

SYLLABUS
HOMOLOGICAL ALGEBRA (MATH 128)
SPRING 2023

CONTENTS

1. General information	1
2. Course description	3
3. Resources	4
4. Miscellanea	5

1. GENERAL INFORMATION

1.1. **Instructor.**

name: Aaron Mazel-Gee

email: mazelgee@caltech.edu

office hours: Mondays 11am-12noon in 285 Linde¹

pronouns: he/him

1.2. **Course logistics.**

lectures: MWF at 1-2pm in 255 Linde

course website: etale.site/teaching/s23-128/

textbook: We will follow the lecture notes posted on the course website.² Possible supplementary texts include Weibel's *Homological algebra*, Loday's *Cyclic homology*, Kashiwara & Schapira's *Sheaves on manifolds*, Lurie's *Higher algebra*, and Gaitsgory–Rozenblyum's *A study in derived algebraic geometry*.

prerequisites: basic algebra (e.g. rings and modules) and basic category theory (e.g. co/limits and adjunctions); familiarity with basic algebraic topology (e.g. the fundamental group and homology) will also be helpful

Canvas: yes, lightly (see §4.1)

last updated: May 8, 2023

¹If my office hours conflict with your schedule, I am also available to meet by appointment.

²They are based on my lecture notes from [a previous iteration](#) of the same course. I will be revising and polishing these notes through the quarter, and afterwards they will be published as a book in Cambridge University Press's *Studies in Advanced Mathematics* series.

1.3. Homework, participation, and grades. Homework is described in §1.3.1, and participation is described in §1.3.2. Your final grade in the course will be obtained by subtracting N letters from your homework grade, where N is determined by your participation in lectures (of which there are 26 total) according to the following table.

minimum participation	18	12	8	4
N	0	1	2	3

If you are taking the course for a pass/fail grade, you will receive a “pass” as long as you earn a C- or better.

1.3.1. Homework. Homework problems will be interwoven through the lecture notes. They will have various purposes: to check your understanding, to illustrate an example or technique, or to offer a further avenue of exploration.

Each homework problem will be assigned a point value.³ Your homework grade will translate into a letter grade according to the following cutoffs.⁴

cutoff	180	150	140	130	100	90	80	50	40
letter grade	A+	A	A-	B+	B	B-	C+	C	C-

Homework will be due at 11:59pm on Friday 6/9. However, you are strongly encouraged to turn in problems early. The main reason for this is that working on homework problems as they appear is the best way of staying on top of the material. But moreover, I will grade the problems as you submit them, and you will be free to correct your solutions as necessary in order to receive full credit.

Depending on interest and on our progress through the material, I may also make available the possibility of giving a presentation on an advanced topic towards the end of the quarter as an alternate way of accumulating homework points.

Additionally, you will receive homework points for informing me of any typos or mistakes in the lecture notes (with the point value depending on the severity of the issue).⁵

Please read the brief document [Guidelines for Good Mathematical Writing](#), and adhere to its suggestions in your homework problem submissions to the greatest extent possible.

You are encouraged to write your homework in LaTeX (pronounced “LAH-teck”). This is a versatile and easy-to-use typesetting system, which is the industry standard in mathematics and adjacent fields. Some resources to help you get started writing in LaTeX are outlined in §4.4.

You will submit your homework simply by emailing it to me. If you write your homework in LaTeX, you may simply submit the pdf. Alternatively, you may write your homework out

³Please note that I may change the point-values slightly from time to time; I apologize for any inconvenience that this causes.

⁴Note that even an A+ homework grade does not require you to solve all of the problems.

⁵However, there will often be links to later material that will be broken until the later material appears in the document. So, please simply ignore such apparent typos.

by hand and submit photos of it, assembled as a single pdf (e.g. using an online photo-to-pdf converter). At times, your solution to a problem may involve a picture or diagram; if you are not up for creating this via LaTeX, you may either embed a photo into your LaTeX file or simply attach a separate photo to your email.

1.3.2. *Participation.* In my own experience, mathematics is most exciting and most rewarding when it is pursued as an interpersonal endeavor: as an interesting and unique way to interact with other human beings. And of course, active participation is an important prerequisite for engaging with the material. Moreover, the homework load for this course is purposely relatively light and flexible. For these reasons, your participation will be factored into your final grade.

The primary form of participation is attendance in lectures: your participation grade will not be lower than the fraction of lectures that you attend. However, you may also earn participation credit through various other means of engagement, e.g.:

- asking and answering questions in Canvas, office hours, or lectures;
- exploring related topics or supplementary resources and sharing your findings with me and/or your classmates.

It is worth acknowledging that such participation can only reasonably be measured subjectively, and so your participation grade may ultimately depend on my subjective evaluation of your engagement. If you are concerned about your participation grade (e.g. due to missing a large number of lectures), it is your responsibility to communicate *promptly and directly* with me to discuss the situation.

2. COURSE DESCRIPTION

Consider the two-element set $\{x, y\}$. Suppose that we impose the relation “ $x = y$ ”. This gives us a singleton, i.e. a set with one element. Moreover, if we impose the relation “ $x = y$ ” again, we get the same result: a singleton admits no nontrivial quotients. Such an outcome is often undesirable: experience teaches us that it is often helpful to remember not simply “the fact that x and y are equal”, but instead “the ways in which x and y are equal”.

In this simple situation, we can record “the ways in which x and y are equal” through the following general construction. Suppose we are given a set S and a set R of (possibly apparently redundant) relations, i.e. a set R equipped with a function

$$R \xrightarrow{f=(f_0, f_1)} S \times S$$

to the cartesian product of S with itself.⁶ Then, rather than form the quotient set S/R , we may form a graph $S//R$ whose vertices are the elements of S and whose edges are the

⁶These relations are irredundant if and only if the function f is injective. In this case, f may be considered as the inclusion of a subset, which reduces us to the more familiar definition of a(n equivalence) relation on a set.

elements of R , such that the endpoints of the edge corresponding to an element $r \in R$ are the vertices $f_0(r)$ and $f_1(r)$. Note that the path components of $S//R$ are precisely the elements of S/R , i.e. the equivalence classes of S under (the equivalence relation generated by) R . However, the graph $S//R$ has not forgotten about the multiplicity of ways in which the elements of S are related by R . For instance, in the above example we obtain the graphs $x \text{ --- } y$ and $x \bigcirc y$, which are respectively shaped like a closed interval and a circle.⁷

Broadly speaking, **homological algebra** is the field of mathematics that arises from applying this same line of reasoning in the context of algebra (e.g. involving groups, rings, and modules, instead of just sets). For instance, given an abelian group A and a pair of elements $a, b \in A$, if we impose the relation “ $a = b$ ” twice in the context of homological algebra, we will end up with “a circle’s worth of elements” in the resulting equivalence class. This turns out to be a remarkably fruitful perspective, whose influence has only grown in the half-century or so since it first appeared.

In this course, we will study homological algebra and its applications, with a focus on examples and computations. Possible topics to be covered, roughly ordered from most likely to least likely, include:

- abelian categories and derived functors (e.g. Tor and Ext);
- dg-categories and stable ∞ -categories;
- filtrations and spectral sequences;
- model categories and nonabelian derived functors (e.g. co/homology, the cotangent complex), the Dold–Kan correspondence, and the Dold–Thom theorem;
- sheaves, the six functor formalism, Verdier duality;
- spectra and generalized co/homology theories;
- Hochschild homology, cyclic homology, and algebraic K-theory.

3. RESOURCES

Collaboration with your fellow students can be a very effective way of learning, and is highly encouraged. Often, we may think we understand something very well, but in trying to explain it to somebody else we find the gaps in our understanding.⁸ However, *everyone must write up their own solutions separately*: you may not copy each other’s work. Moreover, you must indicate any help that you received on each problem.⁹

⁷The graph $S//R$ may be referred to as the *homotopy quotient* of S by R .

⁸There was some research that came out about this recently. Subjects were asked to rate their understanding of some everyday mechanism (e.g. a ballpoint pen or the flushing mechanism of a toilet). Then, they were asked to explain it. Then, they were asked to rate their understanding again. As you might imagine, their own estimation of their understanding dropped significantly from the first round of rating to the second.

⁹For instance, as a footnote to Problem N you might write “I worked on this problem with Alice and Bob.” or “I found a hint/solution to this problem at <http://mathhelp4u.blogspot.com>.”.

As a guideline for this collaboration policy, keep in mind that **you should be able to reproduce any solution you hand in without assistance.**

Here is a list of allowed and disallowed resources. This is intended to be reasonably comprehensive, but the fact that something is not on this list does *not* mean that it is permitted. If you are ever in doubt about whether something is allowed, it is **your responsibility** to ask.

you may consult:	
textbooks	yes
solution manuals (official or unofficial)	no
internet	yes
notes from lecture (yours or others')	yes
course handouts	yes
communications (e.g. emails) from instructor	yes
you may:	
discuss problems with others	yes
look at communal materials (e.g. (e-)whiteboard) when writing up solutions	yes
look at written work of others	no
post about problems online (e.g. at math.stackexchange)	yes

4. MISCELLANEA

4.1. **Canvas.** Much of the relevant information for this course will be posted on the course website. However, we will use Canvas for a few purposes, e.g. for any material that may be considered as sensitive or unpolished – lecture slides, discussion boards, grades, etc.

4.2. **Students with documented disabilities.** If you need an academic accommodation based on the impact of a disability, you must initiate the request with [Caltech Accessibility Services for Students](#) (CASS). Professional staff will evaluate your request with required documentation, recommend reasonable accommodations, and prepare an Accommodation Letter for the quarter in which the request is being made. In this case, you should contact CASS as soon as possible, since timely notice is needed to coordinate accommodations. Undergraduate students should contact Dr. Lesley Nye, Associate Dean of Undergraduate Students (administrative contact: Beth Larranaga) and graduate students should contact Dr. Kate McNulty, Associate Dean of Graduate Studies (administrative contact: Jacob Dalton).

4.3. Academic integrity. You must abide by Caltech’s honor code: *No member of the Caltech community shall take unfair advantage of any other member of the Caltech community.* General guidelines for how this applies to plagiarism are given [here](#). Specific guidelines for this course are described in §3.

4.4. LaTeX resources. LaTeX code is stored as plain text, and therefore can be written and saved with the filename extension `txt` (e.g. as `my-LaTeX-code.txt`). This raw code can then be *compiled* into a `pdf` file (such as the one you are reading now). You can compile LaTeX code directly on your computer after a somewhat lengthy download/installation process. Alternatively, you can write and compile LaTeX code online through the website [Overleaf](#). I would recommend the latter option, especially if you are having difficulty with the former option (which is somewhat nontrivial).

A simple template that you can use to get started on your homework is available [here](#); when compiled, that LaTeX code produces [this](#) file. Of course, you are also welcome to start from scratch. Along with occasional online searches, that template should suffice to get you started.¹⁰ However, if you would like a further introduction to LaTeX you can check out some of the instructions provided by Overleaf: [this](#) shorter one or [this](#) longer one.

Of course, there are many online LaTeX resources. In particular, there are many “crib sheets”, see e.g. [here](#) and [here](#). Another great resource is [Detexify](#): if you draw a mathematical symbol, it will attempt to suggest how to write it in LaTeX.

4.5. Revisions to this syllabus. This syllabus is subject to minor changes throughout the semester, as needed. In the interest of transparency, all versions will remain available on the course website.

¹⁰The diagram is drawn using the `tikz-cd` package, more information about which can be found [here](#).