C : it is a point of C_1 that admits left and right homotopy inverses. This notion is clearly invariant under equivalence (i.e. paths in C_1), so that we obtain a subspace $C_1^{\operatorname{ho.eq.}} \subseteq C_1$ of homotopy equivalences. Moreover, it is immediate that we have a factorization

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta} & C_1 \\ & \searrow & \nearrow \\ & C_1^{\operatorname{ho.eq.}} & \end{array} :$$

every identity morphism is trivially a homotopy equivalence. We say that the Segal space C is **complete** if the morphism $C_0 \rightarrow C_1^{\operatorname{ho.eq.}}$ is an equivalence.²¹¹ Heuristically, this says that the “externally-defined” space of objects C_0 is equivalent to the space of equivalences in the “internally-defined” ∞ -category theory of C . We write

$$\mathbf{CSS} \subset \mathbf{SS}$$

for the full subcategory on the complete Segal spaces.

²¹¹In the context of homotopy type theory [Pro13], this condition is also referred to as **univalence**.

10.2.4. Due to the equivalence $[1]^{\mathbf{gpd}} \xrightarrow{\sim} [0]$, the ∞ -nerve of an ∞ -category is in fact a *complete* Segal space. In fact, the ∞ -nerve functor defines an equivalence

$$\mathbf{Cat} \xrightarrow[\sim]{N_\infty} \mathbf{CSS} .$$

Its inverse functor is the restriction to the subcategory $\mathbf{CSS} \subset \mathbf{sS}$ of the left Kan extension

$$\begin{array}{ccc} \Delta & \xleftarrow{[\bullet]} & \mathbf{Cat} \\ \Delta \bullet \downarrow & \nearrow L & \\ \mathbf{sS} & & \end{array} ,$$

which carries a simplicial space $X \in \mathbf{sS}$ to the colimit

$$L(X) := \operatorname{colim} \left(\Delta_{/X} \xrightarrow{\operatorname{fgt}} \Delta \xrightarrow{[\bullet]} \mathbf{Cat} \right) \in \mathbf{Cat} .$$

So, the ∞ -category $L(X) \in \mathbf{Cat}$ is built by gluing together copies of the ∞ -categories $[n] \in \mathbf{Cat}$, one for each point of $X_n \in \mathcal{S}$. Indeed, the functor L can also be thought of as defining a left adjoint

$$\begin{array}{ccccc} & & L & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathbf{sS} & \xrightarrow{\quad \perp \quad} & \mathbf{CSS} & \xrightleftharpoons[\sim]{L} & \mathbf{Cat} . \end{array}$$

This gives an ∞ -categorical perspective on the Joyal model structure: the localization functor $\mathbf{sSet} \rightarrow \mathbf{sSet}[\mathbf{W}_{\mathbf{Joyal}}^{-1}] =: \mathbf{Cat}$ coincides with the composite

$$\mathbf{sSet} \hookrightarrow \mathbf{sS} \xrightarrow{L} \mathbf{Cat} . \quad 212$$

As it turns out, ∞ -groupoids admit a particularly tautological description under the presentation of ∞ -categories as complete Segal spaces.

Exercise 10.5 (12 points).

- Prove that if $\mathcal{G} \in \mathbf{Gpd}$ is an ∞ -groupoid, then $N_\infty(\mathcal{G}) \in \mathbf{sS}$ is the constant simplicial space $\Delta^{\operatorname{op}} \xrightarrow{\operatorname{const}_{\mathcal{G}}} \mathcal{G}$.
- Prove directly that constant simplicial spaces are complete Segal spaces.
- Prove that the functor $\mathcal{S} \xrightarrow{\operatorname{const}} \mathbf{sS}$ is fully faithful.
- Prove that if a Segal space corresponds to an ∞ -groupoid, then it is a complete Segal space if and only if it is constant.

²¹²At the model-categorical level, there are Quillen equivalences *in both directions* between $\mathbf{sSet}_{\mathbf{Joyal}}$ and $\mathbf{ssSet}_{\mathbf{Rezk}}$ [JT07]. The left Quillen equivalence $\mathbf{sSet}_{\mathbf{Rezk}} \rightarrow \mathbf{sSet}_{\mathbf{Joyal}}$ (or really its precomposite $\mathbf{s}(\mathbf{sSet}_{\mathbf{KQ}})_{\mathbf{Reedy}} \rightarrow \mathbf{ssSet}_{\mathbf{Rezk}} \rightarrow \mathbf{sSet}_{\mathbf{Joyal}}$ with a certain left Quillen functor) is a point-set analog of the functor $\mathbf{sS} \xrightarrow{L} \mathbf{Cat}$. The right Quillen equivalence $\mathbf{sSet}_{\mathbf{Joyal}} \leftarrow \mathbf{ssSet}_{\mathbf{Rezk}}$ is particularly remarkable: it is given by taking the levelwise 0-simplices (considering $\mathbf{ssSet} \simeq \mathbf{s}(\mathbf{sSet})$). (The reasonableness of this construction at the point-set level is in part afforded by the fibrancy conditions in $\mathbf{sSet}_{\mathbf{Rezk}}$: in order to be fibrant, not only must a bisimplicial set $(X_\bullet)_\bullet \in \mathbf{ssSet}_{\mathbf{Rezk}}$ satisfy the evident point-set completeness and Segal conditions, but it must be *Reedy fibrant*, so that various maps (e.g. the map $(X_1)_\bullet \rightarrow (X_0)_\bullet \times (X_0)_\bullet$) are fibrations in $\mathbf{sSet}_{\mathbf{KQ}}$.)

Altogether, Exercise 10.5 yields a commutative diagram

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow[\sim]{N_\infty} & \mathbf{CSS} \\ \uparrow & & \uparrow \text{const} \\ \mathbf{Gpd} & \xrightarrow[\sim]{N_\infty} & \mathcal{S} . \end{array}$$

Exercise 10.6 (6 points). Describe the adjoints

$$\begin{array}{ccc} & (-)^{\mathbf{gpd}} & \\ & \text{---} \text{---} \text{---} & \\ \mathbf{Cat} & \xleftarrow{\perp} & \mathcal{S} \\ & \text{---} \text{---} \text{---} & \\ & \iota_0 & \end{array}$$

in terms of complete Segal spaces.

Exercise 10.7 (4 points). Given a space $X \in \mathcal{S}$, construct a Segal space with 0th space X whose corresponding ∞ -category has all hom-spaces contractible.²¹³

10.2.5. The presentation of ∞ -categories as complete Segal spaces is also compatible with the internal hom of simplicial spaces, which is defined by the formula

$$\underline{\mathrm{hom}}_{\mathbf{sS}}(X, Y)_n := \mathrm{hom}_{\mathbf{sS}}(\Delta^n \times X, Y) .$$

Namely, given ∞ -categories $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$, we have an equivalence

$$\mathbf{N}_\infty(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \underline{\mathrm{hom}}_{\mathbf{sS}}(\mathbf{N}_\infty(\mathcal{C}), \mathbf{N}_\infty(\mathcal{D}))$$

of complete Segal spaces. In fact, more is true.

Exercise 10.8 (8 points). Fix a simplicial space $X \in \mathbf{sS}$ and a complete Segal space $C \in \mathbf{CSS}$.

- (a) Show that the internal hom-object $\underline{\mathrm{hom}}_{\mathbf{sS}}(X, C) \in \mathbf{sS}$ is again a complete Segal space.²¹⁴
- (b) Show that $\underline{\mathrm{hom}}_{\mathbf{sS}}(X, C) \simeq \mathbf{N}_\infty(\mathrm{Fun}(L(X), L(C)))$.

Exercise 10.8(b) should be seen as an analog of the corresponding fact for quasicategories discussed in §9.5.4.

²¹³Thus, this Segal space presents the (-1) -truncation

$$\tau_{\leq (-1)} X := \begin{cases} \mathbf{pt} , & X \not\simeq \emptyset \\ \emptyset , & X \simeq \emptyset \end{cases} \in \mathcal{S} \subset \mathbf{Cat} .$$

²¹⁴Because of this, one may say that the subcategory $\mathbf{CSS} \subset \mathbf{sS}$ is an *exponential ideal*.

10.2.6. In fact, Segal spaces (which are not necessarily complete) *also* carry intrinsic homotopical meaning: namely, a Segal space $C \in \mathbf{SS}$ is equivalent data to its underlying ∞ -category $L(C) \in \mathbf{Cat}$ along with the data of the resulting functor $C_0 \rightarrow L(C)$ from its 0th space. Conversely, any surjective functor from an ∞ -groupoid to an ∞ -category arises in this way. Indeed, writing $\mathbf{Ar}^{\text{surj}}(\mathbf{Cat}) \subset \mathbf{Ar}(\mathbf{Cat})$ for the full subcategory on the surjective functors and writing

$$\begin{array}{ccc} \mathbf{Ar}^{\text{surj}}(\mathbf{Cat})|_{\mathbf{Gpd}} & \xrightarrow{\quad} & \mathbf{Gpd} \\ \downarrow & & \downarrow \\ \mathbf{Ar}^{\text{surj}}(\mathbf{Cat}) & \hookrightarrow \mathbf{Ar}(\mathbf{Cat}) \xrightarrow{s} & \mathbf{Cat} \end{array}$$

for the pullback, we have a diagram

$$\begin{array}{ccccc} & & \mathbf{Ar}^{\text{surj}}(\mathbf{Cat})|_{\mathbf{Gpd}} & \xrightleftharpoons[(\iota_0 \mathcal{C} \rightarrow \mathcal{C}) \hookleftarrow \mathcal{C}]{(\mathcal{G} \rightarrow \mathcal{C}) \mapsto \mathcal{C}} & \mathbf{Cat} \\ & \swarrow s & \downarrow \wr & & \downarrow \wr \mathbf{N}_\infty \\ \mathbf{Gpd} & & & & \mathbf{Cat} \\ & \searrow (-)_0 & \downarrow & & \downarrow \\ & & \mathbf{SS} & \xrightleftharpoons{\perp} & \mathbf{CSS} \end{array}$$

that commutes in the evident senses.

Exercise 10.9 (6 points). Construct the equivalence $\mathbf{Ar}^{\text{surj}}(\mathbf{Cat})|_{\mathbf{Gpd}} \xrightarrow{\sim} \mathbf{SS}$.

In fact, the inclusion $\mathbf{SS} \subset \mathbf{sS}$ admits a left adjoint, so that we obtain a diagram of adjunctions

$$\mathbf{sS} \overset{\perp}{\dashv} \mathbf{SS} \overset{\perp}{\dashv} \mathbf{CSS} \quad .$$

10.3. Monoidal ∞ -categories.

10.3.1. Recall from §10.1 that by definition, a **monoidal ∞ -category** is a reduced Segal object in simplicial ∞ -categories.

Fix a monoidal ∞ -category $\mathcal{V}_\bullet \in \mathbf{Mon}(\mathbf{Cat}) \subset \mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Cat})$. We refer to the bifunctor

$$\mathcal{V} \times \mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_1 \xleftarrow{\sim} \mathcal{V}_2 \xrightarrow{\mu} \mathcal{V}_1 =: \mathcal{V}$$

as its **monoidal structure**; we generally denote this by $(-) \boxtimes (-)$. Recall that this is homotopy-coherently associative, as encoded by the simplicial object $\mathcal{V}_\bullet$. We refer to the object selected by the functor

$$\text{pt} \xleftarrow{\sim} \mathcal{V}_0 \xrightarrow{\eta} \mathcal{V}_1 =: \mathcal{V}$$

as the **unit object**. We generally denote this by $\mathbb{1}_{\mathcal{V}}$, or possibly simply by $\mathbb{1} := \mathbb{1}_{\mathcal{V}}$ if $\mathcal{V}$ itself is clear.

Exercise 10.10 (2 points). Prove that the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{(\text{const}_{1_{\mathcal{V}}}, \text{id}_{\mathcal{V}})} & \mathcal{V} \times \mathcal{V} \\ & \searrow \text{id}_{\mathcal{V}} & \downarrow \boxtimes \\ & & \mathcal{V} \end{array}$$

canonically commutes.

A morphism in $\text{Mon}(\text{Cat})$ is called a *monoidal functor*. We will use the fact that $\text{Mon}(\text{Cat})$ is actually an $(\infty, 2)$ -category: given monoidal ∞ -categories $\mathcal{V}, \mathcal{W} \in \text{Mon}(\text{Cat})$, there is an ∞ -category $\text{Fun}^{\otimes}(\mathcal{V}, \mathcal{W})$ whose objects are monoidal functors and whose morphisms are *monoidal* natural transformations.²¹⁵

10.3.2. Given a monoidal category $\mathcal{V}$, there is a classical notion of an *algebra object* in $\mathcal{V}$: this is an object $A \in \mathcal{V}$ equipped with morphisms $A \boxtimes A \xrightarrow{\mu} A$ and $1_{\mathcal{V}} \xrightarrow{\eta} A$, subject to the usual associativity and unitality conditions. For example, algebra objects in $\mathbb{k}$ -modules are $\mathbb{k}$ -algebras.

Here we introduce the ∞ -category $\text{Alg}(\mathcal{V}) := \text{Alg}(\mathcal{V}^{\boxtimes})$ of algebra objects in a monoidal ∞ -category $\mathcal{V} := (\mathcal{V}, \boxtimes)$. We will eventually see that this generalizes the notion of monoid objects introduced in §10.1 to the case of an arbitrary (not necessarily cartesian) monoidal structure, in the sense that when $\mathcal{X} := (\mathcal{X}, \times)$ is cartesian monoidal then we have an equivalence $\text{Alg}(\mathcal{X}) := \text{Alg}(\mathcal{X}^{\times}) \simeq \text{Mon}(\mathcal{X})$.²¹⁶ (Note that we have not yet endowed $\mathcal{X}$ with a cartesian monoidal structure per se.)

We present the simplest definition; an equivalent but simultaneously more complex and more economical definition is discussed in ???. Let us write $\Delta_+ \in \text{Cat}_1$ for the *augmented simplex category*, i.e. the category of finite totally ordered sets. This contains $\Delta \subset \Delta_+$ as a full subcategory; we may write $[-1] := \emptyset$ for the unique object not in its image. This is a monoidal category under the *join* functor $([m], [n]) \mapsto ([m] \star [n]) \cong [m + n + 1]$.

Exercise 10.11 (6 points).

- (a) Endow the object $[0] \in \Delta_+$ with the structure of an algebra object.
- (b) Show that this is the *universal* algebra object in a monoidal category, i.e. that for any $\mathcal{V} \in \text{Mon}(\text{Cat}_1)$ the composite functor

$$\text{Fun}^{\otimes}(\Delta_+, \mathcal{V}) \longrightarrow \text{Fun}(\text{Alg}(\Delta_+), \text{Alg}(\mathcal{V})) \xrightarrow{\text{ev}_{[0]}} \text{Alg}(\mathcal{V})$$

is an equivalence of categories.

²¹⁵These notions will be implicitly defined in ??.

²¹⁶The linguistic distinction is primarily one of emphasis: traditionally the word “algebra object” is used in algebraic contexts (e.g. in $(\text{Mod}_{\mathbb{k}}, \otimes)$), where the monoidal structure is *not* the cartesian product.

From Exercise 10.11(a), we obtain a monoidal ∞ -category $\Delta_+ := (\Delta_+, \star) \in \mathbf{Mon}(\mathbf{Cat})$ (by applying the product-preserving inclusion $\mathbf{Cat}_1 \hookrightarrow \mathbf{Cat}$). Inspired by Exercise 10.11(b), we then define

$$\mathbf{Alg}(\mathcal{V}) := \mathbf{Fun}^\otimes(\Delta_+, \mathcal{V}) .$$

The fact that this is a robust definition is a philosophical assertion that is strongly justified by [Lur].²¹⁷ It follows immediately from the definition that a monoidal functor $\mathcal{V} \rightarrow \mathcal{W}$ determines a functor $\mathbf{Alg}(\mathcal{V}) \rightarrow \mathbf{Alg}(\mathcal{W})$ – indeed, that the construction $\mathbf{Alg}$ assembles as a functor $\mathbf{Mon}(\mathbf{Cat}) \xrightarrow{\mathbf{Alg}} \mathbf{Cat}$.

Exercise 10.12 (2 points). For any monoidal ∞ -category $\mathcal{V} \in \mathbf{Mon}(\mathbf{Cat})$, endow the unit object $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$ with the structure of an algebra object.

There are alternative presentations of the augmented simplex category (see also Exercise 10.19).

Exercise 10.13 (4 points). Construct a faithful embedding $\Delta_+ \hookrightarrow \Delta^{\mathrm{op}}$, and describe the resulting monoidal structure on its image.

10.3.3. One source of monoidal ∞ -categories is monoidal model categories. Given a monoidal model category $\mathbf{V}$, the subcategory $\mathbf{V}^c \subseteq \mathbf{V}$ of cofibrant objects is a monoidal subcategory, which is compatible with the weak equivalences (in the sense that it is a monoid object in relative categories). It follows that the localization $\mathbf{V}^c[\![\mathbf{W}^{-1}]\!]$ acquires the canonical structure of a monoidal ∞ -category (because the localization functor from relative ∞ -categories to ∞ -categories preserves finite products). Moreover, the inclusion $\mathbf{V}^c \hookrightarrow \mathbf{V}$ in $\mathbf{cat}$ induces a canonical equivalence $\mathbf{V}^c[\![\mathbf{W}^{-1}]\!] \xrightarrow{\sim} \mathbf{V}[\![\mathbf{W}^{-1}]\!]$ in $\mathbf{Cat}$ on localizations, and so this is indeed a monoidal structure on the underlying ∞ -category of $\mathbf{V}$ (up to equivalence). Of course, the monoidal structure in $\mathbf{V}$ only computes that of $\mathbf{V}[\![\mathbf{W}^{-1}]\!]$ when restricted to cofibrant objects.

10.4. Commutative monoids and symmetric monoidal ∞ -categories.

10.4.1. We now discuss *commutative* monoid objects, and in particular *symmetric* monoidal ∞ -categories. The basic facts largely parallel the associative analogs, and so we give fewer details and leave some of the corresponding assertions implicit.

10.4.2. The appropriate analog of Δ^{op} is the category $\mathbf{Fin}_*$ of finite pointed sets. We write $S_+ \in \mathbf{Fin}_*$ for a typical object, i.e. $S \in \mathbf{Fin}$ is a finite set and $S_+ := S \sqcup \{+\}$ is obtained by

²¹⁷This is an instance of the general paradigm that once ∞ -category theory has been set up in a sufficiently homotopy-coherent way, homotopy-coherent objects can be defined as representations of their 1-categorical (and even strict) analogs. (The first example of this was the notion of a commutative triangle in an ∞ -category $\mathcal{C}$, which we have ultimately simply defined as a functor $[2] \rightarrow \mathcal{C}$.)

adjoining a basepoint. We write $\underline{n} := \{1, \dots, n\} \in \mathbf{Fin}$. For any $s \in S$, we write $S_+ \xrightarrow{\rho_s} \underline{1}_+$ for the morphism in $\mathbf{Fin}_*$ defined by the formula

$$\rho_s(t) := \begin{cases} 1, & t = s \\ +, & t \neq s \end{cases}.$$

To emphasize the dependence on $s \in S$, we may write $S_+ \xrightarrow{\rho_s} \{s\}_+$ for this morphism. We also write $S_+ \xrightarrow{\alpha} \underline{1}_+$ for the unique morphism in $\mathbf{Fin}_*$ such that $\alpha^{-1}(+) = \{+\}$.

10.4.3. Let $\mathcal{X}$ be an ∞ -category that admits finite products. Given a functor $\mathbf{Fin}_* \xrightarrow{M} \mathcal{X}$, we often write $M_n := M(\underline{n}_+)$ for simplicity; correspondingly, we may write $M_\bullet := M$ to emphasize the fact that M is a functor. A **commutative monoid object** in $\mathcal{X}$ is a functor $\mathbf{Fin}_* \xrightarrow{M_\bullet} \mathcal{X}$ satisfying the following conditions:

- (1) $M_\bullet$ is **reduced**, i.e. the canonical morphism $M_0 \rightarrow \mathbf{pt}_{\mathcal{X}}$ is an equivalence.
- (2) $M_\bullet$ is **Segal**, i.e. for any $n \geq 2$ the n^{th} **Segal map**

$$M_n := M(\underline{n}_+) \xrightarrow{(M(\rho_1), \dots, M(\rho_n))} M(\{1\}_+) \times \cdots \times M(\{n\}_+) =: (M_1)^{\times n}$$

is an equivalence.

Its underlying object is $M := \mathbf{fgt}(M_\bullet) := M_1 \in \mathcal{X}$, and the morphisms

$$\mu : M_1 \times M_1 \xleftarrow{\sim} M_2 \xrightarrow{M(2_+ \xrightarrow{\alpha} 1_+)} M_1 \quad \text{and} \quad \eta : \mathbf{pt} \xleftarrow{\sim} M_0 \xrightarrow{M(0_+ \xrightarrow{\alpha} 1_+)} M_1$$

are respectively its multiplication and unit maps. We write $\mathbf{CMon}(\mathcal{X}) \subseteq \mathbf{Fun}(\mathbf{Fin}_*, \mathcal{X})$ for the full subcategory on the commutative monoid objects. We simply write $\mathbf{CMon} := \mathbf{CMon}(\mathcal{S})$ and refer to its objects as **commutative monoids** or **commutative ∞ -monoids**. A commutative monoid object in $\mathbf{Cat}$ is called a **symmetric monoidal ∞ -category**. Morphisms in $\mathbf{CMon}(\mathbf{Cat})$ are called **symmetric monoidal functors**. We will use the fact that $\mathbf{CMon}(\mathbf{Cat})$ is actually an $(\infty, 2)$ -category, and simply write $\mathbf{Fun}^{\otimes}(-, -)$ for its hom-objects in $\mathbf{Cat}$.

Exercise 10.14 (2 points). Prove that the multiplication of a commutative monoid object $M \in \mathbf{CMon}(\mathcal{X})$ is indeed commutative, i.e. that the diagram

$$\begin{array}{ccc} M^{\times 2} & \xrightarrow{\tau} & M^{\times 2} \\ & \searrow \mu & \swarrow \mu \\ & M & \end{array}$$

commutes (where τ denotes the transposition of factors).

Exercise 10.15 (4 points). Prove that precomposition with the finite pointed simplicial set

$$\Delta^{\text{op}} \xrightarrow{\Delta^1/\partial\Delta^1} \mathbf{Fin}_*$$

induces a functor

$$\mathbf{Mon}(\mathcal{X}) \longleftarrow \mathbf{CMon}(\mathcal{X})$$

carrying a commutative monoid object to its underlying monoid object.

10.4.4. Let $\mathcal{V} \in \mathbf{CMon}(\mathbf{Cat})$ be a symmetric monoidal ∞ -category. We define the ∞ -category $\mathbf{CAlg}(\mathcal{V}) \in \mathbf{Cat}$ of *commutative algebra objects* in $\mathcal{V}$ via the following (which uses the corresponding notion in 1-categories). First of all, we endow the category $\mathbf{Fin}$ of finite sets with the coproduct symmetric monoidal structure.

Exercise 10.16 (6 points).

- (a) Endow the object $\underline{1} \in \mathbf{Fin}$ with the structure of a commutative algebra object.
- (b) Show that this is the *universal* commutative algebra object in a symmetric monoidal category, i.e. that for any $\mathcal{V} \in \mathbf{CMon}(\mathbf{Cat}_1)$ the composite functor

$$\mathbf{Fun}^{\otimes}(\mathbf{Fin}, \mathcal{V}) \longrightarrow \mathbf{Fun}(\mathbf{CAlg}(\mathbf{Fin}), \mathbf{CAlg}(\mathcal{V})) \xrightarrow{\mathrm{ev}_{\underline{1}+}} \mathbf{CAlg}(\mathcal{V})$$

is an equivalence.

For an arbitrary symmetric monoidal ∞ -category $\mathcal{V} \in \mathbf{CMon}(\mathbf{Cat})$, we then define

$$\mathbf{CAlg}(\mathcal{V}) := \mathbf{Fun}^{\otimes}(\mathbf{Fin}, \mathcal{V}) .$$

Exercise 10.17 (8 points). Construct a functor

$$\mathbf{CAlg}(\mathcal{V}) \longrightarrow \mathbf{Alg}(\mathcal{V}) := \mathbf{Alg}(\mathbf{fgt}(\mathcal{V}))$$

carrying a commutative algebra object to its underlying algebra object (where $\mathbf{fgt}(\mathcal{V}) \in \mathbf{Mon}(\mathbf{Cat})$ arises from Exercise 10.15).

10.5. Factorization homology.

10.5.1. As an application of the foregoing discussion, we briefly discuss the theory of *factorization homology*, which combines higher algebra with differential topology. Namely, it gives a pairing between manifolds and higher-algebraic data; for instance, it gives a way of integrating an $\mathbb{E}_n$ -algebra [AF15] or an (∞, n) -category [AFR18] over an n -manifold.²¹⁸ We outline the former variant of factorization homology.²¹⁹

10.5.2. We define the ∞ -category $\mathcal{Mfld}_n$ to be that underlying the following topological category: its objects are (finitary smooth) manifolds, and its hom-objects are topological spaces of (smooth) embeddings (equipped with the compact-open topology).²²⁰ Because

²¹⁸Actually, there are many variants of factorization homology, and in these various variants the n -manifold may be required to satisfy certain conditions (e.g. compactness) and/or to be equipped with certain structure (e.g. a framing).

²¹⁹The latter variant is substantially more powerful and at the same substantially more technical. Indeed, even the mechanism by which the relevant ∞ -categories of manifolds are defined is itself nontrivial [AFR19].

²²⁰This definition can be interpreted as the philosophical assertion that topological spaces of embeddings are most mathematically meaningful when considered as spaces (i.e. after passing to their fundamental ∞ -groupoids). Indeed, in practice one primarily studies their homotopical features (e.g. their homotopy groups (and in particular their nonemptiness)), not their point-set features.

embeddings respect tangent spaces, there is a functor

$$\mathcal{M}\mathrm{fld}_n \longrightarrow \mathcal{S}_{/\mathbf{BGL}_n(\mathbb{R})}$$

carrying a manifold M to the morphism $\Pi_\infty(M) \xrightarrow{TM} \mathbf{BGL}_n(\mathbb{R})$ of spaces that classifies its tangent bundle. Thereafter, for any morphism $B \rightarrow \mathbf{BGL}_n(\mathbb{R})$, we obtain the ∞ -category of B -framed n -manifolds as the pullback

$$\begin{array}{ccc} \mathcal{M}\mathrm{fld}_n^B & \longrightarrow & \mathcal{S}_{/B} \\ \downarrow & & \downarrow \\ \mathcal{M}\mathrm{fld}_n & \longrightarrow & \mathcal{S}_{/\mathbf{BGL}_n(\mathbb{R})} \end{array} .$$

So, an object of $\mathcal{M}\mathrm{fld}_n^B$ is an n -manifold M equipped with a lift

$$\begin{array}{ccc} & & B \\ & \nearrow \text{dashed} & \downarrow \\ \Pi_\infty(M) & \xrightarrow{TM} & \mathbf{BGL}_n(\mathbb{R}) \end{array}$$

of its tangent classifier, and a morphism in $\mathcal{M}\mathrm{fld}_n^B$ is a morphism $M \xrightarrow{f} N$ in $\mathcal{M}\mathrm{fld}_n$ along with the data of a commutative diagram

$$(41) \quad \begin{array}{ccccc} & & & & B \\ & & & & \downarrow \\ & & & & \mathbf{BGL}_n(\mathbb{R}) \\ & \nearrow & \nearrow & \nearrow & \\ \Pi_\infty(M) & \xrightarrow{TM} & \Pi_\infty(N) & \xrightarrow{TN} & \mathbf{BGL}_n(\mathbb{R}) \\ \downarrow \Pi_\infty(f) & & & & \\ & & & & \end{array}$$

in $\mathcal{S}$ (involving the given lifts of TM and TN). For example, if $B = \mathbf{BGL}_n^+(\mathbb{R})$ then a B -framing is an orientation, and if $B = \mathbf{B}\{e\} \simeq \mathbf{pt}$ then a B -framing is a framing in the usual sense (namely a trivialization of the tangent bundle).²²¹ We use the suggestive notations $\mathcal{M}\mathrm{fld}_n^{\mathrm{or}} := \mathcal{M}\mathrm{fld}_n^{\mathbf{BGL}_n^+(\mathbb{R})}$ and $\mathcal{M}\mathrm{fld}_n^{\mathrm{fr}} := \mathcal{M}\mathrm{fld}_n^{\mathbf{pt}}$.

²²¹It is of course possible to present these data in the context of topological spaces (modeling $B \rightarrow \mathbf{BGL}_n(\mathbb{R})$ by a fibration). Still, it is appropriate to require the diagram (41) to be homotopy-coherently commutative instead of strictly commutative, which is exactly the sort of maneuver that is much more easily handled in ∞ -category theory.

10.5.3. We endow $\mathcal{M}\mathbf{fld}_n^B$ with a symmetric monoidal structure given by disjoint union.²²² We write $\mathcal{D}\mathbf{isk}_n^B \subseteq \mathcal{M}\mathbf{fld}_n^B$ for the full (symmetric monoidal) subcategory on those objects of the form $(\mathbb{R}^n)^{\sqcup i}$ for $i \geq 0$. We likewise use the distinguished superscripts $B \in \{\emptyset, \text{or}, \text{fr}\}$ in the evident way.

Fix a symmetric monoidal ∞ -category $\mathcal{V}$ (e.g. the derived ∞ -category $\mathbf{D}_{\mathbb{k}}$). A **B -framed $\mathbb{E}_n$ -algebra** (or simply an $\mathbb{E}_n^B$ -algebra) in $\mathcal{V}$ is a symmetric monoidal functor

$$\mathcal{D}\mathbf{isk}_n^B \xrightarrow{A} \mathcal{V} ;$$

these assemble into an ∞ -category

$$\mathbf{Alg}_{\mathbb{E}_n^B}(\mathcal{V}) := \mathbf{Fun}^{\otimes}(\mathcal{D}\mathbf{isk}_n^B, \mathcal{V}) .$$

So by definition, $\mathcal{D}\mathbf{isk}_n^B$ is the free symmetric monoidal ∞ -category containing a B -framed $\mathbb{E}_n$ -algebra. Given $A \in \mathbf{Alg}_{\mathbb{E}_n^B}(\mathcal{V})$, one thinks of $A := A(\mathbb{R}^n) \in \mathcal{V}$ as its underlying object; for each $i \geq 0$, this comes equipped with a parametrized family

$$\mathbf{hom}_{\mathcal{D}\mathbf{isk}_n^B}((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n) \xrightarrow{A} \mathbf{hom}_{\mathcal{V}}(A^{\boxtimes i}, A)$$

of i -ary operations (using the symmetric monoidality of A), with compatibilities according to composition in $\mathcal{D}\mathbf{isk}_n^B$. We write

$$\mathbf{Alg}_{\mathbb{E}_n}(\mathcal{V}) := \mathbf{Alg}_{\mathbb{E}_n^{\text{fr}}}(\mathcal{V}) := \mathbf{Fun}^{\otimes}(\mathcal{D}\mathbf{isk}_n^{\text{fr}}, \mathcal{V}) ,$$

and refer to objects of this ∞ -category simply as $\mathbb{E}_n$ -algebras in $\mathcal{V}$.²²³

Exercise 10.18 (12 points).

- (a) Prove that the ∞ -categories $\mathcal{D}\mathbf{isk}_1$ and $\mathcal{D}\mathbf{isk}_1^{\text{fr}}$ are (equivalent to) 1-categories.
- (b) Give combinatorial descriptions of these categories (e.g. as strict categories), including descriptions of their symmetric monoidal structures.
- (c) Prove that $\mathcal{D}\mathbf{isk}_1^{\text{fr}}$ is the free symmetric monoidal category containing an associative algebra object.
- (d) Formulate and prove an analogous universal characterization of $\mathcal{D}\mathbf{isk}_1$.

Exercise 10.19 (6 points).

²²²Note that the functor $\mathbf{cat}(\mathbf{Top}) \rightarrow \mathbf{cat}(\mathbf{Top})[\mathbf{W}^{-1}] \simeq \mathbf{Cat}$ preserves products and so determines a functor $\mathbf{CMon}(\mathbf{cat}(\mathbf{Top})) \rightarrow \mathbf{CMon}(\mathbf{Cat})$.

²²³Beware that this (extremely well-established) convention does *not* agree with the above convention that $\mathcal{D}\mathbf{isk}_n := \mathcal{D}\mathbf{isk}_n^{\mathbf{BGL}_n(\mathbb{R})}$. Even more confusingly, $\mathbb{E}_n^{\mathbf{BGL}_n(\mathbb{R})}$ -algebras are sometimes referred to as “framed $\mathbb{E}_n$ -algebras”. This refers to the fact that morphisms in $\mathcal{D}\mathbf{isk}_n^{\mathbf{BGL}_n(\mathbb{R})} \simeq \mathcal{D}\mathbf{isk}_n$ are (∞ -categorically) equivalent to the data of their images along with “framings”; for instance, differentiation (and in particular evaluation) at the origin determines an equivalence

$$\mathbf{hom}_{\mathcal{D}\mathbf{isk}_n}(\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\sim} \{(p \in \mathbb{R}^n, T_0 \mathbb{R}^n \xrightarrow{\cong} T_p \mathbb{R}^n)\}$$

with the underlying space of the frame bundle of $\mathbb{R}^n$, ultimately yielding an equivalence $\mathbf{hom}_{\mathcal{D}\mathbf{isk}_n}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathbf{GL}_n(\mathbb{R})$ of ∞ -monoids. We suggest the terminologies “unoriented $\mathbb{E}_n$ -algebras” or “ribbon $\mathbb{E}_n$ -algebras” for $\mathbb{E}_n^{\mathbf{BGL}_n(\mathbb{R})}$ -algebras.

- (a) Identify the category Δ_+ in terms of the category $\mathcal{D}\text{isk}_1^{\text{fr}}$.
- (b) Give a topological description of the embedding $\Delta_+ \hookrightarrow \Delta^{\text{op}}$ of Exercise 10.13.

Exercise 10.20 (2 points). Give an explicit description of the ∞ -category $\text{Alg}_{\mathbb{E}_0}(\mathcal{V})$.

Exercise 10.21 (6 points). For any framed n -manifold $M \in \mathcal{M}\text{fld}_n^{\text{fr}}$, prove that evaluation at origins determines an equivalence

$$\text{hom}_{\mathcal{M}\text{fld}_n^{\text{fr}}}((\mathbb{R}^n)^{\sqcup i}, M) \xrightarrow{\sim} \Pi_{\infty}(\text{Conf}_n(M))$$

with the underlying space of the topological space $\text{Conf}_n(M) \in \mathbf{Top}$ of configurations of n distinct points in M (a subspace of $M^{\times n} \in \mathbf{Top}$).

For any $n \geq 0$, there is a canonical symmetric monoidal functor

$$\mathcal{D}\text{isk}_n^{\text{fr}} \longrightarrow \mathcal{D}\text{isk}_{n+1}^{\text{fr}}$$

given by thickening embeddings of n -dimensional disks into embeddings of $(n+1)$ -dimensional disks.²²⁴ Clearly, this extends to a commutative triangle

$$\begin{array}{ccc} \mathcal{D}\text{isk}_n^{\text{fr}} & \longrightarrow & \mathcal{D}\text{isk}_{n+1}^{\text{fr}} \\ & \searrow \pi_0 & \downarrow \pi_0 \\ & & \mathbf{Fin} \end{array}$$

of symmetric monoidal ∞ -categories. In fact, these symmetric monoidal functors determine an equivalence

$$\text{colim}^{\mathbf{CMon}(\mathbf{Cat})} (\mathcal{D}\text{isk}_0^{\text{fr}} \longrightarrow \mathcal{D}\text{isk}_1^{\text{fr}} \longrightarrow \mathcal{D}\text{isk}_2^{\text{fr}} \longrightarrow \cdots) \xrightarrow{\sim} \mathbf{Fin}$$

of symmetric monoidal ∞ -categories.²²⁵ Hence, for any $\mathcal{V} \in \mathbf{CAlg}(\mathbf{Cat})$ we obtain an equivalence

$$\mathbf{CAlg}(\mathcal{V}) \xrightarrow{\sim} \lim^{\mathbf{Cat}} (\cdots \longrightarrow \text{Alg}_{\mathbb{E}_2}(\mathcal{V}) \longrightarrow \text{Alg}_{\mathbb{E}_1}(\mathcal{V}) \longrightarrow \text{Alg}_{\mathbb{E}_0}(\mathcal{V}))$$

²²⁴More precisely, there is a contractible topological space of such functors at the level of topological categories, given by making continuous choices of radii for thickenings. (The necessity of such choices can be avoided at the expense of using a more rigid topological model for $\mathcal{D}\text{isk}_n^{\text{fr}}$ (as in the usual definition of the “little n -disks operad”).)

²²⁵This is because the path components of the space $\text{hom}_{\mathcal{D}\text{isk}_n^{\text{fr}}}((\mathbb{R}^n)^{\sqcup i}, (\mathbb{R}^n)^{\sqcup j})$ become increasingly highly connected as n grows, yielding an equivalence

$$\text{colim}^s \left(\text{hom}_{\mathcal{D}\text{isk}_0^{\text{fr}}}((\mathbb{R}^0)^{\sqcup i}, (\mathbb{R}^0)^{\sqcup j}) \longrightarrow \text{hom}_{\mathcal{D}\text{isk}_1^{\text{fr}}}((\mathbb{R}^1)^{\sqcup i}, (\mathbb{R}^1)^{\sqcup j}) \longrightarrow \text{hom}_{\mathcal{D}\text{isk}_2^{\text{fr}}}((\mathbb{R}^2)^{\sqcup i}, (\mathbb{R}^2)^{\sqcup j}) \longrightarrow \cdots \right) \longrightarrow \text{hom}_{\mathbf{Fin}}(\underline{i}, \underline{j})$$

of spaces. For instance, taking $i = 2$ and $j = 1$, by Exercise 10.21 we have an equivalence

$$\text{hom}_{\mathcal{D}\text{isk}_n^{\text{fr}}}((\mathbb{R}^n)^{\sqcup 2}, \mathbb{R}^n) \simeq \Pi_{\infty}(\text{Conf}_2(\mathbb{R}^n)) \simeq S^{n-1}$$

of spaces. (Note too that the forgetful functor $\mathbf{CMon}(\mathbf{Cat}) \rightarrow \mathbf{Cat}$ is conservative, and that the extraction of hom-spaces in ∞ -categories commutes with filtered colimits (as can be seen via complete Segal spaces).)

of ∞ -categories. For this reason, it is common to refer to commutative algebras as $\mathbb{E}_\infty$ -*algebras*.²²⁶ Note that an $\mathbb{E}_n$ -algebra has an underlying $\mathbb{E}_i$ -algebra for any $0 \leq i \leq n \leq \infty$.

Exercise 10.22 (6 points). Prove that if $\mathcal{V}$ is a symmetric monoidal category and $n \geq 2$ then the forgetful functor

$$\mathrm{CAlg}(\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{V})$$

is an equivalence of categories.

10.5.4. We discuss $\mathbb{E}_n$ -algebras in spaces (i.e. taking $(\mathcal{V}, \boxtimes) = (\mathcal{S}, \times)$). These are closely related to *n-fold loopspaces*. Namely, one-point compactification (and passage to underlying pointed spaces) determines a canonical functor

$$\mathrm{Disk}_n^{\mathrm{fr}} \xrightarrow{(-)^+} \mathcal{S}_*^{\mathrm{op}} .^{227}$$

Moreover, this functor is canonically symmetric monoidal: it carries disjoint unions to wedge sums. Thereafter, a based space $X \in \mathcal{S}_*$ yields an $\mathbb{E}_n$ -algebra

$$\Omega^n X := \left(\mathrm{Disk}_n^{\mathrm{fr}} \xrightarrow{(-)^+} \mathcal{S}_*^{\mathrm{op}} \xrightarrow{\mathrm{hom}_{\mathcal{S}_*}(-, X)} \mathcal{S} \right) \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}) ,$$

whose underlying object is its n -fold loopspace

$$\mathrm{hom}_{\mathcal{S}_*}((\mathbb{R}^n)^+, X) := \mathrm{hom}_{\mathcal{S}_*}(\Pi_\infty((\mathbb{R}^n)^+), X) \simeq \mathrm{hom}_{\mathcal{S}_*}(S^n, X) =: \Omega^n X \in \mathcal{S} .$$

For $n \geq 1$, we say that an $\mathbb{E}_n$ -algebra is *grouplike* if its image under the functor

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}) \xrightarrow{\mathrm{Alg}_{\mathbb{E}_n}(\pi_0)} \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Set}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Set}) \simeq \mathrm{Alg}(\mathrm{Set}) \simeq \mathrm{Mon}(\mathrm{Set})$$

is a group (using Exercise 10.18(c) and the evident equivalence $\mathrm{Alg}(\mathrm{Set}) := \mathrm{Alg}(\mathrm{Set}^\times) \simeq \mathrm{Mon}(\mathrm{Set})$ (which is as-yet unproved for ∞ -categories) to identify the target). We write

$$\mathrm{Alg}_{\mathbb{E}_n}^{\mathrm{gp}}(\mathcal{S}) \subset \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$$

for the full subcategory on the grouplike $\mathbb{E}_n$ -algebras.²²⁸ Clearly, the $\mathbb{E}_n$ -algebra $\Omega^n X$ is grouplike, as $\pi_0(\Omega^n X) =: \pi_n(X)$ is a group. A fundamental theorem in homotopy theory [BV73, May72] asserts that *all* grouplike $\mathbb{E}_n$ -algebras arise in this way: indeed, we have a diagram

$$\mathrm{Alg}_{\mathbb{E}_n}^{\mathrm{gp}}(\mathcal{S}) \xrightleftharpoons[\Omega^n]{B^n} \mathcal{S}_*^{\geq n} \xrightleftharpoons[\perp]{} \mathcal{S}_* .^{229}$$

²²⁶More precisely, there is a fairly evident symmetric monoidal ∞ -category $\mathrm{Disk}_\infty^{\mathrm{fr}}$ of infinite-dimensional framed disks, and this is also equivalent to $\mathbf{Fin}$ as such.

²²⁷Such notions are addressed systematically in [AF].

²²⁸It will be clear that the equivalence $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Mon}(\mathcal{S}) =: \mathrm{Mon}$ factors as an equivalence $\mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Grp}(\mathcal{S}) =: \mathrm{Grp}$.

²²⁹This should be compared with the discussion of §10.1.3: more generally, $\mathbb{E}_n$ -algebras are essentially equivalent to pointed (∞, n) -categories with a single equivalence class of object.

10.5.5. Given a B -framed $\mathbb{E}_n$ -algebra $A \in \mathbf{Alg}_{\mathbb{E}_n^B}(\mathcal{V})$, its **factorization homology** functor is the left Kan extension

$$\begin{array}{ccc} \mathcal{Disk}_n^B & \xrightarrow{A} & \mathcal{V} \\ \downarrow & \nearrow \scriptstyle \text{(-)}^A & \\ \mathcal{Mfld}_n^B & & \end{array} .$$

In other words, its factorization homology over a B -framed n -manifold $M \in \mathcal{Mfld}_n^B$ is the colimit

$$\int_M A := \operatorname{colim} \left((\mathcal{Disk}_n^B)_M \xrightarrow{\operatorname{fgt}} \mathcal{Disk}_n^B \xrightarrow{A} \mathcal{V} \right) .$$

This has a number of pleasant features. Most importantly for us, it admits a characterization akin to the Eilenberg–Steenrod axioms: every “homology theory for B -framed n -manifolds” H is given by factorization homology, with coefficients in the $\mathbb{E}_n$ -algebra $H(\mathbb{R}^n)$ [AF15].²³⁰

10.5.6. A special case of classical interest is **Hochschild homology**, which is factorization homology over $S^1 \in \mathcal{Mfld}_1^{\operatorname{fr}}$. We briefly explain the original definition (as discussed e.g. in [Lod98]) and how factorization homology recovers it.

To simplify our notation, we write

$$\Delta_{\mathcal{G}}^{\operatorname{op}} := (\mathcal{Disk}_1^{\operatorname{fr}})_{/S^1} .$$

Exercise 10.23 (8 points).

- (a) Prove that the ∞ -category $\Delta_{\mathcal{G}}^{\operatorname{op}}$ is a 1-category.
- (b) Give a combinatorial description of this category.

The category $\Delta_{\mathcal{G}}^{\operatorname{op}}$ is called the **paracyclic category**.²³¹

Using Exercise 10.23 (which guarantees that no issues of homotopy-coherence arise), one can construct a canonical functor

$$(42) \quad \Delta^{\operatorname{op}} \longrightarrow \Delta_{\mathcal{G}}^{\operatorname{op}} ,$$

which carries $[n]^{\circ} \in \Delta^{\operatorname{op}}$ to a configuration of $n+1$ framed 1-disks embedded in S^1 , with face maps given by colliding adjacent disks and with degeneracy maps given by allowing new disks to appear. Then, for any $\mathbb{E}_1$ -algebra $A \in \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{V})$, we define its **cyclic bar construction** to be the simplicial object

$$\mathbf{B}^{\operatorname{cyc}}(A) : \Delta^{\operatorname{op}} \longrightarrow \Delta_{\mathcal{G}}^{\operatorname{op}} \xrightarrow{\operatorname{fgt}} \mathcal{Disk}_1^{\operatorname{fr}} \xrightarrow{A} \mathcal{V} .$$

²³⁰In particular, this notion of “homology theory” involves a *multiplicative* form of excision (inspired by TQFT).

²³¹Beware that the usual definition of the paracyclic category does not include the initial object ($\emptyset \hookrightarrow S^1$). We elide this distinction here to simplify the discussion (which is mathematically harmless because the inclusion of the usual paracyclic category into the one discussed here is final).

It turns out that the functor (42) is *final*: colimits over $\Delta_{\mathcal{C}}^{\text{op}}$ can be computed as colimits over Δ^{op} .²³² Hence, the canonical morphism

$$|\mathbf{B}^{\text{cyc}}(A)| := \text{colim}_{\Delta^{\text{op}}}(\mathbf{B}^{\text{cyc}}(A)) \longrightarrow \text{colim}\left(\Delta_{\mathcal{C}}^{\text{op}} \xrightarrow{\text{fgt}} \mathcal{D}\text{isk}_1^{\text{fr}} \xrightarrow{A} \mathcal{V}\right) =: \int_{S^1} A$$

is an equivalence.

Now, the classical definition is as follows: given an associative $\mathbb{k}$ -algebra $A \in \mathbf{Alg}_{\mathbb{k}} := \mathbf{Alg}(\mathbf{Mod}_{\mathbb{k}}) \simeq \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Mod}_{\mathbb{k}})$, its n^{th} **Hochschild homology** is the $\mathbb{k}$ -module

$$\mathbf{HH}_n(A) := \mathbf{H}_n(\mathbf{C}_{\bullet}(\mathbf{B}^{\text{cyc}}(A))) \cong \pi_n(\mathbf{B}^{\text{cyc}}(A)) \in \mathbf{Mod}_{\mathbb{k}}$$

(recall §8.4.4).

Exercise 10.24 (2 points). Give an explicit description of $\mathbf{HH}_0(A)$ in terms of A .

Exercise 10.25 (4 points). Compute $\mathbf{HH}_*(\mathbb{k}[\varepsilon]/\varepsilon^2)$.

Exercise 10.26 (6 points). Prove that for any commutative $\mathbb{k}$ -algebra $A \in \mathbf{CAlg}_{\mathbb{k}}$ there is a canonical isomorphism $\mathbf{HH}_1(A) \cong \Omega_{A|\mathbb{k}}^1$ (recall §8.6.4).²³³

Of central importance is an action of the circle group on Hochschild homology.²³⁴ In terms of factorization homology, this is immediate: it follows from the fact that factorization homology is a functor

$$\mathcal{M}\text{fld}_1^{\text{fr}} \xrightarrow{j_{(-)}^A} \mathcal{V},$$

as the circle group acts on the object $S^1 \in \mathcal{M}\text{fld}_1^{\text{fr}}$. By contrast, in the classical definition, this circle action must be built in “by hand” via simplicial methods.

10.5.7. Whereas ordinary co/homology of manifolds adheres to Poincaré duality, factorization homology participates in **nonabelian Poincaré duality**. Fix a framed n -manifold $M \in \mathcal{M}\text{fld}_n^{\text{fr}}$. Given a based space $X \in \mathcal{S}_*$, there is a canonical map

$$(43) \quad \int_M \Omega^n X \longrightarrow \text{map}_c(M, X)$$

²³²More precisely, a functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{J}$ is **final** if for any functor $\mathcal{J} \xrightarrow{F} \mathcal{C}$, the canonical morphism

$$\text{colim}_{\mathcal{J}}(\varphi^* F) \longrightarrow \text{colim}_{\mathcal{J}}(F)$$

in $\mathcal{C}$ is an equivalence (and in particular both colimits exist if either one does). We discuss finality more thoroughly below.

²³³This fact is the basis of the celebrated *Hochschild–Kostant–Rosenberg* (or simply *HKR*) theorem [HKR62]: assuming that A is smooth over $\mathbb{k}$, there is a canonical isomorphism $\Omega_{A|\mathbb{k}}^{\bullet} \cong \mathbf{HH}_{\bullet}(A)$ of graded $\mathbb{k}$ -algebras. A derived version of the HKR theorem (using the cotangent complex) is established in [TV11].

²³⁴In terms of the HKR theorem, this implements the de Rham differential.

to the space of *compactly-supported maps* from M to X .²³⁵ This can be described informally as follows: a point of $\int_M \Omega^n X$ is the data of a framed embedding $(\mathbb{R}^n)^{\sqcup i} \hookrightarrow M$ along with labels $\gamma_1, \dots, \gamma_i \in \Omega^n X$, and the map (43) carries this to the composite

$$\Pi_\infty(M^+) \longrightarrow \Pi_\infty(((\mathbb{R}^n)^{\sqcup i})^+) \simeq (S^n)^{\vee i} \xrightarrow{(\gamma_1, \dots, \gamma_i)} X$$

where the first map is the Pontrjagin–Thom collapse. Assuming that X is n -connected (i.e. that $\pi_j(X) = 0$ for $j < n$), the map (43) is an equivalence of pointed spaces (where the basepoints are the empty configuration and the constant map).²³⁶ This may be thought of as a sort of “cellular approximation in families” result (as n -manifolds are n -dimensional cell complexes (e.g. via Morse theory)); its proof consists in showing that the functor

$$\mathcal{M}\mathrm{fld}_n^{\mathrm{fr}} \xrightarrow{\mathrm{map}_c(-, X)} \mathcal{S}$$

satisfies the same axioms characterizing $\int_{(-)} \Omega^n X$ that were alluded to in §10.5.5.

Given an abelian group A , this recovers ordinary Poincaré duality by taking $X = K(A, n)$ to be the corresponding Eilenberg–MacLane space, so that we have an equivalence $\Omega^n X \simeq A$ of $\mathbb{E}_n$ -algebras. On the one hand, passing from embedded disks to their origins yields an equivalence

$$\int_M A \xrightarrow{\sim} \Pi_\infty(\mathbb{Z}\{M\} \otimes_{\mathbb{Z}} A)$$

with the space of configurations of points in M labeled by elements of A , and hence we obtain isomorphisms

$$\pi_j \left(\int_M A \right) \cong H_j(M; A)$$

(as in §8.4.5). Meanwhile, we have isomorphisms

$$\pi_j(\mathrm{map}_c(M, K(A, n))) \cong \pi_0(\mathrm{map}_c(M, \Omega^j K(A, n))) \cong \pi_0(\mathrm{map}_c(M, K(A, n-j))) \cong H_c^{n-j}(M; A).$$

Hence, applying π_j to the equivalence (43) in $\mathcal{S}_*$ (in the case that $X = K(A, n)$) yields the Poincaré duality isomorphism

$$H_j(M; A) \xrightarrow{\cong} H_c^{n-j}(M; A).$$

²³⁵Because we have assumed that M is finitary, we have an equivalence

$$\mathrm{map}_c(M, X) \simeq \mathrm{hom}_{\mathcal{S}_*}(\Pi_\infty(M^+), X).$$

More generally, given a topological space $T \in \mathbf{Top}$, writing $\mathbf{Cpct}(T)$ for its poset of compact subsets we have a composite functor

$$\mathbf{Cpct}(T)^{\mathrm{op}} \xrightarrow{K \mapsto T/(T \setminus K)} \mathbf{Top}_* \xrightarrow{\Pi_\infty} \mathcal{S}_*,$$

and using this we define the space of compactly-supported maps from T to the based space $X \in \mathcal{S}_*$ to be

$$\mathrm{map}_c(T, X) := \mathrm{colim}_{K \in \mathbf{Cpct}(T)} \mathrm{hom}_{\mathcal{S}_*}(\Pi_\infty(T/(T \setminus K)), X) \in \mathcal{S}.$$

²³⁶Of course, there is a generalization for B -framed n -manifolds. For this, one must replace compactly-supported maps with compactly-supported sections of a certain bundle (which is trivialized by a framing).

10.5.8. There is a largely analogous theory based on a corresponding ∞ -category $\mathcal{M}\mathrm{fld}_n^{\mathrm{top}}$ of (finitary) *topological* n -manifolds. These only have tangent *microbundles*, which are classified by maps to $\mathbf{B}\mathrm{Top}(n)$ (the classifying space of the topological group $\mathrm{Top}(n)$ of self-homeomorphisms of $\mathbb{R}^n$). These can likewise be equipped with B -framings for morphisms $B \rightarrow \mathbf{B}\mathrm{Top}(n)$, yielding an ∞ -category $\mathcal{M}\mathrm{fld}_n^{\mathrm{top}, B}$ of B -framed topological n -manifolds. In fact, taking $B = \mathbf{BGL}_n(\mathbb{R})$, for $n \neq 4$ the forgetful functor

$$\mathcal{M}\mathrm{fld}_n \longrightarrow \mathcal{M}\mathrm{fld}_n^{\mathrm{top}, \mathbf{BGL}_n(\mathbb{R})}$$

is an equivalence.

10.5.9. In a different direction, one can also incorporate manifolds with boundary (or corners). Let us write $\mathcal{M}\mathrm{fld}_n^{\partial}$ for the analogous symmetric monoidal ∞ -category of smooth manifolds with boundary, and $\mathrm{Disk}_n^{\partial} \subseteq \mathcal{M}\mathrm{fld}_n^{\partial}$ for the full (symmetric monoidal) subcategory on those objects of the form

$$(\mathbb{R}^n)^{\sqcup i} \sqcup (\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0})^{\sqcup j}$$

for $i, j \geq 0$.

Exercise 10.27 (6 points).

- (a) Prove that the ∞ -category $\mathrm{Disk}_1^{\partial, \mathrm{fr}}$ is a 1-category.
- (b) Give a combinatorial description of it, including a description of its symmetric monoidal structure.

In fact, the paper [AFT17] develops the theory of factorization homology for *stratified spaces*. Of course, it continues to be the case that the local models determine the nature of the coefficients.

11. Notes for weeks 7-9

In these last three weeks, I'll give a rapid introduction to the general theory of ∞ -categories, building on our discussion of $\mathbb{k}$ -linear ∞ -categories. Then, I'll proceed to discuss **stable ∞ -categories**, and connect them back to (stable) $\mathbb{k}$ -linear ∞ -categories. From there, we'll cover additional related topics as time allows.

In order to accomplish this, we'll have a substantially less thorough discussion of ∞ -categories than was given in a previous iteration of this course (on the notes from which these notes are based). However, it doesn't really make sense for me to write completely thorough notes that adhere to our current course, and in any case it should be valuable for you to have complementary reading material. So instead, I have simply included all of the relevant material from the previous iteration above, and in this section I'll merely summarize what was discussed in each lecture along with specific references to material above and/or to

external references.²³⁷ It is worth noting that the lecture slides (available from the Home tab in our Canvas page) may be more valuable now than before for reviewing what was covered in lecture.

11.1. **Monday 5/15.** Today's lecture was split between two topics.

- We first discussed Postnikov towers and k-invariants for derived R -modules (see §7.3.3). This involved the n -**truncation** functor $\mathbf{D}_R \xrightarrow{\tau_{\leq n}} \mathbf{D}_R^{\leq n}$, which is left adjoint to the inclusion. A slight variant, the n -**cotruncation** functor $\mathbf{D}_R \xrightarrow{\tau_{\geq n}} \mathbf{D}_R^{\geq n}$, is discussed in §8.4.2; this is *right* adjoint to the inclusion. These are closely related: we have

$$\tau_{\geq n} \simeq \text{coker} \left(i_{\geq n+1} \tau_{\geq n+1} \xrightarrow{\varepsilon} \text{id} \right)$$

in $\text{Fun}(\mathbf{D}_R, \mathbf{D}_R)$.

This concluded our discussion of derived functors (i.e. we skipped §§7.4-7.5).

From here, we transitioned into the brand-new topic of ∞ -categories (although these will take a little while to reach). Though we covered it differently, our approach most closely follows that material in §9.2.1.

- Let $\mathbf{cat}$ denote the category of (small) categories. For each $n \geq 0$, we have the totally ordered set $[n] := \{0 < 1 < \cdots < n\}$, which we can consider as a category $[n] \in \mathbf{cat}$. These assemble into a full subcategory $\Delta \subset \mathbf{cat}$, which is called the **simplicial indexing category**. This has a generators-and-relations presentation, see e.g. Proposition 2.5 [here](#).
- A **simplicial object** in a category $\mathcal{C}$ is by definition a functor $\Delta^{\text{op}} \rightarrow \mathcal{C}$; these form a category $\mathbf{s}\mathcal{C} := \text{Fun}(\Delta^{\text{op}}, \mathcal{C})$. Typically, given $X \in \mathbf{s}\mathcal{C}$ we write $X_n := X([n]^\circ)$. In particular, we have the category $\mathbf{sSet}$ of simplicial sets, which will play a central role for us.
- Given a category $\mathcal{C} \in \mathbf{cat}$, its **nerve** is the simplicial set

$$\mathbf{N}(\mathcal{C}) : \Delta^{\text{op}} \hookrightarrow \mathbf{cat}^{\text{op}} \xrightarrow{\text{hom}_{\mathbf{cat}}(-, \mathcal{C})} \mathbf{Set}.$$

The set $\mathbf{N}(\mathcal{C})_n$ consists of the sequences of n composable morphisms in $\mathcal{C}$ (with $\mathbf{N}(\mathcal{C})_0$ being the set of objects).

11.2. **Wednesday 5/17.** Today's lecture roughly corresponds to the material in §9.2.1 and §§8.3.3-8.3.5.

- Given a category $\mathcal{C} \in \mathbf{cat}$, its nerve $\mathbf{N}(\mathcal{C}) \in \mathbf{sSet}$ “knows everything there is to know” about the original category. For instance, $\mathbf{N}(\mathcal{C})_0$ is its set of objects, $\mathbf{N}(\mathcal{C})_1$ is its set of morphisms, and composition is recorded by a structure map $\mathbf{N}(\mathcal{C})_2 \rightarrow \mathbf{N}(\mathcal{C})_1$

²³⁷Recall from §0.2.5 that we use the letters “T” and “A” to respectively refer to Lurie’s books [Higher Topos Theory](#) and [Higher Algebra](#): [§T.1] is the canonical expository introduction to ∞ -categories (and is extremely accessible), while [§A.1] is the canonical reference for stable ∞ -categories.

(namely that opposite to the map $[2] \xleftarrow{0 \leq 2} [1]$). Associativity is baked in as well, by considering a square

$$\begin{array}{ccc} N(\mathcal{C})_3 & \longrightarrow & N(\mathcal{C})_2 \\ \downarrow & & \downarrow \\ N(\mathcal{C})_2 & \longrightarrow & N(\mathcal{C})_1 \end{array} .$$

Altogether, the functor $\mathbf{sSet} \xleftarrow{N} \mathbf{cat}$ is fully faithful.

- Another way of thinking about Δ is as *topological simplices*: we have a faithful inclusion $\Delta \xleftarrow{\Delta_{\text{top}}^{\bullet}} \mathbf{Top}$, with maps being the vertex-preserving linear functions.
- We can use this to consider a simplicial set $X \in \mathbf{sSet}$ as a “gluing recipe” for a topological space, namely its *geometric realization*

$$|X| := \text{colim}_{(\Delta^n \rightarrow X) \in \Delta_{/X}} (\Delta_{\text{top}}^n) \in \mathbf{Top} .$$

- Relatedly, restricted Yoneda along $\Delta_{\text{top}}^{\bullet}$ carries a topological space $Y \in \mathbf{Top}$ to its simplicial set

$$\text{Sing}(Y) : \Delta^{\text{op}} \xrightarrow{\Delta_{\text{top}}^{\bullet}} \mathbf{Top}^{\text{op}} \xrightarrow{\text{hom}_{\mathbf{Top}}(-, Y)} \mathbf{Set}$$

of *singular simplices*. (So, $\text{Sing}(Y)_n := \text{hom}_{\mathbf{Top}}(\Delta_{\text{top}}^n, Y)$ is the set of singular n -simplices in Y .) Namely, these functors assemble into an adjunction

$$\mathbf{sSet} \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow[\text{Sing}]{\perp} \end{array} \mathbf{Top} .$$

- This is a very general phenomenon. Namely, the functor $\Delta \xrightarrow{\Delta^{\bullet}} \mathbf{sSet}$ is the *free cocompletion*: for any category $\mathcal{D}$ with all (small) colimits, restriction defines an equivalence

$$\text{Fun}(\Delta, \mathcal{D}) \xleftarrow{\sim} \text{Fun}(\mathbf{sSet}, \mathcal{D})^{\text{colim-pres}} ,$$

with the inverse given by “left Kan extension” along the inclusion $\Delta \xleftarrow{\Delta^{\bullet}} \mathbf{sSet}$: for any functor $\Delta \xrightarrow{\Delta^{\bullet}} \mathcal{D}$, the corresponding colimit-preserving functor $\mathbf{sSet} \rightarrow \mathcal{D}$ carries $X \in \mathbf{sSet}$ to

$$\text{colim}_{(\Delta^n \rightarrow X) \in \Delta_{/X}} (\Delta_{\mathcal{D}}^n) \in \mathcal{D} .$$

- In particular, this was the case with the nerves of categories: we have an adjunction

$$\mathbf{sSet} \begin{array}{c} \xrightarrow{\text{ho}} \\ \xleftarrow[\mathbf{N}]{\perp} \end{array} \mathbf{cat} .$$

So we can *also* think of a simplicial set as a recipe for building a category. (The notation ho will be explained later.) More broadly, a simplicial set is “an abstract gluing recipe” for building objects in *any* category $\mathcal{D}$ that admits all colimits, implemented as described just above.

- To characterize the nerves of categories, we define the i^{th} **horn** of the n -simplex to be the maximal subobject $\Lambda_i^n \subset \Delta^n$ not containing the i^{th} face (opposite to the i^{th} vertex). This is an *inner* horn if $0 < i < n$, and an *outer* horn otherwise. Then, a simplicial set $X \in \mathbf{sSet}$ is the nerve of a category iff it admits unique inner horn fillers, i.e. for every $n \geq 0$ and every $0 < i < n$ and every map $\Lambda_i^n \rightarrow X$ there exists a unique extension

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array} .$$

11.3. **Friday 5/19.** Today's lecture roughly corresponds to the material in §§9.1-9.2.

- Recall that a simplicial set is the nerve of a *category* iff it has unique fillers for all *inner* horns. Furthermore, it's the nerve of a *groupoid* iff it has unique fillers for *outer* horns as well. For instance, filling an outer horn $\Lambda_0^2 \rightarrow \mathbf{N}(\mathcal{C})$ amounts to choosing an extension

$$\begin{array}{ccc} & c_1 & \\ f \nearrow & & \searrow \text{dashed} \\ c_0 & \xrightarrow{g} & c_2 \end{array}$$

and if f is an isomorphism then there is a unique such extension, namely $g \circ f^{-1}$.

- Given a topological space $Y \in \mathbf{Top}$, its singular simplicial set $\mathbf{Sing}(Y)_\bullet := \mathbf{hom}_{\mathbf{Top}}(\Delta_\bullet^{\text{top}}, Y)$ admits *all* horn fillers, but these are no longer unique. The inner horn fillers come from “composition of paths up to homotopy”, and the outer horn fillers are similar but use the fact that paths can be reversed – just as isomorphisms in a groupoid can be reversed.
- We can *force* the horn fillers in $\mathbf{Sing}(Y)$ to be unique by applying $\mathbf{ho}$, and we obtain the **fundamental groupoid** of Y :

$$\mathbf{ho}(\mathbf{Sing}(Y)) \cong \Pi_1(Y) .$$

Its objects are the points of Y , and its morphisms are homotopy classes of paths.

- Of course, the fundamental groupoid doesn't capture all the homotopical information contained in an arbitrary topological space, but it does if the space is a *1-truncated*, i.e. $\pi_{\geq 2} = 0$ (at all basepoints). Ultimately, we get an equivalence of categories

$$\begin{array}{ccccc} \mathbf{Top} & \xrightarrow{\quad \mathbf{Sing} \quad} & \mathbf{sSet} & & \\ \uparrow & \searrow \Pi_1 & \downarrow \mathbf{ho} & & \\ \mathbf{Top}^{\leq 1} & \xrightarrow{\quad \Pi_1 \quad} & \mathbf{gpd} & \xleftarrow{\quad} & \mathbf{cat} \\ \downarrow & & \downarrow & & \\ \mathbf{Top}^{\leq 1}[\{\text{w.h.e.'s}\}^{-1}] & \xrightarrow{\sim \Pi_1} & \mathbf{gpd}[\{\text{equivces}\}^{-1}] & & \end{array}$$

between the category of (*homotopy*) *1-types* – 1-truncated spaces up to weak homotopy equivalence – and the category of groupoids up to equivalence.

- One can hope to extend this to higher dimensions. Whatever the “correct” notion of “*n*-category” is, the *n-groupoids* among them – i.e. the *n*-categories in which all *i*-morphisms are invertible for all $i > 0$ – should be equivalent to the category $\mathbf{Top}^{\leq n}[\{\text{w.h.e.'s}\}^{-1}]$ of *n-types*; this idea is called *Grothendieck’s homotopy hypothesis*.
- The naive definition is as follows: a *strict n-category* is a category strictly enriched in strict $(n-1)$ -categories. For instance, associativity in a strict 2-category $\mathcal{C}$ involves commutative squares

$$\begin{array}{ccc} \mathrm{hom}_{\mathcal{C}}(Y, Z) \times \mathrm{hom}_{\mathcal{C}}(X, Y) \times \mathrm{hom}(W, X) & \longrightarrow & \mathrm{hom}_{\mathcal{C}}(Y, Z) \times \mathrm{hom}_{\mathcal{C}}(W, Y) \\ \downarrow & & \downarrow \\ \mathrm{hom}_{\mathcal{C}}(X, Z) \times \mathrm{hom}_{\mathcal{C}}(W, X) & \longrightarrow & \mathrm{hom}_{\mathcal{C}}(W, Z) \end{array}$$

in *cat*, where we ask the square to commute “on the nose” – i.e. the composite functors are exactly equal. But we could *weaken* this notion to only ask it to commute up to a specified natural (“coherence”) isomorphism, and this leads to the notion of *weak 2-category* – and thereafter, *weak 3-category*.

- It turns out that all weak 2-categories are equivalent (in an appropriate sense) to strict 2-categories, but not all weak 3-categories can be strictified. As a concrete example, the fundamental 3-category of the 2-sphere $\Pi_3(S^2)$ is *not* equivalent to a strict 3-groupoid.²³⁸ So, strict categories are not the “correct” notion”, at least from the point of view of the homotopy hypothesis. However, these coherence conditions explode in complexity as we move upwards, though, and humanity seems to have stalled in pursuing this approach around $n = 3$ or $n = 4$ (with something like 30 coherence conditions).
- As a point of terminology, define a (weak or strict) (n, k) -*category* to be an *n*-category in which all *i*-morphisms are invertible for $i > k$. So a general *n*-category is an (n, n) -category, and an *n*-groupoid is an $(n, 0)$ -category. Then, an (n, k) -category should be (weakly or strictly) enriched in $(n-1, k-1)$ -categories. This allows us to make sense of the case where $k = 1$, in which case we can even take $n = \infty$: a weak $(\infty, 1)$ -category should be a category weakly enriched in $(\infty, 0)$ -categories, i.e. ∞ -groupoids, i.e. homotopy ∞ -types – more often just called “homotopy types”.
- Jumping ahead to the conclusion, it turns out that $(\infty, 1)$ -categories *can* all be strictified. Thus, we define an $(\infty, 1)$ -*category* to be a weak equivalence class of *Top*-enriched category (defining weak equivalence just as we did for dg-categories, only

²³⁸This contains (the equivalence class of) the Hopf fibration $S^3 \rightarrow S^2$ as a 3-isomorphism.

replacing quasi-isomorphisms with weak homotopy equivalence). That is, the $(\infty, 1)$ -category $\mathbf{Cat} := \mathbf{Cat}_\infty$ of $(\infty, 1)$ -categories will have its homotopy category given by

$$\mathrm{ho}(\mathbf{Cat}) = \mathrm{cat}(\mathbf{Top})[\{\text{w.e.'s}\}^{-1}] .$$

- This is a sufficient definition, but it's not a “good” one – it's “too strict”. It turns out to be better to extract “only the homotopy-coherent information” out of a $\mathbf{Top}$ -enriched category: this makes it so “it's impossible to speak in strict terms”. Namely, the *homotopy-coherent nerve* functor

$$\mathrm{cat}(\mathbf{Top}) \xrightarrow{\mathbf{N}^{\mathrm{h.c.}}} \mathbf{sSet}$$

is given by a certain restricted Yoneda embedding

$$\mathbf{N}^{\mathrm{h.c.}}(\mathcal{C})_\bullet := \mathrm{hom}_{\mathrm{cat}(\mathbf{Top})}(|\mathcal{C}[\bullet]|, \mathcal{C}) .$$

This object $|\mathcal{C}[n]| \in \mathrm{cat}(\mathbf{Top})$ corepresents “homotopy-coherent compositions of n composable morphisms”. More precisely, for $i, j \in [n]$, we write $P_{i,j}$ for the poset of sequences $i < k_0 < \cdots < k_\ell < j$ in $[n]$, ordered by inclusion. Then, we get a $\mathbf{sSet}$ -enriched category $\mathcal{C}[n]$ by defining $\mathrm{hom}_{\mathcal{C}[n]}(i, j) := \mathbf{N}(P_{i,j})$, and we get the $\mathbf{Top}$ -enriched category $|\mathcal{C}[n]|$ by taking the geometric realizations of its hom-objects. For example, $\mathcal{C}[2]$ has the objects $0, 1, 2$, and $\mathrm{hom}(0, 1)$ and $\mathrm{hom}(1, 2)$ both consist of a single vertex (since there are no factorizations of $i < i + 1$), whereas $\mathrm{hom}(0, 2)$ is the nerve of the poset $\{0 < 2\} \rightarrow \{0 < 1 < 2\}$ – in other words, it's a 1-simplex. Pictorially, we can illustrate $\mathcal{C}[2]$ as

$$\begin{array}{ccc} & 1 & \\ \nearrow & & \searrow \\ 0 & \xrightarrow{\quad} & 2 \end{array}$$

(which we saw in §6.2.3).

11.4. Monday 5/22. Today's lecture roughly corresponds to the material in §§9.1-9.3 and §§9.6.1-9.6.2.

- We fleshed out the 2×2 grid in §9.2.3. In particular, the prototypical Kan complex is $\mathbf{Sing}$ of a topological space, while the prototypical quasicategory is $\mathbf{N}^{\mathrm{h.c.}}$ of a $\mathbf{Top}$ -enriched category.
- Whereas we previously defined an ∞ -category (meaning an $(\infty, 1)$ -category) to be a weak equivalence class of $\mathbf{Top}$ -enriched category, we can equivalently define it to be a weak equivalence class of quasicategory. Once again, this means “homotopically fully faithful” and “homotopically essentially surjective” (or just “essentially surjective on homotopy categories”), using the basic notions described in §9.3.

- Worth highlighting is the fact that “strict composition” is *not* well-defined in a quasicategory $\mathcal{C}$. Rather, a pair of composable morphisms is given by a map $\Lambda_1^2 \rightarrow \mathcal{C}$, and *any* extension to Δ^2 may be called “a homotopy-coherent composition” – any “any of these is just as good as any other”. In particular, if $\mathcal{C} = \mathbf{N}^{\text{h.c.}}(\mathcal{C})$ for some $\mathcal{C} \in \mathbf{cat}(\mathbf{Top})$, in the quasicategory $\mathcal{C}$ we are *forgetting* the strict composition in $\mathcal{C}$ and “putting all homotopy-coherent compositions on equal footing”.

11.5. **Wednesday 5/24.** For the most part, today’s lecture roughly corresponds to §§9.6-9.8. However, we moved into the topic of *stable ∞ -categories*, which are not currently discussed elsewhere in this text. See [§A.1.1.1] (the primary source) for the fundamental notions.

- By definition, the homotopy-coherent nerve functor passes through **Kan**-enriched categories (where $\mathbf{Kan} \subset \mathbf{sSet}$ denotes the category of Kan complexes). One can *also* take the homotopy-coherent nerve of these, and they (up to weak equivalence) *also* give an equivalent definition of “ ∞ -category”. In particular, we have a self-enrichment $\mathbf{Kan} \in \mathbf{cat}(\mathbf{Kan})$.
- Let $\mathcal{CW} \in \mathbf{cat}(\mathbf{Top})$ denote the **Top**-enriched category of CW-complexes. We define the *∞ -category of spaces* (or *∞ -groupoids* – building the homotopy hypothesis into our definitions) to be

$$\mathcal{S} := \mathbf{N}^{\text{h.c.}}(\mathcal{CW}) \simeq \mathbf{N}^{\text{h.c.}}(\mathbf{Kan}) .$$

In particular, we will henceforth say “*topological space*” to refer to an object of **Top** (which has an “underlying space”, namely that represented by any CW-complex that is weak homotopy equivalent to it).

- Quasicategories have an *internal hom*, which is just the hom-simplicial set: we define $\mathbf{Fun}(\mathcal{C}, \mathcal{D}) := \mathbf{hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D})$.²³⁹ This allows us to *also* obtain a Kan complex enrichment $\mathbf{QCat} \in \mathbf{cat}(\mathbf{Kan})$, and we define *the ∞ -category of ∞ -categories* to be

$$\mathbf{Cat} := \mathbf{Cat}_\infty := \mathbf{N}^{\text{h.c.}}(\mathbf{QCat}) .$$

We will mostly work “model-independently”, but if push comes to shove, this is our ambient context.

- Here is a first convenience of working in the ∞ -category of ∞ -categories (akin to our motivation for passing from dg-categories to their underlying $\mathbb{k}$ -linear ∞ -categories). Note that Kan complexes are quasicategories: we have an inclusion $\mathbf{Kan} \hookrightarrow \mathbf{QCat}$. Morally, this “should” have a left adjoint, the “ ∞ -groupoid completion” functor (taking an ∞ -category and freely adjoining inverses to all of its morphisms). However,

²³⁹This is a fundamental technical advantage of quasicategories over **Top**- or **Kan**-enriched categories: this gives the “correct” hom-object (which e.g. respects weak equivalences in both variables). (This technical advantage can be seen as a direct consequence of the philosophical advantage that quasicategories *don’t* admit strictly-defined composition. Again, see [MG] for an elaboration of this point.)

this adjoint does *not* exist at the point-set level: there is not an initial Kan complex under a given quasicategory. But it *does* exist at the ∞ -categorical level. That is, inside of $\mathbf{Cat}$ we have a left adjoint

$$\mathbf{Cat} \overset{(-)^{\mathbf{gpd}}}{\underset{\perp}{\dashrightarrow}} \mathcal{S}.$$

- Let's take a look at the ∞ -category of spaces, modeled as $\mathbf{N}^{\mathbf{h.c.}}(\mathcal{CW})$. Actually, it will be convenient to look at the ∞ -category of *pointed* spaces, modeled as pointed CW-complexes. Then, given two objects $X, Y \in \mathcal{S}_* := \mathcal{S}_{\mathbf{pt}/}$, let's consider commutative squares

$$(44) \quad \begin{array}{ccc} X & \longrightarrow & \mathbf{pt} \\ \downarrow & \searrow & \downarrow \\ \mathbf{pt} & \longrightarrow & Y \end{array}$$

in $\mathcal{S}_* := \mathbf{N}^{\mathbf{h.c.}}(\mathcal{CW}_*)$. Since the one-point space $\mathbf{pt} \in \mathcal{S}_*$ is both terminal and initial (i.e. it is a **zero object**), the outer maps in the diagram are *unique*. However, remember that a square in $\mathcal{S}_*$ is represented by a *homotopy-coherent* square in $\mathcal{CW}_*$. Thus, we must also specify the diagonal map $X \rightarrow Y$ (which is included in the diagram merely for emphasis), along with homotopies that make the two triangles commute. This is much more interesting than asking for (strictly) commutative squares in $\mathcal{CW}_*$!

- Such a square (44) is equivalently a based loop in $\mathbf{hom}_{\mathcal{CW}_*}(X, Y) \in \mathbf{Top}$: a path of maps $X \rightarrow Y$ that starts and ends at the constant map. (We can take the diagonal map to be the value at the midpoint of the loop; then, the two halves of the path pick out the two homotopies.) This can be equivalently expressed as a map $\Sigma X \rightarrow Y$ from the *suspension*

$$\Sigma X := (X \times [0, 1]) / \{(x, 0) \sim (x', 0) \text{ and } (x, 1) \sim (x', 1) \text{ for all } x, x' \in X\}$$

of X to Y . But since this is true for *all* objects $Y \in \mathcal{S}_*$, we have just found that ΣX is the *pushout* of the upper-left span, i.e. it is the ∞ -categorical colimit

$$\Sigma X \simeq \mathbf{colim}^{\mathcal{S}_*} \left(\begin{array}{ccc} X & \longrightarrow & \mathbf{pt} \\ \downarrow & & \\ \mathbf{pt} & \longrightarrow & \end{array} \right).$$

Said differently, it is the *initial cocone* under the span: in the ∞ -category $\mathcal{S}_*$ (where commutative squares are represented by *homotopy-coherently* commutative squares

in $\mathcal{CW}_*$), for every solid outer commutative quadrilateral in the diagram

$$\begin{array}{ccc}
 X & \longrightarrow & \text{pt} \\
 \downarrow & & \downarrow \\
 \text{pt} & \longrightarrow & \Sigma X \\
 & \searrow & \swarrow \text{dashed} \\
 & & Y
 \end{array}$$

(which encodes a loop of maps $X \rightarrow Y$ based at the constant map), there exists a unique dashed morphism making the entire diagram commute. Here, uniqueness must be taken *in the ∞ -categorical sense*: there is a *contractible ∞ -groupoid* of such data.

- Dually, we can also equivalently express such a commutative square (44) as a map $X \rightarrow \Omega Y$ to the (**based**) **loopspace**

$$\Omega Y := \{[0, 1] \xrightarrow{\gamma} Y : \gamma(0) = \gamma(1) = y_0\}$$

of Y from X . Since this is true for *all* objects $X \in \mathcal{S}_*$, it follows that ΩY is the *pullback* of the bottom-right cospan, i.e. it is the ∞ -categorical limit

$$\Omega Y \simeq \lim^{\mathcal{S}_*} \left(\begin{array}{ccc} & & \text{pt} \\ & & \downarrow \\ \text{pt} & \longrightarrow & Y \end{array} \right).$$

- Altogether, we have found an *adjunction*

$$\mathcal{S}_* \begin{array}{c} \xrightarrow{\Sigma} \\ \xleftarrow{\perp} \\ \xrightarrow{\Omega} \end{array} \mathcal{S}_* .$$

That is, for any $X, Y \in \mathcal{S}_*$ we have an equivalence

$$\text{hom}_{\mathcal{S}_*}(X, \Omega Y) \simeq \text{hom}_{\mathcal{S}_*}(\Sigma X, Y) ,$$

and *both* of these hom-spaces are equivalent to the space of commutative squares (44) in $\mathcal{S}_*$.

- Here are a few auxiliary definitions, which give a flavor of the general notion of universal properties in ∞ -category theory.
 - An object $X \in \mathcal{C}$ is called **terminal** if for all $T \in \mathcal{C}$ the space $\text{hom}_{\mathcal{C}}(T, X)$ is contractible, and **initial** if for all $T \in \mathcal{C}$ the space $\text{hom}_{\mathcal{C}}(X, T)$ is contractible.
 - In case $\mathcal{C}$ is an ordinary category, these reduce to the more familiar notions: a set is contractible if and only if it is a singleton. More generally, in ∞ -category theory, the notion of “unique up to unique isomorphism” is replaced by “a contractible space” (or ∞ -groupoid). For instance, given any ∞ -category $\mathcal{C}$, the

subcategory (i.e. sub- ∞ -category) of terminal objects is an empty or contractible ∞ -groupoid.

- In the category **Set** of sets, the empty set $\emptyset \in \mathbf{Set}$ is initial, and the singleton $\mathbf{pt} \in \mathbf{Set}$ is terminal. These are also respectively the initial and terminal spaces. As a result, in any ∞ -category $\mathcal{C}$, we may write $\emptyset_{\mathcal{C}}$ for an initial object and $\mathbf{pt}_{\mathcal{C}}$ for a terminal object.
- We have already seen a more exotic example above: the suspension ΣX is an initial object in the ∞ -category of cocones under the span $\mathbf{pt} \leftarrow X \rightarrow \mathbf{pt}$. This is, after all, the definition of a colimit: given an arbitrary cocone under the span, there’s a unique map of cocones from this one. Dually, ΩY is a terminal object in the ∞ -category of cones over the cospan $\mathbf{pt} \rightarrow Y \leftarrow \mathbf{pt}$.
- An object in an ∞ -category is called a **zero object** if it is both initial and terminal. The ∞ -categories **Set** and **S** don’t have zero objects, but their pointed variants **Set**_{*} and **S**_{*} do: in both cases the singleton is a zero object. In any ∞ -category $\mathcal{C}$, we may write $0_{\mathcal{C}}$ for a zero object. [changed 5/26] If $\mathcal{C}$ has a zero object, we may refer to it as **zero-pointed**, or **pointed** for short.²⁴⁰
- If $\mathcal{C}$ has a zero object, then for every pair of objects $A, B \in \mathcal{C}$ there is a canonical basepoint

$$(A \rightarrow 0_{\mathcal{C}} \rightarrow B) \in \mathrm{hom}_{\mathcal{C}}(A, B) ,$$

which we refer to as the *zero morphism*, [changed 5/26] which we denote by 0 (or occasionally by $0_{A \rightarrow B}$ for emphasis). A homotopy from the zero morphism to an arbitrary morphism $A \rightarrow B$ is called a *nullhomotopy* of the latter.

- [changed 5/26] In a zero-pointed ∞ -category $\mathcal{C}$, a **triangle** is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow g \\ 0_{\mathcal{C}} & \longrightarrow & C \end{array} .$$

That is, a triangle consists of the data of a pair of composable morphisms along with a nullhomotopy of their composite. We may depict such a triangle merely as $A \xrightarrow{f} B \xrightarrow{g} C$, leaving the nullhomotopy of gf implicit.

²⁴⁰The term “pointed” is common but potentially misleading: in general, a pointing of an object should be a morphism from the terminal object, but a functor $\mathbf{pt}_{\mathbf{Cat}} \rightarrow \mathcal{C}$ need not select a zero object.

- If $\mathcal{C}$ has a zero object, then the **fiber** and **cofiber** of a morphism $A \xrightarrow{f} B$ (if they exist) are by definition the limit and colimit

$$\mathrm{fib}(f) := \lim \left(\begin{array}{ccc} & A & \\ & \downarrow f & \\ 0_{\mathcal{C}} & \xrightarrow{0} & B \end{array} \right) \quad \text{and} \quad \mathrm{cofib}(f) := \mathrm{colim} \left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ 0 \downarrow & & \\ 0_{\mathcal{C}} & & \end{array} \right). \quad ^{241}$$

Thus, a map $T \rightarrow \mathrm{fib}(f)$ is equivalent to the data of a map $T \xrightarrow{t} A$ and a nullhomotopy of the composite $T \xrightarrow{t} A \xrightarrow{f} B$, while a map $\mathrm{cofib}(f) \rightarrow T$ is equivalent to the data of a map $B \xrightarrow{t} T$ and a nullhomotopy of the composite $A \xrightarrow{f} B \xrightarrow{t} T$. [changed 5/26] Said differently, we have e.g.

$$\mathrm{hom}_{\mathcal{C}}(\mathrm{cofib}(f), T) \simeq \lim^s \left(\begin{array}{ccc} \mathrm{hom}_{\mathcal{C}}(A, T) & \xleftarrow{f^*} & \mathrm{hom}_{\mathcal{C}}(B, T) \\ 0_{A \rightarrow T} \uparrow & & \\ \mathrm{hom}_{\mathcal{C}}(0_{\mathcal{C}}, T) & & \\ \wr \downarrow & & \\ \mathrm{pt}_s & & \end{array} \right) :$$

mapping out of a cofiber gives a(n ∞ -categorical) fiber on hom-spaces, taken over the zero morphism. (Of course, this is generally true: mapping out of a colimit *or* into a limit always gives a *limit* of hom-spaces.) In particular, we write

$$\Sigma := \Sigma_{\mathcal{C}} := \mathrm{cofib}((-) \longrightarrow 0_{\mathcal{C}}) \quad \text{and} \quad \Omega := \Omega_{\mathcal{C}} := \mathrm{fib}(0_{\mathcal{C}} \longrightarrow (-))$$

for the indicated functors, and refer to them as the **suspension** and (**based**) **loops** endofunctors of $\mathcal{C}$.

- We now arrive at our main definition: an ∞ -category $\mathcal{C}$ is **stable** if the following conditions are satisfied.
 - It has a zero object $0_{\mathcal{C}} \in \mathcal{C}$.
 - It admits all fibers and cofibers.
 - A triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow g \\ 0_{\mathcal{C}} & \longrightarrow & C \end{array}$$

is a fiber square (i.e. $A \xrightarrow{\sim} \mathrm{fib}(g)$) if and only if it is a cofiber square (i.e. $\mathrm{cofib}(f) \xrightarrow{\sim} C$).

²⁴¹We previously called these “ ∞ -categorical co/kernels” (in the context of dg-categories and their underlying $\mathbb{k}$ -linear ∞ -categories). The present terminology is standard in the literature.

Of course, this is strongly reminiscent of our discussion of the $\mathbb{k}$ -linear ∞ -categories $\mathbf{K}_R^\infty$, $\mathbf{P}_R^\infty$, $\mathbf{I}_R^\infty$, $\mathbf{A}_R^\infty$, and $\mathbf{D}_R^\infty$ (which are all also stable). We will soon make this connection explicit.

• [changed 6/2]

Exercise 11.1 (2 points). Show that an ∞ -category $\mathcal{C}$ is stable iff its opposite $\mathcal{C}^{\text{op}}$ is stable.

Exercise 11.2 (2 points). Show that in a stable ∞ -category $\mathcal{C}$, the adjunction $\Sigma \dashv \Omega$ is an equivalence.²⁴²

As a result, in a stable ∞ -category we may write $\Sigma^{-1} := \Omega$, and refer to this endofunctor as *desuspension*.

Exercise 11.3 (2 points). Given a morphism $A \xrightarrow{f} B$ in a stable ∞ -category $\mathcal{C}$, show that its fiber and cofiber are related by the formula

$$\Sigma \text{fib}(f) \simeq \text{cofib}(f) .$$

Exercise 11.4 (2 points). Show that a morphism in a stable ∞ -category is an equivalence iff its cofiber is zero,²⁴³ and that this holds iff its fiber is zero.

Exercise 11.5 (4 points). Given a cospan

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \\ C & & \end{array}$$

in a stable ∞ -category, prove that its pushout is computed by

$$\text{cofib}(\text{fib}(f) \longrightarrow C) .$$

In particular, stable ∞ -categories admit all pushouts. They also admit all pullbacks by a dual argument (or by Exercise 11.1). In fact, it turns out that stable ∞ -categories admit *all* finite co/limits (not just pushouts and pullbacks).

Exercise 11.6 (3 points). Prove that the terminal ∞ -category $\mathbf{pt}_{\text{Cat}} \in \mathbf{Cat}$ is stable, and moreover that it is a zero object for the ∞ -category $\mathbf{St}$.

²⁴²In fact, a zero-pointed ∞ -category $\mathcal{C}$ is stable iff it admits finite colimits and Σ is an equivalence, or equivalently iff it admits finite limits and Ω is an equivalence [A.1.4.2.27].

²⁴³This will recover Exercise 3.10 (when applied to the underlying stable ∞ -categories of the $\mathbb{k}$ -linear ∞ -categories $\mathbf{D}_R$ and $\mathbf{K}_R$).

11.6. **Friday 5/26.** Today's lecture continued with stable ∞ -categories: general properties, and then some first examples. Along the way, we discussed material from §8.4 (regarding the Dold–Kan correspondence, homology, and the Dold–Thom theorem). Moreover, we touched on material (regarding (commutative) ∞ -monoids and ∞ -groups) that is covered in §10 (particularly §10.1 and §10.4).

We began by discussing some more general features of stable ∞ -categories.

- [changed 6/2]

Given a commutative square

$$(45) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

in a zero-pointed ∞ -category $\mathcal{C}$, its **total fiber** and **total cofiber** are respectively

$$\mathrm{tfib}(45) := \mathrm{fib} \left(A \longrightarrow B \times_D C \right) \quad \text{and} \quad \mathrm{tcofib}(45) := \mathrm{cofib} \left(B \coprod_A C \longrightarrow D \right) .$$

By Exercise 11.4, assuming that $\mathcal{C}$ is stable, the total fiber is zero iff (45) is a pullback square, while the total cofiber is zero if and only if (45) is a pushout square.

Exercise 11.7 (6 points).

- (a) Given a commutative square (45) in a zero-pointed ∞ -category, show that its total fiber and total cofiber can be computed by the formulas

$$\mathrm{tfib}(45) \simeq \mathrm{fib}(\mathrm{fib}(f) \longrightarrow \mathrm{fib}(g)) \quad \text{and} \quad \mathrm{tcofib}(45) \simeq \mathrm{cofib}(\mathrm{cofib}(f) \longrightarrow \mathrm{cofib}(g)) .$$

- (b) Given a commutative square (45) in a stable ∞ -category, show that its total fiber and total cofiber are related by the equivalence

$$\Sigma \mathrm{tfib}(45) \simeq \Sigma^{-1} \mathrm{tcofib}(45) .$$

Hence, Exercise 11.7(b) (combined with Exercise 11.2) shows that in any stable ∞ -category, pushout and pullback squares coincide (which is guaranteed by definition in the special case the square is a triangle). It follows that a zero-pointed ∞ -category is stable if and only if

- it admits finite co/limits, and
 - every commutative square is a pullback iff it's a pushout.
- In general, products of pullback squares are pullback squares, since products and pullback are both limits. This implies that for any pair of objects $A, B \in \mathcal{C}$ in a

stable ∞ -category, we have a pullback square

$$\left(\begin{array}{ccc} 0_{\mathcal{C}} & \xrightarrow{0} & A \\ \wr \downarrow & & \downarrow \wr \text{id}_A \\ 0_{\mathcal{C}} & \xrightarrow{0} & A \end{array} \right) \times \left(\begin{array}{ccc} 0_{\mathcal{C}} & \xrightarrow{\sim} & 0_{\mathcal{C}} \\ 0 \downarrow & & \downarrow 0 \\ B & \xrightarrow[\text{id}_B]{\sim} & B \end{array} \right) \simeq \left(\begin{array}{ccc} 0_{\mathcal{C}} & \xrightarrow{0} & A \\ 0 \downarrow & & \downarrow (\text{id}_A, 0) \\ B & \xrightarrow[(0, \text{id}_B)]{} & A \times B \end{array} \right).$$

But since $\mathcal{C}$ is stable this is also a pushout square, i.e. the map

$$A \sqcup B \xrightarrow{\begin{pmatrix} \text{id}_A & 0 \\ 0 & \text{id}_B \end{pmatrix}} A \times B$$

from the coproduct to the product is an equivalence. We may summarize this situation by saying that $\mathcal{C}$ is **preadditive** (a notion that can be applied to any zero-pointed ∞ -category). To emphasize the agreement of (finite) coproducts and products, we denote them by $\oplus$ and refer to them as (**direct**) **sums**.

- In particular, this implies that we have a *commutative* ∞ -monoid structure on the hom-space $\text{hom}_{\mathcal{C}}(A, B)$, given by defining $f + g$ to be

$$A \xrightarrow{\Delta} A \times A =: A \oplus A \xrightarrow{f \oplus g} B \oplus B := B \sqcup B \xrightarrow{\nabla} B.$$

In fact, we will soon see (when we discuss hom-spectra) that these are *commutative* ∞ -groups: all morphisms admit inverses. Said differently, stable ∞ -categories are not merely preadditive but **additive**.

- This gives one explanation for why the existence of fibers and finite products in a stable ∞ -category implies the existence of all finite limits: a (finite) limit can be expressed as an equalizer between (resp. finite) products, which is equivalent to the fiber of the difference between the two maps. For instance, a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow[i]{} & D \end{array}$$

is a pullback if and only if the triangle

$$\begin{array}{ccc} A & \xrightarrow{(f, g)} & B \oplus C \\ \downarrow & & \downarrow (h, -i) \\ 0_{\mathcal{C}} & \longrightarrow & D \end{array}$$

is a fiber sequence. (Here, the composition is given by

$$(h, -i) \circ (f, g) \simeq (h \circ f) + ((-i) \circ g) \simeq (h \circ f) - (i \circ g),$$

a nullhomotopy of which is equivalent data to a homotopy $(h \circ f) \sim (i \circ g)$ making the original square commute.)

- A remark: the homotopy category of a stable ∞ -category is canonically a *triangulated category*.²⁴⁴ If you don't know what this is, don't look it up; and if you do, forget it. There are a number of problems with triangulated categories, which are repaired by stable ∞ -categories. For instance, “cones” in a triangulated category aren't generally functorial, but they are the image of cofibers, which are ∞ -categorically functorial. Indeed, the failure of functoriality in the homotopy category arises precisely because one is forgetting the *data* of the nullhomotopy of the composite.
- [changed 5/31] A functor between stable ∞ -categories is called **exact** if it preserves zero objects and co/fiber sequences. With these morphisms, stable ∞ -categories assemble into a subcategory

$$\mathbf{St} \subset \mathbf{Cat}$$

of the ∞ -category of ∞ -categories.

- [changed 6/2] The subcategory $\mathbf{St} \subset \mathbf{Cat}$ is stable under limits and filtered colimits, because passage to hom-spaces commutes with these (and the subcategory $\{\mathbf{pt}\} \subset \mathcal{S}$ is stable under these). For example, given a scheme X , we can define a stable ∞ -category

$$\mathbf{D}_X := \lim_{\mathbf{Spec}(R) \rightarrow X} \mathbf{D}_R ,$$

where the limit is taken in $\mathbf{Cat}$ but lies in $\mathbf{St}$ (as does the projection functor $\mathbf{D}_X \rightarrow \mathbf{D}_R$ associated to each map $\mathbf{Spec}(R) \rightarrow X$).²⁴⁵

- In general, if an ∞ -category $\mathcal{C}$ admits co/limits of a certain shape, then so does the functor ∞ -category $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$, and they are computed pointwise (i.e. for each $i \in \mathcal{J}$ the functor $\mathbf{Fun}(\mathcal{J}, \mathcal{C}) \xrightarrow{\mathrm{ev}_i} \mathcal{C}$ preserves them). It follows that if $\mathcal{C}$ is stable then so is $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$.

Exercise 11.8 (3 points). Show that if $\mathcal{B}, \mathcal{C} \in \mathbf{St}$ are stable ∞ -categories, then the full subcategory

$$\mathbf{Fun}^{\mathrm{ex}}(\mathcal{B}, \mathcal{C}) \subseteq \mathbf{Fun}(\mathcal{B}, \mathcal{C})$$

on the exact functors is also a stable ∞ -category.

Of course, the definition of a stable ∞ -category is strongly reminiscent of the features that we saw of the $\mathbb{k}$ -linear ∞ -categories $\mathbf{K}_R$, $\mathbf{D}_R \simeq \mathbf{P}_R \simeq \mathbf{I}_R$, and $\mathbf{A}_R$. We now connect these ideas, by first discussing the forgetful functor $\mathbf{Cat}_{\mathbb{k}} \xrightarrow{\mathrm{fgt}} \mathbf{Cat}$ taking a $\mathbb{k}$ -linear ∞ -category to its underlying ∞ -category. (It is here that §8.4 is relevant.)

²⁴⁴Its shift functor is Σ , and the distinguished triangles are the images of co/fiber sequences. The octahedral axiom follows from pushout-pasting.

²⁴⁵This may be referred to as the stable ∞ -category of *derived quasicohherent* (∞ -)sheaves on X . One can use Godement resolutions to show that this coincides with the derived ∞ -category of the ordinary abelian category of quasicohherent sheaves of $\mathcal{O}_X$ -modules, at least on bounded objects.

- The forgetful functor $\mathbf{Cat}_{\mathbb{k}} \xrightarrow{\text{fgt}} \mathbf{Cat}$ amounts to applying a forgetful functor on hom-objects. And in fact, it is instructive to describe this forgetful functor first at the point-set level as well. Thus, we will have forgetful functors

$$\begin{array}{ccc} \mathbf{Ch}_{\mathbb{k}} & \xrightarrow{\text{fgt}} & \mathbf{Kan} \\ \downarrow & & \downarrow \\ \mathbf{D}_{\mathbb{k}} & \xrightarrow{\text{fgt}} & \mathcal{S} \end{array},$$

leading to forgetful functors

$$\begin{array}{ccc} \mathbf{cat}^{\text{dg}} & \xrightarrow{\text{fgt}} & \mathbf{cat}(\mathbf{Kan}) \\ \downarrow & & \downarrow \\ \mathbf{Cat}_{\mathbb{k}} & \xrightarrow{\text{fgt}} & \mathbf{Cat}(\mathcal{S}) \simeq \mathbf{Cat} \end{array}.$$

- The key input is the *Dold–Kan equivalence* between the category $\mathbf{sMod}_R := \mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Mod}_R)$ of simplicial R -modules and the category $\mathbf{Ch}_R^{\geq 0}$ of nonnegatively-graded chain complexes. The most familiar direction is (up to natural quasi-isomorphism) the functor

$$\mathbf{sMod}_R \xrightarrow{\mathbf{C}_{\bullet}} \mathbf{Ch}_R^{\geq 0}$$

given by defining $\mathbf{C}_n(M) := M_n$ (i.e. preserving the levelwise R -modules) and defining the differential $\mathbf{C}_n(M) \xrightarrow{d_n} \mathbf{C}_{n-1}(M)$ to be given by

$$\mathbf{C}_n(M) := M_n \xrightarrow{\sum_{i=0}^n \delta_n^i} M_{n-1} =: \mathbf{C}_{n-1}(M),$$

the alternating sum of the simplicial face maps. Let us simply write

$$\mathbf{Ch}_R^{\geq 0} \xrightarrow[\sim]{\text{DK}} \mathbf{sMod}_R$$

for the inverse equivalence. This yields the composite right adjoint

$$\begin{array}{ccccccc} \mathbf{sSet} & \xleftarrow[\text{fgt}]{R\{-\}} & \mathbf{sMod}_R & \xleftarrow[\text{DK}]{\bar{\mathbf{C}}_{\bullet}} & \mathbf{Ch}_R^{\geq 0} & \xleftarrow[\tau_{\geq 0}]{\perp} & \mathbf{Ch}_R \\ \uparrow & & \nwarrow \text{fgt} & & & & \\ \mathbf{Kan} & & & & & & \end{array} \quad . \quad 246$$

At the ∞ -categorical level, this becomes a composite right adjoint

$$\mathcal{S} \xleftarrow[\text{fgt}]{R\{-\}} \mathbf{Mod}_R(\mathcal{S}) \simeq \mathbf{D}_R^{\geq 0} \xleftarrow[\tau_{\geq 0}]{\perp} \mathbf{D}_R \quad .$$

Our forgetful functors $\mathbf{cat}^{\text{dg}} \xrightarrow{\text{fgt}} \mathbf{cat}(\mathbf{Kan})$ and $\mathbf{Cat}_{\mathbb{k}} \xrightarrow{\text{fgt}} \mathbf{Cat}$ are given by applying these respective composite right adjoints to hom-objects.

²⁴⁶It is an exercise to show that the underlying simplicial set of a simplicial R -module is in fact a Kan complex.

- Note that right adjoints preserve limits. This implies that universal properties in $\mathbb{k}$ -linear (i.e. $\mathbf{D}_{\mathbb{k}}$ -enriched) ∞ -categories yield universal properties in their corresponding unenriched (i.e. $\mathcal{S}$ -enriched) ∞ -categories. In particular, we can now check that $\mathbf{fgt}(\mathbf{K}_R) \in \mathbf{Cat}$ is indeed a stable ∞ -category.
 - The zero object $0 \in \mathbf{K}_R$ clearly defines a zero object $0 \in \mathbf{fgt}(\mathbf{K}_R)$: the hom-derived- $\mathbb{k}$ -module into it or out of it is $0 \in \mathbf{D}_{\mathbb{k}}$, the underlying space of which is $\mathbf{pt} \in \mathcal{S}$. That is, $\mathbf{hom}_{\mathbf{fgt}(\mathbf{K}_R)}(0, -)$ and $\mathbf{hom}_{\mathbf{fgt}(\mathbf{K}_R)}(-, 0)$ are both constant at $\mathbf{pt} \in \mathcal{S}$.
 - Let us temporarily distinguish between “co/kernels” in a $\mathbb{k}$ -linear ∞ -category and “co/fibers” in a (zero-pointed) ∞ -category. Now, given a cokernel sequence in $\mathbf{K}_R$, applying $\mathbf{hom}_{\mathbf{K}_R}(-, T)$ yields a kernel sequence in $\mathbf{D}_{\mathbb{k}}$, which in turn yields a fiber sequence in $\mathcal{S}$ (where $\mathbf{hom}_{\mathbf{K}_R}(\mathbf{coker}, T)$ is based at the zero map). So indeed, a cokernel sequence in $\mathbf{K}_R \in \mathbf{Cat}_{\mathbb{k}}$ determines a cofiber sequence in $\mathbf{fgt}(\mathbf{K}_R) \in \mathbf{Cat}$. Likewise, a kernel sequence in $\mathbf{K}_R \in \mathbf{Cat}_{\mathbb{k}}$ determines a fiber sequence in $\mathbf{fgt}(\mathbf{K}_R) \in \mathbf{Cat}$.
 - Now, since every map in $\mathbf{K}_R$ has both a kernel and a cokernel, then every map in $\mathbf{fgt}(\mathbf{K}_R)$ has both a fiber and a cofiber.
 - Moreover, since kernel and cokernel sequences in $\mathbf{K}_R$ coincide, then fiber and cofiber sequences in $\mathbf{fgt}(\mathbf{K}_R)$ coincide.

Moreover, the underlying ∞ -categories of the full $\mathbb{k}$ -linear subcategories $\mathbf{P}_R, \mathbf{I}_R, \mathbf{A}_R \subseteq \mathbf{K}_R$ are clearly full stable subcategories of $\mathbf{fgt}(\mathbf{K}_R)$, since they all contain the zero object and are closed under co/fibers. And the equivalences $\mathbf{P}_R \simeq \mathbf{D}_R \simeq \mathbf{I}_R$ imply that the ∞ -category $\mathbf{fgt}(\mathbf{D}_R)$ is also stable.

- The Dold–Kan equivalence aligns homology with homotopy: the homotopy groups of a simplicial R -module (computed using simplicial homotopy theory) are canonically R -modules and coincide with the homology groups of the corresponding nonnegatively-graded complex of R -modules. In other words, for each $n \geq 0$ we have a commutative diagram

$$\begin{array}{ccc}
 \mathbf{sMod}_R & \begin{array}{c} \xrightarrow{\quad \overline{\mathbf{c}} \cdot \quad} \\ \sim \\ \xrightarrow{\quad \text{DK} \quad} \end{array} & \mathbf{Ch}_R^{\geq 0} \\
 & \begin{array}{c} \searrow \pi_* \\ \swarrow H_n \end{array} & \\
 & \mathbf{Mod}_R &
 \end{array}$$

In particular, this leads to the **Dold–Thom theorem**: for any cell complex $X \in \mathbf{Top}$ we have an isomorphism

$$\pi_*(R\{X\}) \cong H_*^{\text{sing}}(X; R) ,$$

where $R\{X\}$ denotes the free topological R -module on X . (See §8.4 for further details.) The slogan is that *homology is linearized homotopy*. Or better, without

actually passing to homotopy/homology *groups*, the slogan is that *homology* (i.e. singular chains) *is linearization*.

11.7. Wednesday 5/31. Today’s lecture introduced spectra. This was roughly based on §A.1.4, but with some simplifications (e.g. our very definition of spectra is derived from A.1.4.2.24-25).

- There are a number of interrelated motivations for the ∞ -category $\mathbf{Sp}$ of spectra, which are a fundamental example of a stable ∞ -category.
 - Stable ∞ -categories are canonically enriched in spectra, just as abelian categories are canonically enriched in abelian groups. (This is our primary motivation for introducing spectra.)
 - The ∞ -category $\mathbf{St}$ of stable ∞ -categories and exact functors admits a symmetric monoidal structure categorifying that of $\mathbb{k}$ -modules: a functor $\mathcal{C} \times \mathcal{D} \xrightarrow{F} \mathcal{E}$ is called *biexact* if it is exact separately in each variable (i.e. for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$ the functors $\mathcal{D} \xrightarrow{F(C,-)} \mathcal{E}$ and $\mathcal{C} \xrightarrow{F(-,D)} \mathcal{E}$ are exact), and we define $\mathcal{C} \otimes \mathcal{D}$ to be the initial stable ∞ -category admitting a biexact functor from $\mathcal{C} \times \mathcal{D}$.²⁴⁷ The subcategory $\mathbf{Sp}^{\text{fin}} \subset \mathbf{Sp}$ of finite spectra is the unit for this symmetric monoidal structure.²⁴⁸
 - Spectra have been called “the abelian groups of the 21st century”. This is to say that anywhere one sees abelian groups (or a sequence of abelian groups), one might profitably wonder whether there is a spectrum in the background, whose homology groups recover them. Here are a few examples.
 - * As we’ll see, derived R -modules have underlying spectra. In the case of stable $\mathbb{k}$ -linear ∞ -categories (such as $\mathbf{K}_R$), the hom-spectra are precisely the underlying spectra of their hom-derived- $\mathbb{k}$ -modules (see Exercise 11.9).
 - * Topological K-theory assembles into a spectrum. As we’ll see, this is something like “the spectrum of virtual $\mathbb{C}$ -vector spaces”.
 - * The algebraic K-theory of a ring (or more generally of a stable ∞ -category) is a spectrum. (This clarifies the assertion of §7.5.5, where we asserted that this was “approximately a derived $\mathbb{Z}$ -module”.)

²⁴⁷This is adjoint to the internal hom bifunctor $\mathbf{St}^{\text{op}} \times \mathbf{St} \xrightarrow{\text{Fun}^{\text{ex}}(-,-)} \mathbf{St}$ of Exercise 11.8.

²⁴⁸Thus, for example, just as we can act on any object $M \in \mathcal{A}$ of an abelian category by a finitely presented abelian group (e.g. giving $\mathbb{Z}/n \odot M := \text{coker}(M \xrightarrow{n} M)$, satisfying the universal property that $\text{hom}_{\text{Ab}}(\mathbb{Z}/n, \text{hom}_{\mathcal{A}}(M, -)) \cong \text{hom}_{\mathcal{A}}(\mathbb{Z}/n \odot M, -)$), so can we act on any object $C \in \mathcal{C}$ of a stable ∞ -category by a finite spectrum $E \in \mathbf{Sp}$: this is the image of the object $(E \otimes C)$ under the equivalence $\mathbf{Sp}^{\text{fin}} \otimes \mathcal{C} \xrightarrow[~]{(-) \odot (-)} \mathcal{C}$.

- * Manifolds and cobordisms between them assemble into a ***cobordism spectrum***,²⁴⁹ whose homotopy groups are the cobordism ring. In Thom’s revolutionary thesis, he used this to prove e.g. that the unoriented cobordism ring is a polynomial algebra

$$(\mathbb{Z}/2)[\{t_i : i \neq 2^k - 1\}] = (\mathbb{Z}/2)[t_2, t_4, t_5, t_6, t_8, t_9, \dots]$$

on a generator t_i of degree i for every $i \neq 2^k - 1$. Observing that this ring is zero in degree 3, this recovers the classical result in differential topology that every compact 3-manifold is the boundary of a 4-manifold; and e.g. it implies that the abelian group of unoriented 10-manifolds up to cobordism is a 7-dimensional $\mathbb{Z}/2$ -vector space (namely that generated by the set $\{t_2^5, t_2^3 t_4, t_2^2 t_6, t_2 t_4^2, t_2 t_8, t_4 t_6, t_5^2\}$ of degree-10 monomials).

- * Elliptic curves correspond to *elliptic cohomology theories*, and assembling these together yields a spectrum of ***topological modular forms*** (whose homotopy recovers the ring of integral modular forms – at least after inverting the primes 2 and 3).
 - * The Khovanov homology of a knot in $\mathbb{R}^3$ is by definition the homology of a graded chain complex, which is well-defined up to quasi-isomorphism. This can be refined to a ***Khovanov spectrum***, which is a strictly more sensitive invariant of knots in $\mathbb{R}^3$.
 - * Seiberg–Witten Floer homology can be refined to a ***Seiberg–Witten Floer spectrum***, an invariant of homology 3-spheres.
- Spectra are essentially equivalent to ***extraordinary co/homology theories*** for spaces. Under the composite $\mathbf{Ab} \simeq \mathbf{Mod}_{\mathbb{Z}} \rightarrow \mathbf{D}_{\mathbb{Z}} \xrightarrow{\text{fgt}} \mathbf{Sp}$, an abelian group $A \in \mathbf{Ab}$ corresponds to the ordinary cohomology theory $H^*(-; A)$, as well as to the ordinary homology theory $H_*(-; A)$. We will only discuss the former, since it requires slightly less machinery.

- We begin with the definition: the ∞ -category of ***spectra*** is

$$\mathbf{Sp} := \lim \left(\dots \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \dots \right) .$$

We consider this diagram as a functor $\mathbb{Z}^{\text{op}} \rightarrow \mathbf{Cat}$ (where $\mathbb{Z}$ denotes the integers with their usual ordering), so that a spectrum $E \in \mathbf{Sp}$ is given by the data of a $\mathbb{Z}$ -indexed set of based spaces $\{E_n \in \mathcal{S}_*\}_{n \in \mathbb{Z}}$ as well as equivalences $E_n \simeq \Omega E_{n+1}$ for all $n \in \mathbb{Z}$. We may refer to the latter as the *structural equivalences* of the spectrum E . Schematically, we may depict this data as in the diagram

$$\dots \longleftarrow E_{-1} \simeq \Omega E_0 \longleftarrow E_0 \simeq \Omega E_1 \longleftarrow E_1 \simeq \Omega E_2 \longleftarrow \dots .$$

²⁴⁹The simplest example is the *unoriented cobordism spectrum* MO , which refers to compact unoriented manifolds. But one can also construct cobordism spectra for compact manifolds with other tangential structures (e.g. oriented, almost complex, spin, string, stably framed, etc.).

We will usually just write $\{E_n\}_{n \in \mathbb{Z}}$ for a spectrum, leaving the structural equivalences implicit.

- In particular, given a spectrum $E \in \mathbf{Sp}$, each based space E_n is an *infinite loopspace*: it is equipped with a system of k -fold *deloopings* for all $k \geq 0$ (namely $E_n \simeq \Omega^k E_{n+k}$). Said differently, for any fixed $n \in \mathbb{Z}$, the based space E_n determines the based spaces E_i for all $i \leq n$ by $E_i := \Omega^{n-i} E_n$. Thus, the “hard part” of defining a spectrum is defining its n^{th} spaces for arbitrarily large values of n .
- As the ∞ -category $\mathbf{Sp} \in \mathbf{Cat}$ is defined as a limit, its hom-spaces are given by limits as well: for any $E, F \in \mathbf{Sp}$ we have

$$\mathrm{hom}_{\mathbf{Sp}}(E, F) \simeq \lim (\cdots \longleftarrow \mathrm{hom}_{\mathbf{S}_*}(E_{-1}, F_{-1}) \simeq \mathrm{hom}_{\mathbf{S}_*}(\Omega E_0, \Omega F_0) \longleftarrow \mathrm{hom}_{\mathbf{S}_*}(E_0, F_0) \simeq \mathrm{hom}_{\mathbf{S}_*}(\Omega E_1, \Omega F_1) \longleftarrow \cdots)$$

(where the morphisms are the induced action on hom-spaces of the functor $\mathbf{S}_* \xleftarrow{\Omega} \mathbf{S}_*$).

- Given a stable ∞ -category $\mathcal{C}$, its $\mathbf{Sp}$ -enrichment is given as follows: we define the hom-spectrum $\underline{\mathrm{hom}}_{\mathcal{C}}(C, D) \in \mathbf{Sp}$ by declaring its n^{th} space to be the hom-space

$$\underline{\mathrm{hom}}_{\mathcal{C}}(C, D)_n := \mathrm{hom}_{\mathcal{C}}(C, \Sigma^n D),$$

pointed at the zero morphism $C \rightarrow 0_{\mathcal{C}} \rightarrow \Sigma^n D$. To verify that this assembles into a spectrum, we compute that

$$\begin{aligned} \Omega \underline{\mathrm{hom}}_{\mathcal{C}}(C, D)_n &:= \Omega \mathrm{hom}_{\mathcal{C}}(C, \Sigma^n D) \simeq \lim \left(\begin{array}{ccc} & \mathrm{pt} & \\ & \downarrow_0 & \\ \mathrm{pt} & \xrightarrow{0} & \mathrm{hom}_{\mathcal{C}}(C, \Sigma^n D) \end{array} \right) \simeq \lim^{\mathbf{S}_*} \left(\begin{array}{ccc} & \mathrm{pt} & \\ & \downarrow_0 & \\ \mathrm{pt} & \xrightarrow{0} & \mathrm{hom}_{\mathcal{C}}(C, \Sigma^n D) \end{array} \right) \\ &\simeq \lim^{\mathbf{S}_*} \left(\begin{array}{ccc} & \mathrm{hom}_{\mathcal{C}}(C, 0_{\mathcal{C}}) & \\ & \downarrow_{0_*} & \\ \mathrm{hom}_{\mathcal{C}}(C, 0_{\mathcal{C}}) & \xrightarrow{0_*} & \mathrm{hom}_{\mathcal{C}}(C, \Sigma^n D) \end{array} \right) \simeq \mathrm{hom}_{\mathcal{C}} \left(C, \lim^{\mathcal{C}} \left(\begin{array}{ccc} & 0_{\mathcal{C}} & \\ & \downarrow_0 & \\ 0_{\mathcal{C}} & \xrightarrow{0} & \Sigma^n D \end{array} \right) \right) \\ &=: \mathrm{hom}_{\mathcal{C}}(C, \Omega(\Sigma^n D)) \simeq \mathrm{hom}_{\mathcal{C}}(C, \Sigma^{n-1} D) =: \underline{\mathrm{hom}}_{\mathcal{C}}(C, D)_{n-1} \end{aligned}$$

(using the fact that the adjunction $\Sigma \dashv \Omega$ in $\mathcal{C}$ is an equivalence).

- Let us confirm that $\mathbf{Sp}$ is a stable ∞ -category. First of all, it is a general fact that if $\mathbf{Cat}' \subset \mathbf{Cat}$ denotes the subcategory on those ∞ -categories that admit any fixed limits or colimits and functors that preserve these, then $\mathbf{Cat}' \subset \mathbf{Cat}$ is stable under limits.²⁵⁰ This applies analogously to ∞ -categories admitting a zero object (i.e. an initial object and a terminal object such that the canonical morphism from the former to the latter is an equivalence). In the present context, writing $\mathbf{Cat}' \subset \mathbf{Cat}$ for the subcategory on those ∞ -categories that admit limits and a zero object and functors that preserve these, we find that $\mathbf{Sp}$ has a zero object whose terms are all $0_{\mathbf{S}_*} \simeq \mathrm{pt}_{\mathbf{S}}$

²⁵⁰Again, this is because limits and colimits are all defined in terms of *limits* of hom-spaces, and these are respected by limits in $\mathbf{Cat}$.

(i.e. $0_{\mathcal{S}\mathfrak{p}} \simeq \{\mathbf{pt}_{\mathcal{S}}\}_{n \in \mathbb{Z}}$) and it admits limits computed pointwise. In particular, given a spectrum $E \in \mathcal{S}\mathfrak{p}$, its based loop object $\Omega E := \Omega_{\mathcal{S}\mathfrak{p}} E$ is given by

$$(\Omega_{\mathcal{S}\mathfrak{p}} E)_n \simeq \Omega_{\mathcal{S}*}(E_n) \simeq E_{n-1} .$$

In other words, $\Omega_{\mathcal{S}\mathfrak{p}}$ is simply given by shifting the terms in the spectrum to the right! This is obviously an equivalence, with inverse (which must also be the left adjoint $\Sigma_{\mathcal{S}\mathfrak{p}}$) given by shifting the terms to the left.²⁵¹ And an ∞ -category $\mathcal{C}$ admitting a zero object is stable as soon as it admits finite limits and $\Omega_{\mathcal{C}}$ is an equivalence, or dually as soon as it admits finite colimits and $\Sigma_{\mathcal{C}}$ is an equivalence [A.1.4.2.27].

- We now discuss the forgetful functor $\mathbf{D}_R \xrightarrow{\text{fgt}} \mathcal{S}\mathfrak{p}$ carrying a derived R -module to its underlying spectrum. First of all, the truncation functors assemble to given an equivalence

$$\mathbf{D}_R \xrightarrow{\sim} \lim \left(\cdots \xleftarrow{\tau_{\geq 2}} \mathbf{D}_R^{\geq 1} \xleftarrow{\tau_{\geq 1}} \mathbf{D}_R^{\geq 0} \xleftarrow{\tau_{\geq 0}} \mathbf{D}_R^{\geq -1} \xleftarrow{\tau_{\geq -1}} \cdots \right) . \quad ^{252}$$

Moreover, for each $n \in \mathbb{Z}$ we claim that we have an identification

$$\begin{array}{ccc} \mathbf{D}_R^{\geq n+1} & \xleftarrow{\tau_{\geq n+1}} & \mathbf{D}_R^{\geq n} \\ \Sigma_{\mathbf{D}_R}^{-n-1} \downarrow \wr & & \wr \downarrow \Sigma_{\mathbf{D}_R}^{-n} \\ \mathbf{D}_R^{\geq 0} & \xleftarrow{\Omega_{\mathbf{D}_R^{\geq 0}}} & \mathbf{D}_R^{\geq 0} \end{array}$$

of the truncation functor with based loops in $\mathbf{D}_R^{\geq 0}$ under the indicated equivalences. To simplify notation, let us verify this just in the case that $n = 0$. Then, writing

$$\mathbf{D}_R^{\geq 0} \xleftarrow[\tau_{\geq 0}]{\text{---} \overset{i_{\geq 0}}{\perp} \text{---}} \mathbf{D}_R$$

for the indicated left adjoint, for any $M \in \mathbf{D}_R^{\geq 0}$ we have

$$\Omega_{\mathbf{D}_R^{\geq 0}}(M) \simeq \Omega_{\mathbf{D}_R^{\geq 0}}(\tau_{\geq 0} i_{\geq 0}(M)) \simeq \tau_{\geq 0}(\Omega_{\mathbf{D}_R}(i_{\geq 0}(M))) \simeq \tau_{\geq 0} \Sigma_{\mathbf{D}_R}^{-1}(M) \simeq \Sigma_{\mathbf{D}_R}^{-1} \tau_{\geq 1}(M) .$$

Altogether, then, we have a commutative diagram

$$\begin{array}{ccccccc} \cdots & \xleftarrow{\tau_{\geq 2}} & \mathbf{D}_R^{\geq 1} & \xleftarrow{\tau_{\geq 1}} & \mathbf{D}_R^{\geq 0} & \xleftarrow{\tau_{\geq 0}} & \mathbf{D}_R^{\geq -1} \xleftarrow{\tau_{\geq -1}} \cdots \\ & & \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\ \cdots & \xleftarrow{\Omega} & \mathbf{D}_R^{\geq 0} & \xleftarrow{\Omega} & \mathbf{D}_R^{\geq 0} & \xleftarrow{\Omega} & \mathbf{D}_R^{\geq 0} \xleftarrow{\Omega} \cdots \\ & & \text{fgt} \downarrow & & \text{fgt} \downarrow & & \text{fgt} \downarrow \\ \cdots & \xleftarrow{\Omega} & \mathcal{S}_* & \xleftarrow{\Omega} & \mathcal{S}_* & \xleftarrow{\Omega} & \mathcal{S}_* \xleftarrow{\Omega} \cdots \end{array}$$

²⁵¹In particular, note that $\Sigma_{\mathcal{S}\mathfrak{p}}$ does *not* commute with the projections $\mathcal{S}\mathfrak{p} \xrightarrow{(-)_n} \mathcal{S}_*$ to the terms in the diagram.

²⁵²This is not completely formal, but follows from the same reasoning as in Footnote 121.

(since $\mathcal{S}_* \xleftarrow{\text{fgt}} \mathbf{Mod}_R(\mathcal{S}) \simeq \mathbf{D}_R^{\geq 0}$ commutes with limits, being a right adjoint), and taking limits of the rows we obtain our desired functor $\mathbf{D}_R \xrightarrow{\sim} \mathbf{D}_R \xrightarrow{\text{fgt}} \mathbf{Sp}$. In other words, given a derived R -module $M \in \mathbf{D}_R$, the n^{th} space of its underlying spectrum $\text{fgt}(M) \in \mathbf{Sp}$ is the space $\Sigma^n \tau_{\geq -n}(M)$ (i.e. the underlying space of this connective R -module via the Dold–Kan equivalence).

[changed 6/2]

Exercise 11.9 (3 points). Verify that for any complexes $M, N \in \mathbf{K}_R$, the underlying spectrum of $\text{hom}_{\mathbf{K}_R}(M, N) \in \mathbf{D}_k$ coincides with the spectrum $\text{hom}_{\text{fgt}(\mathbf{K}_R)}(M, N) \in \mathbf{Sp}$ arising from the fact that the ∞ -category $\text{fgt}(\mathbf{K}_R)$ is stable.

Exercise 11.10 (3 points). Verify that the forgetful functor $\mathbf{D}_R \xrightarrow{\text{fgt}} \mathbf{Sp}$ is exact.

- To be more concrete, let us examine a particularly simple case of the forgetful functor $\mathbf{D}_R \xrightarrow{\text{fgt}} \mathbf{Sp}$, namely the composite

$$H : \mathbf{Ab} \simeq \mathbf{Mod}_{\mathbb{Z}} \rightarrow \mathbf{D}_{\mathbb{Z}} \xrightarrow{\text{fgt}} \mathbf{Sp} ,$$

which is referred to as the **Eilenberg–MacLane spectrum** functor. Given an abelian group $A \in \mathbf{Ab}$, recalling that the Dold–Kan equivalence aligns homotopy groups with homology groups, the Eilenberg–MacLane spectrum $HA \in \mathbf{Sp}$ has that

$$\pi_i((HA)_n) \cong \begin{cases} A , & i = n \\ 0 , & i \neq n \end{cases} .$$

In other words, for $n < 0$ the space $(HA)_n$ is contractible, and for $n \geq 0$ the space $(HA)_n$ is an **Eilenberg–MacLane space** of type $K(A, n)$.²⁵³ A classical theorem in algebraic topology asserts that these spaces represent cohomology in the homotopy category of spaces: for any $X \in \mathcal{S}$ we have

$$H^n(X; A) \cong \text{hom}_{\text{ho}(\mathcal{S})}(X, K(A, n)) =: [X, K(A, n)] .$$

More generally, these represent *reduced* cohomology in the homotopy category of *based* spaces: for any $X \in \mathcal{S}_*$ we have

$$\tilde{H}^n(X; A) \cong \text{hom}_{\text{ho}(\mathcal{S}_*)}(X, K(A, n)) =: [X, K(A, n)]_* . \quad ^{254}$$

- This inspires our definition of the (**extraordinary**) **cohomology theory** corresponding to an *arbitrary* spectrum $E \in \mathbf{Sp}$: for any based space $X \in \mathcal{S}_*$ we define

$$\tilde{E}^n(X) := \text{hom}_{\text{ho}(\mathcal{S}_*)}(X, E_n) =: [X, E_n]_* .$$

Thus, we have $\tilde{H}^n(X; A) \cong HA^n(X)$.

²⁵³Whereas most (pointed, and even connected) spaces are *not* determined (up to equivalence) by their homotopy groups, Eilenberg–MacLane spaces are, because they admit no nontrivial k -invariants.

²⁵⁴This recovers the unreduced cohomology of an unbased space $X \in \mathcal{S}$ via the isomorphism $H^n(X; A) \cong \tilde{H}^n(X_+; A)$.

Exercise 11.11 (6 points). Show that for any spectrum $E \in \mathcal{S}\mathbf{p}$, this functor $(\mathcal{S}_*)^{\text{op}} \xrightarrow{E^*} \mathbf{grAb}$ to the category of graded abelian groups satisfies the following *Eilenberg–Steenrod axioms*.

(i) For every $n \in \mathbb{Z}$ and every $X \in \mathcal{S}_*$ there is a natural *suspension isomorphism*

$$\tilde{E}^n(X) \xrightarrow{\cong} \tilde{E}^{n+1}(\Sigma X) .$$

(ii) For every set $\{X_\alpha \in \mathcal{S}_*\}_{\alpha \in A}$ of based spaces, we have

$$\tilde{E}^n\left(\bigvee_{\alpha \in A} X_\alpha\right) \xrightarrow{\cong} \prod_{\alpha \in A} \tilde{E}^n(X_\alpha) .^{255}$$

(iii) For every cofiber square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

in $\mathcal{S}_*$ and every $n \in \mathbb{Z}$, the sequence

$$\tilde{E}^n(X) \longleftarrow \tilde{E}^n(Y) \longleftarrow \tilde{E}^n(Z)$$

in $\mathbf{Ab}$ is exact.

Moreover, for every such cofiber square in $\mathcal{S}_*$, construct a natural long exact sequence

$$\cdots \longleftarrow \tilde{E}^{n+1}(Z) \xleftarrow{\delta} \tilde{E}^n(X) \longleftarrow \tilde{E}^n(Y) \longleftarrow \tilde{E}^n(Z) \xleftarrow{\delta} \tilde{E}^{n-1}(X) \longleftarrow \cdots .$$

Indeed, one can *define* an extraordinary cohomology theory to be such a functor satisfying the Eilenberg–Steenrod axioms,²⁵⁶ and the ***Brown representability theorem*** guarantees that every such functor is represented by a spectrum [§A.1.4.1].

- [changed 6/2] For any $n \in \mathbb{Z}$, the n^{th} ***homotopy group*** of a spectrum $E \in \mathcal{S}\mathbf{p}$ to be the abelian group

$$\pi_n(E) := \pi_0(E_{-n}) \cong \pi_1(E_{-n+1}) \cong \pi_2(E_{-n+2}) \cong \cdots ,$$

where the isomorphisms arise from the structural equivalences

$$E_{-n} \simeq \Omega E_{-n+1} \simeq \Omega^2 E_{-n+2} \simeq \cdots .$$

Meanwhile, for any based space $X \in \mathcal{S}_*$ we can form the ***mapping spectrum*** $\underline{\text{hom}}(X, E) \in \mathcal{S}\mathbf{p}$ by defining

$$\underline{\text{hom}}(X, E)_n := \text{hom}_{\mathcal{S}_*}(X, E_n) ,$$

²⁵⁵In particular, in the case that $A = \emptyset$ we obtain that $\tilde{E}^n(\text{pt}) \cong 0_{\mathbf{Ab}}$.

²⁵⁶More precisely, one should ask for such a functor equipped with suspension isomorphisms, which are additional data; by contrast, axioms (2) and (3) are merely conditions. (Moreover, the long exact sequence follows by combining axioms (1) and (3).)

with structural equivalences arising from those of E . Then, we have the evident identification

$$\tilde{E}^n(X) \cong \pi_{-n} \underline{\mathrm{hom}}(X, E) .$$

Exercise 11.12 (3 points). For any co/fiber sequence $E \rightarrow F \rightarrow G$ of spectra, construct a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(G) \xrightarrow{\delta} \pi_n(E) \longrightarrow \pi_n(F) \longrightarrow \pi_n(G) \xrightarrow{\delta} \pi_{n-1}(E) \longrightarrow \cdots .$$

Exercise 11.13 (3 points). Show that for any derived R -module $M \in \mathbf{D}_R$ and any $n \in \mathbb{Z}$, there is a natural isomorphism

$$\mathrm{H}_n(M) \cong \pi_n(\mathrm{fgt}(M))$$

between its n^{th} homology group and the n^{th} homotopy group of its underlying spectrum $\mathrm{fgt}(M) \in \mathcal{S}\mathbf{p}$.

In particular, Exercise 11.13 shows that (as asserted in Footnote 50) the long exact sequence in homology groups from a co/fiber sequence of derived R -modules is indeed a special case of the long exact sequence in homotopy groups of spectra of Exercise 11.12. (That the underlying spectra indeed form a co/fiber sequence follows from Exercise 11.10.)

- We conclude by giving a nontrivial example of a spectrum and its corresponding cohomology theory.

Recall that a **complex vector bundle** over a topological space $T \in \mathbf{Top}$ is a continuous function $E \xrightarrow{p} T$ equipped with a locally trivial $\mathbb{C}$ -vector space structure on each fiber $E_t := p^{-1}(t)$. The **rank** of a vector bundle is the dimension (over $\mathbb{C}$) of its fibers.²⁵⁷

Now, isomorphism classes of complex vector bundles over T assemble into a commutative monoid under (fiberwise) direct sum. We define the 0^{th} **complex K -theory** of T , denoted $KU^0(T)$, to be its group completion, i.e. the abelian group of pairs $[E_0, E_1]$ of vector bundles over T (which will represent the formal difference $[E_0] - [E_1]$) modulo the relation that $[E_0, E_1] \sim [E'_0, E'_1]$ whenever there exists an isomorphism $E_0 \oplus E'_1 \cong E'_0 \oplus E_1$ (with addition given componentwise). This will be the 0^{th} abelian group in our cohomology theory (at least for compact spaces).

To continue, let us recall that if T is paracompact, then we have an isomorphism

$$\mathrm{VBd}_{\mathbb{C}}^{\mathrm{rk}=n}(T)/\mathrm{iso.} \cong [T, \mathrm{BU}_n]$$

between rank- n vector bundles over T up to isomorphism and homotopy classes of maps into the *classifying space* for such.²⁵⁸ Under the further assumption that T is

²⁵⁷By the local triviality condition, this defines a continuous function $T \xrightarrow{\mathrm{rk}(E)} \mathbb{N}$.

²⁵⁸Concretely, the space $\mathrm{BU}_n \in \mathcal{S}$ (which is equivalent to $\mathrm{BGL}_n(\mathbb{C})$, since the inclusion $\mathrm{U}_n \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is a homotopy equivalence) can be constructed as the Grassmannian $\mathrm{Gr}_n(\mathbb{C}^\infty)$ of complex n -planes in $\mathbb{C}^\infty :=$

compact, any vector bundle can be embedded into a trivial bundle $\underline{\mathbb{C}}^N := (\mathbb{C}^N \times T \xrightarrow{\text{pr}} T)$ for some $N \gg 0$, which implies that every element of $KU^0(T)$ can be represented by a pair of the form $[E, \underline{\mathbb{C}}^N]$.²⁵⁹ Of course, we have further identifications

$$[E, \underline{\mathbb{C}}^N] \sim [(E \oplus \underline{\mathbb{C}}^1), \underline{\mathbb{C}}^{N+1}] \sim [(E \oplus \underline{\mathbb{C}}^2), \underline{\mathbb{C}}^{N+2}] \simeq \dots ,$$

which imply that this K-theory class is entirely captured by the vector bundle E up to the equivalence relation $E \sim E \oplus \underline{\mathbb{C}}^1 \sim E \oplus \underline{\mathbb{C}}^2 \sim \dots$ along with its *virtual rank*, which for a K-theory class represented by the pair $[E_0, E_1]$ is the function $T \xrightarrow{\text{vrk}[E_0, E_1]} \mathbb{Z}$ defined by $\text{vrk}[E_0, E_1](t) := \text{rk}_t(E_0) - \text{rk}_t(E_1)$. Altogether, in this case we have

$$KU^0(T) \cong [T, \mathbb{Z} \times \text{BU}] ,$$

where $\text{BU} := \text{colim}_n(\text{BU}_n)$ is the colimit along the maps $\text{BU}_n \rightarrow \text{BU}_{n+1}$ classifying the operation $(-) \oplus \underline{\mathbb{C}}^1$,²⁶⁰ and the first factor keeps track of the virtual rank. This, finally, defines the 0th space of the complex K-theory spectrum: we have

$$KU_0 := \mathbb{Z} \times \text{BU} .$$

Of course, for all $n < 0$ this determines the space $KU_n := \Omega^n KU_0$. But in order to define a full spectrum, we appeal to ***Bott periodicity***: a canonical equivalence $\Omega^2(\mathbb{Z} \times \text{BU}) \simeq \mathbb{Z} \times \text{BU}$. Said differently, for any space X we have a canonical isomorphism

$$KU^0(\Sigma^2 X) \cong KU^0(X) .$$

This allows us to define the ***complex K-theory spectrum***, denoted $KU \in \text{Sp}$, by setting

$$KU_{2j} := \mathbb{Z} \times \text{BU} \quad \text{and} \quad KU_{2j+1} := \Omega(\mathbb{Z} \times \text{BU}) \simeq \text{U}$$

(where $\text{U} := \text{colim}_n(\text{U}_n)$ denotes the infinite unitary group).²⁶¹

11.8. Friday 6/2. Today's lecture introduced ***recollections***, which can be thought of as “categorified extension sequences” among stable ∞ -categories. In addition to their intrinsic interest, these are a nice context for practicing basic manipulations in stable ∞ -categories. A more detailed discussion (along with a plethora of examples and generalizations) can be found e.g. in [AMGRa, §1].

$\text{colim}_n(\mathbb{C}^n)$. More abstractly, this is the space of n -dimensional complex vector spaces and isomorphisms between them (i.e. the ∞ -groupoid underlying the topological category of such).

²⁵⁹Indeed, given any representative $[0, E]$ and an embedding $E \hookrightarrow \underline{\mathbb{C}}^N$, we have $[0, E] \simeq [E^\perp, \underline{\mathbb{C}}^N]$, where $E^\perp \subseteq \underline{\mathbb{C}}^N$ denotes the orthogonal subbundle with respect to an arbitrary Hermitian metric (which can be chosen using a partition of unity).

²⁶⁰Equivalently, this map is B of the inclusion $\text{U}_n \hookrightarrow \text{U}_{n+1}$ given by block sum with the 1×1 identity matrix. Hence, BU might be referred to as the space of “stabilized” complex vector spaces.

²⁶¹The multiplication $KU_0 \times KU_0 \rightarrow KU_0$ coming from the equivalence $KU_0 \simeq \Omega \text{U}$ is given by addition of virtual dimensions in the $\mathbb{Z}$ factor and direct sum of (“stabilized”) vector spaces in the BU factor.

- A *recollement* is a diagram

$$(46) \quad \begin{array}{ccccc} & \overset{i_L}{\curvearrowright} & & \overset{p_L}{\curvearrowright} & \\ & \perp & & \perp & \\ \mathcal{K} & \xleftarrow{y} & \mathcal{X} & \xleftarrow{\nu} & \mathcal{Q} \\ & \perp & & \perp & \\ & \underset{i_R}{\curvearrowright} & & \underset{p_R}{\curvearrowright} & \end{array}$$

of stable ∞ -categories and adjunctions between them such that all three rows are *exact*, i.e.

$$\mathrm{im}(i_L) = \ker(p_L) , \quad \mathrm{im}(\nu) = \ker(y) , \quad \text{and} \quad \mathrm{im}(i_R) = \ker(p_R) .^{262}$$

(In particular, the functors i_L , ν , and i_R are all required to be fully faithful.) We may say that this is a recollement of $\mathcal{X}$ by $\mathcal{Q}$ and $\mathcal{K}$.

- Since i_L , ν , and i_R are fully faithful inclusions, it can be useful to think of them as defining subcategories $i_L(\mathcal{K}), \nu(\mathcal{Q}), i_R(\mathcal{K}) \subseteq \mathcal{X}$. In these terms, observe that the objects $K \in i_L(\mathcal{K}) \subseteq \mathcal{X}$ are in the *left orthogonal* of the subcategory $\nu(\mathcal{Q}) \subseteq \mathcal{X}$, i.e. we have

$$i_L(\mathcal{K}) \subseteq {}^\perp \nu(\mathcal{Q}) := \{X \in \mathcal{X} \text{ such that } \underline{\mathrm{hom}}_{\mathcal{X}}(X, Q) \simeq 0 \text{ for all } Q \in \nu(\mathcal{Q})\} .^{263}$$

Equivalently, the objects $Q \in \nu(\mathcal{Q}) \subseteq \mathcal{X}$ are in the *right orthogonal* of $i_L(\mathcal{K}) \subseteq \mathcal{X}$, i.e. we have

$$\nu(\mathcal{Q}) \subseteq i_L(\mathcal{K})^\perp := \{X \in \mathcal{X} \text{ such that } \underline{\mathrm{hom}}_{\mathcal{X}}(K, X) \simeq 0 \text{ for all } K \in i_L(\mathcal{K})\} .$$

Exercise 11.14 (2 points). Prove the following orthogonality relations:

$$i_L(\mathcal{K}) = {}^\perp \nu(\mathcal{Q}) , \quad i_L(\mathcal{K})^\perp = \nu(\mathcal{Q}) = {}^\perp i_R(\mathcal{K}) , \quad \text{and} \quad i_R(\mathcal{K}) = \nu(\mathcal{Q})^\perp .$$

Thus, the recollement is entirely determined by any one of these three full subcategories of $\mathcal{X}$. (The adjoints to the inclusions are uniquely determined if they exist.)

Exercise 11.15 (4 points). Prove that the composite adjunction

$$\mathcal{K} \begin{array}{c} \xleftarrow{i_L} \\ \perp \\ \xrightarrow{y} \end{array} \mathcal{X} \begin{array}{c} \xleftarrow{y} \\ \perp \\ \xrightarrow{i_R} \end{array} \mathcal{K}$$

is an adjoint equivalence.²⁶⁴

²⁶²The letters $\mathcal{K}$ and $\mathcal{Q}$ stand for “kernel” and “quotient”, although in fact both play both roles (see Exercise 11.17). Also, the notation y is meant to indicate that this is the restricted Yoneda functor (with respect to i_L), and the notation ν is meant to indicate that this is the inclusion of the null objects with respect to y (see Exercise 11.14).

²⁶³Since the subcategory $\nu(\mathcal{Q}) \subseteq \mathcal{X}$ is closed under de/suspension, it is equivalent to demand only that the hom-spaces $\mathrm{hom}_{\mathcal{X}}(X, Q) \simeq \underline{\mathrm{hom}}_{\mathcal{X}}(X, Q)_0$ are contractible.

²⁶⁴Of course, this is an abstraction of Exercise 6.2 (see Exercise 11.18).

- Given an object $X \in \mathcal{X}$, its unit and counit maps for the various adjunctions participate in canonical triangles

$$\begin{array}{ccc}
 i_L y X & \xrightarrow{\varepsilon_X^{i_L - y}} & X \\
 \downarrow & & \downarrow \eta_X^{p_L - \nu} \\
 0 & \longrightarrow & \nu p_L X
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 i_R y & \xleftarrow{\varepsilon_X^{y - i_R}} & X \\
 \uparrow & & \uparrow \varepsilon_X^{\nu - p_R} \\
 0 & \longleftarrow & \nu p_R X
 \end{array},$$

in which the nullhomotopies arise from Exercise 11.14.

Exercise 11.16 (2 points). Show that these triangles are co/fiber sequences.

Exercise 11.17 (6 points). Prove that each of the three rows in a recollement is canonically a cofiber sequence in $\mathbf{St}$, i.e. that they define pushout squares

$$\begin{array}{ccc}
 \mathcal{K} & \xrightarrow{i_L} & \mathcal{X} \\
 \downarrow & & \downarrow p_L \\
 0_{\mathbf{St}} & \longrightarrow & \mathcal{Q}
 \end{array}, \quad
 \begin{array}{ccc}
 \mathcal{K} & \xleftarrow{y} & \mathcal{X} \\
 \uparrow & & \uparrow \nu \\
 0_{\mathbf{St}} & \longleftarrow & \mathcal{Q}
 \end{array}, \quad \text{and} \quad
 \begin{array}{ccc}
 \mathcal{K} & \xrightarrow{i_R} & \mathcal{X} \\
 \downarrow & & \downarrow p_R \\
 0_{\mathbf{St}} & \longrightarrow & \mathcal{Q}
 \end{array}.^{265}$$

Taken together, Exercises 11.16 and 11.17 can be read as an instance of the “macrocosm/microcosm principle”: a decomposition of the ambient stable ∞ -category determines a decomposition of each of its objects.

- We give some examples of recollements.

Exercise 11.18 (3 points). Prove that the diagram

$$\begin{array}{ccccc}
 & \overset{i_L}{\curvearrowright} & & \overset{a_L}{\curvearrowright} & \\
 & \perp & & \perp & \\
 \mathbf{D}_R & \xleftarrow{\pi} & \mathbf{K}_R & \xleftarrow{i_A} & \mathbf{A}_R \\
 & \perp & & \perp & \\
 & \underset{i_R}{\curvearrowleft} & & \underset{a_R}{\curvearrowleft} &
 \end{array}$$

in $\mathbf{Cat}_{\mathbf{k}}$ of §6.3 defines a recollement on underlying stable ∞ -categories.

Exercise 11.19 (3 points). Given stable ∞ -categories $\mathcal{A}, \mathcal{B} \in \mathbf{St}$, prove that the diagram

$$(47) \quad
 \begin{array}{ccccc}
 & \overset{(\mathrm{id}_{\mathcal{A}}, 0)}{\curvearrowright} & & \overset{\mathrm{pr}_{\mathcal{B}}}{\curvearrowright} & \\
 & \perp & & \perp & \\
 \mathcal{A} & \xleftarrow{\mathrm{pr}_{\mathcal{A}}} & \mathcal{A} \times \mathcal{B} & \xleftarrow{(0, \mathrm{id}_{\mathcal{B}})} & \mathcal{B} \\
 & \perp & & \perp & \\
 & \underset{(\mathrm{id}_{\mathcal{A}}, 0)}{\curvearrowleft} & & \underset{\mathrm{pr}_{\mathcal{B}}}{\curvearrowleft} &
 \end{array}$$

defines a recollement of their product.

²⁶⁵Note that the functors p_L , y , and p_R are all co/reflective localizations. The additional content of these assertions amounts to the fact that e.g. an exact functor $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ to a stable ∞ -category factors through p_L if and only if the composite $F i_L$ is zero.

Another source of examples arises as follows. Given any ring homomorphism $S \rightarrow R$, as in Exercise 5.13(a) we have a diagram

$$\begin{array}{ccc} & \xrightarrow{(-) \otimes_S^{\mathbb{L}} R} & \\ \mathbf{D}_S & \xleftarrow{\text{fgt}} \mathbf{D}_R & \\ & \xleftarrow{\text{hom}_{\mathbf{D}_S}(R, -)} & \end{array} \quad .$$

Let us say that R is a **derived-idempotent** S -algebra if the derived multiplication map

$$R \otimes_S^{\mathbb{L}} R \longrightarrow R \otimes_S R \longrightarrow R$$

is an equivalence. In this case, the upper adjunction is a reflective localization (recall Exercise 6.5), so that the diagram defines the right side of a recollement.

Here are two illustrative examples.

Exercise 11.20 (4 points).

- (a) Given a field $\mathbb{k}$, show that the inclusion $\mathbb{k}[x] \hookrightarrow \mathbb{k}[x^\pm]$ makes $\mathbb{k}[x^\pm]$ into a derived-idempotent $\mathbb{k}[x]$ -algebra.
- (b) Given a prime p , show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[\frac{1}{p}]$ makes $\mathbb{Z}[\frac{1}{p}]$ into a derived-idempotent $\mathbb{Z}$ -algebra.

These derived-idempotent algebras define recollements with rather interesting left sides, namely

$$\begin{array}{ccc} \mathbf{D}_{\mathbb{k}[x]}^{(x)\text{-torsion}} & \xleftarrow{\perp} & \mathbf{D}_{\mathbb{k}[x]} \\ \uparrow \wr & \nearrow \text{fgt} & \uparrow \\ \mathbf{D}_{\mathbb{k}[x]}^{(x)\text{-complete}} & \xleftarrow{(-) \otimes_{\mathbb{k}[x]}^{\mathbb{L}} \mathbb{k}[x]} & \mathbf{D}_{\mathbb{k}[x]} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{D}_{\mathbb{Z}}^{(p)\text{-torsion}} & \xleftarrow{\perp} & \mathbf{D}_{\mathbb{Z}} \\ \uparrow \wr & \nearrow \text{fgt} & \uparrow \\ \mathbf{D}_{\mathbb{Z}_p^\wedge}^{(p)\text{-complete}} & \xleftarrow{(-) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}_p^\wedge} & \mathbf{D}_{\mathbb{Z}} \end{array} ,$$

where the vertical equivalences arise from Exercise 11.15.²⁶⁶

- Let us now return to the context of a general recollement. The following may be referred to as **microcosm reconstruction** for objects of $\mathcal{X}$.

²⁶⁶These notations refer to a (derived) module being *torsion* or *complete* with respect to an ideal.

Exercise 11.21 (4 points). Given an object $X \in \mathcal{X}$, show that the natural commutative square

$$(48) \quad \begin{array}{ccc} X & \xrightarrow{\eta_X^{p_L \dashv \nu}} & \nu p_L X \\ \eta_X^{y \dashv i_R} \downarrow & & \downarrow \nu p_L(\eta_X^{y \dashv i_R}) \\ i_R y X & \xrightarrow[\eta_{i_R y X}^{p_L \dashv \nu}]{} & \nu p_L i_R y X \end{array}$$

is a pullback.

Thus, we can recover the object X from the cospan. For example, a derived $\mathbb{Z}$ -module $M \in \mathbf{D}_{\mathbb{Z}}$ can be recovered as a pullback

$$(49) \quad \begin{array}{ccc} M & \longrightarrow & M[\frac{1}{p}] := M \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] \\ \downarrow & & \downarrow \\ M \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p^{\wedge} =: M_p^{\wedge} & \longrightarrow & (M_p^{\wedge})[\frac{1}{p}] \end{array}$$

(recall Exercise 11.20(b)).

However, in order to recover X it actually suffices to remember strictly less data. First of all, the functors i_R and ν are fully faithful, so to remember the three terms it suffices to record just the objects $yX \in \mathcal{K}$ and $p_L X \in \mathcal{Q}$. Moreover, the lower morphism is determined by yX (being the component of a natural transformation at its image $i_R y X$). However, the right morphism is additional data: it is determined by X , and not merely by yX or $p_L X$. On the other hand, since the functor ν is fully faithful, it suffices to remember just the morphism $p_L(\eta_X^{y \dashv i_R})$ in $\mathcal{Q}$. We refer to this as the *gluing morphism* for X .

So, in order to record the object X , it suffices to record the data of

- the object $yX \in \mathcal{K}$,
- the object $p_L X \in \mathcal{Q}$, and
- the morphism $\gamma_X := p_L(\eta_X^{y \dashv i_R})$ in $\mathcal{Q}$.

In other words, X is *uniquely determined* by this data as the resulting pullback. On the other hand, this process can be reversed.

Exercise 11.22 (4 points). Fix any triple of

- an object $K \in \mathcal{K}$,
- an object $Q \in \mathcal{Q}$, and
- a morphism $Q \xrightarrow{\gamma} p_L i_R K$ in $\mathcal{Q}$,

and denote by

$$X := \lim \left(\begin{array}{ccc} & & \nu Q \\ & & \downarrow \nu(\gamma) \\ i_R K & \xrightarrow[\eta_{i_R K}^{p_L \dashv \nu}]{} & \nu p_L i_R K \end{array} \right)$$

the indicated pullback. Prove that the corresponding pullback square is precisely (48).

In particular, *any* such morphism γ arises as the gluing morphism of an object of $\mathcal{X}$. Altogether, this leads to **macrocosm reconstruction** for $\mathcal{X}$ itself.

Exercise 11.23 (6 points). Prove that the above construction defines an equivalence

$$(50) \quad \mathcal{X} \xrightarrow{\sim} \lim \left(\begin{array}{ccc} & & \text{Fun}([1], \mathcal{Q}) \\ & & \downarrow t \\ \mathcal{K} & \xrightarrow{p_L i_R} & \mathcal{Q} \end{array} \right).$$

We may refer to the functor $p_L i_R$ as the **gluing functor** of the recollement; this is the additional “extension data” needed to reconstruct $\mathcal{X}$ beyond just the data of $\mathcal{K}$ and $\mathcal{Q}$.

This process, too, can be reversed.

Exercise 11.24 (6 points). Fix any exact functor $\mathcal{K} \xrightarrow{F} \mathcal{Q}$ between stable ∞ -categories, and denote by

$$\mathcal{X} := \lim \left(\begin{array}{ccc} & & \text{Fun}([1], \mathcal{Q}) \\ & & \downarrow t \\ \mathcal{K} & \xrightarrow{F} & \mathcal{Q} \end{array} \right)$$

the indicated pullback. Prove that this determines a recollement (46) for which the definition of $\mathcal{X}$ coincides with the equivalence (50) (so that in particular we have an equivalence $F \simeq p_L i_R$).

In particular, *any* exact functor between stable ∞ -categories arises as the gluing functor of a recollement.

Exercise 11.25 (4 points). Prove that the gluing functor of a recollement is zero if and only if the images of i_L and i_R coincide, in which case the recollement is of the form (47).

- Let us discuss the algebro-geometric content of the macrocosm reconstruction of $\mathbf{D}_{\mathbb{k}[x]}$ resulting from Exercise 11.20(a). We introduce the notation

$$\begin{array}{ccc}
 \mathbb{A}^1 \setminus \mathbb{A}^0 & & \mathrm{Spec}(\mathbb{k}[x^\pm]) \\
 \downarrow j & & \downarrow \\
 \mathbb{A}^0 & \xrightarrow{i} & \mathbb{A}^1 \\
 \searrow & \nearrow \iota & \\
 & \hat{\mathbb{A}}^1 &
 \end{array}
 \quad := \quad
 \begin{array}{ccc}
 & & \mathrm{Spec}(\mathbb{k}[x^\pm]) \\
 & & \downarrow \\
 \mathrm{Spec}(\mathbb{k}) & \xrightarrow{\quad} & \mathrm{Spec}(\mathbb{k}[x]) \\
 \searrow & \nearrow & \\
 & \mathrm{Spf}(\mathbb{k}[[x]]) &
 \end{array}$$

$\mathbb{A}^0$ denotes the origin in the affine line $\mathbb{A}^1$, and $\hat{\mathbb{A}}^1$ denotes the formal completion of the latter along the former.²⁶⁷ We then can identify the relevant adjoint functors as the indicated pullback and pushforward functors

$$\begin{array}{ccc}
 & \begin{array}{ccc}
 (-) \otimes_{\mathbb{k}[x]} \mathbb{k}[x^\pm] & & \\
 \downarrow \perp & & \\
 \mathbf{D}_{\mathbb{k}[x]} & \xleftarrow{\mathrm{fgt}} & \mathbf{D}_{\mathbb{k}[x^\pm]}
 \end{array} & \begin{array}{ccc}
 & j^* & \\
 & \downarrow \perp & \\
 \mathrm{QCoh}(\mathbb{A}^1) & \xleftarrow{j_*} & \mathrm{QCoh}(\mathbb{A}^1 \setminus \mathbb{A}^0)
 \end{array} \\
 \swarrow \begin{array}{l} \mathbb{k}[[x]] \otimes_{\mathbb{k}[x]} (-) \\ \downarrow \perp \\ \mathrm{fgt} \end{array} & \simeq & \swarrow \begin{array}{l} \hat{\iota}^* \\ \downarrow \perp \\ \hat{\iota}_* \end{array} \\
 \mathbf{D}_{\mathbb{k}[[x]]}^{(x)\text{-complete}} & & \mathrm{QCoh}(\hat{\mathbb{A}}^1)
 \end{array}$$

among stable ∞ -categories of (derived) quasicoherent sheaves. Thus, a quasicoherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathbb{A}^1)$ is uniquely specified by the data of

- its restriction $\mathcal{F}_0 := \hat{\iota}^* \mathcal{F} \in \mathrm{QCoh}(\hat{\mathbb{A}}^1)$,
- its restriction $\mathcal{F}_1 := j^* \mathcal{F} \in \mathrm{QCoh}(\mathbb{A}^1 \setminus \mathbb{A}^0)$, and
- its gluing morphism

$$(51) \quad \mathcal{F}_1 \xrightarrow{j^*(\eta_{\mathcal{F}}^{i^*} \dashv i_*)} j^* \hat{\iota}_* \mathcal{F}_0 ,$$

and conversely any such triple determines a quasicoherent sheaf: we have a pullback square

$$\begin{array}{ccc}
 \mathrm{QCoh}(\mathbb{A}^1) & \xrightarrow{j^*(\eta^{i^*} \dashv i_*)} & \mathrm{Fun}([1], \mathrm{QCoh}(\mathbb{A}^1 \setminus \mathbb{A}^0)) \\
 \downarrow \hat{\iota}^* & & \downarrow t \\
 \mathrm{QCoh}(\hat{\mathbb{A}}^1) & \xrightarrow{j^* \hat{\iota}_*} & \mathrm{QCoh}(\mathbb{A}^1 \setminus \mathbb{A}^0)
 \end{array}$$

One may think of the gluing morphism (51) as encoding gluing data over the intersection of $\hat{\mathbb{A}}^1$ and $\mathbb{A}^1 \setminus \mathbb{A}^0$, even though this is not literally well-defined.

²⁶⁷This is a *formal* scheme, and should be thought of as an “infinitesimal neighborhood”.

Of course, the macrocosm reconstruction of $\mathbf{D}_{\mathbb{Z}}$ resulting from Exercise 11.20(b) has a completely analogous interpretation in terms of the scheme $\mathrm{Spec}(\mathbb{Z})$, its closed subscheme $\mathrm{Spec}(\mathbb{Z}/p)$, and the formal completion $\mathrm{Spf}(\mathbb{Z}_p^\wedge)$.

- Let us think of a recollement (46) as a one-step filtration of $\mathcal{X}$. More generally, given a suitable n -step filtration

$$\mathcal{K}_0 \hookrightarrow \mathcal{K}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{K}_n = \mathcal{X}$$

we can reconstruct $\mathcal{X}$ in terms of the filtration quotients $\mathcal{Q}_n := \mathcal{K}_n/\mathcal{K}_{n-1}$, although now the relevant gluing data are substantially more involved.

Even more generally, in favorable situations one can reconstruct $\mathcal{X}$ from a filtration over a general poset $\mathbf{P}$ from the filtration quotients $\mathcal{Q}_p := \mathcal{K}_p/\langle\{\mathcal{K}_q\}_{q < p}\rangle$; this is afforded by the theory of *stratifications* introduced in [AMGRa]. This formalism admits generalizations of

- Exercise 11.21 (microcosm reconstruction of objects in a $\mathbf{P}$ -stratified stable ∞ -category),
- Exercise 11.23 (macrocosm reconstruction of a $\mathbf{P}$ -stratified stable ∞ -category), and
- Exercise 11.24 (“metacosm reconstruction” of the ∞ -category of $\mathbf{P}$ -stratified stable ∞ -categories),

as articulated in [AMGRa, Theorem A]. For example, the scheme $\mathrm{Spec}(\mathbb{Z})$ admits an *adelic stratification*, which allows us to recover any quasicoherent sheaf $M \in \mathrm{QCoh}(\mathrm{Spec}(\mathbb{Z})) \simeq \mathbf{D}_{\mathbb{Z}}$ as a pullback

$$\begin{array}{ccc} M & \longrightarrow & M \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_{p \text{ prime}} M_p^\wedge & \longrightarrow & \left(\prod_{p \text{ prime}} M_p^\wedge \right) \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

(refining the pullback (49)) [AMGRa, §1.6].

- Recollements admit an intrinsic symmetry, which we have broken beginning in Exercise 11.21 – leading to the choice of the functor $\mathcal{K} \xrightarrow{p_L i_R} \mathcal{Q}$ as the gluing functor.

Exercise 11.26 (12 points). Formulate and prove dual versions of Exercises 11.21–11.24, obtaining in particular a pullback square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{p_R(\varepsilon^i_L \dashv y)} & \mathrm{Fun}([1], \mathcal{Q}) \\ y \downarrow & & \downarrow s \\ \mathcal{K} & \xrightarrow{p_R i_L} & \mathcal{Q} \end{array}$$

associated to a recollement (46) as the dual of Exercise 11.23.

This duality is referred to as *reflection*, since it recovers the theory of reflection functors in representation theory; when applied to stratifications over more general posets, it is closely related to Verdier duality and gives a categorification of Möbius inversion [AMGRa, §1.10].

Exercise 11.27 (3 points). Express a derived $\mathbb{Z}$ -module $M \in \mathbf{D}_{\mathbb{Z}}$ as a pushout based on the reflected microcosm reconstruction of Exercise 11.26 applied to the recollement arising from Exercise 11.20(b).

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