HOMOLOGICAL ALGEBRA

AARON MAZEL-GEE

Abstract. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the winter 2021 quarter. They are **under construction**, and will be updated at the course website at the end of each lecture.

Contents

0. Miscellanea

0.1. **Exercises.**

0.1.1. In order to obtain a grade of 100% on the homework by the end of the quarter, you will need to earn 120 points.

0.1.2. Partial solutions may be submitted for partial credit.

0.1.3. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a "yes or no" question).

---

last updated: January 21, 2021

0.1.4. In preparing your homework, please copy down the problem statement, since it is possible that the numbering may change.

0.2. **Conventions.**

0.2.1. As this document is a work in progress, I will frequently want to make changes to existing material (in addition to adding material as we cover it during lecture). Of course, it would be quite difficult for a reader to spot such changes, especially as the document grows. Therefore, any substantial such changes will be temporarily flagged by their change date (for easy searching) and additionally will be colored as follows: changes that have occurred since the most recent lecture are **red**, changes made between the past two lectures are **blue**, and changes made between two and three lectures ago are **green**. Older changes are no longer colored.

0.2.2. We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and Set denotes the category of sets. However, notation will very often only be "local": the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.3. We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We mostly ignore set-theoretic issues.[1]

0.2.4. The term "(commutative) ring" means "associative unital (resp. commutative) ring". Likewise, modules are always unital (meaning that the unit element acts as the identity).

0.2.5. The term "natural number" (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.6. In the interest of brevity, universal quantifiers will often be dropped. For instance, an assertion involving an integer n should generally be understood to refer to *all* integers n unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

1. SOME MOTIVATION FOR HOMOLOGICAL ALGEBRA

1.1. **Intersection theory.** A basic endeavor in geometry is to understand *intersections*. For example, given a (smooth) manifold M and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} .$$

---

[1] Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

1.1.1. If N_0 and N_1 intersect *transversely* (i.e. $T_pN_0 + T_pN_1 = T_pM$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

1.1.2. However, if the intersection of N_0 and N_1 is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either N_0 or N_1 that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.[2] However, this approach has a number of (related) drawbacks.

(1) Perturbations are noncanonical.
(2) Perturbations will generally destroy the inherent symmetries of the situation.[3]
(3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally only transcendental.[4]

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. **Tensor products.** We first recall the notion of tensor product.

1.2.1. Let R be a commutative ring, and let M and N be R-modules. The (***relative***) ***tensor product*** of M and N over R,[5] denoted $M \otimes_R N$, is the universal abelian group equipped with an R-balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N \ ,$$

i.e. a function satisfying the following axioms:

(1) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$ and $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$;
(2) $\varphi(m \cdot r, n) = \varphi(m, r \cdot n)$.[6]

In other words, for any abelian group A, precomposition with φ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \to A\} \xleftarrow{\cong} \{\text{abelian group homomorphisms } M \otimes_R N \to A\} \ .$$

---

[2] A good introduction to these ideas is [GP74].

[3] For instance, perturbations to transverse intersections need not exist in the equivariant context.

[4] It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, where "perturb" means "change to a new but rationally equivalent algebraic cycle". It is fundamental in the classical approach to intersection theory in algebraic geometry [EH16].

[5] The word "relative" here is meant to emphasize that R is an arbitrary commutative ring. By contrast, the term "absolute tensor product" would emphasize that $R = \mathbb{Z}$.

[6] The notation here stems from the fact that more generally, we can define the relative tensor product when R is merely an associative ring, M is a right R-module, and N is a left R-module.

In the case that R is understood (and particularly when $R = \mathbb{Z}$ or when R is a field), we may simply write $\otimes := \otimes_R$.

1.2.2. The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

1. $(m + m', n) \sim (m, n) + (m', n)$ and $(m, n + n') \sim (m, n) + (m, n')$;
2. $(m \cdot r, n) \sim (m, r \cdot n)$.

Exercise 1.1 (2 points)**.** For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\mathsf{gcd}(m, n)$.

1.3. **Basic principles of algebraic geometry.** In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §1.1].

1.3.1. The polynomial functions on $\mathbb{R}^n$ are the n-variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving n implicit).

1.3.2. By definition, an ***algebraic subset*** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.[7] In this case we write $Z = V(S)$, and we say that Z is the ***vanishing locus*** of the elements of S. If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.[8]

1.3.3. We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along Z. Then, the ring of polynomial functions on Z is

$$\mathcal{O}(Z) = R/I(Z) \ .$$

1.3.4. Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in I\} \subseteq \mathbb{R}^n \ .$$

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{ideals in } R\} \quad .^{9}$$

---

[7] By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

[8] Since R is noetherian, any ideal is finitely generated. In other words, we may always take S to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

[9] Over an algebraically closed field $\Bbbk$, this construction restricts to a bijection between *closed* subsets of $\Bbbk^n$ (with respect to the Zariski topology) and *radical* ideals of $\Bbbk[x_1, \ldots, x_n]$. The composite $V \circ I$ carries a subset $Y \subseteq \Bbbk^n$ to its closure $\overline{Y} \subseteq \Bbbk^n$, while the composite $I \circ V$ carries an ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ to its radical $\sqrt{I} = \{f \in \Bbbk[x_1, \ldots, x_n] : \exists\ n > 0 \text{ s.t. } f^n \in I\} \subseteq R$. By contrast, over $\mathbb{R}$ the function V fails to be injective, e.g. $V(\mathbb{R}[x]) = V(x^2 + 1) = \varnothing$.

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 \ .$$

1.4. **Intersections via tensor products.** We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

Example 1.2 (a transverse intersection)**.** Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x-1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \qquad \text{and} \qquad Z_1 = \{(x,y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \qquad \text{and} \qquad I_1 = I(Z_1) = (y - x) \subseteq R \ .$$

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned} \mathcal{O}(Z_0 \cap Z_1) &\cong R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y - x^2, y - x) \cong \mathbb{R}[x]/(x - x^2) \\ &= \mathbb{R}[x]/(x \cdot (1-x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1-x) \cong \mathbb{R} \times \mathbb{R} \ , \end{aligned}$$

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a PID, in fact it is a Euclidean domain).[10] The fact that this is a 2-dimensional $\mathbb{R}$-algebra corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements.

Example 1.3 (a non-transverse intersection)**.** Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in R. These correspond to the curves $y = x^2$ and $y = 0$. These intersect "twice" at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y - x^2, y) \cong \mathbb{R}[x]/(x^2) \ .$$

---

[10] An explicit inverse is given by carrying the pair $(a,b) \in \mathbb{R} \times \mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b-a) \cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x \cdot (1-x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0),g(1)}$ lies in the ideal generated by $x \cdot (x-1)$. Observe that $g - f_{g(0),g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

The 2-dimensionality of this $\mathbb{R}$-algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect "with multiplicity two". Namely, this $\mathbb{R}$-algebra corresponds to "the origin along with infinitesimal fuzz in the direction of the x-axis". This is in contrast with the previous example, where the tensor product split as a cartesian product.

These techniques are quite robust.

Exercise 1.4 (2 points)**.** Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

Example 1.5 (another non-transverse intersection)**.** Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x-a) \subseteq R$ and $I_1 = (x-b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x-a, x-b) \cong \mathbb{R}/(a-b) \cong \begin{cases} \mathbb{R} \ , & a = b \\ 0 \ , & a \neq b \end{cases} .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$-algebra.

Using homological algebra, namely the notion of ***derived tensor products***, we will be able to obtain the expected intersection number of 0 even when $a = b$.

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

Exercise 1.6 (6 points)**.** Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

(a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines "just barely avoid intersecting": morally they should intersect "at infinity".[11] This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3 \backslash \{0\})/\mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have

---

[11] A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \to 1^+$ their intersection point has $x \to -\infty$, while as $t \to 1^-$ their intersection point has $x \to +\infty$. This suggests that there should be a *single* point "at infinity" where they intersect in the case that $t = 1$.

$[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.[12]

(b) Show that a *homogenous* polynomial $g \in \mathbb{R}[x, y, z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.

(c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogenous polynomials $g_1, g_2 \in \mathbb{R}[x, y, z]$ such that $g_i([x : y : 1]) = f_i(x, y)$.

(d) Compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. Here is a more interesting non-generic situation where derived tensor products will give the correct answer where ordinary tensor products will not.

Example 1.7. Generically, two lines in $\mathbb{R}^2$ intersect in a point. This fails if the lines are parallel, but as we saw in Exercise 1.6 this failure is repaired by working in $\mathbb{RP}^2$ (and taking the closures of the lines therein). However, this still gives the wrong answer if the two lines are *equal*: of course, the intersection of a line with itself is itself.

Using derived tensor products, we will be able to obtain the expected intersection number of 1 when intersecting a (projective) line with itself in $\mathbb{RP}^2$.

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

2. Fundamentals of homological algebra

In this section we introduce the basic notions of homological algebra. A standard reference for this material is [Wei94, §§1-2]. However, we will take a rather different approach that emphasizes homotopy-coherence, following [RP, §§9-13].

[changed 1/14] For concreteness, we work in the context of ordinary algebra. Namely, we fix a commutative ring $\Bbbk$ and a $\Bbbk$-algebra R. For the most part, we will work in Mod_R, the category of (right) R-modules, and one may take $\Bbbk$ to be $\mathbb{Z}$. However, at times we will want to specialize to a commutative ring, and for this it is convenient for $\Bbbk$ to be arbitrary.[13] Moreover, we will study some interactions between $\Bbbk$-modules and R-modules. At the level of ordinary (i.e. non-homological) algebra, these are encapsulated by the following facts.

(1) The category $\mathsf{Mod}_\Bbbk$ is symmetric monoidal via the tensor product, which we denote by $\otimes := \otimes_\Bbbk$; its unit object is $\Bbbk$.

---

[12] So, the "points at infinity" are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2 \backslash \{0\})/\mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1 \backslash \mathbb{R}^1$).

[13] Of course, we will apply results developed for R-modules to $\Bbbk$-modules without comment.

(2) The category Mod_R is naturally enriched in $\mathsf{Mod}_{\Bbbk}$. In other words, for any R-modules $M, N \in \mathsf{Mod}_R$, the set $\mathsf{hom}_{\mathsf{Mod}_R}(M, N)$ of R-linear homomorphisms carries the natural structure of a $\Bbbk$-module, and moreover composition in Mod_R is $\Bbbk$-multilinear.

(3) Moreover, $\Bbbk$-modules naturally act on R-modules in two different ways: for any $\Bbbk$-module $T \in \mathsf{Mod}_{\Bbbk}$ and any R-modules $M, N \in \mathsf{Mod}_R$ we have R-modules

$$T \otimes_{\Bbbk} M \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Mod}_{\Bbbk}}(T, N) \ ,$$

where the (right) R-actions are induced from those on M and N, and these constructions participate in natural isomorphisms

$$\mathsf{hom}_{\mathsf{Mod}_{\Bbbk}}(T, \mathsf{hom}_{\mathsf{Mod}_R}(M, N) \cong \mathsf{hom}_{\mathsf{Mod}_R}(T \otimes_{\Bbbk} M, N) \cong \mathsf{hom}_{\mathsf{Mod}_R}(M, \mathsf{hom}_{\mathsf{Mod}_{\Bbbk}}(T, N)) \ .$$

Of course, one may take $R = \Bbbk$ as a special case.[14] As a result, the notions that we will develop relating to the interactions between $\Bbbk$-modules and R-modules will all be *generalizations* of the notions that we develop relating to $\Bbbk$-modules alone.

As we will see later, most of the theory works equally well for a general abelian category, although there will be some additional hiccups that do not arise when studying modules.[15]

2.1. **Chain complexes, homology, and tensor products.**

2.1.1. A ***chain complex*** of R-modules is a diagram

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of R-modules such that for all $n \in \mathbb{Z}$, the composite $d_n \circ d_{n+1} = 0$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that "all indices are being referred to at once". To emphasize the differentials, one may write $(M_\bullet, d_\bullet)$. Also, one may simply refer to a chain complex as a "complex".[16] On the other hand, we also may omit the bullet and simply write $M := M_\bullet$ for simplicity. The integer n is called the ***degree*** or the ***dimension***. For an element $m \in M_n$, we may write $\deg(m) := n$.

The morphisms d_n are called the ***differentials*** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of d_n is M_n). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as d_n^M.

---

[14] The above facts then reduce to the assertion that $\mathsf{Mod}_{\Bbbk}$ is a *closed* symmetric monoidal category, i.e. that it carries a self-enrichment that is compatible with its symmetric monoidal structure.

[15] On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into Mod_R for some ring R (although the choice of such a ring R is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

[16] The word "chain" here is historical: the first example of a chain complex has in degree n the "n^{th} chain group" of a simplicial complex X, i.e. the group of chains (i.e. formal linear combinations) of n-simplices of X. (It was only later realized that chain complexes are worth studying in their own right.)

In this notation, a morphism of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of R-modules such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d^M_{n+2}"] & M_{n+1} \arrow[r, "d^M_{n+1}"] \arrow[d, "f_{n+1}"] & M_n \arrow[r, "d^M_n"] \arrow[d, "f_n"] & M_{n-1} \arrow[r, "d^M_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^N_{n+2}"'] & N_{n+1} \arrow[r, "d^N_{n+1}"'] & N_n \arrow[r, "d^N_n"'] & N_{n-1} \arrow[r, "d^N_{n-1}"'] & \cdots
\end{tikzcd}$$

commutes.[17] These morphisms are often referred to as ***chain maps***. We write Ch_R for the category of chain complexes of R-modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline when appropriate.

Any R-module $M \in \mathsf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \utilde{M} \longrightarrow 0 \longrightarrow \cdots \ .$$

This construction determines a fully faithful embedding

$$\mathsf{Mod}_R \hookrightarrow \mathsf{Ch}_R \ .$$

As a result, we may not notationally distinguish between data in Mod_R and its image in Ch_R.

When a complex only has a few nonzero terms, for brevity one may omit the zero terms. For instance, the above complex may also be written as $\utilde{M}$.

2.1.2. Fix a chain complex $M_\bullet$. Its n-***cycles*** and n-***boundaries*** are the submodules

$$\mathsf{Z}_n(M_\bullet) := \mathsf{ker}(d_n) \subseteq M_n \qquad \text{and} \qquad \mathsf{B}_n(M_\bullet) := \mathsf{im}(d_{n+1}) \subseteq M_n \ .^{18}$$

Note that $\mathsf{B}_n(M_\bullet) \subseteq \mathsf{Z}_n(M_\bullet)$ because $d^2 = 0$. Then, the n^{th} ***homology*** of $M_\bullet$ is the quotient R-module

$$\mathsf{H}_n(M_\bullet) := \frac{\mathsf{Z}_n(M_\bullet)}{\mathsf{B}_n(M_\bullet)} := \frac{\mathsf{ker}(d_n)}{\mathsf{im}(d_{n+1})} \ .$$

Although $\mathsf{H}_n(M_\bullet)$ is an R-module, it is common to refer to it merely as a *homology group*.

We say that $M_\bullet$ is ***acyclic*** if $\mathsf{H}_n(M_\bullet) = 0$ for all n.

Exercise 2.1 (2 points)**.** Verify that the constructions Z_n, B_n, and H_n define functors

$$\mathsf{Ch}_R \longrightarrow \mathsf{Mod}_R \ .$$

---

[17] For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

[18] The German words for "cycle" and "boundary" respectively begin with the letters "Z" and "B".

2.1.3. A morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R is called a ***quasi-isomorphism*** if the induced morphisms $H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We may indicate that a morphism is a quasi-isomorphism by decorating the arrow as $\xrightarrow{\approx}$.

By and large, we would like to think of quasi-isomorphic chain complexes as "essentially interchangeable", with some representatives of a given quasi-isomorphism class (namely the *projective* and *injective* complexes introduced below) being "well-adapted" for certain purposes.[19] In other words, one should think of quasi-isomorphisms as if they are actual isomorphisms.

[changed 1/14] This can be made literally true by ***localizing*** the category Ch_R at the quasi-isomorphisms, i.e. by adjoining formal inverses for them. This yields a category that (for reasons that will become clear later) we will denote by $\mathsf{H}_0(\mathbf{D}_R)$ and refer to as ***the derived category of R-modules***; its objects are called ***derived R-modules***.[20] So by definition, there is a canonical functor

$$\mathsf{Ch}_R \longrightarrow \mathsf{H}_0(\mathbf{D}_R)$$

that carries all quasi-isomorphisms to isomorphisms, and moreover it is universal with respect to this requirement. Indeed, for any category $\mathcal{C}$, the restriction functor

$$\mathsf{Fun}(\mathsf{Ch}_R, \mathcal{C}) \longleftarrow \mathsf{Fun}(\mathsf{H}_0(\mathbf{D}_R), \mathcal{C})$$

is a fully faithful inclusion, whose image consists of those functors $\mathsf{Ch}_R \to \mathcal{C}$ that carry quasi-isomorphisms to isomorphisms.

Note that a derived R-module is a "purely homotopical" object: while it can by definition be presented by a chain complex of R-modules, one cannot speak e.g. of its underlying R-module in dimension 0, as this notion is not preserved under quasi-isomorphisms.[21] On the other hand, one *can* speak e.g. of its n^{th} homology, as this notion is by definition preserved under quasi-isomorphisms.

Essentially by construction, given two complexes $M, N \in \mathsf{Ch}_R$, morphisms from M to N in the derived category are given by equivalence classes of zigzags

$$M \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \cdots \longrightarrow \bullet \xleftarrow{\approx} N$$

(in which all backwards maps are quasi-isomorphisms). Thankfully, it will turn out that every equivalence class contains representatives of the forms

$$M \xleftarrow{\approx} \bullet \longrightarrow N \qquad \text{and} \qquad M \longrightarrow \bullet \xleftarrow{\approx} N \ ,$$

---

[19] This is very closely akin to how one should think of equivalent categories as "essentially interchangeable", even when they are not isomorphic. However, in a precise sense, *all* categories are "equally well-adapted" for all purposes (in contrast with chain complexes).

[20] The placement of the word "derived" is admittedly slightly unfortunate, but this terminology is quite common.

[21] Likewise, one cannot speak of the underlying set of a weak homotopy equivalence class of topological spaces, nor can one speak of the set of objects of an equivalence class of categories.

which makes the situation substantially more manageable.[22]

Although we introduce the derived category now, we will not have much use for it: it contains too little information. The richer and more primitive object is $\mathbf{D}_R$, the ***derived ∞-category*** of R. This is a mathematical entity whose objects are still the derived R-modules, but whose hom-objects are more elaborate: namely, they are *derived* $\Bbbk$*-modules*. Of course, passing from $\mathbf{D}_R$ to $\mathsf{H}_0(\mathbf{D}_R)$ amounts to entracting only the 0^{th} homology groups of these hom-objects.

While quasi-isomorphic complexes have isomorphic homology groups, we will see that the converse is generally false: the obstruction will be encoded by ***k-invariants***.[23] That is, a quasi-isomorphism class of complexes is equivalent data to its homology groups along with all of its k-invariants. For this reason, we will generally consider (quasi-isomorphism classes of) complexes themselves as the "true" mathematical objects of lasting interest, while their homology groups are merely algebraic invariants that can be extracted therefrom.

2.1.4. Given complexes $M, N \in \mathsf{Ch}_\Bbbk$ of $\Bbbk$-modules, we define their ***tensor product*** complex

$$(M \otimes N)_\bullet := (M \otimes_\Bbbk N)_\bullet$$

as follows. First of all, we define its k^{th} term to be

$$(M \otimes N)_k := \bigoplus_{i+j=k} (M_i \otimes N_j) := \bigoplus_{i+j=k} (M_i \otimes_\Bbbk N_j) \ .$$

Then, the differential is characterized by the fact that it carries a pure tensor

$$m \otimes n \in (M_i \otimes N_j) \subseteq (M \otimes N)_k$$

to the sum of pure tensors

$$d(m \otimes n) := d(m) \otimes n + (-1)^j \cdot m \otimes d(n) \ ,$$[24]

(an element of $((M_{i-1} \otimes N_j) \oplus (M_i \otimes N_{j-1})) \subseteq (M \otimes N)_{k-1}$). More elaborately, this may be written as

$$d_k^{M\otimes N}(m \otimes n) := d_i^M(m) \otimes n + (-1)^j \cdot m \otimes d_j^N(n) \ .$$

Exercise 2.2 (2 points)**.** Verify that this formula defines a complex.

---

[22] These reductions are guaranteed by the existence of two different ***model structures*** on Ch_R, which respectively have the features that all objects are fibrant and that all objects are cofibrant.

[23] This same name is given to the (closely analogous) obstructions to spaces with the same homotopy groups being weak homotopy equivalent.

[24] The factor $(-1)^j$ is determined by the *Koszul sign rule*, which is a general principle asserting that in commuting graded quantities past one another of degrees $k, j \in \mathbb{Z}$ one should pick up a factor of $(-1)^{k \cdot j}$. Namely, we consider the symbol "d" as an expression of degree -1 (which makes sense since it changes dimensions by 1).

In particular, in solving Exercise 2.2 you will see why the signs are necessary in the definition of the tensor product of complexes. In fact, many sign conventions are possible (and all give equivalent symmetric monoidal categories), but it is impossible to remove all signs from the theory (unless one works over $\mathbb{F}_2$).

From here, it is straightforward to see that the above construction defines a monoidal structure

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_{\Bbbk} \xrightarrow{\otimes} \mathsf{Ch}_{\Bbbk} ,$$

with unit object $\Bbbk := \underset{\sim}{\Bbbk} \in \mathsf{Ch}_{\Bbbk}$. In fact, this is a *symmetric* monoidal structure, with symmetry isomorphisms

$$M \otimes N \xrightarrow{\cong} N \otimes M$$

determined by the formula

$$m \otimes n \longmapsto (-1)^{\deg(m)\cdot\deg(n)} \cdot n \otimes m \ .$$[25]

[changed 1/14] More generally, this same construction defines an action

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_{R} \xrightarrow{\otimes} \mathsf{Ch}_{R}$$

of the symmetric monoidal category $(\mathsf{Ch}_{\Bbbk}, \otimes, \Bbbk)$ on the category Ch_R.

Exercise 2.3 (8 points)**.** Fix two complexes $M, N \in \mathsf{Ch}_{\Bbbk}$.

(a) Verify that the formula $[m] \otimes [n] \mapsto [m \otimes n]$ determines a morphism

$$H_i(M) \otimes H_j(N) \longrightarrow H_{i+j}(M \otimes N)$$

of $\Bbbk$-modules.

It follows that we obtain a morphism

$$\bigoplus_{i+j=k} (H_i(M) \otimes H_j(N)) \longrightarrow H_k(M \otimes N)$$

of $\Bbbk$-modules.

(b) Prove that this is an isomorphism under the assumption that $\Bbbk$ is a field.
(c) Find an example where this is not an isomorphism.

2.2. **Homotopies, homotopy co/kernels, and exact sequences.**

---

[25] This formula is another instance of the Koszul sign rule.

2.2.1. Let $M_1 \xrightarrow{f} M_0$ be a morphism of R-modules. This gives us a complex $M_\bullet := (M_1 \xrightarrow{f} \utilde{M_0})$. Observe that this has a canonical morphism

$$\begin{tikzcd}
M_\bullet \arrow[d] & & M_1 \arrow[r, "f"] \arrow[d] & \utilde{M_0} \arrow[d] \\
\mathsf{coker}(f) & := & 0 \arrow[r] & \utilde{\mathsf{coker}(f)}
\end{tikzcd}$$

to the cokernel of f (considered as a complex in degree 0). Observe further that

$$\mathsf{H}_n(M_\bullet) \cong \begin{cases} \mathsf{coker}(f)\ , & n = 0 \\ \mathsf{ker}(f)\ , & n = 1 \\ 0\ , & \text{otherwise} \end{cases} .$$

Hence, the above map is a quasi-isomorphism iff f is an injection. One might think of $M_\bullet$ as a "presentation" of the underlying R-module $\mathsf{H}_0(M_\bullet) \cong \mathsf{coker}(f)$: the generators are M_0, the relations are M_1 (i.e. each $m \in M_1$ gives a relation $d(m) \sim 0$), but then $\mathsf{H}_1(M_\bullet)$ furthermore measures the "redundancy" of the relations. Said differently, $M_\bullet$ is a "homotopically correct" version of the cokernel of f, which remembers not only the literal cokernel but also the extent to which the relations are overdetermined. Indeed, it will be the *homotopy cokernel* of the morphism f.

2.2.2. Let $M_\bullet, N_\bullet \in \mathsf{Ch}_R$ be complexes and let $f_\bullet, g_\bullet \in \mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ be morphisms. A (***chain***) ***homotopy*** from $f_\bullet$ to $g_\bullet$ is a set of morphisms

$$M_n \xrightarrow{h_n} N_{n+1}$$

satisfying the condition that

$$g_n - f_n = d^N_{n+1} \circ h_n + h_{n-1} \circ d^M_n \ .$$

We may write this as $f_\bullet \overset{h_\bullet}{\Rightarrow} g_\bullet$. A ***nullhomotopy*** of $g_\bullet$ is a homotopy $0 \overset{h_\bullet}{\Rightarrow} g_\bullet$ from the zero map. A ***contraction*** of a complex is a nullhomotopy of its identity map. If a complex admits a contraction, we say that it is ***contractible***.

Exercise 2.4 (3 points)**.** Show that the relation of homotopy on $\mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ is an equivalence relation.

In fact, the argument of Exercise 2.4 easily upgrades to imply that we can enhance Ch_R from an ordinary category to a category enriched in groupoids (a.k.a. a $(2,1)$-category):[26] its objects are chain complexes, its 1-morphisms are chain maps, and its 2-morphisms are chain homotopies.[27] Indeed, the arguments for transitivity, reflexivity, and symmetry of

---

[26] As we will see, this is the homotopy $(2,1)$-category of a more fundamental object, namely an ∞-category (meaning an $(\infty,1)$-category).

[27] This explains the notation $f_\bullet \Rightarrow g_\bullet$ just introduced.

the relation of homotopy respectively endow these hom-categories with their composition laws, identity morphisms, and inverses (so that they are indeed hom-groupoids). It is moreover clear that homotopies may be composed appropriately, either by definition or using Exercise 2.5 below.

For present and future use, we introduce the complex $\mathbb{I} \in \mathsf{Ch}_{\Bbbk}$ and the morphisms $i_0, i_1 \in \mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(\Bbbk, \mathbb{I})$ according to the diagram

$$\begin{tikzcd}
\Bbbk \arrow[d, hook, "i_0"'] & & 0 \arrow[r] \arrow[d] & \utilde{\Bbbk} \arrow[d, "{(-\mathsf{id}_{\Bbbk},0)}"] \\
\mathbb{I} & := & \Bbbk \arrow[r, "{(-\mathsf{id}_{\Bbbk},\mathsf{id}_{\Bbbk})}"] & \utilde{\Bbbk \oplus \Bbbk} & . \\
\Bbbk \arrow[u, hook, "i_1"] & & 0 \arrow[r] \arrow[u] & \utilde{\Bbbk} \arrow[u, "{(0,\mathsf{id}_{\Bbbk})}"']
\end{tikzcd}$$

This object $\mathbb{I} \in \mathsf{Ch}_{\Bbbk}$ (along with the two maps i_0 and i_1) is an ***interval object*** for the homotopy theory of chain complexes (which explains the notation).[28] We will see the general definition of an interval object later. In the present setting, this assertion amounts to the following result.

Exercise 2.5 (4 points)**.** Given morphisms $f, g \in \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$, prove that a homotopy $f \Rightarrow g$ is equivalent data to a morphism $\mathbb{I} \otimes M \to N$ that makes the diagram

$$\begin{tikzcd}
\Bbbk \otimes M \cong M \arrow[d, hook, "i_0 \otimes \mathsf{id}_M"'] \arrow[drr, "f"] & & \\
\mathbb{I} \otimes M \arrow[rr, dashed, "h"] & & N \\
\Bbbk \otimes M \cong M \arrow[u, hook, "i_1 \otimes \mathsf{id}_M"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

commute.

---

[28] This is a particularly natural choice of an interval object: as we will see, it is the simplicial chains (with coefficients in $\Bbbk$) on the 1-simplex Δ^1 (whose underlying topological space is a closed interval), and the two maps i_0 and i_1 are the simplicial chains on the inclusions of its two 0-simplices (i.e. the endpoints of the closed interval).

A morphism $M \xrightarrow{f} N$ in Ch_R is a ***homotopy equivalence*** if there exists a morphism $N \xrightarrow{g} M$ and homotopies $\mathsf{id}_M \Rightarrow g \circ f$ and $f \circ g \Rightarrow \mathsf{id}_N$.[29] We may indicate that a morphism is a homotopy equivalence by decorating it as $\xrightarrow{\sim}$.

As a special case, a complex $M_\bullet$ is contractible if and only if either unique map $0 \to M_\bullet$ or $M_\bullet \to 0$ is a homotopy equivalence.

Exercise 2.6 (3 points)**.** Show that homotopic maps on complexes give equal maps on homology.

It follows immediately from Exercise 2.6 that homotopy equivalences are quasi-isomorphisms, and in particular that contractible complexes are acyclic. However, the converse is false.

Exercise 2.7 (10 points)**.**

(a) Show that the complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots$$

of $\mathbb{Z}/4$-modules is acyclic but not contractible.

(b) Show that the complex

$$\cdots \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \cdots$$

of $\Bbbk[x]/x^2$-modules is acyclic. Show that it is contractible as a complex of $\Bbbk$-modules, but not as a complex of $\Bbbk[x]/x^2$-modules (assuming that $\Bbbk \neq 0$).

We will generally consider homotopic maps as "essentially interchangeable". On the other hand, rather than merely positing the *existence* of a homotopy between two maps, we will always want to *keep track* of the homotopy that witnesses them as being homotopic.

2.2.3. Recall that the cokernel of a morphism $M \xrightarrow{f} N$ in Mod_R is by definition an R-module $\mathsf{coker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$N \xrightarrow{u} \mathsf{coker}(f)$$

satisfying the universal property that precomposition with u determines a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(\mathsf{coker}(f), T) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N \to T \text{ such that the} \\ \text{composite } M \xrightarrow{f} N \to T \text{ is zero} \end{array} \right\}.$$[30]

---

[29] The directions of these homotopies are intended to be suggestive of adjunctions (with f functioning as the left adjoint), but by Exercise 2.4 they are irrelevant. (Likewise, equivalences of categories are both left adjoints and right adjoints.)

[30] Said differently, the cokernel of f is the pushout

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ 0 \arrow[r] & \mathsf{coker}(f) \end{tikzcd}.$$

From here, the above principles lead directly to the definition of a ***homotopy cokernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R:[31] this is an object $\mathsf{hcoker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$N_\bullet \xrightarrow{u_\bullet} \mathsf{hcoker}(f_\bullet)$$

satisfying the universal property that precomposition with $u_\bullet$ determines a bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f_\bullet), T_\bullet) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N_\bullet \to T_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } M_\bullet \xrightarrow{f_\bullet} N_\bullet \to T_\bullet \end{array} \right\} .$$ [32]

In fact, we claim that we have already seen an example of a homotopy cokernel: namely, for any morphism $M \xrightarrow{f} N$ in Mod_R, the complex $(M \xrightarrow{f} \utilde{N}) \in \mathsf{Ch}_R$ equipped with the map

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \utilde{N} \arrow[d, "\mathsf{id}_N"] \\
M \arrow[r, "f"'] & \utilde{N}
\end{tikzcd}$$

is a homotopy cokernel of f (when considered in Ch_R). More generally, given a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R, consider the diagram

$$\begin{tikzcd}[column sep=large]
\cdots \arrow[r, "\begin{pmatrix} -d^M_1 & 0 \\ f_1 & d^N_2 \end{pmatrix}"] & \begin{array}{c} M_0 \\ \oplus \\ N_1 \end{array} \arrow[r, "\begin{pmatrix} -d^M_0 & 0 \\ f_0 & d^N_1 \end{pmatrix}"] & \utilde{\begin{array}{c} M_{-1} \\ \oplus \\ N_0 \end{array}} \arrow[r, "\begin{pmatrix} -d^M_{-1} & 0 \\ f_{-1} & d^N_0 \end{pmatrix}"] & \begin{array}{c} M_{-2} \\ \oplus \\ N_{-1} \end{array} \arrow[r, "\begin{pmatrix} -d^M_{-2} & 0 \\ f_{-2} & d^N_{-1} \end{pmatrix}"] & \cdots
\end{tikzcd}$$

of R-modules.

Exercise 2.8 (6 points)**.** Verify that the above diagram indeed defines a complex and moreover is a homotopy cokernel of $f_\bullet$.

In particular, we find that for a morphism $M \xrightarrow{f} N$ in Mod_R, we have a canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f)$, and this is a quasi-isomorphism iff f is injective. This is an instance of the general principle that homotopically sensitive constructions are equivalent (in this case

---

[31] This is more often referred to as the *cone*, but we prefer the term "homotopy cokernel" as it is more directly suggestive of the object's "role in life" (a.k.a. its *raison d'être*). It is also common to refer to this as the *homotopy cofiber*, but this deviates from the more standard terminology of "cokernel" (as opposed to "cofiber") in the context of an abelian category.

[32] Similarly, the homotopy cokernel of f may be characterized as the *homotopy* pushout

$$\begin{tikzcd}
M_\bullet \arrow[r, "f_\bullet"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & N_\bullet \arrow[d] \\
0 \arrow[r] & \mathsf{hcoker}(f_\bullet)
\end{tikzcd} ,$$

i.e. the *initial homotopy-coherent cocone* over the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$. Here, the symbol $\Rightarrow$ indicates that the square only commutes up to a (specified) homotopy.

quasi-isomorphic) to their ordinary variants in "simple" situations (in this case, when there is no redundacy in the relations). More generally, we have the following.

Exercise 2.9 (6 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R that is injective in each dimension.

(a) Show that the canonical morphism

$$\mathsf{hcoker}(f) \longrightarrow \mathsf{coker}(f)$$

to the levelwise cokernel is a quasi-isomorphism.

(b) Give an example showing that this map need not be a homotopy equivalence.

As a special case of a homotopy cokernel, we simply write $\Sigma M_\bullet := \mathsf{hcoker}(M_\bullet \to 0)$,[33] and refer to this as the ***suspension*** of $M_\bullet$.[34] So by definition, giving a chain map $\Sigma M_\bullet \to T_\bullet$ is equivalent to giving a nullhomotopy of the composite map $M_\bullet \to 0 \to T_\bullet$.

It is evident from the construction that suspension defines an autoequivalence

$$\mathsf{Ch}_R \xrightarrow{\Sigma} \mathsf{Ch}_R \ .$$

Namely,

$$\Sigma\left(\cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} \underset{\sim}{M_0} \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots\right) \cong \left(\cdots \xrightarrow{-d_1} M_0 \xrightarrow{-d_0} \underset{\sim}{M_{-1}} \xrightarrow{-d_{-1}} M_{-2} \xrightarrow{-d_{-2}} \cdots\right) :$$

the operation of suspension simply shifts all terms up by one and negates all differentials. We write Σ^{-1} for its inverse, which we refer to as ***desuspension***. More generally, for any $k \in \mathbb{N}$ we write $\Sigma^k := \Sigma^{\circ k}$ and $\Sigma^{-k} := (\Sigma^{-1})^{\circ k}$. We also note for future reference the evident natural isomorphisms

$$\mathsf{H}_n \circ \Sigma^k \cong \mathsf{H}_{n-k}$$

for all $n, k \in \mathbb{Z}$.

Exercise 2.10 (6 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R.

(a) Using the formula for the homotopy cokernel above, establish a canonical homotopy equivalence

$$\Sigma M \simeq \mathsf{hcoker}(N \xrightarrow{u} \mathsf{hcoker}(f)) \ .^{35}$$

(b) Establish this same homotopy equivalence using the universal characterization of homotopy cokernels (as well as previously established properties of homotopies).

---

[33] Not coincidentally, when R is commutative this admits a canonical identification

$$\Sigma M_\bullet \cong (R \to \underset{\sim}{0}) \otimes M_\bullet$$

with the tensor product of $M_\bullet$ with the reduced simplicial chains on the simplicial circle $\Delta^1/\partial\Delta^1$. (Note that this is consistent with our sign convention for tensor products; the complex $M_\bullet \otimes (R \to \underset{\sim}{0})$ is different (although naturally isomorphic)).

[34] This is more often denoted $M_\bullet[1]$ and referred to as the *shift* of M, but we prefer the more blatantly topological notation and terminology.

[35] That these are quasi-isomorphic follows from Exercise 2.9(a).

Exercise 2.11 (4 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R.

(a) Prove that f is a quasi-isomorphism iff $\mathsf{hcoker}(f)$ is acyclic.
(b) Prove that f is a homotopy equivalence iff $\mathsf{hcoker}(f)$ is contractible.

2.2.4. Dually, recall that the kernel of a morphism $M \xrightarrow{f} N$ is Mod_R is by definition an R-module $\mathsf{ker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$\mathsf{ker}(f) \xrightarrow{v} M$$

satisfying the universal property that postcomposition with v determines a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(T, \mathsf{ker}(f)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T \to M \text{ such that the} \\ \text{composite } T \to M \xrightarrow{f} N \text{ is zero} \end{array} \right\} .$$[36]

This leads to the dual notion of a ***homotopy kernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R: this is an object $\mathsf{hker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$\mathsf{hker}(f_\bullet) \xrightarrow{v_\bullet} M_\bullet$$

satisfying the universal property that postcomposition with $v_\bullet$ determines a bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(T_\bullet, \mathsf{hker}(f_\bullet)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T_\bullet \to M_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } T_\bullet \to M_\bullet \xrightarrow{f_\bullet} N_\bullet \end{array} \right\} .$$[37]

Exercise 2.12 (4 points)**.** Given a morphism $M \xrightarrow{f} N$ in Ch_R, prove that the evident levelwise projection map

$$\Sigma^{-1}\mathsf{hcoker}(f) \longrightarrow M$$

is a homotopy kernel of f.[38]

---

[36] Said differently, the kernel of f is the pullback

$$\begin{tikzcd} \mathsf{ker}(f) \arrow[r] \arrow[d] & 0 \arrow[d] \\ M \arrow[r, "f"'] & N \end{tikzcd} .$$

[37] Similarly, the homotopy kernel of f may be characterized as the *homotopy* pullback

$$\begin{tikzcd} \mathsf{hker}(f_\bullet) \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ M_\bullet \arrow[r, "f_\bullet"'] & N_\bullet \end{tikzcd} ,$$

i.e. the *terminal homotopy-coherent cone* over the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$.

[38] There are unfortunately some signs that should arise here. They could be removed by tweaking the construction of hker (giving a different but isomorphic formula), but they would then arise elsewhere.

By combining Exercises 2.9(a) and 2.12, it follows that if a morphism $M \xrightarrow{f} N$ in Ch_R is surjective in each dimension, then the canonical morphism

$$\mathsf{ker}(f) \longrightarrow \mathsf{hker}(f)$$

from the levelwise kernel is a quasi-isomorphism (though again it is not necessarily a homotopy equivalence).

Although it will take some time to see why, the following feature is in some sense the *fundamental advantage* of working in Ch_R instead of in Mod_R.[39]

Exercise 2.13 (6 points)**.** Prove that for any morphism $M \xrightarrow{f} N$ in Ch_R the dashed canonical morphisms

$$\begin{tikzcd}
M \arrow[rrd, bend left, "f"] \arrow[rd, dashed] \arrow[rdd, bend right] & & \\
& \mathsf{hker}(u) \arrow[r] \arrow[d] & N \arrow[d, "u"] \\
& 0 \arrow[r] & \mathsf{hcoker}(f)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathsf{hker}(f) \arrow[r] \arrow[d, "v"] & 0 \arrow[d] \arrow[rdd, bend left] & \\
M \arrow[r] \arrow[rrd, bend right, "f"] & \mathsf{hcoker}(v) \arrow[rd, dashed] & \\
& & N
\end{tikzcd}$$

are homotopy equivalences.[40]

Namely, this implies that up to homotopy equivalence, every homotopy cokernel sequence is a homotopy kernel sequence, and conversely. Of course, this fails drastically in Mod_R: a cokernel sequence is a kernel sequence iff the original map is injective, and a kernel sequence is a cokernel sequence iff the original map is surjective. (So, the only co/kernel sequences in Mod_R that are also homotopy co/kernel sequences in Ch_R are trivial: those for which the original map is an isomorphism.)

2.2.5. A complex $M_\bullet \in \mathsf{Ch}_R$ is said to be ***exact*** at M_n if $\mathsf{H}_n(M_\bullet) = 0$. In particular, an acyclic complex is also called an ***exact sequence***, or sometimes a ***long exact sequence*** to emphasize that it is (potentially) infinite in one or both directions.

As a special case, a ***short exact sequence*** is a three-term acyclic complex

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 .$$

So, a short exact sequence is such a diagram satisfying the conditions that f is injective, g is surjective, and $\mathsf{ker}(g) = \mathsf{im}(f)$. In this case, one may also say that M is an ***extension*** of

---

[39] Namely, this is the key property that makes the underlying ∞-category of Ch_R into a *stable* ∞-category.

[40] Note that these diagrams are only *homotopy-coherently* commutative (indeed, the canonical morphisms are induced by homotopy-coherent universal properties).

N by L. For instance,

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules, which expresses $\mathbb{Z}/4$ as an extension of $\mathbb{Z}/2$ by itself.

More generally, we may refer to any (possibly finite) sequence of morphisms in Mod_R as an ***exact sequence*** if it is exact at all interior terms. So for example, one may refer to the diagram

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

as an exact sequence.[41]

The following is the source of almost every single long exact sequence in mathematics.[42]

Exercise 2.14 (2 points)**.** For any morphism $M \xrightarrow{f} N$ in Ch_R, show that the sequence

$$\mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f))$$

is exact.

Namely, from Exercises 2.10, 2.12, and 2.13 we obtain an infinite sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\Sigma^{-1}v"] & \Sigma^{-1}M \arrow[r, "\Sigma^{-1}f"] & \Sigma^{-1}N \arrow[r, "\Sigma^{-1}u"] & \Sigma^{-1}\mathsf{hcoker}(f) \arrow[d, phantom, sloped, "\simeq"] & & & & & & \\
& & & \mathsf{hker}(f) \arrow[r, "v"] & M \arrow[r, "f"] & N \arrow[r, "u"] & \mathsf{hcoker}(f) \arrow[d, phantom, sloped, "\simeq"] & & & \\
& & & & & & \Sigma\mathsf{hker}(f) \arrow[r, "\Sigma v"] & \Sigma M \arrow[r, "\Sigma f"] & \Sigma N \arrow[r, "\Sigma u"] & \cdots
\end{tikzcd}$$

of morphisms in Ch_R in which every composable pair of morphisms is a homotopy cokernel sequence up to homotopy equivalence. Thereafter, by Exercise 2.14, applying H_0 yields a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{H}_1(M) \arrow[r] & \mathsf{H}_1(N) \arrow[r] & \mathsf{H}_1(\mathsf{coker}(f)) \arrow[d, phantom, sloped, "\cong"] & & & & & & \\
& & & \mathsf{H}_0(\mathsf{hker}(f)) \arrow[r] & \mathsf{H}_0(M) \arrow[r] & \mathsf{H}_0(N) \arrow[r] & \mathsf{H}_0(\mathsf{hcoker}(f)) \arrow[d, phantom, sloped, "\cong"] & & & \\
& & & & & & \mathsf{H}_{-1}(\mathsf{hker}(f)) \arrow[r] & \mathsf{H}_{-1}(M) \arrow[r] & \mathsf{H}_{-1}(N) \arrow[r] & \cdots
\end{tikzcd}$$

in Mod_R.

Just as (quasi-isomorphism classes of) complexes encode more information than their homology groups, so does a (quasi-isomorphism class of) morphism in Ch_R encode more information than the corresponding long exact sequence. Therefore, we view the homotopy co/kernel sequence as the more fundamental notion.

---

[41] Note in particular that we are *not* implicitly extending the sequence by zero here.

[42] A notable exception is the long exact sequence on homotopy groups.

The following exercise illustrates the more classical approach to constructing long exact sequences.

Exercise 2.15 (6 points)**.** Suppose that $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a sequence of morphisms in Ch_R that is exact in each dimension.[43] Construct a long exact sequence

$$\begin{tikzcd}
& \cdots \arrow[r, "\mathsf{H}_{n+1}(g)"] & \mathsf{H}_{n+1}(N) \arrow[dll, rounded corners] \\
\mathsf{H}_n(L) \arrow[r, "\mathsf{H}_n(f)"] & \mathsf{H}_n(M) \arrow[r, "\mathsf{H}_n(g)"] & \mathsf{H}_n(N) \arrow[dll, rounded corners] \\
\mathsf{H}_{n-1}(L) \arrow[r, "\mathsf{H}_{n-1}(f)"] & \cdots &
\end{tikzcd}$$

in Mod_R.

2.3. The dg-category of complexes.

2.3.1. Given complexes $M, N \in \mathsf{Ch}_R$, we have seen that morphisms $M \to N$ may be related by *homotopies*. In fact, these homotopies can be related by *higher homotopies*, ad infinitum. Altogether, these data organize neatly into the ***hom-complex*** from M to N, denoted

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \in \mathsf{Ch}_{\Bbbk} .$$[44]

Namely, in dimension n the hom-complex is given by

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)_n := \prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \ ,$$

and its differential

$$\prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \xrightarrow{d_n} \prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1})$$

carries a tuple $\{f_i \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n})\}_{i \in \mathbb{Z}}$ to the tuple

$$\left\{(d^N_{i+n} \circ f_i + (-1)^{n-1} \cdot f_{i-1} \circ d^M_i) \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1})\right\}_{i \in \mathbb{Z}} \ .$$

Exercise 2.16 (2 points)**.** Verify that this formula defines a complex.

---

[43] Recall from Exercise 2.9(a) that this implies that the canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f) \cong N$ is a quasi-isomorphism.

[44] In fact, the hom-complex also contains "lower homotopies", in the sense that complexes are "like spaces, but with the possibility of cells in negative dimensions".

Directly from the definition, we have that $\mathsf{Z}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M,N)$. Moreover, given any morphism $M \xrightarrow{f} N$ in Ch_R, a nullhomotopy $0 \overset{h}{\Rightarrow} f$ is equivalent data to an element $h \in \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)$ such that $d_1(h) = f$. So, $\mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N))$ is canonically isomorphic to the abelian group of *homotopy classes* of morphisms $M \to N$. Note too that

$$\Sigma^i \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(\Sigma^{-i}M,N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,\Sigma^i N) \ ;$$

this gives an analogous description of all homology groups of the hom-complex, as $\mathsf{H}_n \cong \mathsf{H}_0 \circ \Sigma^{-n}$. On the other hand, these homology groups can also be understood in a more homotopical (although closely related) manner.

Exercise 2.17 (4 points)**.** Given a chain map $M \xrightarrow{f} N$ and two nullhomotopies $0 \overset{h_0}{\Rightarrow} f$ and $0 \overset{h_1}{\Rightarrow} f$, define a notion of a homotopy $h_0 \Rightarrow h_1$ between homotopies, and prove that such a higher homotopy always exists precisely when $\mathsf{H}_1(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) = 0$.

If one views the complexes M and N as 0-cells, the maps $M \xrightarrow{0} N$ and $M \xrightarrow{f} N$ as 1-cells, and the homotopies h_i as 2-cells, then such a higher homotopy should be viewed as a 3-cell. Of course, there are analogs of this same interpretation for all the homology groups of $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)$.

For brevity, we may simply write

$$\underline{\mathsf{hom}}(M,N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N) \ .$$

2.3.2. These hom-complexes, in turn, can be naturally assembled into a single object.

Exercise 2.18 (8 points)**.** Fix any complexes $L, M, N, O \in \mathsf{Ch}_R$.

(a) Construct a map

$$\underline{\mathsf{hom}}(L,M) \otimes \underline{\mathsf{hom}}(M,N) \xrightarrow{\chi_{L,M,N}} \underline{\mathsf{hom}}(L,N)$$

in $\mathsf{Ch}_{\Bbbk}$ that encodes the composition of morphisms of complexes.

(b) Verify that these composition morphisms are associative, in the sense that the diagram

$$\begin{tikzcd}
\underline{\mathsf{hom}}(L,M) \otimes \underline{\mathsf{hom}}(M,N) \otimes \underline{\mathsf{hom}}(N,O) \arrow[r, "\chi_{L,M,N} \otimes \mathsf{id}"] \arrow[d, "\mathsf{id} \otimes \chi_{M,N,O}"'] & \underline{\mathsf{hom}}(L,N) \otimes \underline{\mathsf{hom}}(N,O) \arrow[d, "\chi_{L,N,O}"] \\
\underline{\mathsf{hom}}(L,M) \otimes \underline{\mathsf{hom}}(M,O) \arrow[r, "\chi_{L,M,O}"'] & \underline{\mathsf{hom}}(L,O)
\end{tikzcd}$$

commutes.[45]

---

[45] This implicitly uses the associativity of the operation $\otimes$ in $\mathsf{Ch}_{\Bbbk}$.

(c) Construct a map

$$\Bbbk \xrightarrow{\iota_M} \underline{\mathsf{hom}}(M, M)$$

using the identity morphism of $M \in \mathsf{Ch}_R$, and verify that it defines a two-sided identity for the above composition in the sense that the diagrams

$$\begin{tikzcd}
\underline{\mathsf{hom}}(L, M) \otimes \Bbbk \arrow[r, "\mathsf{id} \otimes \iota_M"] \arrow[dr, "\cong"'] & \underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, M) \arrow[d, "\chi_{L,M,M}"] \\
& \underline{\mathsf{hom}}(L, M)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Bbbk \otimes \underline{\mathsf{hom}}(M, N) \arrow[r, "\iota_M \otimes \mathsf{id}"] \arrow[dr, "\cong"'] & \underline{\mathsf{hom}}(M, M) \otimes \underline{\mathsf{hom}}(M, N) \arrow[d, "\chi_{M,M,N}"] \\
& \underline{\mathsf{hom}}(M, N)
\end{tikzcd}$$

commute (where the isomorphisms are the canonical ones coming from the fact that $\Bbbk \in \mathsf{Ch}_\Bbbk$ is the unit object).

Altogether, Exercise 2.18 yields a ($\Bbbk$***-linear***) ***dg-category***,[46] i.e. a category enriched in the symmetric monoidal category $(\mathsf{Ch}_\Bbbk, \otimes, \Bbbk)$:[47] its objects are the chain complexes of R-modules and its hom-objects are the hom-complexes between them. We denote this dg-category by $\mathbf{K}_R$ and refer to it as ***the dg-category of complexes of*** R***-modules***.[48] In particular, we may also write $\mathsf{hom}_{\mathbf{K}_R}(M, N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$; our chosen notation will depend on our desired emphasis.

The fundamental purpose of the dg-category $\mathbf{K}_R$ is to assemble the data of chain maps, homotopies, and higher (and lower) homotopies into a single object. In particular, it is the homology groups of the hom-complexes in $\mathbf{K}_R$ that are relevant. Therefore, it will be natural to view the hom-complexes in $\mathbf{K}_R$ as "only being important up to quasi-isomorphism".

---

[46] Here, "dg" is short for "differential graded". (In general, it is common to refer to a chain complex of R-modules as a *dg-*R*-module*.)

[47] Of course, a dg-category is a particular instance of a more general notion. Namely, given a monoidal category $\mathcal{V} := (\mathcal{V}, \otimes_\mathcal{V}, \mathbb{1}_\mathcal{V})$, there is a natural notion of a ***category enriched in*** $\mathcal{V}$, or simply a $\mathcal{V}$***-enriched category***: a $\mathcal{V}$-enriched category $\mathcal{C}$ consists of a set of objects, the data of hom-objects $\underline{\mathsf{hom}}_\mathcal{C}(X, Y) \in \mathcal{V}$ for all $X, Y \in \mathcal{C}$, and the data of composition and identity morphisms

$$\underline{\mathsf{hom}}_\mathcal{C}(X, Y) \otimes_\mathcal{V} \underline{\mathsf{hom}}_\mathcal{C}(Y, Z) \xrightarrow{\chi_{X,Y,Z}} \underline{\mathsf{hom}}_\mathcal{C}(X, Z) \qquad \text{and} \qquad \mathbb{1}_\mathcal{V} \xrightarrow{\iota_Y} \underline{\mathsf{hom}}_\mathcal{C}(Y, Y)$$

in $\mathcal{V}$ for all $X, Y, Z \in \mathcal{C}$, subject to the evident associativity and unitality conditions. (As indicated, one sometimes uses an underline to emphasize that these are *enriched* hom-objects (as opposed to mere hom-sets). On the other hand, note that the notation $\mathsf{hom}_\mathcal{C}(X, Y)$ is already unambiguous, as $\mathcal{C}$ is a $\mathcal{V}$-enriched category.

[48] The German word for "complex" begins with the letter "K".

As a special case, note that for any complex $M \in \mathsf{Ch}_R$, composition in $\mathbf{K}_R$ makes the complex $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ into an *associative algebra object*, i.e. it is a ***dg-algebra*** (or ***dga*** for short) over $\Bbbk$.

Exercise 2.19 (4 points)**.** Show that the following conditions are equivalent.

(i) The complex $M \in \mathsf{Ch}_R$ is contractible.
(ii) The complex $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ is acyclic.
(iii) The complex $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ has that $\mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M)) = 0$.
(iv) The element $[\mathsf{id}_M] \in \mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M))$ has that $[\mathsf{id}_M] = 0$.

[changed 1/21] Of course, this construct can be reversed. Given a dg-algebra $A \in \mathsf{Alg}(\mathsf{Ch}_{\Bbbk})$, we can form a dg-category $\mathfrak{B}A \in \mathsf{Cat}^{\mathsf{dg}}$ as follows: it has a single object $*$, and we declare that $\mathsf{hom}_{\mathfrak{B}A}(*,*) := A$ (with composition and identity respectively defined as the multiplication and unit in A).

2.3.3. In order to to understand the hom-complexes in the dg-category $\mathbf{K}_R$, it is helpful to understand how they interact with other complexes of $\Bbbk$-modules.

We take our motivation from the expected tensor-hom adjunction for complexes of $\Bbbk$-modules: for any $T, M, N \in \mathsf{Ch}_{\Bbbk}$ we have a natural isomorphism

$$\mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(T \otimes_{\Bbbk} M, N) \cong \mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(M,N)) \ .$$

The general situation is as follows. For any complex $T \in \mathsf{Ch}_{\Bbbk}$ of $\Bbbk$-modules and any complexes $M, N \in \mathsf{Ch}_R$ of R-modules, we may form the complexes

$$T \otimes_{\Bbbk} M \qquad \text{and} \qquad \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N)$$

of R-modules, where the (right) R-actions are induced from those on M and N. These satisfy the universal properties that

$$\mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N)) \ .$$[49]

In fact, these satisfy enriched universal properties.

Exercise 2.20 (4 points)**.** Prove that for any $T \in \mathsf{Ch}_{\Bbbk}$ and any $M, N \in \mathsf{Ch}_R$ we have natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N))$$

in $\mathsf{Ch}_{\mathbb{Z}}$.

We can now show that the hom-complexes in R preserve homotopy co/kernel sequences separately in each variable.

---

[49] In the general context of enriched category theory, one says that $T \otimes M$ is the *tensoring* of M by T and that $\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N)$ is the *cotensoring* of N by T.

Exercise 2.21 (8 points)**.** Choose any complexes $M, N, T \in \mathsf{Ch}_R$ and any morphism $M \xrightarrow{f} N$. Using Exercise 2.20 and the universal property of the homotopy co/kernel, construct natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(T, \mathsf{hker}\left(M \xrightarrow{f} N\right)\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, M) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T,f)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, N)\right)$$

and

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(\mathsf{hcoker}\left(M \xrightarrow{f} N\right), T\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(N, T) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(f,T)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, T)\right)$$

in $\mathsf{Ch}_{\Bbbk}$.

Of course, essentially identical reasoning shows that the hom-complex bifunctor

$$\mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.[50]

Exercise 2.22 (4 points)**.** Show that the tensor product bifunctor

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.

2.4. **Projective and injective resolutions.**

2.4.1. The original motivation for homological algebra is the fact that many natural functors on ordinary modules do not preserve exact sequences. Equivalently but more fundamentally, they do not respect short exact sequences, i.e. they do not respect both kernels and cokernels.[51]

Exercise 2.23 (6 points)**.**

(a) Show that the functor

$$\mathsf{Mod}_{\Bbbk} \times \mathsf{Mod}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Mod}_R$$

does not generally preserve exact sequences in either variable.

---

[50] Alternatively, this follows from Exercise 2.21, the commutative square

$$\begin{tikzcd}
\mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_R \arrow[r, "\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)"] \arrow[d, "\mathsf{id}\times\mathsf{fgt}"'] & \mathsf{Ch}_R \arrow[d, "\mathsf{fgt}"] \\
\mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_{\Bbbk} \arrow[r, "\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)"'] & \mathsf{Ch}_{\Bbbk}
\end{tikzcd},$$

and the fact that the forgetful functor both preserves and detects homotopy kernel sequences.

[51] Here we use the word "respect" instead of "preserve" due to the contravariance of $\mathsf{hom}_{\mathsf{Mod}_R}(-, M)$: recognizing that $\mathsf{Mod}_R^{\mathsf{op}}$ is also an abelian category, one might hope that this would carry kernels to cokernels and cokernels to kernels.

(b) Show that the functor

$$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\mathsf{hom}_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_R$$

does not generally preserve exact sequences in either variable.

Namely, applying either bifunctor appearing in Exercise 2.23 to an exact sequence in one of its slots, one obtains a "half-exact" sequence: $(-) \otimes_{\Bbbk} (-)$ preserves cokernels separately in each variable, $\mathsf{hom}_{\mathsf{Mod}_R}(M, -)$ preserves kernels, and $\mathsf{hom}_{\mathsf{Mod}_R}(-, M)$ carries cokernels to kernels. It was originally desired for these half-exact sequences to extend to long exact sequences, whose additional terms would quantify these various failures of exactness.

As illustrated by Exercises 2.21 and 2.22, towards resolving these issues it is fruitful to pass from ordinary modules and ordinary co/kernels to complexes of modules and homotopy co/kernels; the desired long exact sequences would then be those on homology discussed in §2.2.5, although of course we will take the perspective that the homotopy co/kernel sequences themselves are the more fundamental objects. However, given that we would like to consider quasi-isomorphisms as isomorphisms, the following results show that this maneuver does not suffice on its own.

Exercise 2.24 (6 points)**.**

(a) Show that the functor

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Ch}_R$$

does not generally preserve acyclic objects in either variable.

(b) Show that the functor

$$\mathsf{Ch}_R^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(-,-)} \mathsf{Ch}_{\Bbbk}$$

does not generally preserve acyclic objects in either variable.

Clearly, preservation of acyclics is necessary for the preservation of quasi-isomorphisms. But in fact, the converse is guaranteed by Exercise 2.11(a) (combined with Exercises 2.21 and 2.22). This motivates the notions that we introduce now.

2.4.2. We write $\mathbf{A}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the acyclic complexes.

We say that a complex $P \in \mathbf{K}_R$ is ***projective*** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(P, A) \in \mathsf{Ch}_{\mathbb{Z}}$ is acyclic. Because acyclic complexes are preserved under de/suspensions, we may equivalently demand simply that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(P, A)) = 0$ for every acyclic complex $A \in \mathbf{A}_R$. In other words, P is projective iff every morphism $P \to A$ to an acyclic complex admits a nullhomotopy. In turn, this is equivalent to the condition that for every solid diagram

$$\begin{tikzcd} & \mathsf{hker}(\mathsf{id}_A) \arrow[d, "v"] \\ P \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

where $A \in \mathbf{A}_R$ is acyclic there exists a lift making the diagram commute. We write $\mathbf{P}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the projective complexes.

We observe for future reference that v is a levelwise surjective quasi-isomorphism: indeed, $\mathsf{hker}(\mathsf{id}_A)$ is acyclic by the long exact sequence in homology, and v is surjective by construction (or by its defining universal property). So, in order for a complex to be projective it suffices for it to have the analogous lifting property with respect to *all* levelwise surjective quasi-isomorphisms.

Dually, we say that a complex $I \in \mathbf{K}_R$ is ***injective*** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(A, I) \in \mathsf{Ch}_{\mathbb{Z}}$ is acyclic. Likewise, we may equivalently demand for all acyclic complexes $A \in \mathbf{A}_R$ that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(A, I)) = 0$, or that every morphism $A \to I$ admits a nullhomotopy, or that for every solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & I \\ \mathsf{hcoker}(\mathsf{id}_A) \arrow[ur, dashed] & \end{tikzcd}$$

there exists an extension making the diagram commute. We write $\mathbf{I}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the injective complexes.

It is easy to deduce the following facts directly from the definitions.

Exercise 2.25 (6 points)**.**

(a) Show that all quasi-isomorphisms in $\mathbf{P}_R$ are homotopy equivalences.[52]

(b) Show that all quasi-isomorphisms in $\mathbf{I}_R$ are homotopy equivalences.

In particular, clearly the zero complex is projective (resp. injective), so that an acyclic projective (resp. injective) complex must be contractible.

(c) Show that projective complexes are preserved under tensor product: if $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$ then $P \otimes Q \in \mathbf{P}_R$.

(d) Show that for any projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P,-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(Q,-)} \mathbf{K}_{\Bbbk}$$

preserve quasi-isomorphisms.

(e) Show that for any injective complex $I \in \mathbf{I}_R$, the functors

$$\mathbf{K}_{\Bbbk}^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(-,I)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(-,I)} \mathbf{K}_{\Bbbk}$$

---

[52] This is formally analogous to *Whitehead's theorem*, which states that a weak homotopy equivalence between cell complexes (or retracts thereof) is necessarily a homotopy equivalence.

preserve quasi-isomorphisms.

Of course, the definitions of projective and injective complexes on their own are not so useful. What gives them their power is that every complex $M \in \mathbf{K}_R$ admits both a ***projective resolution*** $P \xrightarrow{\approx} M$ and an ***injective resolution*** $M \xrightarrow{\approx} I$ (i.e. quasi-isomorphisms as indicated). These are the promised representatives of the quasi-isomorphism class of M that are "well-adapted" for certain purposes. Specifically, we will respectively view the functors

$$(-) \otimes_{\Bbbk} P \ , \qquad \mathsf{hom}_{\mathbf{K}_R}(P, -) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-, I)$$

as "corrected" (a.k.a. "derived") versions of the functors

$$(-) \otimes_{\Bbbk} M \ , \qquad \mathsf{hom}_{\mathbf{K}_R}(M, -) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-, I)$$

(and similarly for projective resolutions of complexes of $\Bbbk$-modules).

As we will see, it is relatively straightforward to construct projective resolutions of *bounded-below* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \ll 0$) and to construct injective resolutions of *bounded-above* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \gg 0$). Note in particular that this will apply to ordinary R-modules via the inclusion $\mathsf{Mod}_R \subseteq \mathsf{Ch}_R$.

As for the general (i.e. unbounded) situation, it turns out to be much easier (and more conceptually satisfying) to construct projective resolutions than injective resolutions. On the other hand, we will not have any specific need for injective resolutions in general, beyond their existence. Therefore, we will discuss only projective resolutions in general, and refer the reader to [Hov99, §2.3] for the construction of injective resolutions in general.

It is easy to deduce the following facts directly from the existence of projective and resolutions.

Exercise 2.26 (4 points)**.**

(a) Show that if the complex $M \in \mathbf{K}_R$ has that $\underline{\mathsf{hom}}_{\mathbf{K}_R}(P, M)$ is acyclic for every projective complex $P \in \mathbf{P}_R$, then M is acyclic.[53]

(b) Show that if the complex $M \in \mathbf{K}_R$ has that $\underline{\mathsf{hom}}_{\mathbf{K}_R}(M, I)$ is acyclic for every injective complex $I \in \mathbf{I}_R$, then M is acyclic.

(c) Show that for any projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{P \otimes_{\Bbbk} (-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_{\Bbbk} \xrightarrow{(-) \otimes_{\Bbbk} Q} \mathbf{K}_R$$

preserve quasi-isomorphisms.

In particular, the tensor product of a projective complex and an acyclic complex is acyclic.

We will eventually organize many of the facts enumerated in Exercises 2.25 and 2.26 in a more systematic way.

---

[53] Evidently, it is already sufficient just to take $P = R$ (which is projective by Exercises 2.27 and 2.32).

2.4.3. Recall that an R-module $P \in \mathsf{Mod}_R$ is called ***projective*** if mapping out of it preserves surjections, i.e. if for any surjection $M \twoheadrightarrow N$ in Mod_R the induced map $\mathsf{hom}_{\mathsf{Mod}_R}(P, M) \to \mathsf{hom}_{\mathsf{Mod}_R}(P, N)$ is a surjection. Said differently, given any solid diagram

$$\begin{tikzcd}
& M \arrow[d, two heads] \\
P \arrow[r] \arrow[ur, dashed] & N
\end{tikzcd}$$

there exists a dashed lift making the diagram commute.

Exercise 2.27 (2 points)**.** Show that an R-module $P \in \mathsf{Mod}_R$ is projective iff it is a summand of a free module (i.e. there exists an R-module $Q \in \mathsf{Mod}_R$ and an isomorphism $P \oplus Q \cong R^{\oplus S}$ with the free R-module on a set S).

Exercise 2.28 (2 points)**.** Show that every projective $\mathbb{Z}$-module is free.

Exercise 2.29 (4 points)**.** Give necessary and sufficient conditions on $n \in \mathbb{N}$ such that every projective $\mathbb{Z}/n$-module is free.

Exercise 2.30 (4 points)**.** Show that an R-module $M \in \mathsf{Mod}_R$ is projective iff the corresponding complex $\underset{\sim}{M} \in \mathsf{Ch}_R$ is projective.

We now turn from projective modules back to projective complexes.

Exercise 2.31 (4 points)**.** Show that if $P \in \mathbf{P}_R$ is a projective complex, then $P_n \in \mathsf{Mod}_R$ is a projective R-module for all $n \in \mathbb{Z}$.

We have the following partial converse.

Exercise 2.32 (6 points)**.** Show that if $P \in \mathbf{K}_R$ is a bounded-below complex such that each $P_n \in \mathsf{Mod}_R$ is projective, then $P \in \mathbf{P}_R$ is projective.

However, the bounded-below hypothesis in Exercise 2.32 is necessary: both parts of Exercise 2.7 give examples of unbounded complexes which are levelwise free (hence levelwise projective by Exercise 2.27) that cannot be projective. Indeed, an acyclic projective complex must be contractible, as its identity map must admit a nullhomotopy.

Exercise 2.33 (4 points)**.** Given a bounded-below complex $M \in \mathbf{K}_R$, construct a projective resolution $P \xrightarrow{\approx} M$.

2.4.4. Recall that an R-module $I \in \mathsf{Mod}_R$ is called ***injective*** if mapping into it carries injections to surjections, i.e. if for any injection $M \hookrightarrow N$ in Mod_R the induced map $\mathsf{hom}_{\mathsf{Mod}_R}(N, I) \to \mathsf{hom}_{\mathsf{Mod}_R}(M, I)$ is surjective. Said differently, given any solid diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, hook] & I \\
N \arrow[ur, dashed] &
\end{tikzcd}$$

there exists a dashed extension making the diagram commute.

Injective modules are much more bizarre than projective modules.

Exercise 2.34 (2 points)**.** Assuming that R is a PID, show that an R-module $M \in \mathsf{Mod}_R$ is injective iff it is *divisible*, i.e. for every nonzero element $r \in R$ the map $M \xrightarrow{r} M$ is surjective.

So for instance, $\mathbb{Q} \in \mathsf{Mod}_\mathbb{Z}$ is injective while $\mathbb{Z} \in \mathsf{Mod}_\mathbb{Z}$ is not.

It is easy to dualize the arguments of Exercises 2.31 and 2.32: injective complexes are levelwise injective, and bounded-above levelwise injective complexes are injective.

The argument of Exercise 2.33 uses the fact that every R-module admits a surjection from a projective R-module; one says in this situation that Mod_R ***has enough projectives***. It can be easily dualized to show that any bounded-above complex $M \in \mathbf{K}_R$ admits an injective resolution $M \xrightarrow{\approx} I$, using the following result showing that Mod_R also ***has enough injectives***.

Exercise 2.35 (12 points)**.**

(a) For any ring homomorphism $S \to R$, verify that the functors

$$\begin{tikzcd} \mathsf{Mod}_S \arrow[rr, bend left, "(-)\otimes_S R"] \arrow[rr, bend right, "\hom_{\mathsf{Mod}_S}(R{,}-)"'] & \perp \\ \mathsf{fgt} \\ \perp & \mathsf{Mod}_R \arrow[ll] \end{tikzcd}$$

participate in adjunctions as indicated.[54]

(b) Given modules $M \in \mathsf{Mod}_R$ and $N \in \mathsf{Mod}_S$ and an injection

$$\mathsf{fgt}(M) \hookrightarrow N$$

in Mod_S, show that the corresponding morphism

$$M \longrightarrow \hom_{\mathsf{Mod}_S}(R, N)$$

in Mod_R is also an injection.

(c) Deduce from the fact that fgt preserves injections that the functor $\hom_{\mathsf{Mod}_S}(R, -)$ preserves injective objects.[55]

(d) Show that injective objects are preserved under products.[56]

(e) Fix an abelian group $A \in \mathsf{Ab}$. Show that for every $a \in A$, there exists a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ carrying a to a nonzero element.[57] Deduce that there exists an

---

[54] In defining the functor $(-) \otimes_S R$ (resp. $\hom_{\mathsf{Mod}_S}(R, -)$), we use that R is an (S, R)-bimodule (resp. an (R, S)-bimodule).

[55] Dually, the functor $(-) \otimes_S R$ preserves projective objects because the functor fgt also preserves surjections.

[56] Dually, projective objects are preserved under coproducts.

[57] This uses Exercise 2.34.

injection

$$A \hookrightarrow \prod_A \mathbb{Q}/\mathbb{Z}\ .$$[58]

Namely, suppose we are given an R-module $M \in \mathsf{Mod}_R$. By (d) we have an injection

$$\mathsf{fgt}(M) \hookrightarrow \prod_M \mathbb{Q}/\mathbb{Z} =: I$$

in Ab, and by (d) and Exercise 2.34 we see that $I \in \mathsf{Ab}$ is injective. Thereafter, by (b) we obtain an injection

$$M \hookrightarrow \mathsf{hom}_{\mathsf{Ab}}(R, I) =: J$$

in Mod_R, and by (c) it follows that $J \in \mathsf{Mod}_R$ is injective.

2.4.5. For the purpose of constructing projective resolutions, we first introduce some auxiliary ideas.

For any $n \in \mathbb{Z}$, we define the objects and morphism

$$\begin{tikzcd} S^n \arrow[d, hook, "i_n"'] & & 0 \arrow[r] \arrow[d] & R \arrow[d, "\mathsf{id}_R"] \\ D^{n+1} & := & R \arrow[r, "\mathsf{id}_R"'] & R \end{tikzcd}$$

in Ch_R, where the columns are in dimensions $n+1$ and n.[59] For any complex $M \in \mathsf{Ch}_R$, we have natural isomorphisms

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \cong \mathsf{Z}_n(M) \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \cong M_{n+1}\ ;$$

precomposition with i_n induces the morphism

$$\mathsf{Z}_n(M) \xleftarrow{d_{n+1}} M_{n+1}\ .$$[60]

---

[58] One says that $\mathbb{Q}/\mathbb{Z} \in \mathsf{Ab}$ is an ***injective cogenerator***. Dually, $R \in \mathsf{Mod}_R$ is a ***projective generator***: it is projective, and moreover for any $M \in \mathsf{Mod}_R$ there exists a surjection $R^{\oplus S} \twoheadrightarrow M$ for some set S.

[59] Of course, this morphism will function as "the inclusion of the n-sphere into the $(n+1)$-disk": indeed, for any $n \geq 0$ it is the reduced simplicial chains on the inclusion $\Delta^{\{0,\ldots,n\}}/\partial\Delta^{\{0,\ldots,n\}} \hookrightarrow \Delta^n/\Lambda^n_0$ (which is a particularly small simplicial model for this morphism). Note too that $S^n \cong \Sigma^n S^0$ and $D^n \cong \Sigma^n D^0$. While the former isomorphism is always an equality (so that we may take $\Sigma^n S^0$ to be the definition of S^n), the latter introduces an inconvenient sign when n is odd (which is why we do not take $\Sigma^n D^0$ to be the definition of D^n).

[60] Categorically speaking, the reason that the morphism

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, -) \xleftarrow{(-)\circ i_n} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, -)$$

in $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_{\mathbb{Z}})$ lifts to $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_R)$ is that the morphism i_n is in fact a morphism of complexes of (R,R)-bimodules.

We say that a morphism $M \xrightarrow{f} N$ has the ***right lifting property*** with respect to i_n (or equivalently that i_n has the ***left lifting property*** with respect to f) if for every solid commutative diagram

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, "i_n"'] & M \arrow[d, "f"] \\ D^{n+1} \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

there exists a dashed lift that makes the diagram commute. This is equivalent to saying that the canonical dashed morphism in the pullback

$$\begin{tikzcd} M_{n+1} \arrow[dr, dashed] \arrow[drr, bend left, "d^M_{n+1}"] \arrow[ddr, bend right, "f_{n+1}"'] & & \\ & N_{n+1} \times_{\mathsf{Z}_n(N)} \mathsf{Z}_n(M) \arrow[r] \arrow[d] & \mathsf{Z}_n(M) \arrow[d, "\mathsf{Z}_n(f)"] \\ & N_{n+1} \arrow[r, "d^N_{n+1}"'] & \mathsf{Z}_n(N) \end{tikzcd}$$

is surjective. We may abbreviate this by writing that $f \in \mathsf{rlp}(i_n)$. We also introduce the notation $I := \{i_n\}_{n\in\mathbb{Z}}$ and $\mathsf{rlp}(I) := \bigcap_{n\in\mathbb{Z}} \mathsf{rlp}(i_n)$.

Exercise 2.36 (4 points)**.** Prove that if $f \in \mathsf{rlp}(i_n)$ then $\mathsf{H}_n(f)$ is injective and $\mathsf{Z}_{n+1}(f)$ is surjective.

Now, if $f \in \mathsf{rlp}(I)$, then Exercise 2.36 immediately implies that f is a quasi-isomorphism. On the other hand, suppose that $f \in \mathsf{rlp}(i_{n-1}) \cap \mathsf{rlp}(i_n)$. Then, $\mathsf{Z}_n(f)$ is surjective by Exercise 2.36. It follows that f_{n+1} is surjective by the above reformulation of $\mathsf{rlp}(i_n)$. So, $f \in \mathsf{rlp}(I)$ implies as well that f is levelwise surjective.

In fact, the converse is also true.

Exercise 2.37 (4 points)**.** Prove that if $M \xrightarrow{f} N$ is a quasi-isomorphism that is levelwise surjective, then $f \in \mathsf{rlp}(I)$.

Given a morphism $M \xrightarrow{f} N$ in Ch_R, we write

$$I(f)_n := \mathsf{hom}_{\mathsf{Ar}(\mathsf{Ch}_R)}(i_n, f) \cong \mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \times_{\mathsf{hom}_{\mathsf{Ch}_R}(S^n,N)} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, N)$$

for the set of morphisms

$$\begin{tikzcd} S^n \arrow[r, dashed] \arrow[d, "i_n"'] & M \arrow[d, "f"] \\ D^{n+1} \arrow[r, dashed] & N \end{tikzcd}$$

in the arrow category $\mathsf{Ar}(\mathsf{Ch}_R) := \mathsf{Fun}([1], \mathsf{Ch}_R)$ of Ch_R (i.e. pairs of morphisms $S^n \to M$ and $D^{n+1} \to N$ making the square commute), and we write

$$I(f) := \coprod_{n \in \mathbb{Z}} I(f)_n$$

for their disjoint union over all $n \in \mathbb{Z}$. Given an element $\alpha \in I(f)$, we write $n_\alpha := n$ if $\alpha \in I(f)_n$. (That is, we write $I(f) \xrightarrow{n_{(-)}} \mathbb{Z}$ for the evident function from the disjoint union.)

2.4.6. We now construct projective resolutions.[61] In fact, given a morphism $M \xrightarrow{f} N$ in Ch_R, we will construct a factorization

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, dashed, "c^{(\infty)}"'] & & N \\ & M^{(\infty)} \arrow[ur, dashed, "f^{(\infty)}"'] & \end{tikzcd}$$

satisfying the following conditions:

(1) the morphism $c^{(\infty)}$ has the left lifting property with respect to levelwise surjective quasi-isomorphisms, and

(2) the morphism $f^{(\infty)}$ is a quasi-isomorphism.

Taking $M \xrightarrow{f} N$ to be the map $0 \to N$ then yields a projective resolution of N.

We make the following general construction. First of all, observe that we have a canonical commutative square

$$\begin{tikzcd} \bigoplus_{\alpha \in I(f)} S^{n_\alpha} \arrow[r] \arrow[d, "\bigoplus_{\alpha \in I(f)} i_{n_\alpha}"'] & M \arrow[d, "f"] \\ \bigoplus_{\alpha \in I(f)} D^{n_\alpha + 1} \arrow[r] & N \end{tikzcd} .$$

We write

$$M^{(1)} := \left(M \coprod_{\bigoplus_{\alpha \in I(f)} S^{n_\alpha}} \bigoplus_{\alpha \in I(f)} D^{n_\alpha + 1} \right)$$

for the pushout, and we write

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, dashed, "c^{(1)}"'] & & N \\ & M^{(1)} \arrow[ur, dashed, "f^{(1)}"'] & \end{tikzcd}$$

for the morphisms in the induced factorization.

---

[61] This procedure will be formally analogous to the construction of a cell complex that is weak homotopy equivalent to a given topological space. The general construction is called the *small object argument*, due to its crucial use of a certain "smallness" property of the sources of the elements of the set I. In the present situation, the relevant fact is given as Exercise 2.40.

Exercise 2.38 (2 points)**.** Show that the morphism $c^{(1)}$ has the left lifting property with respect to any levelwise surjective quasi-isomorphism.

Exercise 2.39 (2 points)**.** Show that for every solid commutative diagram

$$\begin{tikzcd}
S^n \arrow[rr] \arrow[dd] & & M \arrow[dl, "c^{(1)}"'] \arrow[dd, "f"] \\
& M^{(1)} \arrow[dr, "f^{(1)}"'] & \\
D^{n+1} \arrow[ur, dashed] \arrow[rr] & & N
\end{tikzcd}$$

there exists a dashed factorization making the diagram commute.

We now apply the above construction again, but to $f^{(1)}$ instead of to f: namely, we observe the canonical commutative square

$$\begin{tikzcd}
\bigoplus_{\alpha \in I(f^{(1)})} S^{n_\alpha} \arrow[r] \arrow[d, "\bigoplus_{\alpha \in I(f^{(1)})} i_{n_\alpha}"'] & M^{(1)} \arrow[d, "f^{(1)}"] \\
\bigoplus_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} \arrow[r] & N
\end{tikzcd} ,$$

we write

$$M^{(2)} := \left(M^{(1)} \coprod_{\bigoplus_{\alpha \in I(f^{(1)})} S^{n_\alpha}} \bigoplus_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} \right)$$

for the pushout, and we write

$$\begin{tikzcd}
M^{(1)} \arrow[rr, "f"] \arrow[dr, dashed, "c^{(2)}"'] & & N \\
& M^{(2)} \arrow[ur, dashed, "f^{(2)}"'] &
\end{tikzcd}$$

for the morphisms in the induced factorization. This gives us a commutative diagram

$$\begin{tikzcd}
M \arrow[rrr, "f"] \arrow[dr, "c^{(1)}"'] & & & N \\
& M^{(1)} \arrow[urr, "f^{(1)}"'] \arrow[dr, "c^{(2)}"'] & & \\
& & M^{(2)} \arrow[uur, "f^{(2)}"'] &
\end{tikzcd} .$$

Of course, we continue to iterate this construction in the obvious way. To conclude, we define

$$M^{(\infty)} := \mathsf{colim}\left(M \xrightarrow{c^{(1)}} M^{(1)} \xrightarrow{c^{(2)}} M^{(2)} \xrightarrow{c^{(3)}} \cdots \right) \ ;$$

we define the map $M \xrightarrow{c^{(\infty)}} M^{(\infty)}$ to be the canonical map to the colimit, and we define the map $M^{(\infty)} \xrightarrow{f^{(\infty)}} N$ to be induced by the universal property of the colimit.

We now verify the above two conditions.

(1) Let $B \xrightarrow{g} C$ be a levelwise surjective quasi-isomorphism. Given any commutative square

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "c^{(\infty)}"'] & B \arrow[d, "g"] \\
M^{(\infty)} \arrow[r] & C
\end{tikzcd} ,$$

we enlarge it to the solid commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "c^{(1)}"'] & B \arrow[dddd, "g"] \\
M^{(1)} \arrow[d, "c^{(2)}"'] \arrow[ur, dashed] & \\
M^{(2)} \arrow[d, "c^{(3)}"'] \arrow[uur, dashed] & \\
\vdots \arrow[d] & \\
M^{(\infty)} \arrow[r] & C
\end{tikzcd} .$$

Here, applying Exercise 2.38 repeatedly we may inductively construct dashed lifts as indicated that make the diagram commute. By the universal property of the colimit, together these yield the desired lift in the original diagram.[62]

(2) In fact, we show that $f^{(\infty)} \in \mathsf{rlp}(I)$. For this, suppose we are given any commutative diagram

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d, "i_n"'] & M^{(\infty)} \arrow[d, "f^{(\infty)}"] \\
D^{n+1} \arrow[r] & N
\end{tikzcd} .$$

---

[62] This may be phrased as choosing an element of a codirected limit of *surjective* functions between sets.

By Exercise 2.40 below, there exists some $k \in \mathbb{N}$ and a lift

$$\begin{tikzcd}
 & M^{(k)} \arrow[d] \\
S^n \arrow[ur, dashed] \arrow[r] & M^{(\infty)}
\end{tikzcd}$$

of the given map. Now, just as in Exercise 2.39, we obtain a lift

$$\begin{tikzcd}
S^n \arrow[rr] \arrow[dddd, "i_n"'] & & M^{(k)} \arrow[dl, "c^{(k+1)}"'] \arrow[ddd] \\
 & & \\
 & M^{(k+1)} \arrow[dr] \arrow[ddr, "f^{(k+1)}"'] & \\
 & & M^{(\infty)} \arrow[d, "f^{(\infty)}"] \\
D^{n+1} \arrow[uur, dashed] \arrow[rr] & & N
\end{tikzcd},$$

which proves the claim.

Exercise 2.40 (2 points)**.** Show that the object $S^n \in \mathsf{Ch}_R$ is ***compact***, i.e. that the functor

$$\mathsf{Ch}_R \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(S^n,-)} \mathsf{Set}$$

commutes with filtered colimits.[63]

---

[63] Note that a functor commutes with filtered colimits if and only if it commutes with directed colimits. This is one instance of the useful general principle of "coordinatization" of a class of colimits. As another example, given a functor between categories that admit finite colimits, it preserves them if and only if it preserves the initial object and pushouts. Another related fact is that given a functor between cocomplete categories (i.e. categories admitting all (small) colimits), it preserves all (small) colimits if and only if it preserves finite colimits and filtered colimits.

References

[EH16] David Eisenbud and Joe Harris, 3264 and all that—a second course in algebraic geometry, Cambridge University Press, Cambridge, 2016, available here.

[GP74] Victor Guillemin and Alan Pollack, Differential topology, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.

[Har77] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[Hov99] Mark Hovey, Model categories, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[Lur09] Jacob Lurie, Higher topos theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Mac71] Saunders MacLane, Categories for the working mathematician, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 5.

[MG] Aaron Mazel-Gee, The zen of ∞-categories, available here.

[RP] Sam Raskin and Oron Propp, Lectures on cohomological class field theory, available here.

[Saf] Pavel Safronov, Category theory [lecture notes], available here.

[Wei94] Charles A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.