# HOMOLOGICAL ALGEBRA

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Abstract. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the winter 2021 quarter. They are **under construction**, and will be updated at the course website at the end of each lecture.

## Contents



## 0. Miscellanea

### 0.1. **Exercises.**

0.1.1. Currently, out of **36** points available, the number of points necessary for a grade of 100% on homework is **12**.

0.1.2. Partial solutions may be submitted for partial credit.

0.1.3. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a "yes or no" question).

0.1.4. [changed 1/7] In preparing your homework, please copy down the problem statement, since it is possible that the numbering may change.

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## 0.2. Conventions.

0.2.1. [changed 1/7] As this document is a work in progress, I will frequently want to make changes to existing material (in addition to adding material as we cover it during lecture). Of course, it would be quite difficult for a reader to spot such changes, especially as the document grows. Therefore, any substantial such changes will be temporarily flagged by their change date (for easy searching) and additionally will be colored as follows: changes that have occurred since the most recent lecture are **red**, changes made between the past two lectures are **blue**, and changes made between two and three lectures ago are **green**. Older changes are no longer colored.

0.2.2. We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and $\mathsf{Set}$ denotes the category of sets. However, notation will very often only be "local": the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.3. We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We generally ignore set-theoretic issues.[1]

0.2.4. The term "(commutative) ring" means "associative unital (resp. commutative) ring". Likewise, modules are always unital (meaning that the unit element acts as the identity).

0.2.5. The term "natural number" (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.6. In the interest of brevity, universal quantifiers will often be dropped. For instance, an assertion involving an integer $n$ should generally be understood to refer to *all* integers $n$ unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

# 1. Some motivation for homological algebra

1.1. **Intersection theory.** A basic endeavor in geometry is to understand *intersections*. For example, given a (smooth) manifold $M$ and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} .$$

1.1.1. If $N_0$ and $N_1$ intersect *transversely* (i.e. $T_pN_0 + T_pN_1 = T_pM$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

[1] Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

1.1.2. However, if the intersection of $N_0$ and $N_1$ is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either $N_0$ or $N_1$ that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.[2] However, this approach has a number of (related) drawbacks.

(1) Perturbations are noncanonical.
(2) Perturbations will generally destroy the inherent symmetries of the situation.[3]
(3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally only transcendental.[4]

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. **Tensor products.** We first recall the notion of tensor product.

1.2.1. Let $R$ be a commutative ring, and let $M$ and $N$ be $R$-modules. The (***relative***) ***tensor product*** of $M$ and $N$ over $R$,[5] denoted $M \otimes_R N$, is the universal abelian group equipped with an $R$-balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N \ ,$$

i.e. a function satisfying the following axioms:

(1) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$ and $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$;
(2) $\varphi(m \cdot r, n) = \varphi(m, r \cdot n)$.[6]

In other words, for any abelian group $A$, precomposition with $\varphi$ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \to A\} \xleftarrow{\cong} \{\text{abelian group homomorphisms } M \otimes_R N \to A\} \ .$$

In the case that $R$ is understood (and particularly when $R = \mathbb{Z}$ or when $R$ is a field), we may simply write $\otimes := \otimes_R$.

[2] A good introduction to these ideas is [GP74].

[3] For instance, perturbations to transverse intersections need not exist in the equivariant context.

[4] It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, where "perturb" means "change to a new but rationally equivalent algebraic cycle". It is fundamental in the classical approach to intersection theory in algebraic geometry [EH16].

[5] The word "relative" here is meant to emphasize that $R$ is an arbitrary commutative ring. By contrast, the term "absolute tensor product" would emphasize that $R = \mathbb{Z}$.

[6] The notation here stems from the fact that more generally, we can define the relative tensor product when $R$ is merely an associative ring, $M$ is a right $R$-module, and $N$ is a left $R$-module.

1.2.2. The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

(1) $(m + m', n) \sim (m, n) + (m', n)$ and $(m, n + n') \sim (m, n) + (m, n')$;
(2) $(m \cdot r, n) \sim (m, r \cdot n)$.

**Exercise 1.1** (2 points)**.** For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\mathsf{gcd}(m, n)$.

1.3. **Basic principles of algebraic geometry.** In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §1.1].

1.3.1. The polynomial functions on $\mathbb{R}^n$ are the $n$-variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving $n$ implicit).

1.3.2. By definition, an ***algebraic subset*** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.[7] In this case we write $Z = V(S)$, and we say that $Z$ is the ***vanishing locus*** of the elements of $S$. If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.[8]

1.3.3. We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along $Z$. Then, the ring of polynomial functions on $Z$ is

$$\mathcal{O}(Z) = R/I(Z) \ .$$

1.3.4. Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in I\} \subseteq \mathbb{R}^n \ .$$

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{ideals in } R\} \ .$$ [9]

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 \ .$$

[7] By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

[8] Since $R$ is noetherian, any ideal is finitely generated. In other words, we may always take $S$ to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

[9] This restricts to a bijection between *closed* subsets and *radical* ideals. The composite $V \circ I$ carries a subset $Y \subseteq \mathbb{R}^n$ to its closure $\overline{Y} \subseteq \mathbb{R}^n$, while the composite $I \circ V$ carries an ideal $I \subseteq R$ to its radical $\sqrt{I} = \{f \in R : \exists\, n > 0 \text{ s.t. } f^n \in I\} \subseteq R$.

1.4. **Intersections via tensor products.** We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

**Example 1.2** (a transverse intersection)**.** Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x-1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \qquad \text{and} \qquad Z_1 = \{(x,y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \qquad \text{and} \qquad I_1 = I(Z_1) = (y - x) \subseteq R \ .$$

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned} \mathcal{O}(Z_0 \cap Z_1) &\cong R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y - x^2, y - x) \cong \mathbb{R}[x]/(x - x^2) \\ &= \mathbb{R}[x]/(x \cdot (1-x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1-x) \cong \mathbb{R} \times \mathbb{R} \ , \end{aligned}$$

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a PID, in fact it is a Euclidean domain).[10] The fact that this is a 2-dimensional $\mathbb{R}$-algebra corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements.

**Example 1.3** (a non-transverse intersection)**.** Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in $R$. These correspond to the curves $y = x^2$ and $y = 0$. These intersect "twice" at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y - x^2, y) \cong \mathbb{R}[x]/(x^2) \ .$$

The 2-dimensionality of this $\mathbb{R}$-algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect "with multiplicity two". Namely, this $\mathbb{R}$-algebra corresponds to "the origin along with infinitesimal fuzz in the direction of the $x$-axis". This is in contrast with the previous example, where the tensor product split as a cartesian product.

[10] An explicit inverse is given by carrying the pair $(a,b) \in \mathbb{R} \times \mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b-a) \cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x \cdot (1-x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0),g(1)}$ lies in the ideal generated by $x \cdot (x-1)$. Observe that $g - f_{g(0),g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

These techniques are quite robust.

**Exercise 1.4** (2 points)**.** Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

**Example 1.5** (another non-transverse intersection)**.** Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x-a) \subseteq R$ and $I_1 = (x-b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x-a, x-b) \cong \mathbb{R}/(a-b) \cong \begin{cases} \mathbb{R}\ , & a = b \\ 0\ , & a \neq b \end{cases} .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$-algebra.

Using homological algebra, namely the notion of ***derived tensor products***, we will be able to obtain the expected intersection number of 0 even when $a = b$.

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

**Exercise 1.6** (6 points)**.** Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

(a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines "just barely avoid intersecting": morally they should intersect "at infinity".[11] This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3 \backslash \{0\})/\mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.[12]

[11] A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \to 1^+$ their intersection point has $x \to -\infty$, while as $t \to 1^-$ their intersection point has $x \to +\infty$. This suggests that there should be a *single* point "at infinity" where they intersect in the case that $t = 1$.

[12] So, the "points at infinity" are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2 \backslash \{0\})/\mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1 \backslash \mathbb{R}^1$).

(b) Show that a *homogenous* polynomial $g \in \mathbb{R}[x,y,z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.

(c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogenous polynomials $g_1, g_2 \in \mathbb{R}[x,y,z]$ such that $g_i([x:y:1]) = f_i(x,y)$.

(d) Compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. Here is a more interesting non-generic situation where derived tensor products will give the correct answer where ordinary tensor products will not.

**Example 1.7.** Generically, two lines in $\mathbb{R}^2$ intersect in a point. This fails if the lines are parallel, but as we saw in Exercise 1.6 this failure is repaired by working in $\mathbb{RP}^2$ (and taking the closures of the lines therein). However, this still gives the wrong answer if the two lines are *equal*: of course, the intersection of a line with itself is itself.

Using derived tensor products, we will be able to obtain the expected intersection number of 1 when intersecting a (projective) line with itself in $\mathbb{RP}^2$.

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

## 2. Fundamentals of homological algebra

In this section we introduce the basic notions of homological algebra. A fine reference for this material is [Wei94, §§1-2], although we will take a rather different approach.

For concreteness, we fix a ring $R$ and work in $\mathsf{Mod}_R$, the category of (right) $R$-modules. However, as we will see later, essentially all of the theory works equally well for a general abelian category.[13]

### 2.1. Chain complexes, homology, and tensor products.

2.1.1. A ***chain complex*** of $R$-modules is a diagram

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of $R$-modules such that for all $n \in \mathbb{Z}$, the composite $d_n \circ d_{n+1} = 0$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that "all indices are being referred to at once". To emphasize the differentials, one may write $(M_\bullet, d_\bullet)$. Also, one may simply refer to a chain

[13]On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into $\mathsf{Mod}_R$ for some ring $R$ (although the choice of such a ring $R$ is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

complex as a "complex".[14] On the other hand, we also may omit the bullet and simply write $M := M_\bullet$ for simplicity. The integer $n$ is called the ***degree*** or the ***dimension***.

The morphisms $d_n$ are called the ***differentials*** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of $d_n$ is $M_n$). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as $d_n^M$.

In this notation, a morphism of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of $R$-modules such that the diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d^M_{n+2}} & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \xrightarrow[d^N_{n+2}]{} & N_{n+1} & \xrightarrow[d^N_{n+1}]{} & N_n & \xrightarrow[d^N_n]{} & N_{n-1} & \xrightarrow[d^N_{n-1}]{} & \cdots \end{array}$$

commutes.[15] [changed 1/7] These morphisms are often referred to as ***chain maps***. We write $\mathsf{Ch}_R$ for the category of chain complexes of $R$-modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline when appropriate.

Any $R$-module $M \in \mathsf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \underset{\sim}{M} \longrightarrow 0 \longrightarrow \cdots .$$

This construction determines a fully faithful embedding

$$\mathsf{Mod}_R \hookrightarrow \mathsf{Ch}_R .$$

As a result, we may not notationally distinguish between data in $\mathsf{Mod}_R$ and its image in $\mathsf{Ch}_R$.

When a complex only has a few nonzero terms, for brevity one may omit the zero terms. For instance, the above complex may also be written as $\underset{\sim}{M}$.

2.1.2. Fix a chain complex $M_\bullet$. Its $n$-***cycles*** and $n$-***boundaries*** are the submodules

$$\mathsf{Z}_n(M_\bullet) := \ker(d_n) \subseteq M_n \qquad \text{and} \qquad \mathsf{B}_n(M_\bullet) := \mathrm{im}(d_{n+1}) \subseteq M_n .$$[16]

Note that $\mathsf{B}_n(M_\bullet) \subseteq \mathsf{Z}_n(M_\bullet)$ because $d^2 = 0$. Then, the $n^{th}$ ***homology*** of $M_\bullet$ is the quotient $R$-module

$$\mathsf{H}_n(M_\bullet) := \frac{\mathsf{Z}_n(M_\bullet)}{\mathsf{B}_n(M_\bullet)} := \frac{\ker(d_n)}{\mathrm{im}(d_{n+1})} .$$

[14] The word "chain" here is historical: the first example of a chain complex has in degree $n$ the "$n^{\text{th}}$ chain group" of a simplicial complex $X$, i.e. the group of chains (i.e. formal linear combinations) of $n$-simplices of $X$. (It was only later realized that chain complexes are worth studying in their own right.)

[15] For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

[16] The German words for "cycle" and "boundary" respectively begin with the letters "Z" and "B".

[changed 1/7] Although $\mathsf{H}_n(M_\bullet)$ is an $R$-module, it is common to refer to it merely as a *homology group*.

We say that $M_\bullet$ is ***acyclic*** if $\mathsf{H}_n(M_\bullet) = 0$ for all $n$.

**Exercise 2.1** (2 points)**.** Verify that the constructions $\mathsf{Z}_n$, $\mathsf{B}_n$, and $\mathsf{H}_n$ define functors

$$\mathsf{Ch}_R \longrightarrow \mathsf{Mod}_R \ .$$

2.1.3. A morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in $\mathsf{Ch}_R$ is called a ***quasi-isomorphism*** if the induced morphisms $H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We may indicate that a morphism is a quasi-isomorphism by decorating the arrow as $\xrightarrow{\approx}$.

By and large, one should think of quasi-isomorphic chain complexes as "essentially interchangeable", with some representatives of a given quasi-isomorphism class (e.g. the *projective resolutions* introduced below) being "better adapted" for certain purposes than others.[17] In other words, one should think of quasi-isomorphisms as if they are actual isomorphisms.

[changed 1/7] It will turn out that quasi-isomorphisms do *not* define an equivalence relation in and of themselves: a quasi-isomorphism need not have a ***quasi-inverse*** (i.e. a chain map in the opposite direction that induces inverse maps on homology), so the relation they define is not symmetric. Nevertheless, we say that two complexes $M, N \in \mathsf{Ch}_R$ are ***quasi-isomorphic*** if they are related by the equivalence relation *generated* by the quasi-isomorphisms: that is, there exists a zig-zag

$$M \xrightarrow{\approx} L_1 \xleftarrow{\approx} L_2 \xrightarrow{\approx} \cdots \xleftarrow{\approx} L_k \xrightarrow{\approx} N$$

of quasi-isomorphisms connecting $M$ and $N$ for some $k \in \mathbb{N}$. Thankfully, it will also turn out that one can always reduce zig-zags to *spans*: $M$ and $N$ are quasi-isomorphic if and only if there exists a span

$$M \xleftarrow{\approx} L \xrightarrow{\approx} N$$

of quasi-isomorphisms.

While quasi-isomorphic complexes have isomorphic homology groups (essentially by definition), we will see that the converse is false: the obstruction will be encoded by ***k-invariants***.[18] That is, a quasi-isomorphism class of complex is equivalent data to its homology groups along with all of its k-invariants. For this reason, we will generally consider (quasi-isomorphism classes of) complexes themselves as the "true" mathematical objects of lasting interest, while their homology groups are merely algebraic invariants that can be extracted therefrom.

[17] This is very closely akin to how one should think of equivalent categories as "essentially interchangeable" (even if they are not isomorphic). However, in a precise sense, *all* categories are "equally well-adapted" for all purposes (in contrast with chain complexes).

[18] This same name is given to the (closely analogous) obstructions to spaces with the same homotopy groups being weak homotopy equivalent.

2.1.4. [changed 1/7] In this subsubsection we assume that $R$ is commutative, so that $\mathsf{Mod}_R$ is symmetric monoidal via the tensor product.

Given complexes $M, N \in \mathsf{Ch}_R$, we define their ***tensor product*** complex

$$(M \otimes N)_\bullet := (M \otimes_R N)_\bullet$$

as follows.[19] First of all, we define its $k^{\text{th}}$ term to be

$$(M \otimes N)_k := \bigoplus_{i+j=k} (M_i \otimes N_j) := \bigoplus_{i+j=k} (M_i \otimes_R N_j) \ .$$

Then, the differential is characterized by the fact that it carries a pure tensor

$$m \otimes n \in (M_i \otimes N_j) \subseteq (M \otimes N)_k$$

to the sum of pure tensors

$$d(m \otimes n) := d(m) \otimes n + (-1)^j \cdot m \otimes d(n) \ ,$$ [20]

(an element of $((M_{i-1} \otimes N_j) \oplus (M_i \otimes N_{j-1})) \subseteq (M \otimes N)_{k-1}$). More elaborately, this may be written as

$$d_k^{M \otimes N}(m \otimes n) := d_k^M(m) \otimes n + (-1)^j \cdot m \otimes d_k^N(n) \ .$$

**Exercise 2.2** (2 points)**.** Verify that this formula defines a complex.

From here, it is straightforward to see that the above construction defines a symmetric monoidal structure

$$\mathsf{Ch}_R \times \mathsf{Ch}_R \xrightarrow{\otimes_R} \mathsf{Ch}_R \ ,$$

with unit object $R := \underset{\sim}{R} \in \mathsf{Ch}_R$.

**Exercise 2.3** (8 points)**.** Fix two complexes $M, N \in \mathsf{Ch}_R$.

(a) Verify that the formula $[m] \otimes [n] \mapsto [m \otimes n]$ determines a morphism

$$H_i(M) \otimes_R H_j(N) \longrightarrow H_{i+j}(M \otimes_R N)$$

of $R$-modules.

[19] If we work with chain complexes in a general abelian category which is not assumed to be symmetric monoidal, then of course its category of chain complexes will not be symmetric monoidal either. Rather, it will admit an action of the symmetric monoidal category of bounded chain complexes of finitely-generated $\mathbb{Z}$-modules. (These finiteness hypotheses can be dropped assuming that the abelian category admits all colimits.)

[20] The factor $(-1)^j$ is determined by the *Koszul sign rule*, which is a general principle that in commuting graded quantities past one another of degrees $k, j \in \mathbb{Z}$ one should pick up a factor of $(-1)^{k \cdot j}$. Namely, we consider the symbol "$d$" as an expression of degree $-1$ (which makes sense since it changes dimensions by 1).

It follows that we obtain a morphism

$$\bigoplus_{i+j=k} (H_i(M) \otimes_R H_j(N)) \longrightarrow H_k(M \otimes_R N)$$

of $R$-modules.

(b) Prove that this is an isomorphism under the assumption that $R$ is a field.

(c) Find an example where this is not an isomorphism.

## 2.2. Homotopies, homotopy co/kernels, and exact sequences.

2.2.1. Let $M_1 \xrightarrow{f} M_0$ be a morphism of $R$-modules. This gives us a complex $M_\bullet := (M_1 \xrightarrow{f} \underset{\sim}{M_0})$. Observe that this has a canonical morphism

$$\begin{array}{c} M_\bullet \\ \downarrow \\ \underset{\sim}{\mathsf{coker}(f)} \end{array} \quad := \quad \begin{array}{ccc} M_1 & \xrightarrow{f} & \underset{\sim}{M_0} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \underset{\sim}{\mathsf{coker}(f)} \end{array}$$

to the cokernel of $f$ (considered as a complex in degree 0). Observe further that

$$\mathsf{H}_n(M_\bullet) \cong \begin{cases} \mathsf{coker}(f) \ , & n = 0 \\ \mathsf{ker}(f) \ , & n = 1 \\ 0 \ , & \text{otherwise} \end{cases} .$$

Hence, the above map is a quasi-isomorphism iff $f$ is an injection. One might think of $M_\bullet$ as a "presentation" of the underlying $R$-module $\mathsf{H}_0(M_\bullet) \cong \mathsf{coker}(f)$: the generators are $M_0$, the relations are $M_1$ (i.e. each $m \in M_1$ gives a relation $d(m) \sim 0$), but then $\mathsf{H}_1(M_\bullet)$ furthermore measures the "redundancy" of the relations. Said differently, $M_\bullet$ is a "homotopically correct" version of the cokernel of $f$, which remembers not only the literal cokernel but also the extent to which the relations are overdetermined. Indeed, it will be the *homotopy cokernel* of the morphism $f$.

2.2.2. Let $M_\bullet, N_\bullet \in \mathsf{Ch}_R$ be complexes and let $f_\bullet, g_\bullet \in \mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ be morphisms. A (***chain***) ***homotopy*** from $f_\bullet$ to $g_\bullet$ is a set of morphisms

$$M_n \xrightarrow{h_n} N_{n+1}$$

satisfying the condition that

$$g_n - f_n = d^N_{n+1} \circ h_n + h_{n-1} \circ d^M_n \ .$$

We may write this as $f_\bullet \overset{h_\bullet}{\Rrightarrow} g_\bullet$. A ***nullhomotopy*** of $g_\bullet$ is a homotopy $0 \overset{h_\bullet}{\Rrightarrow} g_\bullet$ from the zero map. A ***contraction*** of a complex is a nullhomotopy of its identity map. If a complex admits a contraction, we say that it is ***contractible***.

**Exercise 2.4** (3 points)**.** Show that the relation of homotopy on $\mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ is an equivalence relation.

In fact, the argument of Exercise 2.4 easily upgrades to imply that we can enhance $\mathsf{Ch}_R$ from an ordinary category to a category enriched in groupoids (a.k.a. a $(2,1)$-category):[21] its objects are chain complexes, its 1-morphisms are chain maps, and its 2-morphisms are chain homotopies.[22] Indeed, the arguments for transitivity, reflexivity, and symmetry of the relation of homotopy respectively endow these hom-categories with their composition laws, identity morphisms, and inverses (so that they are indeed hom-groupoids). It is moreover clear that homotopies may be composed appropriately, either by definition or using Exercise 2.5 below.

For present and future use, we introduce the complex $\mathbb{I} \in \mathsf{Ch}_R$ and the morphisms $i_0, i_1 \in \mathsf{hom}_{\mathsf{Ch}_R}(R, \mathbb{I})$ according to the diagram

$$\begin{array}{ccccc} R & & 0 & \longrightarrow & \underset{\sim}{R} \\ {\scriptstyle i_0}\downarrow & & \downarrow & & \downarrow{\scriptstyle (-\mathsf{id}_R,0)} \\ \mathbb{I} & := & R & \xrightarrow{(-\mathsf{id}_R,\mathsf{id}_R)} & \underset{\sim\sim\sim}{R \oplus R} \\ {\scriptstyle i_1}\uparrow & & \uparrow & & \uparrow{\scriptstyle (0,\mathsf{id}_R)} \\ R & & 0 & \longrightarrow & \underset{\sim}{R} \end{array} .$$

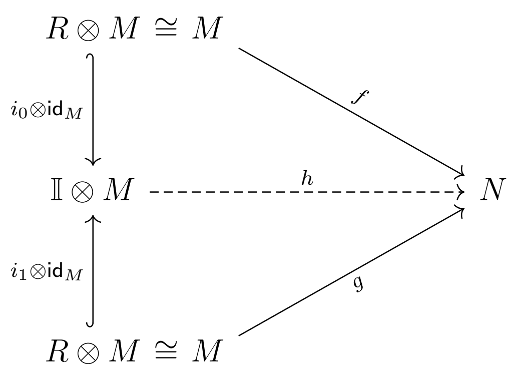
This object $\mathbb{I} \in \mathsf{Ch}_R$ (along with the two maps $i_0$ and $i_1$) is an ***interval object*** for the homotopy theory of chain complexes (which explains the notation).[23] We will see the general definition of an interval object later. In the present setting, this assertion amounts to the following result.

[21] As we will see, this is the homotopy $(2,1)$-category of a more fundamental object, namely an $\infty$-category (meaning an $(\infty,1)$-category).

[22] This explains the notation $f_\bullet \Rightarrow g_\bullet$ just introduced.

[23] This is a particularly natural choice of an interval object: as we will see, it is the simplicial chains on the 1-simplex $\Delta^1$ (whose underlying topological space is a closed interval).

**Exercise 2.5** (4 points)**.** Given morphisms $f, g \in \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$, prove that a homotopy $f \Rightarrow g$ is equivalent data to a morphism $\mathbb{I} \otimes M \to N$ that makes the diagram




commute.

A morphism $M \xrightarrow{f} N$ in $\mathsf{Ch}_R$ is a ***homotopy equivalence*** if there exists a morphism $N \xrightarrow{g} M$ and homotopies $\mathsf{id}_M \Rightarrow g \circ f$ and $f \circ g \Rightarrow \mathsf{id}_N$.[24] We may indicate that a morphism is a homotopy equivalence by decorating it as $\xrightarrow{\sim}$.

As a special case, a complex $M_\bullet$ is contractible if and only if either unique map $0 \to M_\bullet$ or $M_\bullet \to 0$ is a homotopy equivalence.

**Exercise 2.6** (3 points)**.** Show that homotopic maps on complexes give equal maps on homology.

It follows immediately from Exercise 2.6 that homotopy equivalences are quasi-isomorphisms, and in particular that contractible complexes are acyclic. However, the converse is false.

**Exercise 2.7** (4 points)**.** Show that the complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots$$

of $\mathbb{Z}$-modules is acyclic but not contractible.

We will generally consider homotopic maps as "essentially interchangeable". On the other hand, rather than merely positing the *existence* of a homotopy between two maps, we will always want to *keep track* of the homotopy that witnesses them as being homotopic.

2.2.3. Recall that the cokernel of a morphism $M \xrightarrow{f} N$ in $\mathsf{Mod}_R$ is by definition an $R$-module $\mathsf{coker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$N \xrightarrow{u} \mathsf{coker}(f)$$

[24]The directions of these homotopies are intended to be suggestive of adjunctions (with $f$ functioning as the left adjoint), but by Exercise 2.4 they are irrelevant. (Likewise, equivalences of categories are both left adjoints and right adjoints.)

satisfying the universal property that precomposition with $u$ determines a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(\mathsf{coker}(f), Z) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N \to Z \text{ such that the} \\ \text{composite } M \xrightarrow{f} N \to Z \text{ is zero} \end{array} \right\} .$$ [25]

From here, the above principles lead directly to the definition of a ***homotopy cokernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in $\mathsf{Ch}_R$:[26] this is an object $\mathsf{hcoker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$N_\bullet \xrightarrow{u_\bullet} \mathsf{hcoker}(f_\bullet)$$

satisfying the universal property that precomposition with $u_\bullet$ determines a bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f_\bullet), Z_\bullet) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N_\bullet \to Z_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } M_\bullet \xrightarrow{f_\bullet} N_\bullet \to Z_\bullet \end{array} \right\} .$$ [27]

[25]Said differently, the cokernel of $f$ is the pushout

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{coker}(f) \end{array} .$$

[26]This is more often referred to as the *cone*, but we prefer the term "homotopy cokernel" as it is more directly suggestive of the object's "role in life" (a.k.a. its *raison d'être*). It is also common to refer to this as the *homotopy cofiber*, but this deviates from the more standard terminology of "cokernel" (as opposed to "cofiber") in the context of an abelian category.

[27]Similarly, the homotopy cokernel of $f$ may be characterized as the *homotopy* pushout

$$\begin{array}{ccc} M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \\ \downarrow & \Leftrightarrow & \downarrow \\ 0 & \longrightarrow & \mathsf{hcoker}(f_\bullet) \end{array} ,$$

i.e. the *initial homotopy-coherent cocone* over the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$. Here, the symbol $\Leftrightarrow$ indicates that the square only commutes up to a (specified) homotopy.

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