HOMOLOGICAL ALGEBRA

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Abstract. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the winter 2021 quarter. They are **under construction**, and will be updated at the course website at the end of each lecture.

Contents

0. Miscellanea

0.1. Exercises.

0.1.1. Currently, out of **12** points available, the number of points necessary for a grade of 100% on homework is **6**.

0.1.2. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a "yes or no" question).

0.2. Conventions.

0.2.1. We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and Set denotes the category of sets. However, notation will very often only be "local": the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.2. We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We generally ignore set-theoretic issues.[1]

0.2.3. The term "(commutative) ring" means "associative unital (resp. commutative) ring". Likewise, modules are always unital (meaning that the unit element acts as the identity).

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[1]Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

0.2.4. The term "natural number" (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.5. In the interest of brevity, universal quantifiers will often be dropped. For instance, an assertion involving an integer n should generally be understood to refer to *all* integers n unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

1. Some motivation for homological algebra

1.1. **Intersection theory.** A basic endeavor in geometry is to understand *intersections.* For example, given a (smooth) manifold M and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} .$$

1.1.1. If N_0 and N_1 intersect *transversely* (i.e. $T_pN_0 + T_pN_1 = T_pM$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

1.1.2. However, if the intersection of N_0 and N_1 is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either N_0 or N_1 that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.[2] However, this approach has a number of (related) drawbacks.

(1) Perturbations are noncanonical.
(2) Perturbations will generally destroy the inherent symmetries of the situation.[3]
(3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally transcendental.[4]

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. **Tensor products.** We first recall the notion of tensor product.

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[2]A good introduction to these ideas is [GP74].

[3]For instance, perturbations to transverse intersections need not exist in the equivariant context.

[4]It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, where "perturb" means "change to a new but rationally equivalent algebraic cycle". It is fundamental in the classical approach to intersection theory in algebraic geometry [EH16].

1.2.1. Let R be a commutative ring, and let M and N be R-modules. The (***relative***) ***tensor product*** of M and N over R,[5] denoted $M \otimes_R N$, is the universal abelian group equipped with an R-balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N \ ,$$

i.e. a function satisfying the following axioms:

(1) $\varphi(m+m',n) = \varphi(m,n) + \varphi(m',n)$ and $\varphi(m,n+n') = \varphi(m,n) + \varphi(m,n')$;
(2) $\varphi(m \cdot r, n) = \varphi(m, r \cdot n)$.[6]

In other words, for any abelian group A, precomposition with φ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \to A\} \xleftarrow{\cong} \{\text{abelian group homomorphisms } M \otimes_R N \to A\} \ .$$

In the case that R is understood (and particularly when $R = \mathbb{Z}$ or when R is a field), we may simply write $\otimes := \otimes_R$.

1.2.2. The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

(1) $(m+m',n) \sim (m,n) + (m',n)$ and $(m,n+n') \sim (m,n) + (m,n')$;
(2) $(m \cdot r, n) \sim (m, r \cdot n)$.

Exercise 1.1 (2 points)**.** For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\mathsf{gcd}(m,n)$.

1.3. **Basic principles of algebraic geometry.** In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §1.1].

1.3.1. The polynomial functions on $\mathbb{R}^n$ are the n-variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving n implicit).

1.3.2. By definition, an ***algebraic subset*** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.[7] In this case we write $Z = V(S)$, and we say that Z is the ***vanishing locus*** of the elements of S. If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.[8]

[5] The word "relative" here is meant to emphasize that R is an arbitrary commutative ring. By contrast, the term "absolute tensor product" would emphasize that $R = \mathbb{Z}$.

[6] The notation here stems from the fact that more generally, we can define the relative tensor product when R is merely an associative ring, M is a right R-module, and N is a left R-module.

[7] By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

[8] Since R is noetherian, any ideal is finitely generated. In other words, we may always take S to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

1.3.3. We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along Z. Then, the ring of polynomial functions on Z is

$$\mathcal{O}(Z) = R/I(Z) \ .$$

1.3.4. Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in I\} \subseteq \mathbb{R}^n \ .$$

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{ideals in } R\} \ .$$ [9]

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 \ .$$

1.4. **Intersections via tensor products.** We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

Example 1.2 (a transverse intersection)**.** Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x-1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \qquad \text{and} \qquad Z_1 = \{(x,y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \qquad \text{and} \qquad I_1 = I(Z_1) = (y - x) \subseteq R \ .$$

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned} \mathcal{O}(Z_0 \cap Z_1) &\cong R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y - x^2, y - x) \cong \mathbb{R}[x]/(x - x^2) \\ &= \mathbb{R}[x]/(x \cdot (1-x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1-x) \cong \mathbb{R} \times \mathbb{R} \ , \end{aligned}$$

[9]This restricts to a bijection between *closed* subsets and *radical* ideals. The composite $V \circ I$ carries a subset $Y \subseteq \mathbb{R}^n$ to its closure $\overline{Y} \subseteq \mathbb{R}^n$, while the composite $I \circ V$ carries an ideal $I \subseteq R$ to its radical $\sqrt{I} = \{f \in R : \exists\, n > 0 \text{ s.t. } f^n \in I\} \subseteq R$.

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a PID, in fact it is a Euclidean domain).[10] The fact that this is a 2-dimensional $\mathbb{R}$-algebra corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements.

Example 1.3 (a non-transverse intersection)**.** Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in R. These correspond to the curves $y = x^2$ and $y = 0$. These intersect "twice" at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y - x^2, y) \cong \mathbb{R}[x]/(x^2) \ .$$

The 2-dimensionality of this $\mathbb{R}$-algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect "with multiplicity two". Namely, this $\mathbb{R}$-algebra corresponds to "the origin along with infinitesimal fuzz in the direction of the x-axis". This is in contrast with the previous example, where the tensor product split as a cartesian product.

These techniques are quite robust.

Exercise 1.4 (2 points)**.** Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

Example 1.5 (another non-transverse intersection)**.** Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x - a) \subseteq R$ and $I_1 = (x - b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x - a, x - b) \cong \mathbb{R}/(a - b) \cong \begin{cases} \mathbb{R} \ , & a = b \\ 0 \ , & a \neq b \end{cases} .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$-algebra.

Using homological algebra, namely the notion of ***derived tensor products***, we will be able to obtain the expected intersection number of 0 even when $a = b$.

[10] An explicit inverse is given by carrying the pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b - a) \cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x \cdot (1 - x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0),g(1)}$ lies in the ideal generated by $x \cdot (x - 1)$. Observe that $g - f_{g(0),g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

Exercise 1.6 (6 points)**.** Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

(a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines "just barely avoid intersecting": morally they should intersect "at infinity".[11] This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3 \backslash \{0\})/\mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.[12]

(b) Show that a *homogenous* polynomial $g \in \mathbb{R}[x, y, z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.

(c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogenous polynomials $g_1, g_2 \in \mathbb{R}[x, y, z]$ such that $g_i([x : y : 1]) = f_i(x, y)$.

(d) Compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. Here is a more interesting non-generic situation where derived tensor products will give the correct answer where ordinary tensor products will not.

Example 1.7. Generically, two lines in $\mathbb{R}^2$ intersect in a point. This fails if the lines are parallel, but as we saw in Exercise 1.6 this failure is repaired by working in $\mathbb{RP}^2$ (and taking the closures of the lines therein). However, this still gives the wrong answer if the two lines are *equal*: of course, the intersection of a line with itself is itself.

Using derived tensor products, we will be able to obtain the expected intersection number of 1 when intersecting a (projective) line with itself in $\mathbb{RP}^2$.

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

[11] A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \to 1^+$ their intersection point has $x \to -\infty$, while as $t \to 1^-$ their intersection point has $x \to +\infty$. This suggests that there should be a *single* point "at infinity" where they intersect in the case that $t = 1$.

[12] So, the "points at infinity" are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2 \backslash \{0\})/\mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1 \backslash \mathbb{R}^1$).

2. Fundamentals of homological algebra

In this section we introduce the basic notions of homological algebra. A fine reference for this material is [Wei94, §§1-2], although we will take a rather different approach.

For concreteness, we fix a ring R and work in Mod_R, the category of (right) R-modules. However, as we will see later, the entire theory works equally well for a general abelian category.[13]

2.1. **Basic definitions.**

2.1.1. A ***chain complex*** of R-modules is a diagram

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of R-modules such that for all $n \in \mathbb{Z}$, the composite $d_n \circ d_{n+1} = 0$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that "all indices are being referred to at once". Also, one may simply refer to a chain complex as a "complex".[14] The integer n is sometimes called the ***degree*** or the ***dimension***.

The morphisms d_n are called the ***differentials*** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of d_n is M_n). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as d_n^M.

In this notation, a morphism of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of R-modules such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^M_{n+2}} & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow[d^N_{n+2}]{} & N_{n+1} & \xrightarrow[d^N_{n+1}]{} & N_n & \xrightarrow[d^N_n]{} & N_{n-1} & \xrightarrow[d^N_{n-1}]{} & \cdots
\end{array}$$

commutes.[15] We write Ch_R for the category of chain complexes of R-modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline when appropriate.

Any R-module $M \in \mathsf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \underset{\sim}{M} \longrightarrow 0 \longrightarrow \cdots .$$

[13]On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into Mod_R for some ring R (although the choice of such a ring R is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

[14]The word "chain" here is historical: the first example of a chain complex has in degree n the "n^{th} chain group" of a simplicial complex X, i.e. the group of chains (i.e. formal linear combinations) of n-simplices of X. (It was only later realized that chain complexes are worth studying in their own right.)

[15]For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

This construction determines a fully faithful embedding

$$\mathsf{Mod}_R \hookrightarrow \mathsf{Ch}_R \ .$$

When a complex only has a few nonzero terms, for brevity one may omit the zero terms. For instance, the above complex may be written simply as $\underset{\sim}{M}$.

2.1.2. Fix a chain complex $M_\bullet$. Its ***n-cycles*** and ***n-boundaries*** are the submodules

$$\mathsf{Z}_n(M_\bullet) := \ker(d_n) \subseteq M_n \qquad \text{and} \qquad \mathsf{B}_n(M_\bullet) := \mathsf{im}(d_{n+1}) \subseteq M_n \ .$$[16]

Note that $\mathsf{B}_n(M_\bullet) \subseteq \mathsf{Z}_n(M_\bullet)$ because $d^2 = 0$. Then, the n^{th} ***homology*** of $M_\bullet$ is the quotient R-module

$$\mathsf{H}_n(M_\bullet) := \frac{\mathsf{Z}_n(M_\bullet)}{\mathsf{B}_n(M_\bullet)} := \frac{\ker(d_n)}{\mathsf{im}(d_{n+1})} \ .$$

Exercise 2.1 (2 points)**.** Verify that the constructions Z_n, B_n, and H_n define functors

$$\mathsf{Ch}_R \longrightarrow \mathsf{Mod}_R \ .$$

We say that $M_\bullet$ is ***acyclic*** if $\mathsf{H}_n(M_\bullet) = 0$ for all n.

2.1.3. A morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R is called a ***quasi-isomorphism*** if the induced morphisms $H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We may indicate that a morphism is a quasi-isomorphism by decorating the arrow as $\xrightarrow{\approx}$.

By and large, one should think of quasi-isomorphic chain complexes as "essentially interchangeable", with some representatives of a given quasi-isomorphism class (e.g. the *projective resolutions* discussed in **??**) being "better adapted" for certain purposes than others.[17] In other words, one should think of quasi-isomorphisms as if they are actual isomorphisms.

2.2. Homotopies and homotopy co/kernels.

2.2.1. Let $M_1 \xrightarrow{f} M_0$ be a morphism of R-modules. This gives us a complex $M_\bullet := (M_1 \xrightarrow{f} \underset{\sim}{M_0})$. Observe that this has a canonical morphism

$$\begin{array}{ccccc} M_\bullet & := & M_1 & \xrightarrow{f} & \underset{\sim}{M_0} \\ \downarrow & & \downarrow & & \downarrow \\ \underset{\sim}{\mathsf{coker}(f)} & := & 0 & \longrightarrow & \underset{\sim}{\mathsf{coker}(f)} \end{array}$$

[16] The German words for "cycle" and "boundary" respectively begin with the letters "Z" and "B".

[17] This is very closely akin to how one should think of equivalent categories as "essentially interchangeable" (even if they are not isomorphic). However, in a precise sense, *all* categories are "equally well-adapted" for all purposes (in contrast with chain complexes).

to the cokernel of f (considered as a complex in degree 0). Observe further that

$$\mathsf{H}_n(M_\bullet) \cong \begin{cases} \mathsf{coker}(f) \ , & n = 0 \\ \mathsf{ker}(f) \ , & n = 1 \\ 0 \ , & \text{otherwise} \end{cases} .$$

Hence, the above map is a quasi-isomorphism iff f is an injection. One might think of $M_\bullet$ as a "presentation" of the underlying R-module $\mathsf{H}_0(M_\bullet) \cong \mathsf{coker}(f)$: the generators are M_0, the relations are M_1 (i.e. each $m \in M_1$ gives a relation $d(m) \sim 0$), but then $\mathsf{H}_1(M_\bullet)$ furthermore measures the "redundancy" of the relations. Said differently, $M_\bullet$ is a "homotopically correct" version of the cokernel of f, which remembers not only the literal cokernel but also the extent to which the relations are overdetermined. Indeed, it will be the *homotopy cokernel* of the morphism f.

References

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