AN INVITATION TO HIGHER ALGEBRA

AARON MAZEL-GEE

Abstract. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the winter 2021 quarter. They are **under construction**, and will be updated at the course website at the end of each lecture.

Contents

0. Miscellanea 2

Part I. Homological algebra 3

1. Some motivation for homological algebra 3
2. Chain complexes, homology, and tensor products 8
3. Homotopies, homotopy co/kernels, and exact sequences 13
4. The dg-category of complexes 22
5. Projective and injective resolutions 27
6. The derived ∞-category 39
7. Derived functors 49

Part II. Higher category theory 65

8. Model categories 65

References 81

0. Miscellanea

0.1. Exercises.

0.1.1. In order to obtain a grade of 100% on the homework by the end of the quarter, you will need to earn 120 points.

0.1.2. Partial solutions may be submitted for partial credit.

0.1.3. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a "yes or no" question).

last updated: February 11, 2021

0.1.4. In preparing your homework, please copy down the problem statement, since it is possible that the numbering may change.

0.2. **Conventions.**

0.2.1. As this document is a work in progress, I will frequently want to make changes to existing material (in addition to adding material as we cover it during lecture). Of course, it would be quite difficult for a reader to spot such changes, especially as the document grows. Therefore, any substantial such changes will be temporarily flagged by their change date (for easy searching) and additionally will be colored as follows: changes that have occurred since the most recent lecture are **red**, changes made between the past two lectures are **blue**, and changes made between two and three lectures ago are **green**. Older changes are no longer colored.

0.2.2. We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and Set denotes the category of sets. However, notation will very often only be "local": the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.3. The term "natural number" (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.4. We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We consider posets as categories without comment. We write e.g. $\mathbb{N}^{\leq}$ and $\mathbb{Z}^{\leq}$ for the usual poset structures on the natural numbers and the integers. [changed 2/9] Given some datum in a category (e.g. an object or morphism), we may use the superscript $(-)^{\circ}$ to denote the corresponding datum in the opposite category, although we may also omit this superscript when our meaning is sufficiently clear. We mostly ignore set-theoretic issues.[1]

0.2.5. Especially in later sections, we will frequently give references to Lurie's books [Lur09] and [Lur]. Our exposition is nevertheless intended to be self-contained, with these references merely providing the reader with entry points for exploring those books further. For brevity, we will use the abbreviations "T" and "A" to refer to these works, and moreover we will omit environment names (except for the section symbol §). So for instance, we will refer to [Lur09, Theorem 4.1.3.1] simply as [T.4.1.3.1] and to [Lur, §1.3.3] simply as [§A.1.3.3].

0.2.6. The term "(commutative) ring" means "associative unital (resp. commutative) ring". Likewise, modules are always unital (meaning that the unit element acts as the identity).

[1] Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

0.2.7. In the interest of brevity, universal quantifiers will often be dropped. For instance, an assertion involving an integer n should generally be understood to refer to *all* integers n unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

0.2.8. [changed 2/4] For brevity, we will often use a slash to make multiple statements at once. This idiom has two possible meanings; the specific meaning should always be clear from context. On the one hand, we will write e.g. "homotopy co/kernel sequence" as a stand-in for "sequences which are simultaneously homotopy cokernel sequences and homotopy kernel sequences – and let us not forget that it suffices to check either condition in order to deduce both". On the other hand, we will write e.g. "co/limits" as a stand in for "both colimits and limits".

Part I. Homological algebra

1. Some motivation for homological algebra

1.1. **Intersection theory.** A basic endeavor in geometry is to understand *intersections.* For example, given a (smooth) manifold M and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} \ .$$

If N_0 and N_1 intersect *transversely* (i.e. $T_pN_0 + T_pN_1 = T_pM$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

However, if the intersection of N_0 and N_1 is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either N_0 or N_1 that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.[2] However, this approach has a number of (related) drawbacks.

(1) Perturbations are noncanonical.
(2) Perturbations will generally destroy the inherent symmetries of the situation.[3]
(3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally only transcendental.[4]

[2]A good introduction to these ideas is [GP74].

[3]For instance, perturbations to transverse intersections need not exist in the equivariant context.

[4]It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, where "perturb" means "change to a new but rationally equivalent algebraic cycle". It is fundamental in the classical approach to intersection theory in algebraic geometry [EH16].

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. **Tensor products.** We first recall the notion of tensor product.

Let R be a commutative ring, and let M and N be R-modules. The (***relative***) ***tensor product*** of M and N over R,[5] denoted $M \otimes_R N$, is the universal abelian group equipped with an R-balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N \ ,$$

i.e. a function satisfying the following axioms:

(1) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$ and $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$;
(2) $\varphi(m \cdot r, n) = \varphi(m, r \cdot n)$.[6]

In other words, for any abelian group A, precomposition with φ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \to A\} \xleftarrow{\cong} \{\text{abelian group homomorphisms } M \otimes_R N \to A\} \ .$$

In the case that R is understood (and particularly when $R = \mathbb{Z}$ or when R is a field), we may simply write $\otimes := \otimes_R$.

The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

(1) $(m + m', n) \sim (m, n) + (m', n)$ and $(m, n + n') \sim (m, n) + (m, n')$;
(2) $(m \cdot r, n) \sim (m, r \cdot n)$.

Exercise 1.1 (2 points)**.** For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\mathsf{gcd}(m, n)$.

1.3. **Basic principles of algebraic geometry.** In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §I.1].

1.3.1. The polynomial functions on $\mathbb{R}^n$ are the n-variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving n implicit).

[5] The word "relative" here is meant to emphasize that R is an arbitrary commutative ring. By contrast, the term "absolute tensor product" would emphasize that $R = \mathbb{Z}$.

[6] The notation here stems from the fact that more generally, we can define the relative tensor product when R is merely an associative ring, M is a right R-module, and N is a left R-module.

1.3.2. By definition, an ***algebraic subset*** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.[7] In this case we write $Z = V(S)$, and we say that Z is the ***vanishing locus*** of the elements of S. If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.[8]

1.3.3. We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along Z. Then, the ring of polynomial functions on Z is

$$\mathcal{O}(Z) = R/I(Z) \ .$$

1.3.4. Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in I\} \subseteq \mathbb{R}^n \ .$$

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{ideals in } R\} \quad .$$[9]

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 \ .$$

1.4. **Intersections via tensor products.** We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x-1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \qquad \text{and} \qquad Z_1 = \{(x,y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \qquad \text{and} \qquad I_1 = I(Z_1) = (y - x) \subseteq R \ .$$

---

[7]By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

[8]Since R is noetherian, any ideal is finitely generated. In other words, we may always take S to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

[9]Over an algebraically closed field $\Bbbk$, this construction restricts to a bijection between *closed* subsets of $\Bbbk^n$ (with respect to the Zariski topology) and *radical* ideals of $\Bbbk[x_1, \ldots, x_n]$. The composite $V \circ I$ carries a subset $Y \subseteq \Bbbk^n$ to its closure $\overline{Y} \subseteq \Bbbk^n$, while the composite $I \circ V$ carries an ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ to its radical $\sqrt{I} = \{f \in \Bbbk[x_1, \ldots, x_n] : \exists\ n > 0 \text{ s.t. } f^n \in I\} \subseteq R$. By contrast, over $\mathbb{R}$ the function V fails to be injective, e.g. $V(\mathbb{R}[x]) = V(x^2 + 1) = \varnothing$.

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned}\mathcal{O}(Z_0 \cap Z_1) &\cong R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y-x^2, y-x) \cong \mathbb{R}[x]/(x-x^2) \\ &= \mathbb{R}[x]/(x \cdot (1-x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1-x) \cong \mathbb{R} \times \mathbb{R} ,\end{aligned}$$

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a PID, in fact it is a Euclidean domain).[10] The fact that this is a 2-dimensional $\mathbb{R}$-algebra corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements, which can in good situations detect the correct multiplicity of a non-transverse intersection point.

Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in R. These correspond to the curves $y = x^2$ and $y = 0$. These intersect "twice" at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x,y]/(y-x^2, y) \cong \mathbb{R}[x]/(x^2) .$$

The 2-dimensionality of this $\mathbb{R}$-algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect "with multiplicity two". Namely, this $\mathbb{R}$-algebra corresponds to "the origin along with infinitesimal fuzz in the direction of the x-axis". This is in contrast with the previous example, where the tensor product split as a cartesian product.

These techniques are quite robust.

Exercise 1.2 (4 points)**.** Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x - a) \subseteq R$ and $I_1 = (x - b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x-a, x-b) \cong \mathbb{R}/(a-b) \cong \begin{cases} \mathbb{R} , & a = b \\ 0 , & a \neq b \end{cases} .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$-algebra.

[10] An explicit inverse is given by carrying the pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b-a) \cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x \cdot (1-x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0),g(1)}$ lies in the ideal generated by $x \cdot (x-1)$. Observe that $g - f_{g(0),g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

Using homological algebra, namely the notion of ***derived tensor products***, we will be able to obtain the expected intersection number of 0 even when $a = b$.

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

Exercise 1.3 (6 points)**.** Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

(a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines "just barely avoid intersecting": morally they should intersect "at infinity".[11] This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3 \backslash \{0\})/\mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.[12]

(b) Show that a *homogenous* polynomial $g \in \mathbb{R}[x, y, z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.

(c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogenous polynomials $g_1, g_2 \in \mathbb{R}[x, y, z]$ such that $g_i([x : y : 1]) = f_i(x, y)$.

(d) Compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. As we have seen in §1.4.4, given two lines in $\mathbb{R}^2$, we are more likely to get the expected number if we intersect them (or rather their closures) in $\mathbb{RP}^2$: namely, this gives the correct answer even when the lines are parallel. However, this fails to give the correct answer when the two lines are *equal*. Derived tensor products repair this failure. Namely, the derived tensor product of a (projective) line with itself in $\mathbb{RP}^2$ is "a line, but with cardinality equal to that of a single point".

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

[11] A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \to 1^+$ their intersection point has $x \to -\infty$, while as $t \to 1^-$ their intersection point has $x \to +\infty$. This suggests that there should be a *single* point "at infinity" where they intersect in the case that $t = 1$.

[12] So, the "points at infinity" are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2 \backslash \{0\})/\mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1 \backslash \mathbb{R}^1$).

2. Chain complexes, homology, and tensor products

We now proceed to introduce the basic objects of study in homological algebra.

2.1. **Algebra conventions.** For concreteness, we work in the context of ordinary algebra. Namely, we fix a commutative ring $\Bbbk$ and a $\Bbbk$-algebra R. For the most part, we will work in Mod_R, the category of (right) R-modules, and one may take $\Bbbk$ to be $\mathbb{Z}$. However, at times we will want to specialize to a commutative ring, and for this it is convenient for $\Bbbk$ to be arbitrary.[13] Moreover, we will study some interactions between $\Bbbk$-modules and R-modules. At the level of ordinary (i.e. non-homological) algebra, these are encapsulated by the following facts.

(1) The category $\mathsf{Mod}_\Bbbk$ is symmetric monoidal via the tensor product, which we denote by $\otimes := \otimes_\Bbbk$; its unit object is $\Bbbk$.
(2) The category Mod_R is naturally enriched in $\mathsf{Mod}_\Bbbk$. In other words, for any R-modules $M, N \in \mathsf{Mod}_R$, the set $\mathsf{hom}_{\mathsf{Mod}_R}(M, N)$ of R-linear homomorphisms carries the natural structure of a $\Bbbk$-module, and moreover composition in Mod_R is $\Bbbk$-multilinear.
(3) Moreover, $\Bbbk$-modules naturally act on R-modules in two different ways: for any $\Bbbk$-module $T \in \mathsf{Mod}_\Bbbk$ and any R-modules $M, N \in \mathsf{Mod}_R$ we have R-modules
$$T \otimes_\Bbbk M \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N) ,$$
where the (right) R-actions are induced from those on M and N, and these constructions participate in natural isomorphisms
$$\mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, \mathsf{hom}_{\mathsf{Mod}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Mod}_R}(T \otimes_\Bbbk M, N) \cong \mathsf{hom}_{\mathsf{Mod}_R}(M, \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N)) .$$

Of course, one may take $R = \Bbbk$ as a special case.[14] As a result, the notions that we will develop relating to the interactions between $\Bbbk$-modules and R-modules will all be *generalizations* of the notions that we develop relating to $\Bbbk$-modules alone.

As we will see later, most of the theory works equally well for a general abelian category, although there will be some additional hiccups that do not arise when studying modules.[15]

2.2. **Chain complexes.** A ***chain complex*** of R-modules is a diagram
$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$
of R-modules such that for all $n \in \mathbb{Z}$, the composite $d_n \circ d_{n+1} = 0$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that "all indices are being referred to at once". To emphasize the differentials, one may write $(M_\bullet, d_\bullet)$. Also, one may simply refer to a

[13]Of course, we will apply results developed for R-modules to $\Bbbk$-modules without comment.

[14]The above facts then reduce to the assertion that $\mathsf{Mod}_\Bbbk$ is a *closed* symmetric monoidal category, i.e. that it carries a self-enrichment that is compatible with its symmetric monoidal structure.

[15]On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into Mod_R for some ring R (although the choice of such a ring R is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

chain complex as a "complex".[16] On the other hand, we also may omit the bullet and simply write $M := M_\bullet$ for simplicity. The integer n is called the ***degree*** or the ***dimension***. For an element $m \in M_n$, we may write $\deg(m) := n$.

The morphisms d_n are called the ***differentials*** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of d_n is M_n). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as d_n^M.

In this notation, a morphism of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of R-modules such that the diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_{n+2}^M} & M_{n+1} & \xrightarrow{d_{n+1}^M} & M_n & \xrightarrow{d_n^M} & M_{n-1} & \xrightarrow{d_{n-1}^M} & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \xrightarrow[d_{n+2}^N]{} & N_{n+1} & \xrightarrow[d_{n+1}^N]{} & N_n & \xrightarrow[d_n^N]{} & N_{n-1} & \xrightarrow[d_{n-1}^N]{} & \cdots \end{array}$$

commutes.[17] These morphisms are often referred to as ***chain maps***. We write Ch_R for the category of chain complexes of R-modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline when appropriate.

Any R-module $M \in \mathsf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \underset{\sim}{M} \longrightarrow 0 \longrightarrow \cdots .$$

This construction determines a fully faithful embedding

$$\mathsf{Mod}_R \hookrightarrow \mathsf{Ch}_R .$$

As a result, we may not notationally distinguish between data in Mod_R and its image in Ch_R.

When a complex only has a few nonzero terms, for brevity one may omit the zero terms. For instance, the above complex may also be written as $\underset{\sim}{M}$.

2.3. **Homology.** Fix a chain complex $M_\bullet$. Its n-***cycles*** and n-***boundaries*** are the submodules

$$\mathsf{Z}_n(M_\bullet) := \mathsf{ker}(d_n) \subseteq M_n \qquad \text{and} \qquad \mathsf{B}_n(M_\bullet) := \mathsf{im}(d_{n+1}) \subseteq M_n .$$[18]

[16]The word "chain" here is historical: the first example of a chain complex has in degree n the "n^{th} chain group" of a simplicial complex X, i.e. the group of chains (i.e. formal linear combinations) of n-simplices of X. (It was only later realized that chain complexes are worth studying in their own right.)

[17]For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

[18]The German words for "cycle" and "boundary" respectively begin with the letters "Z" and "B".

Note that $\mathsf{B}_n(M_\bullet) \subseteq \mathsf{Z}_n(M_\bullet)$ because $d^2 = 0$. Then, the n^{th} ***homology*** of $M_\bullet$ is the quotient R-module

$$\mathsf{H}_n(M_\bullet) := \frac{\mathsf{Z}_n(M_\bullet)}{\mathsf{B}_n(M_\bullet)} := \frac{\mathsf{ker}(d_n)}{\mathsf{im}(d_{n+1})} \ .$$

Although $\mathsf{H}_n(M_\bullet)$ is an R-module, it is common to refer to it merely as a *homology group*.

Exercise 2.1 (2 points)**.** Verify that the constructions Z_n, B_n, and H_n define functors

$$\mathsf{Ch}_R \longrightarrow \mathsf{Mod}_R \ .$$

A morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R is called a ***quasi-isomorphism*** if the induced morphisms $H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We may indicate that a morphism is a quasi-isomorphism by decorating the arrow as $\xrightarrow{\approx}$.

We say that $M_\bullet$ is ***acyclic*** if $\mathsf{H}_n(M_\bullet) = 0$ for all n. So, $M_\bullet$ is acyclic if and only if the unique map $0 \to M_\bullet$ from the zero complex is a quasi-isomorphism, if and only if the unique map $M_\bullet \to 0$ to the zero complex is a quasi-isomorphism.

2.4. The derived category of R-modules.

2.4.1. By and large, we would like to think of quasi-isomorphic chain complexes as "essentially interchangeable", with some representatives of a given quasi-isomorphism class (namely the *projective* and *injective* complexes introduced below) being "well-adapted" for certain purposes.[19] In other words, one should think of quasi-isomorphisms as if they are actual isomorphisms.

This can be made literally true by ***localizing*** the category Ch_R at the quasi-isomorphisms, i.e. by adjoining formal inverses for them. This yields a category that (for reasons that will become clear later) we will denote by $\mathsf{H}_0(\mathbf{D}_R)$ and refer to as ***the derived category of R-modules***; its objects are called ***derived R-modules***.[20] So by definition, there is a canonical functor

$$\mathsf{Ch}_R \longrightarrow \mathsf{H}_0(\mathbf{D}_R)$$

that carries all quasi-isomorphisms to isomorphisms, and moreover it is universal with respect to this requirement. Indeed, for any category $\mathcal{C}$, the restriction functor

$$\mathsf{Fun}(\mathsf{Ch}_R, \mathcal{C}) \longleftarrow \mathsf{Fun}(\mathsf{H}_0(\mathbf{D}_R), \mathcal{C})$$

is a fully faithful inclusion, whose image consists of those functors $\mathsf{Ch}_R \to \mathcal{C}$ that carry quasi-isomorphisms to isomorphisms.

[19]This is very closely akin to how one should think of equivalent categories as "essentially interchangeable", even when they are not isomorphic. However, in a precise sense, *all* categories are "equally well-adapted" for all purposes (in contrast with chain complexes).

[20]The placement of the word "derived" is admittedly slightly unfortunate, but this terminology is quite common.

Note that a derived R-module is a "purely homotopical" object: while it can by definition be presented by a chain complex of R-modules, one cannot speak e.g. of its underlying R-module in dimension 0, as this notion is not preserved under quasi-isomorphisms.[21] On the other hand, one *can* speak e.g. of its n^{th} homology, as this notion is by definition preserved under quasi-isomorphisms.

2.4.2. Essentially by construction, given two complexes $M, N \in \mathsf{Ch}_R$, morphisms from M to N in the derived category are given by equivalence classes of zigzags

$$M \overset{\approx}{\longleftarrow} \bullet \longrightarrow \bullet \overset{\approx}{\longleftarrow} \cdots \longrightarrow \bullet \overset{\approx}{\longleftarrow} N$$

(in which all backwards maps are quasi-isomorphisms). Thankfully, it will turn out that every equivalence class contains representatives of the forms

$$M \overset{\approx}{\longleftarrow} \bullet \longrightarrow N \qquad \text{and} \qquad M \longrightarrow \bullet \overset{\approx}{\longleftarrow} N ,$$

which makes the situation substantially more manageable.[22]

2.4.3. Although we introduce the derived category now, we will not have much use for it: it contains too little information. The richer and more primitive object is $\mathbf{D}_R$, the ***derived ∞-category*** of R. This is a mathematical entity whose objects are still the derived R-modules, but whose hom-objects are more elaborate: namely, they are *derived* $\Bbbk$*-modules*. Of course, passing from $\mathbf{D}_R$ to $\mathsf{H}_0(\mathbf{D}_R)$ amounts to entracting only the 0^{th} homology groups of these hom-objects.

2.4.4. While quasi-isomorphic complexes have isomorphic homology groups, we will see that the converse is generally false: the obstruction will be encoded by ***k-invariants***, as explained in §7.3.3.[23] That is, a quasi-isomorphism class of complexes is equivalent data to its homology groups along with all of its k-invariants. For this reason, we will generally consider (quasi-isomorphism classes of) complexes themselves as the "true" mathematical objects of lasting interest, while their homology groups are merely algebraic invariants that can be extracted therefrom.

2.5. **Tensor products.** Given complexes $M, N \in \mathsf{Ch}_\Bbbk$ of $\Bbbk$-modules, we define their ***tensor product*** complex

$$(M \otimes N)_\bullet := (M \otimes_\Bbbk N)_\bullet$$

[21] Likewise, one cannot speak of the underlying set of a weak homotopy equivalence class of topological spaces, nor can one speak of the set of objects of an equivalence class of categories.

[22] These reductions are guaranteed by the existence of two different ***model structures*** on Ch_R, which respectively have the features that all objects are fibrant and that all objects are cofibrant.

[23] This same name is given to the (closely analogous) obstructions to spaces with the same homotopy groups being weak homotopy equivalent.

as follows. First of all, we define its k^{th} term to be

$$(M \otimes N)_k := \bigoplus_{i+j=k} (M_i \otimes N_j) := \bigoplus_{i+j=k} (M_i \otimes_{\Bbbk} N_j) \ .$$

Then, the differential is characterized by the fact that it carries a pure tensor

$$m \otimes n \in (M_i \otimes N_j) \subseteq (M \otimes N)_k$$

to the sum of pure tensors

$$d(m \otimes n) := d(m) \otimes n + (-1)^j \cdot m \otimes d(n) \ ,$$ [24]

(an element of $((M_{i-1} \otimes N_j) \oplus (M_i \otimes N_{j-1})) \subseteq (M \otimes N)_{k-1}$). More elaborately, this may be written as

$$d_k^{M \otimes N}(m \otimes n) := d_i^M(m) \otimes n + (-1)^j \cdot m \otimes d_j^N(n) \ .$$

Exercise 2.2 (2 points)**.** Verify that this formula defines a complex.

In particular, in solving Exercise 2.2 you will see why the signs are necessary in the definition of the tensor product of complexes. In fact, many sign conventions are possible (and all give equivalent symmetric monoidal categories), but it is impossible to remove all signs from the theory (unless one works over $\mathbb{F}_2$).

From here, it is straightforward to see that the above construction defines a monoidal structure

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_{\Bbbk} \xrightarrow{\otimes} \mathsf{Ch}_{\Bbbk} \ ,$$

with unit object $\Bbbk := \underset{\sim}{\Bbbk} \in \mathsf{Ch}_{\Bbbk}$. In fact, this is a *symmetric* monoidal structure, with symmetry isomorphisms

$$M \otimes N \xrightarrow{\cong} N \otimes M$$

determined by the formula

$$m \otimes n \longmapsto (-1)^{\deg(m) \cdot \deg(n)} \cdot n \otimes m \ .$$ [25]

More generally, this same construction defines an action

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{\otimes} \mathsf{Ch}_R$$

of the symmetric monoidal category $(\mathsf{Ch}_{\Bbbk}, \otimes, \Bbbk)$ on the category Ch_R.

Exercise 2.3 (8 points)**.** Fix two complexes $M, N \in \mathsf{Ch}_{\Bbbk}$.

---

[24] The factor $(-1)^j$ is determined by the *Koszul sign rule*, which is a general principle asserting that in commuting graded quantities past one another of degrees $k, j \in \mathbb{Z}$ one should pick up a factor of $(-1)^{k \cdot j}$. Namely, we consider the symbol "d" as an expression of degree -1 (which makes sense since it changes dimensions by 1).

[25] This formula is another instance of the Koszul sign rule.

(a) Verify that the formula $[m] \otimes [n] \mapsto [m \otimes n]$ determines a morphism

$$H_i(M) \otimes H_j(N) \longrightarrow H_{i+j}(M \otimes N)$$

of $\Bbbk$-modules.

It follows that we obtain a morphism

$$\bigoplus_{i+j=k} (H_i(M) \otimes H_j(N)) \longrightarrow H_k(M \otimes N)$$

of $\Bbbk$-modules.

(b) Prove that this is an isomorphism under the assumption that $\Bbbk$ is a field.

(c) Find an example where this is not an isomorphism.

3. Homotopies, homotopy co/kernels, and exact sequences

3.1. **Homotopies.**

3.1.1. Let $M_1 \xrightarrow{f} M_0$ be a morphism of R-modules. This gives us a complex $M_\bullet := (M_1 \xrightarrow{f} M_0)$. Observe that this has a canonical morphism

$$\begin{array}{ccc} M_\bullet \\ \downarrow \\ \mathsf{coker}(f) \end{array} \quad := \quad \begin{array}{ccc} M_1 & \xrightarrow{f} & M_0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{coker}(f) \end{array}$$

to the cokernel of f (considered as a complex in degree 0). Observe further that

$$\mathsf{H}_n(M_\bullet) \cong \begin{cases} \mathsf{coker}(f) , & n = 0 \\ \mathsf{ker}(f) , & n = 1 \\ 0 , & \text{otherwise} \end{cases} .$$

Hence, the above map is a quasi-isomorphism iff f is an injection. One might think of $M_\bullet$ as a "presentation" of the underlying R-module $\mathsf{H}_0(M_\bullet) \cong \mathsf{coker}(f)$: the generators are M_0, the relations are M_1 (i.e. each $m \in M_1$ gives a relation $d(m) \sim 0$), but then $\mathsf{H}_1(M_\bullet)$ furthermore measures the "redundancy" of the relations. Said differently, $M_\bullet$ is a "homotopically correct" version of the cokernel of f, which remembers not only the literal cokernel but also the extent to which the relations are overdetermined. Indeed, it will be the *homotopy cokernel* of the morphism f.

3.1.2. Let $M_\bullet, N_\bullet \in \mathsf{Ch}_R$ be complexes and let $f_\bullet, g_\bullet \in \mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ be morphisms. A (***chain***) ***homotopy*** from $f_\bullet$ to $g_\bullet$ is a set of morphisms

$$M_n \xrightarrow{h_n} N_{n+1}$$

satisfying the condition that

$$g_n - f_n = d^N_{n+1} \circ h_n + h_{n-1} \circ d^M_n \ .$$

We may write this as $f_\bullet \overset{h_\bullet}{\Longrightarrow} g_\bullet$. A ***nullhomotopy*** of $g_\bullet$ is a homotopy $0 \overset{h_\bullet}{\Longrightarrow} g_\bullet$ from the zero map. A ***contraction*** of a complex is a nullhomotopy of its identity map. If a complex admits a contraction, we say that it is ***contractible***.

Exercise 3.1 (3 points)**.** Show that the relation of homotopy on $\mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ is an equivalence relation.

In fact, the argument of Exercise 3.1 easily upgrades to imply that we can enhance Ch_R from an ordinary category to a category enriched in groupoids (a.k.a. a $(2,1)$-category):[26] its objects are chain complexes, its 1-morphisms are chain maps, and its 2-morphisms are chain homotopies.[27] Indeed, the arguments for transitivity, reflexivity, and symmetry of the relation of homotopy respectively endow these hom-categories with their composition laws, identity morphisms, and inverses (so that they are indeed hom-groupoids). It is moreover clear that homotopies may be composed appropriately, either by definition or using Exercise 3.2 below.

3.1.3. For present and future use, we introduce the complex $\mathbb{I} \in \mathsf{Ch}_{\Bbbk}$ and the morphisms $i_0, i_1 \in \mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(\Bbbk, \mathbb{I})$ according to the diagram

$$\begin{array}{ccccc}
\Bbbk & & 0 & \longrightarrow & \underset{\sim}{\Bbbk} \\
{\scriptstyle i_0}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle (-\mathsf{id}_{\Bbbk},0)} \\
\mathbb{I} & := & \Bbbk & \xrightarrow{(-\mathsf{id}_{\Bbbk},\mathsf{id}_{\Bbbk})} & \underset{\sim\sim\sim}{\Bbbk \oplus \Bbbk} \\
{\scriptstyle i_1}\big\uparrow & & \big\uparrow & & \big\uparrow{\scriptstyle (0,\mathsf{id}_{\Bbbk})} \\
\Bbbk & & 0 & \longrightarrow & \underset{\sim}{\Bbbk}
\end{array} \quad .$$

This object $\mathbb{I} \in \mathsf{Ch}_{\Bbbk}$ (along with the two maps i_0 and i_1) is an ***interval object*** for the homotopy theory of chain complexes (which explains the notation).[28] The general definition

---

[26]As we will see, this is the homotopy $(2,1)$-category of a more fundamental object, namely an ∞-category (meaning an $(\infty,1)$-category).

[27]This explains the notation $f_\bullet \Rightarrow g_\bullet$ just introduced.

[28]This is a particularly natural choice of an interval object: as we will see, it is the simplicial chains (with coefficients in $\Bbbk$) on the 1-simplex Δ^1 (whose underlying topological space is a closed interval), and the two maps i_0 and i_1 are the simplicial chains on the inclusions of its two 0-simplices (i.e. the endpoints of the closed interval).

of an interval object is suggested by the discussion of §8.1.2. In the present setting, this assertion amounts to the following result.

Exercise 3.2 (4 points)**.** Given morphisms $f, g \in \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$, prove that a homotopy $f \Rightarrow g$ is equivalent data to a morphism $\mathbb{I} \otimes M \to N$ that makes the diagram

$$\begin{array}{ccc}
\Bbbk \otimes M \cong M & & \\
\Big\downarrow{\scriptstyle i_0 \otimes \mathsf{id}_M} & \searrow^{f} & \\
\mathbb{I} \otimes M & \overset{h}{\dashrightarrow} & N \\
\Big\uparrow{\scriptstyle i_1 \otimes \mathsf{id}_M} & \nearrow_{g} & \\
\Bbbk \otimes M \cong M & &
\end{array}$$

commute.

3.1.4. A morphism $M \xrightarrow{f} N$ in Ch_R is a ***homotopy equivalence*** if there exists a morphism $N \xrightarrow{g} M$ and homotopies $\mathsf{id}_M \Rightarrow g \circ f$ and $f \circ g \Rightarrow \mathsf{id}_N$.[29] We may indicate that a morphism is a homotopy equivalence by decorating it as $\xrightarrow{\sim}$.

As a special case, a complex $M_\bullet$ is contractible if and only if either unique map $0 \to M_\bullet$ or $M_\bullet \to 0$ is a homotopy equivalence.

Exercise 3.3 (3 points)**.** Show that homotopic maps on complexes give equal maps on homology.

It follows immediately from Exercise 3.3 that homotopy equivalences are quasi-isomorphisms, and in particular that contractible complexes are acyclic. However, the converse is false.

Exercise 3.4 (10 points)**.**

(a) Show that the complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots$$

of $\mathbb{Z}/4$-modules is acyclic but not contractible.

(b) Show that the complex

$$\cdots \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \Bbbk[x]/x^2 \xrightarrow{x} \cdots$$

of $\Bbbk[x]/x^2$-modules is acyclic. Show that it is contractible as a complex of $\Bbbk$-modules, but not as a complex of $\Bbbk[x]/x^2$-modules (assuming that $\Bbbk \neq 0$).

[29] The directions of these homotopies are intended to be suggestive of adjunctions (with f functioning as the left adjoint), but by Exercise 3.1 they are irrelevant. (Likewise, equivalences of categories are both left adjoints and right adjoints.)

3.1.5. We will generally consider homotopic maps as "essentially interchangeable". On the other hand, rather than merely positing the *existence* of a homotopy between two maps, we will always want to *keep track* of the homotopy that witnesses them as being homotopic.

3.2. Homotopy cokernels.

3.2.1. Recall that the cokernel of a morphism $M \xrightarrow{f} N$ in Mod_R is by definition an R-module $\mathsf{coker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$N \xrightarrow{u} \mathsf{coker}(f)$$

satisfying the universal property that precomposition with u determines a bijection

$$\hom_{\mathsf{Mod}_R}(\mathsf{coker}(f), T) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N \to T \text{ such that the} \\ \text{composite } M \xrightarrow{f} N \to T \text{ is zero} \end{array} \right\} .$$ [30]

From here, the principles indicated in §3.1.5 lead directly to the definition of a ***homotopy cokernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R:[31] this is an object $\mathsf{hcoker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$N_\bullet \xrightarrow{u_\bullet} \mathsf{hcoker}(f_\bullet)$$

satisfying the universal property that precomposition with $u_\bullet$ determines a bijection

$$\hom_{\mathsf{Ch}_R}(\mathsf{hcoker}(f_\bullet), T_\bullet) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } N_\bullet \to T_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } M_\bullet \xrightarrow{f_\bullet} N_\bullet \to T_\bullet \end{array} \right\} .$$ [32]

[30] Said differently, the cokernel of f is the pushout

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{coker}(f) \end{array} .$$

[31] This is more often referred to as the *cone*, but we prefer the term "homotopy cokernel" as it is more directly suggestive of the object's "role in life" (a.k.a. its *raison d'être*). It is also common to refer to this as the *homotopy cofiber*, but this deviates from the more standard terminology of "cokernel" (as opposed to "cofiber") in the context of an abelian category.

[32] Similarly, the homotopy cokernel of f may be characterized as the *homotopy* pushout

$$\begin{array}{ccc} M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \\ \downarrow & \nearrow\!\!\!\!\nearrow & \downarrow \\ 0 & \longrightarrow & \mathsf{hcoker}(f_\bullet) \end{array} ,$$

i.e. the *initial homotopy-coherent cocone* over the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$. Here, the symbol $\Rightarrow$ indicates that the square only commutes up to a (specified) homotopy.

In fact, we claim that we have already seen an example of a homotopy cokernel: namely, for any morphism $M \xrightarrow{f} N$ in Mod_R, the complex $(M \xrightarrow{f} \underset{\sim}{N}) \in \mathsf{Ch}_R$ equipped with the map

$$\begin{array}{ccc} 0 & \longrightarrow & \underset{\sim}{N} \\ \downarrow & & \downarrow{\scriptstyle \mathsf{id}_N} \\ M & \underset{f}{\longrightarrow} & \underset{\sim}{N} \end{array}$$

is a homotopy cokernel of f (when considered in Ch_R). More generally, given a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R, consider the diagram

$$\cdots \xrightarrow{\begin{pmatrix} -d_1^M & 0 \\ f_1 & d_2^N \end{pmatrix}} \begin{matrix} M_0 \\ \oplus \\ N_1 \end{matrix} \xrightarrow{\begin{pmatrix} -d_0^M & 0 \\ f_0 & d_1^N \end{pmatrix}} \underset{\sim\sim\sim}{\begin{matrix} M_{-1} \\ \oplus \\ N_0 \end{matrix}} \xrightarrow{\begin{pmatrix} -d_{-1}^M & 0 \\ f_{-1} & d_0^N \end{pmatrix}} \begin{matrix} M_{-2} \\ \oplus \\ N_{-1} \end{matrix} \xrightarrow{\begin{pmatrix} -d_{-2}^M & 0 \\ f_{-2} & d_{-1}^N \end{pmatrix}} \cdots$$

of R-modules.

Exercise 3.5 (6 points)**.** Verify that the above diagram indeed defines a complex and moreover is a homotopy cokernel of $f_\bullet$.

In particular, we find that for a morphism $M \xrightarrow{f} N$ in Mod_R, we have a canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f)$, and this is a quasi-isomorphism iff f is injective. This is an instance of the general principle that homotopically sensitive constructions are equivalent (in this case quasi-isomorphic) to their ordinary variants in "simple" situations (in this case, when there is no redundancy in the relations). More generally, we have the following.

Exercise 3.6 (6 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R that is injective in each dimension.

(a) Show that the canonical morphism

$$\mathsf{hcoker}(f) \longrightarrow \mathsf{coker}(f)$$

to the levelwise cokernel is a quasi-isomorphism.

(b) Give an example showing that this map need not be a homotopy equivalence.

As the following exercise illustrates, homotopy cokernels give us a way of translating conditions on morphisms between chain complexes to conditions on chain complexes themselves.[33]

Exercise 3.7 (4 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R.

[33] In the paradigm of *Goodwillie calculus*, a category may be thought of as a "categorified manifold": objects and morphisms in the category respectively correspond to points and paths in the manifold. In this analogy, vector spaces correspond to *stable* categories, of which chain complexes give a fundamental example, and passage to homotopy cokernels corresponds to translation to the origin.

(a) Prove that f is a quasi-isomorphism iff $\mathsf{hcoker}(f)$ is acyclic.[34]
(b) Prove that f is a homotopy equivalence iff $\mathsf{hcoker}(f)$ is contractible.

3.2.2. As a special case of a homotopy cokernel, we simply write

$$\Sigma M_\bullet := \mathsf{hcoker}(M_\bullet \longrightarrow 0) \ ,^{35}$$

and refer to this as the ***suspension*** of $M_\bullet$.[36] So by definition, giving a chain map $\Sigma M_\bullet \to T_\bullet$ is equivalent to giving a nullhomotopy of the composite map $M_\bullet \to 0 \to T_\bullet$.

It is evident from the construction that suspension defines an autoequivalence

$$\mathsf{Ch}_R \xrightarrow{\Sigma} \mathsf{Ch}_R \ .$$

Namely,

$$\Sigma\left(\cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} \underset{\sim}{M_0} \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots \right) \cong \left(\cdots \xrightarrow{-d_1} M_0 \xrightarrow{-d_0} \underset{\sim}{M_{-1}} \xrightarrow{-d_{-1}} M_{-2} \xrightarrow{-d_{-2}} \cdots \right) :$$

the operation of suspension simply shifts all terms up by one and negates all differentials. We write Σ^{-1} for its inverse, which we refer to as ***desuspension***. More generally, for any $k \in \mathbb{N}$ we write $\Sigma^k := \Sigma^{\circ k}$ and $\Sigma^{-k} := (\Sigma^{-1})^{\circ k}$. We also note for future reference the evident natural isomorphisms

$$\mathsf{H}_n \circ \Sigma^k \cong \mathsf{H}_{n-k}$$

for all $n, k \in \mathbb{Z}$.

Exercise 3.8 (6 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R.

(a) Using the formula for the homotopy cokernel above, establish a canonical homotopy equivalence

$$\Sigma M \simeq \mathsf{hcoker}(N \xrightarrow{u} \mathsf{hcoker}(f)) \ .^{37}$$

(b) Establish this same homotopy equivalence using the universal characterization of homotopy cokernels (as well as previously established properties of homotopies).

3.3. Homotopy kernels.

[34]The corresponding statement fails e.g. for spaces: there exist spaces X such that the morphism $X \to \mathsf{pt}$ is not a weak homotopy equivalence and yet $\Sigma X := \mathsf{hcoker}(X \to \mathsf{pt})$ is weakly contractible. Such spaces are called *acyclic*, as they are characterized by having the integral homology of a point.

[35]Not coincidentally, when R is commutative this admits a canonical identification

$$\Sigma M_\bullet \cong (R \to \underset{\sim}{0}) \otimes M_\bullet$$

with the tensor product of $M_\bullet$ with the reduced simplicial chains on the simplicial circle $\Delta^1/\partial\Delta^1$. (Note that this is consistent with our sign convention for tensor products; the complex $M_\bullet \otimes (R \to \underset{\sim}{0})$ is different (although naturally isomorphic)).

[36]This is more often denoted $M_\bullet[1]$ and referred to as the *shift* of M, but we prefer the more blatantly topological notation and terminology.

[37]That these are quasi-isomorphic follows from Exercise 3.6(a).

3.3.1. Dually, recall that the kernel of a morphism $M \xrightarrow{f} N$ is Mod_R is by definition an R-module $\mathsf{ker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$\mathsf{ker}(f) \xrightarrow{v} M$$

satisfying the universal property that postcomposition with v determines a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(T, \mathsf{ker}(f)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T \to M \text{ such that the} \\ \text{composite } T \to M \xrightarrow{f} N \text{ is zero} \end{array} \right\} .$$[38]

This leads to the dual notion of a ***homotopy kernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R: this is an object $\mathsf{hker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$\mathsf{hker}(f_\bullet) \xrightarrow{v_\bullet} M_\bullet$$

satisfying the universal property that postcomposition with $v_\bullet$ determines a bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(T_\bullet, \mathsf{hker}(f_\bullet)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T_\bullet \to M_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } T_\bullet \to M_\bullet \xrightarrow{f_\bullet} N_\bullet \end{array} \right\} .$$[39]

Exercise 3.9 (4 points)**.** Given a morphism $M \xrightarrow{f} N$ in Ch_R, prove that the evident levelwise projection map

$$\Sigma^{-1}\mathsf{hcoker}(f) \longrightarrow M$$

is a homotopy kernel of f.[40]

By combining Exercises 3.6(a) and 3.9, it follows that if a morphism $M \xrightarrow{f} N$ in Ch_R is surjective in each dimension, then the canonical morphism

$$\mathsf{ker}(f) \longrightarrow \mathsf{hker}(f)$$

from the levelwise kernel is a quasi-isomorphism (though again it is not necessarily a homotopy equivalence).

[38] Said differently, the kernel of f is the pullback

$$\begin{array}{ccc} \mathsf{ker}(f) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array} .$$

[39] Similarly, the homotopy kernel of f may be characterized as the *homotopy* pullback

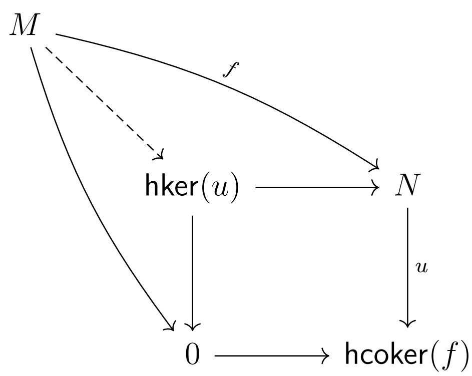
$$\begin{array}{ccc} \mathsf{hker}(f_\bullet) & \longrightarrow & 0 \\ \downarrow & \swarrow & \downarrow \\ M_\bullet & \xrightarrow[f_\bullet]{} & N_\bullet \end{array} ,$$

i.e. the *terminal homotopy-coherent cone* over the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$.

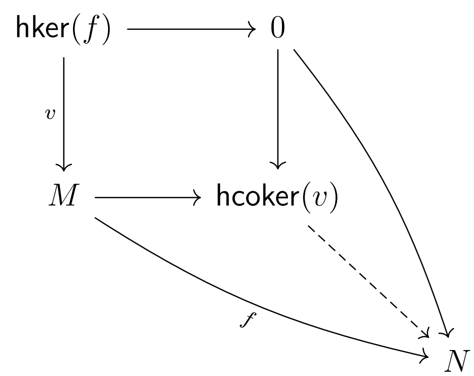
[40] There are unfortunately some signs that should arise here. They could be removed by tweaking the construction of hker (giving a different but isomorphic formula), but they would then arise elsewhere.

3.3.2. Although it will take some time to see why, the following feature is in some sense the *fundamental advantage* of working in Ch_R instead of in Mod_R.[41]

Exercise 3.10 (6 points)**.** Prove that for any morphism $M \xrightarrow{f} N$ in Ch_R the dashed canonical morphisms



and



are homotopy equivalences.[42]

Namely, this implies that up to homotopy equivalence, every homotopy cokernel sequence is a homotopy kernel sequence, and conversely. Of course, this fails drastically in Mod_R: a cokernel sequence is a kernel sequence iff the original map is injective, and a kernel sequence is a cokernel sequence iff the original map is surjective. (So, the only co/kernel sequences in Mod_R that are also homotopy co/kernel sequences in Ch_R are trivial: those for which the original map is an isomorphism.)

3.4. **Exact sequences.**

3.4.1. A complex $M_\bullet \in \mathsf{Ch}_R$ is said to be ***exact*** at M_n if $\mathsf{H}_n(M_\bullet) = 0$. In particular, an acyclic complex is also called an ***exact sequence***, or sometimes a ***long exact sequence*** to emphasize that it is (potentially) infinite in one or both directions.

As a special case, a ***short exact sequence*** is a three-term acyclic complex

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 .$$

So, a short exact sequence is such a diagram satisfying the conditions that f is injective, g is surjective, and $\mathsf{ker}(g) = \mathsf{im}(f)$. In this case, one may also say that M is an ***extension*** of N by L. For instance,

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules, which expresses $\mathbb{Z}/4$ as an extension of $\mathbb{Z}/2$ by itself.

[41]Namely, this is the key property that makes chain complexes into a *stable* category.

[42]Note that these diagrams are only *homotopy-coherently* commutative (indeed, the canonical morphisms are induced by homotopy-coherent universal properties).

More generally, we may refer to any (possibly finite) sequence of morphisms in Mod_R as an ***exact sequence*** if it is exact at all interior terms. So for example, one may refer to the diagram

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

as an exact sequence.[43]

3.4.2. The following is the source of almost every single long exact sequence in mathematics.[44]

Exercise 3.11 (2 points). For any morphism $M \xrightarrow{f} N$ in Ch_R, show that the sequence

$$\mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f))$$

is exact.

Namely, from Exercises 3.8, 3.9, and 3.10 we obtain an infinite sequence

$$\begin{array}{l} \cdots \xrightarrow{\Sigma^{-1}v} \Sigma^{-1}M \xrightarrow{\Sigma^{-1}f} \Sigma^{-1}N \xrightarrow{\Sigma^{-1}u} \Sigma^{-1}\mathsf{hcoker}(f) \\ \qquad\qquad\qquad\qquad\qquad\qquad \wr| \\ \qquad\qquad\qquad\qquad\qquad\quad \mathsf{hker}(f) \xrightarrow{v} M \xrightarrow{f} N \xrightarrow{u} \mathsf{hcoker}(f) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Sigma\mathsf{hker}(f) \xrightarrow{\Sigma v} \Sigma M \xrightarrow{\Sigma f} \Sigma N \xrightarrow{\Sigma u} \cdots \end{array}$$

of morphisms in Ch_R in which every composable pair of morphisms is a homotopy cokernel sequence up to homotopy equivalence. Thereafter, by Exercise 3.11, applying H_0 yields a long exact sequence

$$\begin{array}{l} \cdots \longrightarrow \mathsf{H}_1(M) \longrightarrow \mathsf{H}_1(N) \longrightarrow \mathsf{H}_1(\mathsf{coker}(f)) \\ \qquad\qquad\qquad\qquad\qquad\quad \wr\| \\ \qquad\qquad\qquad\qquad \mathsf{H}_0(\mathsf{hker}(f)) \longrightarrow \mathsf{H}_0(M) \longrightarrow \mathsf{H}_0(N) \longrightarrow \mathsf{H}_0(\mathsf{hcoker}(f)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{H}_{-1}(\mathsf{hker}(f)) \longrightarrow \mathsf{H}_{-1}(M) \longrightarrow \mathsf{H}_{-1}(N) \longrightarrow \cdots \end{array}$$

in Mod_R.

Just as (quasi-isomorphism classes of) complexes encode more information than their homology groups (recall §2.4.4), so does a (quasi-isomorphism class of) morphism in Ch_R encode more information than the corresponding long exact sequence. Therefore, we view the homotopy co/kernel sequence as the more fundamental notion.

The following exercise illustrates the more classical approach to constructing long exact sequences.

[43] Note in particular that we are *not* implicitly extending the sequence by zero here.

[44] A notable exception is the long exact sequence on homotopy groups (of which this is actually a special case).

Exercise 3.12 (6 points)**.** Suppose that $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a sequence of morphisms in Ch_R that is exact in each dimension.[45] Construct a long exact sequence

$$\cdots \xrightarrow{\mathsf{H}_{n+1}(g)} \mathsf{H}_{n+1}(N) \longrightarrow$$

$$\longrightarrow \mathsf{H}_n(L) \xrightarrow{\mathsf{H}_n(f)} \mathsf{H}_n(M) \xrightarrow{\mathsf{H}_n(g)} \mathsf{H}_n(N) \longrightarrow$$

$$\longrightarrow \mathsf{H}_{n-1}(L) \xrightarrow{\mathsf{H}_{n-1}(f)} \cdots$$

in Mod_R.

4. The dg-category of complexes

4.1. **Hom-complexes.** Given complexes $M, N \in \mathsf{Ch}_R$, we have seen that morphisms $M \to N$ may be related by *homotopies*. In fact, these homotopies can be related by *higher homotopies*, ad infinitum. Altogether, these data organize neatly into the ***hom-complex*** from M to N, denoted

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \in \mathsf{Ch}_{\Bbbk} .$$[46]

Namely, in dimension n the hom-complex is given by

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)_n := \prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \ ,$$

and its differential

$$\prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \xrightarrow{d_n} \prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1})$$

carries a tuple $\{f_i \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n})\}_{i \in \mathbb{Z}}$ to the tuple

$$\left\{ (d^N_{i+n} \circ f_i + (-1)^{n-1} \cdot f_{i-1} \circ d^M_i) \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1}) \right\}_{i \in \mathbb{Z}} \ .$$

Exercise 4.1 (2 points)**.** Verify that this formula defines a complex.

Directly from the definition, we have that $\mathsf{Z}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M,N)$. Moreover, given any morphism $M \xrightarrow{f} N$ in Ch_R, a nullhomotopy $0 \overset{h}{\Rightarrow} f$ is equivalent data to an element

[45] Recall from Exercise 3.6(a) that this implies that the canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f) \cong N$ is a quasi-isomorphism.

[46] In fact, the hom-complex also contains "lower homotopies", in the sense that complexes are "like spaces, but with the possibility of cells in negative dimensions".

$h \in \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$ such that $d_1(h) = f$. So, $\mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N))$ is canonically isomorphic to the abelian group of *homotopy classes* of morphisms $M \to N$. Note too that

$$\Sigma^i \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(\Sigma^{-i} M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, \Sigma^i N) \ ;$$

this gives an analogous description of all homology groups of the hom-complex, as $\mathsf{H}_n \cong \mathsf{H}_0 \circ \Sigma^{-n}$. On the other hand, these homology groups can also be understood in a more homotopical (although closely related) manner.

Exercise 4.2 (4 points)**.** Given a chain map $M \xrightarrow{f} N$ and two nullhomotopies $0 \overset{h_0}{\Rightarrow} f$ and $0 \overset{h_1}{\Rightarrow} f$, define a notion of a homotopy $h_0 \Rightarrow h_1$ between homotopies, and prove that such a higher homotopy always exists precisely when $\mathsf{H}_1(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) = 0$.

If one views the complexes M and N as 0-cells, the maps $M \xrightarrow{0} N$ and $M \xrightarrow{f} N$ as 1-cells, and the homotopies h_i as 2-cells, then such a higher homotopy should be viewed as a 3-cell. Of course, there are analogs of this same interpretation for all the homology groups of $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$.

For brevity, we may simply write

$$\underline{\mathsf{hom}}(M, N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \ .$$

4.2. **The dg-category of complexes.** These hom-complexes, in turn, can be naturally assembled into a single object.

Exercise 4.3 (8 points)**.** Fix any complexes $L, M, N, O \in \mathsf{Ch}_R$.

(a) Construct a map

$$\underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, N) \xrightarrow{\chi_{L,M,N}} \underline{\mathsf{hom}}(L, N)$$

in $\mathsf{Ch}_{\Bbbk}$ that encodes the composition of morphisms of complexes.

(b) Verify that these composition morphisms are associative, in the sense that the diagram

$$\begin{array}{ccc} \underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, N) \otimes \underline{\mathsf{hom}}(N, O) & \xrightarrow{\chi_{L,M,N} \otimes \mathsf{id}} & \underline{\mathsf{hom}}(L, N) \otimes \underline{\mathsf{hom}}(N, O) \\ \Big\downarrow {\scriptstyle \mathsf{id} \otimes \chi_{M,N,O}} & & \Big\downarrow {\scriptstyle \chi_{L,N,O}} \\ \underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, O) & \xrightarrow[\chi_{L,M,O}]{} & \underline{\mathsf{hom}}(L, O) \end{array}$$

commutes.[47]

[47] This implicitly uses the associativity of the operation $\otimes$ in $\mathsf{Ch}_{\Bbbk}$.

(c) Construct a map

$$\Bbbk \xrightarrow{\iota_M} \underline{\mathsf{hom}}(M, M)$$

using the identity morphism of $M \in \mathsf{Ch}_R$, and verify that it defines a two-sided identity for the above composition in the sense that the diagrams

$$\begin{array}{ccc} \underline{\mathsf{hom}}(L,M) \otimes \Bbbk & \xrightarrow{\mathsf{id} \otimes \iota_M} & \underline{\mathsf{hom}}(L,M) \otimes \underline{\mathsf{hom}}(M,M) \\ & \searrow^{\cong} & \downarrow \chi_{L,M,M} \\ & & \underline{\mathsf{hom}}(L,M) \end{array}$$

and

$$\begin{array}{ccc} \Bbbk \otimes \underline{\mathsf{hom}}(M,N) & \xrightarrow{\iota_M \otimes \mathsf{id}} & \underline{\mathsf{hom}}(M,M) \otimes \underline{\mathsf{hom}}(M,N) \\ & \searrow^{\cong} & \downarrow \chi_{M,M,N} \\ & & \underline{\mathsf{hom}}(M,N) \end{array}$$

commute (where the isomorphisms are the canonical ones coming from the fact that $\Bbbk \in \mathsf{Ch}_{\Bbbk}$ is the unit object).

Altogether, Exercise 4.3 yields a ($\Bbbk$-***linear***) ***dg-category***,[48] i.e. a category enriched in the symmetric monoidal category $(\mathsf{Ch}_{\Bbbk}, \otimes, \Bbbk)$:[49] its objects are the chain complexes of R-modules and its hom-objects are the hom-complexes between them. We denote this dg-category by $\mathbf{K}_R$ and refer to it as ***the dg-category of complexes of R-modules***.[50] In particular, we may also write $\mathsf{hom}_{\mathbf{K}_R}(M, N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$; our chosen notation will depend on our desired emphasis.

The fundamental purpose of the dg-category $\mathbf{K}_R$ is to assemble the data of chain maps, homotopies, and higher (and lower) homotopies into a single object. In particular, it is the homology groups of the hom-complexes in $\mathbf{K}_R$ that are relevant. Therefore, it will be natural to view the hom-complexes in $\mathbf{K}_R$ as "only being important up to quasi-isomorphism".

[48] Here, "dg" is short for "differential graded". (In general, it is common to refer to a chain complex of R-modules as a *dg-R-module*.)

[49] Of course, a dg-category is a particular instance of a more general notion. Namely, given a monoidal category $\mathcal{V} := (\mathcal{V}, \otimes_{\mathcal{V}}, \mathbb{1}_{\mathcal{V}})$, there is a natural notion of a ***category enriched in*** $\mathcal{V}$, or simply a $\mathcal{V}$-***enriched category***: a $\mathcal{V}$-enriched category $\mathcal{C}$ consists of a set of objects, the data of hom-objects $\underline{\mathsf{hom}}_{\mathcal{C}}(X, Y) \in \mathcal{V}$ for all $X, Y \in \mathcal{C}$, and the data of composition and identity morphisms

$$\underline{\mathsf{hom}}_{\mathcal{C}}(X,Y) \otimes_{\mathcal{V}} \underline{\mathsf{hom}}_{\mathcal{C}}(Y,Z) \xrightarrow{\chi_{X,Y,Z}} \underline{\mathsf{hom}}_{\mathcal{C}}(X,Z) \qquad \text{and} \qquad \mathbb{1}_{\mathcal{V}} \xrightarrow{\iota_Y} \underline{\mathsf{hom}}_{\mathcal{C}}(Y,Y)$$

in $\mathcal{V}$ for all $X, Y, Z \in \mathcal{C}$, subject to the evident associativity and unitality conditions. (As indicated, one sometimes uses an underline to emphasize that these are *enriched* hom-objects (as opposed to mere hom-sets).) On the other hand, note that the notation $\mathsf{hom}_{\mathcal{C}}(X, Y)$ is already unambiguous, as $\mathcal{C}$ is a $\mathcal{V}$-enriched category.

[50] The German word for "complex" begins with the letter "K".

As a special case, note that for any complex $M \in \mathsf{Ch}_R$, composition in $\mathbf{K}_R$ makes the complex $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ into an *associative algebra object*, i.e. it is a ***dg-algebra*** (or ***dga*** for short) over $\Bbbk$.

Exercise 4.4 (4 points)**.** Show that the following conditions are equivalent.

(i) The complex $M \in \mathsf{Ch}_R$ is contractible.
(ii) The complex $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ is acyclic.
(iii) The complex $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M) \in \mathsf{Ch}_{\Bbbk}$ has that $\mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M)) = 0$.
(iv) The element $[\mathsf{id}_M] \in \mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M))$ has that $[\mathsf{id}_M] = 0$.

Of course, this construct can be reversed. Given a dg-algebra $A \in \mathsf{Alg}(\mathsf{Ch}_{\Bbbk})$, we can form a dg-category $\mathfrak{B}A \in \mathsf{Cat}^{\mathsf{dg}}$ as follows: it has a single object $*$, and we declare that $\mathsf{hom}_{\mathfrak{B}A}(*,*) := A$ (with composition and identity respectively defined as the multiplication and unit in A).

On the other hand, among the morphisms in the dg-category $\mathbf{K}_R$, the homotopy equivalences play a distinguished role.

Exercise 4.5 (2 points)**.** Given a homotopy equivalence $M \to N$ in $\mathbf{K}_R$, show that the natural transformations

$$\mathsf{hom}_{\mathbf{K}_R}(-,M) \longrightarrow \mathsf{hom}_{\mathbf{K}_R}(-,N) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(M,-) \longleftarrow \mathsf{hom}_{\mathbf{K}_R}(N,-)$$

are natural homotopy equivalences.

The quasi-isomorphisms cannot play any such distinguished role in $\mathbf{K}_R$. For instance, given any acyclic but noncontractible complex $A \in \mathbf{K}_R$ (recall Exercise 3.4), by Exercise 4.4 the morphism

$$0 \simeq \mathsf{hom}_{\mathbf{K}_R}(A,0) \longrightarrow \mathsf{hom}_{\mathbf{K}_R}(A,A)$$

cannot be a quasi-isomorphism (let alone a homotopy equivalence).

4.3. **Basic features of the dg-category of complexes.** In order to to understand the hom-complexes in the dg-category $\mathbf{K}_R$, it is helpful to understand how they interact with other complexes of $\Bbbk$-modules.

We take our motivation from the expected tensor-hom adjunction for complexes of $\Bbbk$-modules: for any $T, M, N \in \mathsf{Ch}_{\Bbbk}$ we have a natural isomorphism

$$\mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(T \otimes_{\Bbbk} M, N) \cong \mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(M,N)) .$$

The general situation is as follows. For any complex $T \in \mathsf{Ch}_{\Bbbk}$ of $\Bbbk$-modules and any complexes $M, N \in \mathsf{Ch}_R$ of R-modules, we may form the complexes

$$T \otimes_{\Bbbk} M \qquad \text{and} \qquad \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N)$$

of R-modules, where the (right) R-actions are induced from those on M and N. These satisfy the universal properties that

$$\mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N)) \text{ .[51]}$$

In fact, these satisfy enriched universal properties.

Exercise 4.6 (4 points)**.** Prove that for any $T \in \mathsf{Ch}_{\Bbbk}$ and any $M, N \in \mathsf{Ch}_R$ we have natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N))$$

in $\mathsf{Ch}_{\mathbb{Z}}$.

We can now show that the hom-complexes in R preserve homotopy co/kernel sequences separately in each variable.

Exercise 4.7 (8 points)**.** Choose any complexes $M, N, T \in \mathsf{Ch}_R$ and any morphism $M \xrightarrow{f} N$. Using Exercise 4.6 and the universal property of the homotopy co/kernel, construct natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(T, \mathsf{hker}\left(M \xrightarrow{f} N\right)\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T,M) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T,f)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T,N)\right)$$

and

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(\mathsf{hcoker}\left(M \xrightarrow{f} N\right), T\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(N,T) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(f,T)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,T)\right)$$

in $\mathsf{Ch}_{\Bbbk}$.

Of course, essentially identical reasoning shows that the hom-complex bifunctor

$$\mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.[52]

Exercise 4.8 (4 points)**.** Show that the tensor product bifunctor

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.

[51] In the general context of enriched category theory, one says that $T \otimes M$ is the *tensoring* of M by T and that $\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T,N)$ is the *cotensoring* of N by T.

[52] Alternatively, this follows from Exercise 4.7, the commutative square

$$\begin{array}{ccc} \mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_R & \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} & \mathsf{Ch}_R \\ {\scriptstyle \mathsf{id}\times\mathsf{fgt}}\downarrow & & \downarrow{\scriptstyle \mathsf{fgt}} \\ \mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_{\Bbbk} & \xrightarrow[\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)]{} & \mathsf{Ch}_{\Bbbk} \end{array} ,$$

and the fact that the forgetful functor both preserves and detects homotopy kernel sequences.

5. Projective and injective resolutions

5.1. **Motivation for resolutions.** The original motivation for homological algebra is the fact that many natural functors on ordinary modules do not preserve exact sequences. Equivalently but more fundamentally, they do not respect short exact sequences, i.e. they do not respect both kernels and cokernels.[53]

Exercise 5.1 (6 points).
(a) Show that the functor

$$\mathsf{Mod}_{\Bbbk} \times \mathsf{Mod}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Mod}_R$$

does not generally preserve exact sequences in either variable.
(b) Show that the functor

$$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\mathsf{hom}_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_R$$

does not generally preserve exact sequences in either variable.

Namely, applying either bifunctor appearing in Exercise 5.1 to an exact sequence in one of its slots, one obtains a "half-exact" sequence: $(-) \otimes_{\Bbbk} (-)$ preserves cokernels separately in each variable, $\mathsf{hom}_{\mathsf{Mod}_R}(M, -)$ preserves kernels, and $\mathsf{hom}_{\mathsf{Mod}_R}(-, M)$ carries cokernels to kernels. It was originally desired for these half-exact sequences to extend to long exact sequences, whose additional terms would quantify these various failures of exactness.

As illustrated by Exercises 4.7 and 4.8, towards resolving these issues it is fruitful to pass from ordinary modules and ordinary co/kernels to complexes of modules and homotopy co/kernels; the desired long exact sequences would then be those on homology discussed in §3.4, although of course we will take the perspective that the homotopy co/kernel sequences themselves are the more fundamental objects. However, given that we would like to consider quasi-isomorphisms as isomorphisms, the following results show that this maneuver does not suffice on its own.

Exercise 5.2 (6 points).
(a) Show that the functor

$$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Ch}_R$$

does not generally preserve acyclic objects in either variable.
(b) Show that the functor

$$\mathsf{Ch}_R^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(-,-)} \mathsf{Ch}_{\Bbbk}$$

does not generally preserve acyclic objects in either variable.

[53] Here we use the word "respect" instead of "preserve" due to the contravariance of $\mathsf{hom}_{\mathsf{Mod}_R}(-, M)$: recognizing that $\mathsf{Mod}_R^{\mathsf{op}}$ is also an abelian category, one might hope that this would carry kernels to cokernels and cokernels to kernels.

Clearly, preservation of acyclics is necessary for the preservation of quasi-isomorphisms. But in fact, the converse is guaranteed by Exercise 3.7(a) (combined with Exercises 4.7 and 4.8). This motivates the notions that we introduce now.

5.2. Projective and injective complexes.

5.2.1. We write $\mathbf{A}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the acyclic complexes.

We say that a complex $P \in \mathbf{K}_R$ is ***projective*** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(P, A) \in \mathsf{Ch}_{\mathbb{Z}}$ is acyclic. Because acyclic complexes are preserved under de/suspensions, we may equivalently demand simply that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(P, A)) = 0$ for every acyclic complex $A \in \mathbf{A}_R$. In other words, P is projective iff every morphism $P \to A$ to an acyclic complex admits a nullhomotopy. In turn, this is equivalent to the condition that for every solid diagram

$$\begin{array}{ccc} & & \mathsf{hker}(\mathsf{id}_A) \\ & \nearrow & \downarrow v \\ P & \longrightarrow & A \end{array}$$

where $A \in \mathbf{A}_R$ is acyclic there exists a lift making the diagram commute. We write $\mathbf{P}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the projective complexes.

We observe for future reference that v is a levelwise surjective quasi-isomorphism: indeed, $\mathsf{hker}(\mathsf{id}_A)$ is acyclic by the long exact sequence in homology, and v is surjective by construction (or by its defining universal property). So, in order for a complex to be projective it suffices for it to have the analogous lifting property with respect to *all* levelwise surjective quasi-isomorphisms.

Dually, we say that a complex $I \in \mathbf{K}_R$ is ***injective*** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(A, I) \in \mathsf{Ch}_{\mathbb{Z}}$ is acyclic. Likewise, we may equivalently demand for all acyclic complexes $A \in \mathbf{A}_R$ that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(A, I)) = 0$, or that every morphism $A \to I$ admits a nullhomotopy, or that for every solid diagram

$$\begin{array}{ccc} A & \longrightarrow & I \\ u \downarrow & \nearrow & \\ \mathsf{hcoker}(\mathsf{id}_A) & & \end{array}$$

there exists an extension making the diagram commute. We write $\mathbf{I}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the injective complexes.

It is easy to deduce the following facts directly from the definitions.

Exercise 5.3 (6 points)**.**

(a) Show that all quasi-isomorphisms in $\mathbf{P}_R$ are homotopy equivalences.[54]

[54]This is formally analogous to *Whitehead's theorem*, which states that a weak homotopy equivalence between cell complexes (or retracts thereof) is necessarily a homotopy equivalence.

(b) Show that all quasi-isomorphisms in $\mathbf{I}_R$ are homotopy equivalences.

In particular, clearly the zero complex is projective (resp. injective), so that an acyclic projective (resp. injective) complex must be contractible.

(c) Show that projective complexes are preserved under tensor product: if $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$ then $P \otimes Q \in \mathbf{P}_R$.

(d) Show that for any projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P,-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(Q,-)} \mathbf{K}_{\Bbbk}$$

preserve quasi-isomorphisms.

(e) Show that for any injective complex $I \in \mathbf{I}_R$, the functors

$$\mathbf{K}_{\Bbbk}^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(-,I)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(-,I)} \mathbf{K}_{\Bbbk}$$

preserve quasi-isomorphisms.

Moreover, it follows from Exercise 4.5 that the property of projectivity (resp. injectivity) is stable under homotopy equivalence: if a complex is homotopy equivalent to a projective (resp. injective) complex, then it itself is projective (resp. injective).

5.2.2. Of course, the definitions of projective and injective complexes on their own are not so useful. What gives them their power is that every complex $M \in \mathbf{K}_R$ admits both a ***projective resolution*** $P \xrightarrow{\approx} M$ and an ***injective resolution*** $M \xrightarrow{\approx} I$ (i.e. quasi-isomorphisms as indicated). These are the promised representatives of the quasi-isomorphism class of M that are "well-adapted" for certain purposes. Specifically, we will respectively view the functors

$$(-) \otimes_{\Bbbk} P \ , \qquad \mathsf{hom}_{\mathbf{K}_R}(P,-) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-,I)$$

as "corrected" (a.k.a. "derived") versions of the functors

$$(-) \otimes_{\Bbbk} M \ , \qquad \mathsf{hom}_{\mathbf{K}_R}(M,-) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-,I)$$

(and similarly for projective resolutions of complexes of $\Bbbk$-modules).

As we will see, it is relatively straightforward to construct projective resolutions of *bounded-below* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \ll 0$) and to construct injective resolutions of *bounded-above* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \gg 0$). Note in particular that this will apply to ordinary R-modules via the inclusion $\mathsf{Mod}_R \subseteq \mathsf{Ch}_R$.

As for the general (i.e. unbounded) situation, it turns out to be much easier (and more conceptually satisfying) to construct projective resolutions than injective resolutions. On the other hand, we will not have any specific need for injective resolutions in general, beyond their existence. Therefore, we will discuss only projective resolutions in general, and refer the reader to [Spa88] (or to [Hov99, §2.3]) for the construction of injective resolutions in general.

It is easy to deduce the following facts directly from the existence of projective and resolutions.

Exercise 5.4 (4 points)**.**
(a) Show that if the complex $M \in \mathbf{K}_R$ has that $\underline{\mathsf{hom}}_{\mathbf{K}_R}(P, M)$ is acyclic for every projective complex $P \in \mathbf{P}_R$, then M is acyclic.[55]
(b) Show that if the complex $M \in \mathbf{K}_R$ has that $\underline{\mathsf{hom}}_{\mathbf{K}_R}(M, I)$ is acyclic for every injective complex $I \in \mathbf{I}_R$, then M is acyclic.
(c) Show that for any projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{P \otimes_{\Bbbk} (-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_{\Bbbk} \xrightarrow{(-) \otimes_{\Bbbk} Q} \mathbf{K}_R$$

preserve quasi-isomorphisms.

In particular, the tensor product of a projective complex and an acyclic complex is acyclic.

We will eventually organize many of the facts enumerated in Exercises 5.3 and 5.4 in a more systematic way.

5.3. **Projective resolutions in the bounded-below case.** Recall that an R-module $P \in \mathsf{Mod}_R$ is called ***projective*** if mapping out of it preserves surjections, i.e. if for any surjection $M \twoheadrightarrow N$ in Mod_R the induced map $\mathsf{hom}_{\mathsf{Mod}_R}(P, M) \to \mathsf{hom}_{\mathsf{Mod}_R}(P, N)$ is a surjection. Said differently, given any solid diagram

$$\begin{array}{ccc} & & M \\ & \nearrow & \downarrow \\ P & \longrightarrow & N \end{array}$$

there exists a dashed lift making the diagram commute.

Exercise 5.5 (2 points)**.** Show that an R-module $P \in \mathsf{Mod}_R$ is projective iff it is a summand of a free module (i.e. there exists an R-module $Q \in \mathsf{Mod}_R$ and an isomorphism $P \oplus Q \cong R^{\oplus S}$ with the free R-module on a set S).

Exercise 5.6 (2 points)**.** Show that every projective $\mathbb{Z}$-module is free.

Exercise 5.7 (4 points)**.** Give necessary and sufficient conditions on $n \in \mathbb{N}$ such that every projective $\mathbb{Z}/n$-module is free.

Exercise 5.8 (4 points)**.** Show that an R-module $M \in \mathsf{Mod}_R$ is projective iff the corresponding complex $\underset{\sim}{M} \in \mathsf{Ch}_R$ is projective.

We now turn from projective modules back to projective complexes.

Exercise 5.9 (4 points)**.** Show that if $P \in \mathbf{P}_R$ is a projective complex, then $P_n \in \mathsf{Mod}_R$ is a projective R-module for all $n \in \mathbb{Z}$.

[55]Evidently, it is already sufficient just to take $P = R$ (which is projective by Exercises 5.5 and 5.10).

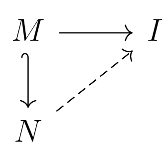
We have the following partial converse.

Exercise 5.10 (6 points)**.** Show that if $P \in \mathbf{K}_R$ is a bounded-below complex such that each $P_n \in \mathsf{Mod}_R$ is projective, then $P \in \mathbf{P}_R$ is projective.

However, the bounded-below hypothesis in Exercise 5.10 is necessary: both parts of Exercise 3.4 give examples of unbounded complexes which are levelwise free (hence levelwise projective by Exercise 5.5) that cannot be projective. Indeed, an acyclic projective complex must be contractible, as its identity map must admit a nullhomotopy.

Exercise 5.11 (4 points)**.** Given a bounded-below complex $M \in \mathbf{K}_R$, construct a projective resolution $P \xrightarrow{\approx} M$.

5.4. **Injective resolutions in the bounded-above case.** Recall that an R-module $I \in \mathsf{Mod}_R$ is called ***injective*** if mapping into it carries injections to surjections, i.e. if for any injection $M \hookrightarrow N$ in Mod_R the induced map $\mathsf{hom}_{\mathsf{Mod}_R}(N, I) \to \mathsf{hom}_{\mathsf{Mod}_R}(M, I)$ is surjective. Said differently, given any solid diagram



there exists a dashed extension making the diagram commute.

Injective modules are much more bizarre than projective modules.

Exercise 5.12 (2 points)**.** Assuming that R is a PID, show that an R-module $M \in \mathsf{Mod}_R$ is injective iff it is *divisible*, i.e. for every nonzero element $r \in R$ the map $M \xrightarrow{r} M$ is surjective.

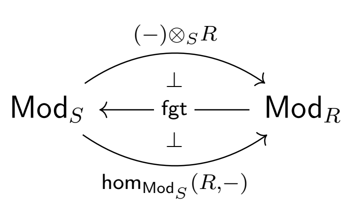
So for instance, $\mathbb{Q} \in \mathsf{Mod}_\mathbb{Z}$ is injective while $\mathbb{Z} \in \mathsf{Mod}_\mathbb{Z}$ is not.

It is easy to dualize the arguments of Exercises 5.9 and 5.10: injective complexes are levelwise injective, and bounded-above levelwise injective complexes are injective.

The argument of Exercise 5.11 uses the fact that every R-module admits a surjection from a projective R-module; one says in this situation that Mod_R ***has enough projectives***. It can be easily dualized to show that any bounded-above complex $M \in \mathbf{K}_R$ admits an injective resolution $M \xrightarrow{\approx} I$, using the following result showing that Mod_R also ***has enough injectives***.

Exercise 5.13 (12 points)**.**

(a) For any ring homomorphism $S \to R$, verify that the functors



participate in adjunctions as indicated.[56]

(b) Given modules $M \in \mathsf{Mod}_R$ and $N \in \mathsf{Mod}_S$ and an injection
$$\mathsf{fgt}(M) \hookrightarrow N$$
in Mod_S, show that the corresponding morphism
$$M \longrightarrow \mathsf{hom}_{\mathsf{Mod}_S}(R, N)$$
in Mod_R is also an injection.

(c) Deduce from the fact that fgt preserves injections that the functor $\mathsf{hom}_{\mathsf{Mod}_S}(R, -)$ preserves injective objects.[57]

(d) Show that injective objects are preserved under products.[58]

(e) Fix an abelian group $A \in \mathsf{Ab}$. Show that for every $a \in A$, there exists a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ carrying a to a nonzero element.[59] Deduce that there exists an injection
$$A \hookrightarrow \prod_A \mathbb{Q}/\mathbb{Z}\ .$$[60]

Namely, suppose we are given an R-module $M \in \mathsf{Mod}_R$. By (d) we have an injection
$$\mathsf{fgt}(M) \hookrightarrow \prod_M \mathbb{Q}/\mathbb{Z} =: I$$
in Ab, and by (d) and Exercise 5.12 we see that $I \in \mathsf{Ab}$ is injective. Thereafter, by (b) we obtain an injection
$$M \hookrightarrow \mathsf{hom}_{\mathsf{Ab}}(R, I) =: J$$
in Mod_R, and by (c) it follows that $J \in \mathsf{Mod}_R$ is injective.

5.5. **Cell complexes and lifting criteria.** For the purpose of constructing projective resolutions, we first introduce some auxiliary ideas.

For any $n \in \mathbb{Z}$, we define the objects and morphism
$$\begin{array}{c} S^n \\ {\scriptstyle i_n} \big\downarrow \\ D^{n+1} \end{array} \quad := \quad \begin{array}{ccc} 0 & \longrightarrow & R \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathsf{id}_R} \\ R & \xrightarrow[\mathsf{id}_R]{} & R \end{array}$$

[56] In defining the functor $(-) \otimes_S R$ (resp. $\mathsf{hom}_{\mathsf{Mod}_S}(R, -)$), we use that R is an (S, R)-bimodule (resp. an (R, S)-bimodule).

[57] Dually, the functor $(-) \otimes_S R$ preserves projective objects because the functor fgt also preserves surjections.

[58] Dually, projective objects are preserved under coproducts.

[59] This uses Exercise 5.12.

[60] One says that $\mathbb{Q}/\mathbb{Z} \in \mathsf{Ab}$ is an ***injective cogenerator***. Dually, $R \in \mathsf{Mod}_R$ is a ***projective generator***: it is projective, and moreover for any $M \in \mathsf{Mod}_R$ there exists a surjection $R^{\oplus S} \twoheadrightarrow M$ for some set S.

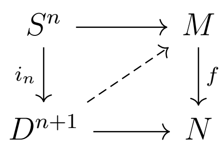
in Ch_R, where the columns are in dimensions $n+1$ and n.[61] For any complex $M \in \mathsf{Ch}_R$, we have natural isomorphisms

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \cong \mathsf{Z}_n(M) \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \cong M_{n+1} \ ;$$

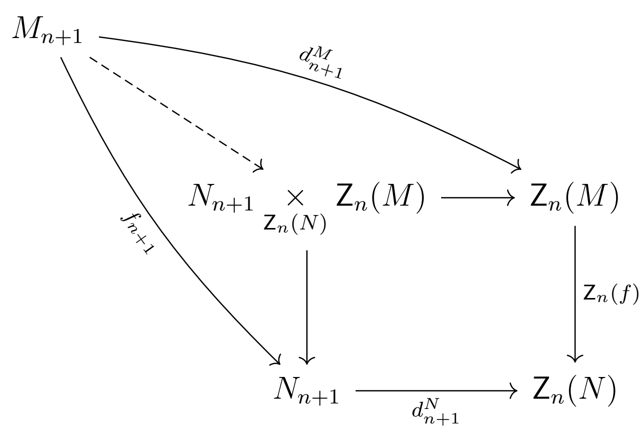
precomposition with i_n induces the morphism

$$\mathsf{Z}_n(M) \xleftarrow{d_{n+1}} M_{n+1} \ .^{62}$$

We say that a morphism $M \xrightarrow{f} N$ has the ***right lifting property*** with respect to i_n (or equivalently that i_n has the ***left lifting property*** with respect to f) if for every solid commutative diagram



there exists a dashed lift that makes the diagram commute. This is equivalent to saying that the canonical dashed morphism in the pullback



is surjective. We may abbreviate this by writing that $f \in \mathsf{rlp}(i_n)$. We also introduce the notation $I := \{i_n\}_{n \in \mathbb{Z}}$ and $\mathsf{rlp}(I) := \bigcap_{n \in \mathbb{Z}} \mathsf{rlp}(i_n)$.

Exercise 5.14 (4 points)**.** Prove that if $f \in \mathsf{rlp}(i_n)$ then $\mathsf{H}_n(f)$ is injective and $\mathsf{Z}_{n+1}(f)$ is surjective.

[61] Of course, this morphism will function as "the inclusion of the n-sphere into the $(n+1)$-disk": indeed, for any $n \geq 0$ it is the reduced simplicial chains on the inclusion $\Delta^{\{0,\ldots,n\}}/\partial\Delta^{\{0,\ldots,n\}} \hookrightarrow \Delta^n/\Lambda^n_0$ (which is a particularly small simplicial model for this morphism). Note too that $S^n \cong \Sigma^n S^0$ and $D^n \cong \Sigma^n D^0$. While the former isomorphism is always an equality (so that we may take $\Sigma^n S^0$ to be the definition of S^n), the latter introduces an inconvenient sign when n is odd (which is why we do not take $\Sigma^n D^0$ to be the definition of D^n).

[62] Categorically speaking, the reason that the morphism

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, -) \xleftarrow{(-)\circ i_n} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, -)$$

in $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_{\mathbb{Z}})$ lifts to $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_R)$ is that the morphism i_n is in fact a morphism of complexes of (R,R)-bimodules.

Now, if $f \in \mathsf{rlp}(I)$, then Exercise 5.14 immediately implies that f is a quasi-isomorphism. On the other hand, suppose that $f \in \mathsf{rlp}(i_{n-1}) \cap \mathsf{rlp}(i_n)$. Then, $\mathsf{Z}_n(f)$ is surjective by Exercise 5.14. It follows that f_{n+1} is surjective by the above reformulation of $\mathsf{rlp}(i_n)$. So, $f \in \mathsf{rlp}(I)$ implies as well that f is levelwise surjective.

In fact, the converse is also true.

Exercise 5.15 (4 points)**.** Prove that if $M \xrightarrow{f} N$ is a quasi-isomorphism that is levelwise surjective, then $f \in \mathsf{rlp}(I)$.

Given a morphism $M \xrightarrow{f} N$ in Ch_R, we write

$$I(f)_n := \mathsf{hom}_{\mathsf{Ar}(\mathsf{Ch}_R)}(i_n, f) \cong \mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \underset{\mathsf{hom}_{\mathsf{Ch}_R}(S^n, N)}{\times} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, N)$$

for the set of morphisms

$$\begin{array}{ccc} S^n & \dashrightarrow & M \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f} \\ D^{n+1} & \dashrightarrow & N \end{array}$$

in the arrow category $\mathsf{Ar}(\mathsf{Ch}_R) := \mathsf{Fun}([1], \mathsf{Ch}_R)$ of Ch_R (i.e. pairs of morphisms $S^n \to M$ and $D^{n+1} \to N$ making the square commute), and we write

$$I(f) := \coprod_{n \in \mathbb{Z}} I(f)_n$$

for their disjoint union over all $n \in \mathbb{Z}$. Given an element $\alpha \in I(f)$, we write $n_\alpha := n$ if $\alpha \in I(f)_n$. (That is, we write $I(f) \xrightarrow{n_{(-)}} \mathbb{Z}$ for the evident function from the disjoint union.)

5.6. Projective resolutions as cellular approximations.

5.6.1. We now construct projective resolutions via a general procedure known as the ***small object argument***.[63] In fact, given a morphism $M \xrightarrow{f} N$ in Ch_R, we will construct a factorization

$$\begin{array}{ccccc} M & & \xrightarrow{\quad f \quad} & & N \\ & {\scriptstyle c^{(\infty)}}\searrow & & \nearrow{\scriptstyle f^{(\infty)}} & \\ & & M^{(\infty)} & & \end{array}$$

satisfying the following conditions:

(1) the morphism $c^{(\infty)}$ has the left lifting property with respect to levelwise surjective quasi-isomorphisms, and
(2) the morphism $f^{(\infty)}$ is a quasi-isomorphism.

[63]This is formally analogous to the construction of a cell complex that is weak homotopy equivalent to a given topological space. The name arises from its crucial use of a certain "smallness" property of the sources of the elements of the set I. In the present situation, the relevant fact is given as Exercise 5.18.

Taking $M \xrightarrow{f} N$ to be the map $0 \to N$ then yields a projective resolution of N.[64]

5.6.2. We make the following general construction. First of all, observe that we have a canonical commutative square

$$\begin{array}{ccc} \bigoplus_{\alpha\in I(f)} S^{n_\alpha} & \longrightarrow & M \\ {\scriptstyle \bigoplus_{\alpha\in I(f)} i_{n_\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1} & \longrightarrow & N \end{array} .$$

We write

$$M^{(1)} := \left(M \coprod_{\bigoplus_{\alpha\in I(f)} S^{n_\alpha}} \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1} \right)$$

for the pushout,[65] and we write

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {\scriptstyle c^{(1)}}\searrow \quad \nearrow{\scriptstyle f^{(1)}} & \\ & M^{(1)} & \end{array}$$

for the morphisms in the induced factorization.

Exercise 5.16 (2 points)**.** Show that the morphism $c^{(1)}$ has the left lifting property with respect to any levelwise surjective quasi-isomorphism.

---

[64]One may interpret the general case as providing a "relatively projective" resolution of N as an object under M. (Whereas a projective complex is like a cell complex (or a retract thereof), a relatively projective complex is like a relative cell complex (or a retract thereof).)

[65]Note that co/limits in chain complexes are levelwise. Indeed, it is a full subcategory of the functor category $\mathsf{Fun}((\mathbb{Z}^{\leq})^{\mathsf{op}}, \mathsf{Mod}_R)$ (where $\mathbb{Z}^{\leq} := \{\cdots \to -1 \to 0 \to 1 \to \cdots\}$), and co/limits in any functor category are computed pointwise. So, it suffices to see that the subcategory $\mathsf{Ch}_R \subseteq \mathsf{Fun}((\mathbb{Z}^{\leq})^{\mathsf{op}}, \mathsf{Mod}_R)$ is stable under co/limits. This follows from the more general fact that the full subcategory of the arrow category $\mathsf{Ar}(\mathsf{Mod}_R) := \mathsf{Fun}([1], \mathsf{Mod}_R)$ on those morphisms that factor through $0 \in \mathsf{Mod}_R$ is stable under co/limits; as co/limits in $\mathsf{Ar}(\mathsf{Mod}_R)$ are of course also computed pointwise, this follows from the fact that the zero object $0 \in \mathsf{Mod}_R$ is stable under co/limits (which follows from the fact that it is both initial and terminal).

Exercise 5.17 (2 points)**.** Show that for every solid commutative diagram

$$\begin{array}{ccccc} S^n & \longrightarrow & & & M \\ \downarrow & & & \swarrow^{c^{(1)}} & \downarrow f \\ & & M^{(1)} & & \\ \downarrow & \nearrow & & \searrow^{f^{(1)}} & \downarrow \\ D^{n+1} & \longrightarrow & & & N \end{array}$$

there exists a dashed factorization making the diagram commute.

We now apply the above construction again, but to $f^{(1)}$ instead of to f: namely, we observe the canonical commutative square

$$\begin{array}{ccc} \bigoplus_{\alpha \in I(f^{(1)})} S^{n_\alpha} & \longrightarrow & M^{(1)} \\ {\scriptstyle \bigoplus_{\alpha \in I(f^{(1)})} i_{n_\alpha}} \downarrow & & \downarrow {\scriptstyle f^{(1)}} \\ \bigoplus_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} & \longrightarrow & N \end{array} ,$$

we write

$$M^{(2)} := \left(M^{(1)} \coprod_{\bigoplus_{\alpha \in I(f^{(1)})} S^{n_\alpha}} \bigoplus_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} \right)$$

for the pushout, and we write

$$\begin{array}{ccc} M^{(1)} & \xrightarrow{f} & N \\ & {\scriptstyle c^{(2)}} \searrow \quad \nearrow {\scriptstyle f^{(2)}} & \\ & M^{(2)} & \end{array}$$

for the morphisms in the induced factorization. This gives us a commutative diagram

$$\begin{array}{ccccc} M & & \xrightarrow{f} & & N \\ & {\scriptstyle c^{(1)}} \searrow & & \nearrow {\scriptstyle f^{(1)}} & \\ & & M^{(1)} & & \uparrow {\scriptstyle f^{(2)}} \\ & & & {\scriptstyle c^{(2)}} \searrow & \\ & & & & M^{(2)} \end{array} .$$

Of course, we continue to iterate this construction in the obvious way. To conclude, we define

$$M^{(\infty)} := \mathsf{colim} \left(M \xrightarrow{c^{(1)}} M^{(1)} \xrightarrow{c^{(2)}} M^{(2)} \xrightarrow{c^{(3)}} \cdots \right) ;$$

we define the map $M \xrightarrow{c^{(\infty)}} M^{(\infty)}$ to be the canonical map to the colimit, and we define the map $M^{(\infty)} \xrightarrow{f^{(\infty)}} N$ to be induced by the universal property of the colimit.

5.6.3. We now verify the two conditions given in §5.6.1.

(1) Let $B \xrightarrow{g} C$ be a levelwise surjective quasi-isomorphism. Given any commutative square

$$\begin{array}{ccc} M & \longrightarrow & B \\ {\scriptstyle c^{(\infty)}}\downarrow & & \downarrow{\scriptstyle g} \\ M^{(\infty)} & \longrightarrow & C \end{array},$$

we enlarge it to the solid commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & B \\ {\scriptstyle c^{(1)}}\downarrow & \nearrow & \\ M^{(1)} & & \\ {\scriptstyle c^{(2)}}\downarrow & \nearrow & \\ M^{(2)} & & \downarrow{\scriptstyle g} \\ {\scriptstyle c^{(3)}}\downarrow & & \\ \vdots & & \\ \downarrow & & \\ M^{(\infty)} & \longrightarrow & C \end{array}.$$

Here, applying Exercise 5.16 repeatedly we may inductively construct dashed lifts as indicated that make the diagram commute. By the universal property of the colimit, together these yield the desired lift in the original diagram.[66]

(2) In fact, we show that $f^{(\infty)} \in \mathsf{rlp}(I)$. For this, suppose we are given any commutative diagram

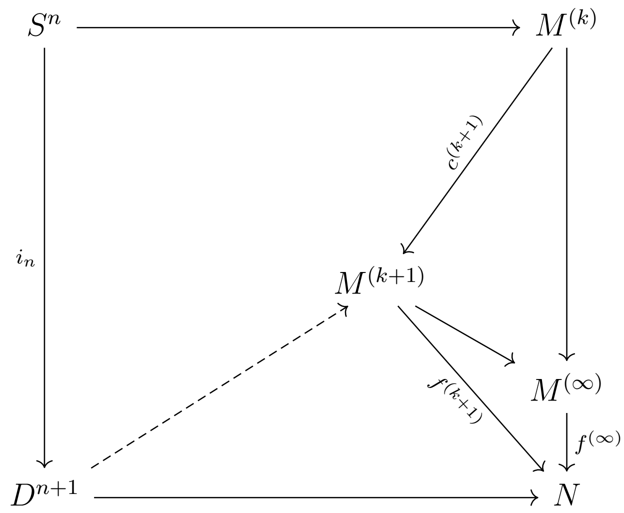
$$\begin{array}{ccc} S^n & \longrightarrow & M^{(\infty)} \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f^{(\infty)}} \\ D^{n+1} & \longrightarrow & N \end{array}.$$

By Exercise 5.18 below, there exists some $k \in \mathbb{N}$ and a lift

$$\begin{array}{ccc} & & M^{(k)} \\ & \nearrow & \downarrow \\ S^n & \longrightarrow & M^{(\infty)} \end{array}$$

[66] This may be phrased as choosing an element of a codirected limit of *surjective* functions between sets.

of the given map. Now, just as in Exercise 5.17, we obtain a lift



,

which proves the claim.

Exercise 5.18 (2 points)**.** Show that the object $S^n \in \mathsf{Ch}_R$ is ***compact***, i.e. that the functor

$$\mathsf{Ch}_R \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(S^n,-)} \mathsf{Set}$$

commutes with filtered colimits.[67]

6. The derived ∞-category

6.1. The homotopy theory of the dg-category of complexes.

6.1.1. We have by now obtained a diagram

$$\begin{array}{ccccc} \mathbf{P}_R & & & & \\ & \searrow & & & \\ & & \mathbf{K}_R & \hookleftarrow & \mathbf{A}_R \\ & \nearrow & & & \\ \mathbf{I}_R & & & & \end{array} \tag{1}$$

of fully faithful inclusions among dg-categories: we began with $\mathbf{A}_R \subseteq \mathbf{K}_R$, and then we defined $\mathbf{P}_R \subseteq \mathbf{K}_R$ (resp. $\mathbf{I}_R \subseteq \mathbf{K}_R$) to be what might be called its *homotopical left* (resp. *right*) *orthogonal*.

[67]Note that a functor commutes with filtered colimits if and only if it commutes with directed colimits. This is one instance of the useful general principle of "coordinatization" of a class of colimits. As another example, given a functor between categories that admit finite colimits, it preserves them if and only if it preserves the initial object and pushouts. Another related fact is that given a functor between cocomplete categories (i.e. categories admitting all (small) colimits), it preserves all (small) colimits if and only if it preserves finite colimits and filtered colimits.

Recall that for any complex $M \in \mathbf{K}_R$ we have constructed a projective resolution $M' \xrightarrow{\approx} M$, i.e. a quasi-isomorphism from a projective object. Consider the resulting homotopy kernel sequence

$$A \longrightarrow M' \longrightarrow M$$

in Ch_R, where we write $A := \mathsf{hker}(M' \to M)$ because this complex is acyclic (by the long exact sequence in homology). Now, for any projective complex $P \in \mathbf{P}_R \subseteq \mathbf{K}_R$, by Exercise 4.7 we obtain a homotopy kernel sequence

$$\underline{\mathsf{hom}}(P, A) \longrightarrow \underline{\mathsf{hom}}(P, M') \longrightarrow \underline{\mathsf{hom}}(P, M)$$

in $\mathsf{Ch}_{\Bbbk}$. Because P is projective, the complex $\underline{\mathsf{hom}}(P, A)$ is acyclic, and hence we find that the morphism

$$\underline{\mathsf{hom}}(P, M') \xrightarrow{\approx} \underline{\mathsf{hom}}(P, M)$$

is a quasi-isomorphism (again by the long exact sequence). Heuristically, we have found that "the morphism $M' \to M$ appears to be an isomorphism when mapping from projective complexes (and considering hom-complexes up to quasi-isomorphism)".

6.1.2. This sort of phenomenon is in fact quite ubiquitous. Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between ordinary categories. Given an object $D \in \mathcal{D}$, a ***pointwise right adjoint*** to F at D consists of an object $C \in \mathcal{C}$ and a morphism $F(C) \to D$ such that the resulting composite morphism

$$\mathsf{hom}_{\mathcal{C}}(-, C) \xrightarrow{F} \mathsf{hom}_{\mathcal{D}}(F(-), F(C)) \longrightarrow \mathsf{hom}_{\mathcal{D}}(F(-), D)$$

in $\mathsf{Fun}(\mathcal{C}, \mathsf{Set})$ is a natural isomorphism. In this situation, we may refer to the morphism $F(C) \to D$ as the ***pointwise counit*** and denote it by ε_D (or simply by ε).

Exercise 6.1 (6 points)**.**

(a) Show that if it exists, a pointwise right adjoint to F at D is unique up to unique isomorphism.[68]

(b) Show that the datum of a right adjoint to F is equivalent data to a choice of pointwise right adjoint to F at every object $D \in \mathcal{D}$.[69]

So, the above discussion may be summarized as saying that in a homotopical sense – that is, if we only consider hom-complexes up to quasi-isomorphism – the fully faithful inclusion $\mathbf{P}_R \hookrightarrow \mathbf{K}_R$ "should" admit a right adjoint whose pointwise counits are projective resolutions. On the other hand, because these morphisms on hom-complexes are only quasi-isomorphisms and not isomorphisms, we should not expect this structure to exist at the level of dg-categories.

[68] Said differently, the category of pointwise right adjoints to F at D is always either an empty or contractible groupoid.

[69] It follows that the category of right adjoints to F is always either an empty or contractible groupoid (because these are preserved under products).

6.1.3. Of course, this discussion immediately dualizes: any complex $M \in \mathbf{K}_R$ has an injective resolution $M \xrightarrow{\approx} M''$ (i.e. a quasi-isomorphism to an injective object), and this has the property that for any injective complex $I \in \mathbf{I}_R \subseteq \mathbf{K}_R$ the morphism

$$\underline{\mathsf{hom}}(M'', I) \xrightarrow{\approx} \underline{\mathsf{hom}}(M, I)$$

is a quasi-isomorphism. Heuristically, "the morphism $M \to M''$ appears to be an isomorphism when mapping to an injective complex (and considering hom-complexes up to quasi-isomorphism)". So, in a homotopical sense the fully faithful inclusion $\mathbf{I}_R \hookrightarrow \mathbf{K}_R$ "should" admit a left adjoint whose pointwise units are injective resolutions.

6.2. A brief introduction to $\Bbbk$-linear ∞-categories.

6.2.1. These adjoints that "should" exist in a homotopical sense actually *do* exist at the level of underlying $\Bbbk$-linear ∞-categories. So, we work within an ∞-categorical context.

However, our present usage of ∞-category theory will be extremely "soft": it will not require any real familiarity with the foundations of the theory. So in the interest of maintaining the narrative thread, here we give an extremely brief summary of the immediately relevant features of ∞-category theory, and pursue a more systematic discussion later.

6.2.2. Here is the most expedient definition of a $\Bbbk$-linear ∞-category.[70] We write cat for the category of ordinary categories, and we write $\mathsf{Cat}^{\mathsf{dg}} \xrightarrow{\mathsf{H}_0} \mathsf{cat}$ for the functor given by applying H_0 to each hom-complex. Then, a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between dg-categories is called a ***weak equivalence*** if the following two conditions hold:

1. it is ***homotopically fully faithful***, i.e. for all $C, C' \in \mathcal{C}$ the induced morphism $\mathsf{hom}_{\mathcal{C}}(C, C') \to \mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$ in $\mathsf{Ch}_{\Bbbk}$ is a quasi-isomorphism; and
2. it is ***essentially surjective***, i.e. the induced functor $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ is essentially surjective.[71]

(Of course, if $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a weak equivalence of dg-categories then $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ is an equivalence of categories.)

We localize the category $\mathsf{Cat}^{\mathsf{dg}}_{\Bbbk}$ of $\Bbbk$-linear dg-categories at the weak equivalences. This yields a category that (for reasons that will become clear later) we will denote by $\mathsf{ho}(\mathsf{Cat}_{\Bbbk})$ and refer to as ***the homotopy category of $\Bbbk$-linear ∞-categories***; its objects are called ***$\Bbbk$-linear ∞-categories***. So by definition, there is a canonical functor

$$\mathsf{Cat}^{\mathsf{dg}} \longrightarrow \mathsf{ho}(\mathsf{Cat}_{\Bbbk})$$

[70] This should be compared with the discussion of §2.4.

[71] It is possible to phrase essential surjectivity at the level of dg-categories, but this requires a bit of enriched category theory.

that carries all weak equivalences to isomorphisms. In particular, this functor is essentially surjective: every $\Bbbk$-linear ∞-category can be represented by a $\Bbbk$-linear dg-category.[72]

Given a dg-category $\mathcal{C}$, for the remainder of this section we will write $\mathcal{C}^\infty$ for its underlying $\Bbbk$-linear ∞-category. However, we will thereafter omit this superscript, as we will have finished drawing distinctions between dg-categories and their underlying $\Bbbk$-linear ∞-categories. We will generally refer to data in $\mathcal{C}$ as "point-set" and data in $\mathcal{C}^\infty$ as "∞-categorical".

Essentially by construction, given two dg-categories $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}^{\mathsf{dg}}$, morphisms $\mathcal{C}^\infty \to \mathcal{D}^\infty$ in $\mathsf{ho}(\mathsf{Cat}_\Bbbk)$ are given by equivalence classes of zigzags

$$\mathcal{C} \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \cdots \longrightarrow \bullet \xleftarrow{\approx} \mathcal{D}$$

in $\mathsf{Cat}^{\mathsf{dg}}$ (in which all backwards maps are weak equivalences). However, it turns out that every equivalence class contains a representative of the form

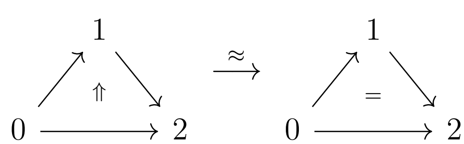
$$\mathcal{C} \xleftarrow{\approx} \mathcal{C}' \longrightarrow \mathcal{D} \ .$$

In fact, dg-categories admit projective resolutions: every equivalence class contains a representative of this form for *any* fixed projective resolution $\mathcal{C}' \xrightarrow{\approx} \mathcal{C}$.

Once one is only considering hom-complexes up to quasi-isomorphism, homotopy equivalences become indistinguishable from isomorphisms. In general, one simply uses the term ***equivalence*** to refer to the ∞-categorical notion of isomorphism. We will denote equivalences in an ∞-category by $\xrightarrow{\sim}$.

For brevity, for the time being we will often simply refer to $\Bbbk$-linear ∞-categories as "∞-categories".

6.2.3. To a first approximation, a $\Bbbk$-linear ∞-category is simply a category enriched in $\mathsf{H}_0(\mathbf{D}_\Bbbk)$, the derived category of $\Bbbk$ introduced in §2.4. In particular, passing from dg-categories to their underlying $\Bbbk$-linear ∞-categories does indeed "only remember their hom-complexes up to quasi-isomorphism". The distinction lies in the higher homotopies (e.g. those recording the homotopy-coherent associativity of composition) which are present in the underlying ∞-category of a dg-category but are lost in passing to its underlying $\mathsf{H}_0(\mathbf{D}_\Bbbk)$-enriched category. Namely, as we will see when we study ∞-categories more generally, given a dg-category $\mathcal{C}$, commutative diagrams in its underlying $\Bbbk$-linear ∞-category are represented by *homotopy-coherently* commutative diagrams in $\mathcal{C}$ itself. For instance, we will see the projective resolution



72 [changed 2/4] This is analogous to the presentation of ordinary ∞-categories as categories enriched in simplicial sets; in particular, in both cases the presentations admit strict compositions (in contrast with the perspective on composition in ∞-categories explained in §6.2.3). A presentation of $\Bbbk$-linear ∞-categories that adheres more closely to e.g. quasicategories or complete Segal spaces is given by ($\Bbbk$-linear) $\mathbb{A}_\infty$-categories.

of the free strictly-commutative triangle $[2] = \{0 < 1 < 2\}$ by the free homotopy-coherently commutative triangle, and this persists upon passing to free dg-categories.[73]

6.2.4. As a particular case, it turns out that homotopy co/kernel squares in dg-categories define co/limit diagrams (namely pushout/pullback squares) in their underlying ∞-categories.[74] For the moment, we will write coker^∞ and ker^∞ to denote ∞-categorical co/kernels.

It will also be convenient to discuss more general ∞-categorical pushout/pullback squares in a $\Bbbk$-linear ∞-category. As an expedient definition, we can declare that a commutative square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[i]{} & Z \end{array}$$

in a $\Bbbk$-linear ∞-category is a ***pushout*** (resp. ***pullback***) if the resulting commutative square

$$\begin{array}{ccc} W & \xrightarrow{(f,g)} & X \oplus Y \\ \downarrow & & \downarrow{\scriptstyle (h,-i)} \\ 0 & \longrightarrow & Z \end{array}$$

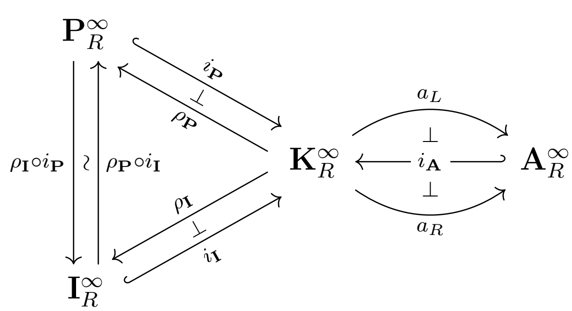
is a cokernel (resp. kernel) square.

It turns out that pushout and pullback squares (and in particular, cokernel and kernel squares) coincide in *any* $\Bbbk$-linear ∞-category.[75] In order to refer to them in an unbiased way, we may refer to them as ***exact squares***.

6.3. **The derived ∞-category.**

6.3.1. As we will explain, the diagram (1) of dg-categories extends to a diagram

(2)



on underlying ∞-categories, in which the triangle involving $\rho_{\mathbf{P}}$, $\rho_{\mathbf{I}}$, and either vertical equivalence commutes. In particular, this yields a canonical equivalence $\mathbf{P}^\infty_R \simeq \mathbf{I}^\infty_R$ of ∞-categories.

[73] To be precise, this is a projective resolution among categories enriched in simplicial sets, and by "the free $\Bbbk$-linear dg-category" on a category enriched in simplicial sets we mean the dg-category obtained by taking $\Bbbk$-linear simplicial chains on its hom-objects.

[74] This should not be surprising, based on our discussion of the former.

[75] This makes crucial use of the $\Bbbk$-linearity: it is certainly not the case in a general ∞-category that pushout and pullback squares coincide.

We write $\mathbf{D}_R$ for their common value, which we refer to as ***the derived ∞-category of R-modules***.[76] (The fact that this is indeed the ∞-categorical localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms will be justified in §6.3.4.) Hence, collapsing the equivalences and using the unbiased notation $\mathbf{D}_R$, the above diagram may be expressed as a diagram

$$\mathbf{D}_R \underset{i_R}{\overset{i_L}{\rightleftarrows}} \xleftarrow{\ \pi\ } \mathbf{K}_R^\infty \xleftarrow{\ i_{\mathbf{A}}\ } \underset{a_R}{\overset{a_L}{\rightrightarrows}} \mathbf{A}_R^\infty \quad , \tag{3}$$

in which i_L (resp. i_R) denotes "the inclusion of $\mathbf{D}_R$ into $\mathbf{K}_R^\infty$ as the full subcategory of projective (resp. injective) complexes", which is the left (resp. right) adjoint to the projection functor π. This latter diagram forms what is called a ***recollement***, a sort of categorified extension sequence, which notion we will discuss more later.

6.3.2. At this point, we can sharpen the perspective on the passage from a dg-category to its underlying ∞-category given in §6.2.3, in a way that will be helpful shortly. Namely, one can also *define* a $\mathbb{k}$-linear ∞-category to be a category enriched in the derived ∞-category $\mathbf{D}_\mathbb{k}$ of $\mathbb{k}$ (using the formalism of enriched ∞-categories of [GH15]). Then, the passage from a $\mathbb{k}$-linear dg-category to its underlying $\mathbb{k}$-linear ∞-category amounts to applying the functor

$$\mathbf{K}_\mathbb{k}^\infty \xrightarrow{\pi} \mathbf{D}_\mathbb{k}$$

to each of its hom-complexes.

It is important to note that there are many commutative squares in $\mathbf{K}_R^\infty$ that may not be co/kernel squares themselves but nevertheless become so in $\mathbf{D}_R$, e.g. any commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{\approx} & Y \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[\approx]{} & Z \end{array} \quad .$$

In general, it is only after projective resolution or injective resolution that such a square becomes a homotopy co/kernel square.

6.3.3. We now obtain the adjunctions and equivalences appearing in diagram (2).

First of all, the existence of the right adjoint $\rho_\mathbf{P}$ follows directly from (the ∞-categorical analog of) Exercise 6.1, and the existence of the left adjoint $\rho_\mathbf{I}$ follows from its dual. We refer to $\rho_\mathbf{P}$ (resp. $\rho_\mathbf{I}$) as the ***projective*** (resp. ***injective***) ***resolution*** functor.

[76] The superscript $(-)^\infty$ need not be applied here, as we have not defined $\mathbf{D}_R$ to be the underlying ∞-category of a specific dg-category: it is only well-defined up to canonical equivalence of ∞-categories in the first place.

Exercise 6.2 (6 points)**.** Prove that the composite adjunction

$$\mathbf{P}_R^\infty \underset{\rho_{\mathbf{P}}}{\overset{i_{\mathbf{P}}}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\bot}}}} \mathbf{K}_R^\infty \underset{i_{\mathbf{I}}}{\overset{\rho_{\mathbf{I}}}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{I}_R^\infty$$

is an adjoint equivalence.

We now construct the adjoint a_L; the adjoint a_R will arises from dual considerations. We refer to a_L (resp. a_R) as the ***left*** (resp. ***right***) ***acyclification*** functor.

For this, let us observe that for any complex $M \in \mathbf{K}_R$, the complex

$$a_L(M) := \mathsf{coker}^\infty \left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M)) \xrightarrow{\varepsilon_M} M \right)$$

is acyclic by Exercise 3.7(a).[77] In other words, the formula

$$a_L := \mathsf{coker}^\infty \left(i_{\mathbf{P}} \circ \rho_{\mathbf{P}} \xrightarrow{\varepsilon} \mathsf{id}_{\mathbf{K}_R} \right)$$

defines a functor $\mathbf{K}_R^\infty \xrightarrow{a_L} \mathbf{A}_R^\infty$.

We now verify that a_L defines a left adjoint to the inclusion $\mathbf{K}_R^\infty \overset{i_{\mathbf{A}}}{\hookleftarrow} \mathbf{A}_R^\infty$. At the ∞-categorical level, this is a straightforward computation. However, it is also a good opportunity to highlight the distinction between working in a dg-category and working in its underlying ∞-category, which makes it somewhat more subtle. Namely, the functor a_L is only well-defined at the ∞-categorical level, which implies that it is not literally meaningful to compute with it at the point-set level. On the other hand, by the dual of Exercise 6.1, it suffices to verify that *any* point-set representative of the morphism $M \to i_{\mathbf{A}} a_L(M)$ in $\mathbf{K}_R^\infty$ is a pointwise unit on underlying ∞-categories. Therefore, let us write

$$i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon'_M} M$$

for a specific but arbitrary point-set representative of the morphism ε_M (i.e. an arbitrary projective resolution of M), and let us write $a_L(M)' := \mathsf{hcoker}(\varepsilon'_M)$; recall that homotopy cokernels in $\mathbf{K}_R$ compute ∞-categorical cokernels in $\mathbf{K}_R^\infty$, so the morphism $M \to a_L(M)'$ in $\mathbf{K}_R$ is indeed a point-set model for the morphism $M \to a_L(M)$ in $\mathbf{K}_R^\infty$. Now, for any acyclic complex $A \in \mathbf{A}_R^\infty$ we compute that

$$\begin{aligned}
\mathsf{hom}_{\mathbf{A}_R^\infty}(a_L(M), A) &:= \pi\left(\mathsf{hom}_{\mathbf{A}_R}(a_L(M)', A)\right) \\
&:= \pi\left(\mathsf{hom}_{\mathbf{K}_R}(a_L(M)', A)\right) \\
&:= \pi\left(\mathsf{hom}_{\mathbf{K}_R}\left(\mathsf{hcoker}\left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon'_M} M\right), A\right)\right) \\
&\simeq \pi\left(\mathsf{hker}\left(\mathsf{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)} \mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right) \qquad (4)
\end{aligned}$$

[77] In view of the adjunction $i_{\mathbf{P}} \dashv \rho_{\mathbf{P}}$, one may think of $a_L(M)$ as being obtained by killing off the part of M that can be seen by projective complexes.

$$\simeq \pi\left(\ker^\infty\left(\mathsf{hom}_{\mathbf{K}_R}(M,A) \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M,A)} \mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))',A)\right)\right) \tag{5}$$

$$\simeq \ker^\infty\left(\pi\left(\mathsf{hom}_{\mathbf{K}_R}(M,A)\right) \xrightarrow{\pi(\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M,A))} \pi\left(\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))',A)\right)\right) \tag{6}$$

$$\simeq \ker^\infty\left(\pi\left(\mathsf{hom}_{\mathbf{K}_R}(M,A)\right) \longrightarrow 0\right) \tag{7}$$

$$\simeq \pi(\mathsf{hom}_{\mathbf{K}_R}(M,A))$$

$$=: \mathsf{hom}_{\mathbf{K}_R^\infty}(M,A)$$

as desired, by the following reasoning.

- Equivalence (4) follows from the fact that hom-complexes respect homotopy co/kernels (Exercise 4.7).
- Equivalence (5) follows from the fact that homotopy kernels in $\mathbf{K}_R$ compute ∞-categorical kernels in $\mathbf{K}_R^\infty$.
- As we have just seen, $\mathbf{K}_\Bbbk^\infty \xrightarrow{\pi} \mathbf{D}_\Bbbk$ is a right adjoint. It therefore preserves ∞-categorical limits, and in particular ∞-categorical kernels. This gives equivalence (6).
- By construction, the complex $i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \in \mathbf{K}_R$ is projective.[78] Hence, the complex $\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))',A) \in \mathbf{K}_\Bbbk$ is acyclic. It follows that we have an equivalence

$$\pi\left(\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))',A)\right) \simeq 0$$

  in $\mathbf{D}_\Bbbk$ (recall Exercise 5.3(a)), which explains equivalence (7).

Because cokernel sequences and kernel sequences in $\mathbf{K}_R^\infty$ agree, note that we can also view the projective (resp. injective) resolution functor as the kernel (resp. cokernel) of the left (resp. right) acyclification functor. Indeed, we have ∞-categorical co/kernel sequences

$$\begin{array}{ccc} i_{\mathbf{P}}\rho_{\mathbf{P}} & \longrightarrow & \mathsf{id}_{\mathbf{K}_R^\infty} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{\mathbf{A}}a_L \end{array} \quad \text{and} \quad \begin{array}{ccc} i_{\mathbf{A}}a_R & \longrightarrow & \mathsf{id}_{\mathbf{K}_R^\infty} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{\mathbf{I}}\rho_{\mathbf{I}} \end{array}$$

in $\mathsf{Fun}(\mathbf{K}_R^\infty, \mathbf{K}_R^\infty)$.

6.3.4. We now simultaneously address the commutativity of the left triangle in diagram (2) and explain why $\mathbf{D}_R$ is the (∞-categorical) localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms.

We say that an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$$

is a ***reflective localization adjunction*** if the right adjoint G is fully faithful. In this situation, we refer to the functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ as a ***reflective localization***.

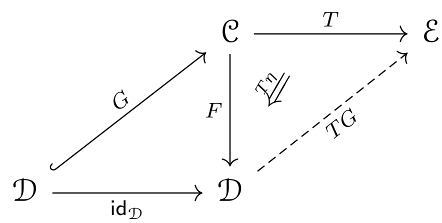
[78] Recall that the property of projectivity is invariant under homotopy equivalence: any complex that is homotopy equivalent to a projective complex is itself projective.

Exercise 6.3 (6 points)**.**

(a) Show that G is fully faithful if and only if the counit $FG \xrightarrow{\varepsilon} \mathsf{id}_{\mathcal{D}}$ is a natural equivalence.

(b) Show that for a reflective localization adjunction $F \dashv G$, the morphism $F \xrightarrow{F\eta} FGF$ is a natural equivalence.

Given such a reflective localization adjunction, let us write $\mathbf{W} \subseteq \mathcal{C}$ for the subcategory consisting of those morphisms that are carried by F to equivalences. Then, we claim that the reflective localization $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is precisely the localization of $\mathcal{C}$ at $\mathbf{W}$.[79] To see this, fix any functor $\mathcal{C} \xrightarrow{T} \mathcal{E}$, and consider the diagram



,

in which the left triangle commutes via the natural equivalence ε of Exercise 6.3(a). Considering just the solid diagram, we see that there is at most one choice of factorization of T through F, namely TG. On the other hand, we *always* have the indicated natural transformation $T \overset{T\eta}{\Longrightarrow} TGF$. So, it remains to check that $T\eta$ is a natural equivalence if and only if T carries the morphisms in $\mathbf{W}$ to equivalences. Now, by Exercise 6.3(b), the components of η lie in $\mathbf{W} \subseteq \mathcal{C}$, which implies the "if" direction. The "only if" direction is handled by the following exercise.

Exercise 6.4 (4 points)**.**

(a) Show that every morphism in the image of $\mathcal{C} \xrightarrow{G} \mathcal{D}$ lies in $\mathbf{W}$.

(b) Use this to show that if T carries all components of η to equivalences then T carries all morphisms in $\mathbf{W}$ to equivalences.

Now, by definition the adjunction $\rho_{\mathbf{I}} \dashv i_{\mathbf{I}}$ is a reflective localization adjunction.

Exercise 6.5 (2 points)**.** Prove that a morphism in $\mathbf{K}_R^{\infty}$ is carried by $\rho_{\mathbf{I}}$ to an equivalence if and only if it is a quasi-isomorphism.

By what we have just seen, this implies that the functor $\mathbf{K}_R^{\infty} \xrightarrow{\rho_{\mathbf{I}}} \mathbf{I}_R^{\infty}$ is indeed the localization of $\mathbf{K}_R^{\infty}$ at the quasi-isomorphisms.

Of course, dually the functor $\mathbf{K}_R^{\infty} \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_R^{\infty}$ is a *coreflective* localization, and so it is *also* the localization of $\mathbf{K}_R^{\infty}$ at the quasi-isomorphisms.

[79] To explain the terminology further, considering $\mathcal{D} \subseteq \mathcal{C}$ as a full subcategory via G, the functor F is also called the *reflector* of $\mathcal{C}$ into $\mathcal{D}$.

So, *both* functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ are the localization at the quasi-isomorphisms. In other words, they share the same universal property. Therefore, they must be canonically equivalent. Moreover, tracing through our proof that a reflective localization is a localization (and dualizing), we find that the canonical morphisms between $\mathbf{P}_R^\infty$ and $\mathbf{I}_R^\infty$ that are induced by their shared universal property (as $\Bbbk$-linear ∞-categories under $\mathbf{K}_R^\infty$) are precisely the composites

$$\begin{array}{ccc} \mathbf{P}_R^\infty & & \\ \uparrow{\scriptstyle \rho_{\mathbf{P}} \circ i_{\mathbf{I}}} & \nwarrow^{\rho_{\mathbf{P}}} & \\ & & \mathbf{K}_R^\infty \\ & \swarrow_{\rho_{\mathbf{I}}} & \\ \mathbf{I}_R^\infty & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{P}_R^\infty & & \\ \downarrow{\scriptstyle \rho_{\mathbf{I}} \circ i_{\mathbf{P}}} & \nwarrow^{\rho_{\mathbf{P}}} & \\ & & \mathbf{K}_R^\infty \\ & \swarrow_{\rho_{\mathbf{I}}} & \\ \mathbf{I}_R^\infty & & \end{array}$$

(which are equivalences by Exercise 6.2). In particular, these triangles commute, as asserted.

It is also not hard to see that these triangles commute using the Yoneda lemma. For instance, given any $P \in \mathbf{P}_R^\infty$ and any $M \in \mathbf{K}_R^\infty$, we have the composite equivalence

$$\mathsf{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}} i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \simeq \mathsf{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \xrightarrow{\sim} \mathsf{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), M) \simeq \mathsf{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}}(M))$$

in $\mathbf{D}_\Bbbk$, using the fact that $\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(P), -)$ carries quasi-isomorphisms to quasi-isomorphisms.

6.3.5. The ∞-category $\mathbf{K}_\Bbbk^\infty$ of complexes over $\Bbbk$ is symmetric monoidal via tensor product, and the ∞-category $\mathbf{K}_R^\infty$ of complexes over R is a left module over it. It follows from Exercise 5.3(c) that the fully faithful inclusions

$$\mathbf{D}_\Bbbk \overset{i_L}{\hookrightarrow} \mathbf{K}_\Bbbk^\infty \qquad \text{and} \qquad \mathbf{D}_R \overset{i_L}{\hookrightarrow} \mathbf{K}_R^\infty$$

endow the derived ∞-categories with the same structures: $\mathbf{D}_\Bbbk$ is symmetric monoidal and $\mathbf{D}_R$ is a left module over it. It is customary to denote these tensor products as

$$(-) \overset{\mathbb{L}}{\otimes} (-) \ ,$$ [80]

in order to emphasize that they are taking place in the context of *derived* ∞-categories.

Note that here we must use the inclusions i_L (not i_R). This is due to the handedness of the tensor-hom adjunction: tensor products satisfying a universal *mapping out* property (instead of a universal *mapping in* property). Said differently, it is the tensor product of *projective* complexes (and not injective complexes) that computes the tensor product of derived modules.

Relatedly, the localization $\mathbf{K}_\Bbbk^\infty \xrightarrow{\pi} \mathbf{D}_\Bbbk$ is *not* a symmetric monoidal localization. For instance, given ordinary $\Bbbk$-modules $M, N \in \mathsf{Mod}_\Bbbk$, as explained further in §7.2, we have a comparison morphism

$$M \overset{\mathbb{L}}{\otimes} N := \rho_{\mathbf{P}}(M) \otimes \rho_{\mathbf{P}}(N) \longrightarrow M \otimes N \tag{8}$$

[80] The notation will be explained in §7.

(note that the inclusion $\mathsf{Mod}_{\Bbbk} \hookrightarrow \mathbf{K}_{\Bbbk}$ is symmetric monoidal), but this morphism is not generally an equivalence. Rather, the functor π is *right-laxly* symmetric monoidal: it comes equipped with natural comparison morphisms (such as the morphism (8)) from the (iterated) tensor product in $\mathbf{D}_{\Bbbk}$ of its values to its value on the corresponding tensor product in $\mathbf{K}_{\Bbbk}^{\infty}$.

Inasmuch as the derived tensor product behaves better than the ordinary tensor product, we view the fact that π is only right-laxly symmetric monoidal as a feature rather than a bug. Moreover, right-laxly symmetric monoidal functors still interact nicely with algebraic structures.[81] For instance, in the present situation, we have a canonical lift

$$\begin{array}{ccc} \mathsf{CAlg}(\mathbf{K}_{\Bbbk}^{\infty}) & \overset{\pi}{\dashrightarrow} & \mathsf{CAlg}(\mathbf{D}_{\Bbbk}) \\ {\scriptstyle \mathsf{fgt}}\downarrow & & \downarrow{\scriptstyle \mathsf{fgt}} \\ \mathbf{K}_{\Bbbk}^{\infty} & \underset{\pi}{\longrightarrow} & \mathbf{D}_{\Bbbk} \end{array} ,$$

so that commutative algebra objects in $\mathbf{K}_{\Bbbk}^{\infty}$ define commutative algebra objects in $\mathbf{D}_{\Bbbk}$, and likewise for associative algebra objects.[82]

6.3.6. Fix an arbitrary abelian category $\mathcal{A}$. Given this, we may form the category $\mathsf{Ch}(\mathcal{A})$ of complexes in $\mathcal{A}$. This admits an enrichment in $\mathsf{Ch}_{\mathbb{Z}}$, i.e. it defines a $\mathbb{Z}$-linear dg-category, whose underlying $\mathbb{Z}$-linear ∞-category we denote by $\mathbf{K}(\mathcal{A}) := \mathbf{K}^{\infty}(\mathcal{A})$. From here, we may form the derived ∞-category $\mathbf{D}(\mathcal{A})$ by freely inverting the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$.

Under suitable hypotheses on $\mathcal{A}$ (namely the existence of enough projectives/injectives and/or the property of being a *Grothendieck* abelian category), the derived ∞-category $\mathbf{D}(\mathcal{A})$ and certain variants (namely the bounded-above and bounded below derived ∞-categories $\mathbf{D}^{-}(\mathcal{A})$ and $\mathbf{D}^{+}(\mathcal{A})$) admit various universal properties.[83] However, these universal properties are somewhat subtle, and make reference to “analytic” notions (in the sense of convergence) involving t-structures on stable ∞-categories. We refer the reader to [§A.1.3] for the specifics, taking the point of view here that the underlying stable ∞-categories are the objects of intrinsic mathematical interest.

7. Derived functors

A substantial vein of current mathematical research takes place entirely within the derived realm, e.g. working only with the derived ∞-category $\mathbf{D}_R$ instead of with the abelian category Mod_R. We will take this point of view when we study sheaf theory. However, it is nevertheless worthwhile to connect back with the original approach to homological algebra. We explain

[81] Dually, left-laxly symmetric monoidal functors interact nicely with *coalgebraic* structures.

[82] It turns out that associative algebra objects in $\mathbf{D}_{\Bbbk}$ always lift to associative algebra objects in $\mathbf{K}_{\Bbbk}$ (i.e. dgas). When $\Bbbk$ has characteristic zero, commutative algebra objects in $\mathbf{D}_{\Bbbk}$ also always lift to commutative algebra objects in $\mathbf{K}_{\Bbbk}$ (i.e. cdgas). However, in general there are commutative algebra objects in $\mathbf{D}_{\Bbbk}$ that do not lift to cdgas; the obstructions effectively arise from the cohomology of symmetric groups (which vanish in characteristic zero).

[83] In the notations $\mathbf{D}^{\pm}(\mathcal{A})$, the superscript indicates the direction in which infinitude is permitted.

the general principles in §7.1, and then illustrate how passing to classical derived functors amounts to working in the derived ∞-category in §§7.2-7.3. We then briefly discuss the example of group co/homology in §7.4. We refer the interested reader to [Wei94] for a more in-depth treatment of group co/homology, as well as for treatments of two other fundamental examples: Lie algebra co/homology and Hochschild co/homology. We conclude with a brief discussion of decategorification in §7.5, which applies to the derived approach to the counting problems described in §1.4.

7.1. **Derived functors.**

7.1.1. As explained in §5.1, the original motivation for homological algebra was to repair certain failures of exactness. Namely, an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories is called

- ***left exact*** if it commutes with finite limits,
- ***right exact*** if it commutes with finite colimits, and
- ***exact*** if it is both left exact and right exact.

These names arise from the following facts (which also serve as mnemonics). Throughout, we refer to an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{9}$$

in $\mathcal{A}$.

Exercise 7.1 (8 points)**.**

(a) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is left exact if and only if it commutes with kernels, i.e. the sequence

$$0 \cong F(0) \longrightarrow F(L) \longrightarrow F(M) \longrightarrow F(N) \tag{10}$$

in $\mathcal{B}$ is also exact.

(b) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is right exact if and only if it commutes with cokernels, i.e. the sequence

$$F(L) \longrightarrow F(M) \longrightarrow F(N) \longrightarrow F(0) \cong 0 \tag{11}$$

in $\mathcal{B}$ is also exact.

So, if F is only left exact, one is inclined to extend the sequence (10) to a long exact sequence

$$
\begin{array}{l}
0 \longrightarrow (\mathbb{R}^0 F)(L) \longrightarrow (\mathbb{R}^0 F)(M) \longrightarrow (\mathbb{R}^0 F)(N) \longrightarrow \\
\hookrightarrow (\mathbb{R}^1 F)(L) \longrightarrow (\mathbb{R}^1 F)(M) \longrightarrow (\mathbb{R}^1 F)(N) \longrightarrow \\
\hookrightarrow (\mathbb{R}^2 F)(L) \longrightarrow \cdots
\end{array}
$$

in $\mathcal{B}$. Here, we refer to $\mathbb{R}^n F$ as the n^{th} ***right derived functor*** of F, and by definition $\mathbb{R}^0 F := F$. Dually, if F is only right exact, one is inclined to the extend the sequence (11) to a long exact sequence

$$
\begin{array}{l}
\cdots \longrightarrow (\mathbb{L}_2 F)(N) \longrightarrow \\
\hookrightarrow (\mathbb{L}_1 F)(L) \longrightarrow (\mathbb{L}_1 F)(M) \longrightarrow (\mathbb{L}_1 F)(N) \longrightarrow \\
\hookrightarrow (\mathbb{L}_0 F)(L) \longrightarrow (\mathbb{L}_0 F)(M) \longrightarrow (\mathbb{L}_0 F)(N) \longrightarrow 0
\end{array}
$$

in $\mathcal{B}$. Here, we refer to $\mathbb{L}_n F$ as the n^{th} ***left derived functor*** of F, and by definition $\mathbb{L}_0 F := F$.[84]

7.1.2. As we saw in §3.4, long exact sequences in algebra arise from homotopy co/kernel sequences in homological algebra. Thus, we should expect to obtain the above long exact sequences from homotopy co/kernel sequences. More specifically, the exact sequence (9) in $\mathcal{A}$ may be thought of as a homotopy co/kernel sequence among complexes in $\mathcal{A}$ (recall Exercise 3.6(a)), to which we wish to apply some functor derived from F to obtain another homotopy co/kernel sequence among complexes in $\mathcal{B}$. We implement this in a slightly more general situation, and then specialize to the case of interest.

Let us write $\mathbf{K}(\mathcal{A})$ for the ($\mathbb{Z}$-linear) ∞-category of complexes in $\mathcal{A}$, and let us write

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\pi} \mathbf{D}(\mathcal{A})$$

[84]A priori, it is not clear that these derived functors are well-defined or uniquely characterized by these conditions, although they do turn out to be so.

for its localization at the quasi-isomorphisms. Then, for any functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{T} \mathcal{C} \ ,$$

we define its ***total right derived functor*** and its ***total left derived functor*** to respectively be the terminal and initial extensions

$$(12) \qquad \begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{T} & \mathcal{C} \\ {\scriptstyle\pi}\big\downarrow & \Swarrow & \nearrow_{\mathbb{R}T} \\ \mathbf{D}(\mathcal{A}) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{T} & \mathcal{C} \\ {\scriptstyle\pi}\big\downarrow & \Nearrow & \nearrow_{\mathbb{L}T} \\ \mathbf{D}(\mathcal{A}) & & \end{array}$$

along π (which may not exist in general). Said differently, these total derived functors participate in comparison maps

$$\pi \circ \mathbb{L}T \longrightarrow T \longrightarrow \pi \circ \mathbb{R}T$$

in $\mathsf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C})$, and by definition they are the universal objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C})$ equipped with such comparison maps. Thereafter, the composites $\pi \circ \mathbb{L}T$ and $\pi \circ \mathbb{R}T$ should be thought of two dual ways of universally forcing the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ to be sent to equivalences in $\mathcal{C}$.

We now return to our additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories. To apply the foregoing discussion, we take T to be the composite functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(F)} \mathbf{K}(\mathcal{B}) \xrightarrow{\pi} \mathcal{D}(\mathcal{B}) \ ,$$

and we denote the resulting total derived functors (which are objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathbf{D}(\mathcal{B}))$) simply by

$$\mathbb{R}F := \mathbb{R}(\pi \circ \mathbf{K}(F)) \qquad \text{and} \qquad \mathbb{L}F := \mathbb{L}(\pi \circ \mathbf{K}(F)) \ .$$

As we will see in Exercise 7.3, we thereafter recover the derived functors of §7.1.1 (which are objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{B})$) as the composites

$$\mathbb{R}^n F \cong \mathsf{H}_{-n} \circ \mathbb{R}F \qquad \text{and} \qquad \mathbb{L}_n F \cong \mathsf{H}_n \circ \mathbb{L}F \ .$$

Examining the universal extensions (12), we see that these assignments $T \mapsto \mathbb{R}T$ and $T \mapsto \mathbb{L}T$ are nothing other than adjoints

$$(13) \qquad \mathsf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C}) \ \underset{\underset{\mathbb{L}(-)}{\dashrightarrow}}{\overset{\overset{\mathbb{R}(-)}{\dashrightarrow}}{\underset{\perp}{\overset{\perp}{\xleftarrow{\mathsf{Fun}(\pi, \mathcal{C})}}}}} \ \mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C})$$

to the restriction functor.[85] (Note the unfortunate conventional clash that taking the *right* derived functor is a *left* adjoint, and reversely.) There are formulas for computing such adjoints in general, but they are extremely easy to compute in certain special cases.

[85]In this generality, these adjoints may not exist. But when a particular total derived functor exists, it is nothing other than a pointwise adjoint as indicated.

Exercise 7.2 (4 points)**.** Given any categories $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$, show that applying the functors $\mathsf{Fun}(\mathcal{F}, -)$ and $\mathsf{Fun}(-, \mathcal{F})$ to an adjunction

$$\mathcal{D} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{E}$$

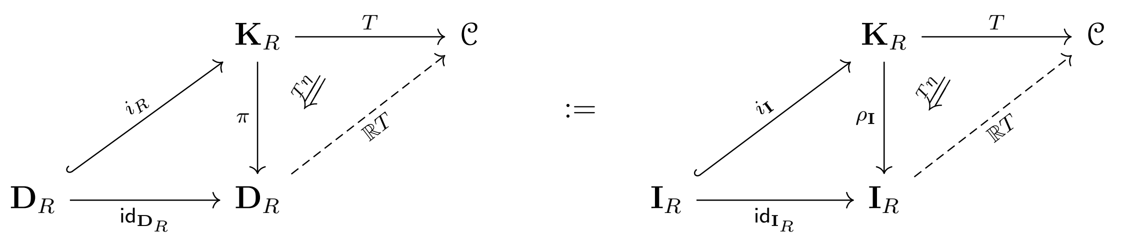
yields adjunctions

$$\mathsf{Fun}(\mathcal{F}, \mathcal{D}) \underset{\mathsf{Fun}(\mathcal{F},R)}{\overset{\mathsf{Fun}(\mathcal{F},L)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Fun}(\mathcal{F}, \mathcal{E}) \qquad \text{and} \qquad \mathsf{Fun}(\mathcal{D}, \mathcal{F}) \underset{\mathsf{Fun}(L,\mathcal{F})}{\overset{\mathsf{Fun}(R,\mathcal{F})}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Fun}(\mathcal{E}, \mathcal{F}) \quad .$$

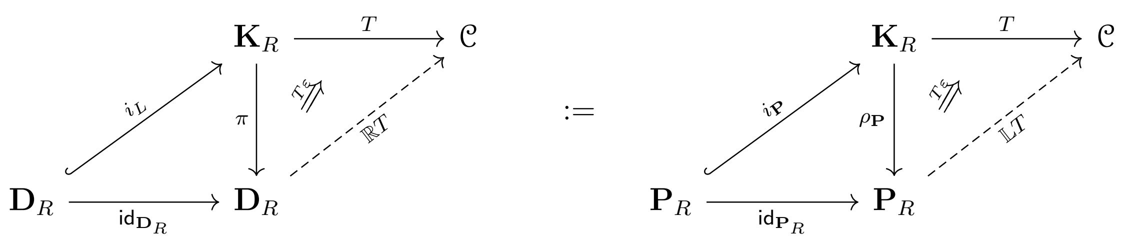
In the particular case that $\mathcal{A} = \mathsf{Mod}_R$ (so that $\mathbf{K}(\mathcal{A}) = \mathbf{K}(\mathsf{Mod}_R) =: \mathbf{K}_R$), the ∞-categorical analog of Exercise 7.2 identifies the adjoints (13) simply as

$$\mathsf{Fun}(\mathbf{K}_R, \mathcal{C}) \; \begin{array}{c} \xrightarrow{\mathsf{Fun}(i_R,\mathcal{C})} \\ \bot \\ \xleftarrow{\mathsf{Fun}(\pi,\mathcal{C})} \\ \bot \\ \xrightarrow[\mathsf{Fun}(i_L,\mathcal{C})]{} \end{array} \; \mathsf{Fun}(\mathbf{D}_R, \mathcal{C}) \quad .$$

In other words, passing to total right derived functors amounts to injective resolution



while passing to total left derived functors amounts to projective resolution



Exercise 7.3 (12 points)**.** Verify that these formulas produce the long exact sequences sought in §7.1.1. In particular,

- in the case that F is left exact, prove that

$$\mathsf{H}_n \circ \mathbb{R}F \cong \begin{cases} F \, , & n = 0 \\ 0 \, , & n > 0 \end{cases} \quad ,$$

and

- in the case that F is right exact, prove that

$$\mathsf{H}_n \circ \mathbb{L}F \cong \begin{cases} F \, , & n = 0 \\ 0 \, , & n < 0 \end{cases} \quad .$$

We have passed from an arbitrary abelian category $\mathcal{A}$ to the abelian category Mod_R in order to guarantee the existence of the adjoints i_L and i_R to the projection π. In particular, in this situation the adjoints (13) exist without any hypotheses on $\mathcal{C}$.

7.2. **Tor.**

7.2.1. We now study Tor, one of two classical examples of a derived functor. By definition, Tor is the left derived functor of relative tensor product. It will be illuminating to work relative to our base commutative ring $\Bbbk$, so we begin in that special case, and then proceed to discuss Tor over the associative $\Bbbk$-algebra R.

7.2.2. The relative tensor product bifunctor

$$\mathsf{Mod}_{\Bbbk} \times \mathsf{Mod}_{\Bbbk} \xrightarrow{(-)\otimes(-)} \mathsf{Mod}_{\Bbbk}$$

is right exact separately in each variable, but it is not left exact in either variable. Let us fix a $\Bbbk$-module $M \in \mathsf{Mod}_{\Bbbk}$ and consider the resulting right exact functor

$$\mathsf{Mod}_{\Bbbk} \xrightarrow{M\otimes(-)} \mathsf{Mod}_{\Bbbk} \ .$$

According to the prescription of §7.1.2, we define its (total) left derived functor $\mathbb{L}(M \otimes (-))$ as the composite

$$\mathbf{D}_{\Bbbk} \simeq \mathbf{P}_{\Bbbk} \xhookrightarrow{i_{\mathbf{P}}} \mathbf{K}_{\Bbbk} \xrightarrow{\mathbf{K}(M\otimes(-))} \mathbf{K}_{\Bbbk} \xrightarrow{\pi} \mathbf{D}_{\Bbbk} \ .$$

In particular, its value on an ordinary $\Bbbk$-module is obtained as the upper composite in the diagram

$$\mathsf{Mod}_{\Bbbk} \hookrightarrow \underbrace{\mathbf{K}_{\Bbbk} \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_{\Bbbk} \xhookrightarrow{i_{\mathbf{P}}} \mathbf{K}_{\Bbbk}}_{\varepsilon \Downarrow \ \ \mathsf{id}_{\mathbf{K}_{\Bbbk}}} \xrightarrow{\mathbf{K}((-)\otimes N)} \mathbf{K}_{\Bbbk} \xrightarrow{\pi} \mathbf{D}_{\Bbbk} \ .$$

Of course, we can equally well perform the same operation in the second slot. We are therefore led to consider *two* possible notions of "the left derived tensor product of M and N". Luckily, they agree: by Exercise 5.4(c), we have natural quasi-isomorphisms fitting into a commutative square

$$\begin{array}{ccc}
 & & \mathbb{L}(M \otimes (-))(N) \\
 & & \| \\
\rho_{\mathbf{P}}(M) \otimes \rho_{\mathbf{P}}(N) & \xrightarrow{\ \approx\ } & M \otimes \rho_{\mathbf{P}}(N) \\
\wr\big\downarrow & & \big\downarrow \\
\rho_{\mathbf{P}}(M) \otimes N & \longrightarrow & M \otimes N \\
\| & & \\
\mathbb{L}((-) \otimes N)(M) & &
\end{array}$$

in $\mathbf{K}_{\Bbbk}$ (omitting the functor $i_{\mathbf{P}}$ for simplicity). It is customary to simply write

$$M \overset{\mathbb{L}}{\otimes} N \in \mathbf{D}_{\Bbbk}$$

for the common value of these derived functors and refer to it as the ***derived tensor product***. Note that this is nothing more than the tensor product in $\mathbf{D}_{\Bbbk}$, as defined in §6.3.5.

For any $n \geq 0$, we define

$$\mathsf{Tor}_n(M,N) := \mathsf{Tor}_n^{\Bbbk}(M,N) := \mathsf{H}_n(M \otimes^{\mathbb{L}} N) \in \mathsf{Mod}_{\Bbbk} \ .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{L}_n(M \otimes (-))(N) \cong \mathsf{Tor}_n(M,N) \cong \mathbb{L}_n((-) \otimes N)(M) \ .$$

7.2.3. The source of the name "Tor" is the following fundamental example.

Exercise 7.4 (6 points)**.** Taking $\Bbbk = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$ and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute $\mathsf{Tor}_i(M,N)$ for all $i \geq 0$.

As each functor $\mathsf{Tor}_i(-,-)$ preserves finite sums separately in each variable, Exercise 7.4 effectively gives a computation of Tor for all finitely generated abelian groups.

7.2.4. We now discuss Tor over the associative $\Bbbk$-algebra R. Of course, it is the left derived functor of the bifunctor

$$\mathsf{Mod}_R \times {}_R\mathsf{Mod} \xrightarrow{(-)\otimes_R(-)} \mathsf{Mod}_{\Bbbk} \ ;$$

the subtlety is simply that this is not a monoidal structure on a single category. Once again, it is sufficient to take a projective resolution in one variable or the other.

7.2.5. It is worth mentioning a particular small (and therefore computable) model for the derived relative tensor product of ordinary R-modules.[86] Namely, given a right R-module $M \in \mathsf{Mod}_R$ and a left R-module $N \in {}_R\mathsf{Mod}$, we define the (***two-sided***) ***bar complex*** $\mathsf{Bar}(M,R,N) \in \mathbf{K}_{\Bbbk}$ to be

$$\cdots \xrightarrow{d_3} M \otimes R^{\otimes 2} \otimes N \xrightarrow{d_2} M \otimes R \otimes N \xrightarrow{d_1} \underset{\sim\sim\sim}{M \otimes N} \longrightarrow 0 \ ,$$

where we define $d_i := \sum_{j=0}^{i} (-1)^j d_i^j$ and we define d_i^j by multiplying the j^{th} and $(j+1)^{\text{st}}$ tensor factors (where we count starting from 0), i.e.

$$d_i^j(m \otimes r_1 \otimes \cdots \otimes r_i \otimes n) := \begin{cases} m \cdot r_1 \otimes r_2 \otimes \cdots \otimes r_i \otimes n \ , & j = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_{j-1} \otimes r_j \cdot r_{j+1} \otimes r_{j+2} \otimes \cdots \otimes r_i \otimes n \ , & 0 < j < i \\ m \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes r_i \cdot n \ , & j = i \end{cases} .$$

Then, we have a canonical equivalence

$$\mathsf{Bar}(M,R,N) \simeq M \underset{R}{\overset{\mathbb{L}}{\otimes}} N \tag{14}$$

[86] This admits a straightforward generalization to complexes of R-modules, which we do not address here.

in $\mathbf{D}_\Bbbk$. [changed 2/9] This is particularly easy to see if $\mathsf{fgt}(M) \in \mathsf{Mod}_\Bbbk$ is projective (or similarly if $\mathsf{fgt}(N) \in \mathsf{Mod}_\Bbbk$ is projective), e.g. if $\Bbbk$ is a field.[87] In general, if N carries a right action S-action that commutes with its left R-action, then we may consider $\mathsf{Bar}(M, R, N) \in \mathbf{K}_S$ as a complex of right S-modules. Taking $N = R = S$, we obtain a complex $\mathsf{Bar}(M, R, R) \in \mathbf{K}_R$ of right R-modules. By Exercise 5.10, this is projective: it is levelwise projective by the dual of Exercise 5.13(c), and moreover it is bounded below.

Exercise 7.5 (2 points)**.** Construct a natural quasi-isomorphism $\mathsf{Bar}(M, R, R) \xrightarrow{\approx} M$ in $\mathbf{K}_R$.

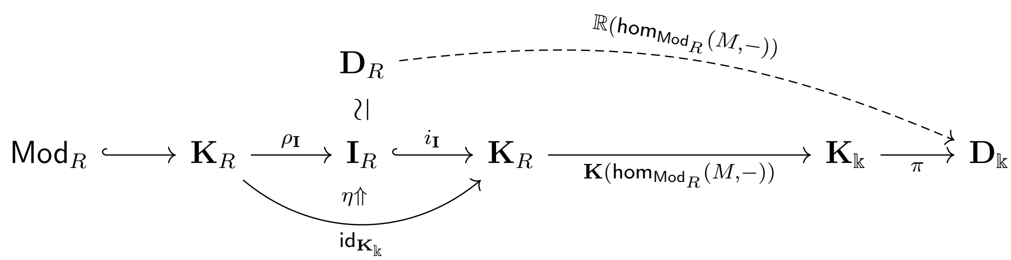
In other words, Exercise 7.5 implies that $\mathsf{Bar}(M, R, R) \xrightarrow{\approx} M$ is a projective resolution. Now, equivalence (14) follows from the evident isomorphism $\mathsf{Bar}(M, R, R) \otimes_R N \cong \mathsf{Bar}(M, R, N)$ in $\mathbf{K}_\Bbbk$.

7.3. **Ext.**

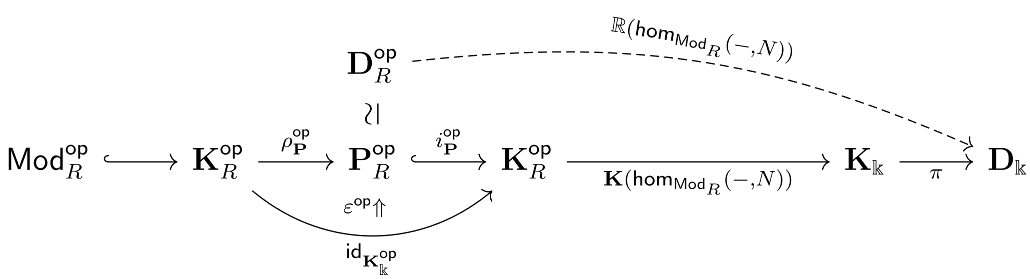
7.3.1. We now study Ext, the other classical example of a derived functor. By definition, Ext is the right derived functor of hom. The hom bifunctor

$$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\mathsf{hom}_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_\Bbbk$$

is left exact separately in each variable, but it is not right exact in either variable.[88] Now, following §7.1.2 we define its two possible right derived functors as in the diagrams



and



.[89]

[87] The proof in the general case requires techniques that we do not discuss here; see [Wei94, §8.6] for an explanation.

[88] When considering contravariant functors, it is customary to put the $(-)^{\mathsf{op}}$ on the source category (e.g. for determining whether the functor $\mathsf{hom}_{\mathsf{Mod}_R}(-, N)$ should be considered as left exact or right exact).

[89] Note that passing to opposites exchanges projectives and injectives.

Once again, these agree: by Exercise 5.3(d)(e), we have natural quasi-isomorphisms fitting into a commutative square

$$\begin{array}{ccc}
 & & \mathbb{R}(\mathsf{hom}_{\mathsf{Mod}_R}(M,-))(N) \\
 & & \| \\
\mathsf{hom}_{\mathsf{Mod}_R}(M,N) & \longrightarrow & \mathsf{hom}_{\mathbf{K}_R}(M,\rho_{\mathbf{I}}(N)) \\
\downarrow & & \downarrow \wr \\
\mathsf{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M),N) & \xrightarrow[\approx]{} & \mathsf{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M),\rho_{\mathbf{I}}(N)) \\
\| & & \\
\mathbb{R}(\mathsf{hom}_{\mathsf{Mod}_R}(-,N))(M) & &
\end{array}$$

in $\mathbf{K}_{\Bbbk}$. It is customary to simply write

$$\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N) \in \mathbf{D}_{\Bbbk}$$

for the common value of these derived functors and refer to it as the ***derived hom***. Note that this is nothing more than the hom in $\mathbf{D}_R$:

$$\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N) \simeq \mathsf{hom}_{\mathbf{D}_R}(M,N) := \mathsf{hom}_{\mathbf{D}_R}(\pi(M),\pi(N)) \ .$$

For any $n \geq 0$, we define

$$\mathsf{Ext}^n(M,N) := \mathsf{Ext}^n_R(M,N) := \mathsf{H}_{-n}(\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N)) \in \mathsf{Mod}_{\Bbbk} \ .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{R}^n(\mathsf{hom}_{\mathsf{Mod}_R}(M,-))(N) \cong \mathsf{Ext}^n(M,N) \cong \mathbb{R}^n(\mathsf{hom}_{\mathsf{Mod}_R}(-,N))(M) \ .$$

7.3.2. The name "Ext" arises from the fact that for any $n \geq 1$, $\mathsf{Ext}^n_R(M,N)$ classifies equivalence classes of exact sequences

$$0 \longrightarrow N \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0 \ , \tag{15}$$

called n***-extensions*** of M by N, under the relation that there exists a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle \mathsf{id}_N}\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathsf{id}_M} & & \\
0 & \longrightarrow & N & \longrightarrow & X'_{n-1} & \longrightarrow & \cdots & \longrightarrow & X'_0 & \longrightarrow & M & \longrightarrow & 0
\end{array} \ . \tag{16}$$

Observe that we may view an n-extension (15) as a sequence of composable morphisms

$$\begin{array}{c} \Sigma^{n-1}N \\ {\scriptstyle f}\downarrow \\ X_\bullet \\ {\scriptstyle g}\downarrow \\ M \end{array} \quad := \quad \begin{array}{ccccccccc} N & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & \underset{\sim}{X_0} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \underset{\sim}{M} \end{array}$$

in Ch_R, such that the composition equals zero. Hence, we obtain a commutative square

$$\begin{array}{ccc} \Sigma^{n-1}N & \xrightarrow{f} & X_\bullet \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & M \end{array}$$

in the ∞-category $\mathbf{K}_R$. This may not be a co/kernel square in $\mathbf{K}_R$, but by the long exact sequence in homology it does become a co/kernel square in $\mathbf{D}_R$. As such, it is completely determined by the morphism $X_\bullet \xrightarrow{g} M$, which in turn is completely specified by the morphism $M \to \mathsf{coker}(g)$. On the other hand, we have a canonical identification

$$\mathsf{coker}(g) \simeq \Sigma^n N \ :$$

both squares in the diagram

$$\begin{array}{ccccc} \Sigma^{n-1}N & \xrightarrow{f} & X_\bullet & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ 0 & \longrightarrow & M & \xrightarrow[k]{} & \mathsf{hcoker}(g) \end{array} \tag{17}$$

in $\mathbf{D}_R$ are pushouts, and so the composite rectangle is also a pushout. Of course, this morphism k is precisely the element of $\mathsf{Ext}^n_R(M,N)$ that is classified by the n-extension (15). Note too that a morphism (16) of n-extensions determines a quasi-isomorphism

$$X_\bullet \xrightarrow{\approx} X'_\bullet$$

in $(\mathsf{Ch}_R)_{\Sigma^{n-1}N//M}$, which gives a homotopy between the corresponding morphisms $M \to \Sigma^n N$ in $\mathbf{D}_R$.

Exercise 7.6 (10 points)**.** Construct an inverse function from $\mathsf{Ext}^n_R(M,N)$ to the set of equivalence classes of n-extensions of M by N.

7.3.3. To simplify our discussion, let us assume that $n \geq 2$. Then, by the long exact sequence in homology, the complex $X \in \mathsf{Ch}_R$ determined by an n-extension (15) has homology groups

$$\mathsf{H}_i(X) \cong \begin{cases} M \ , & i = 0 \\ N \ , & i = n-1 \\ 0 \ , & \text{otherwise} \end{cases} .$$

Said differently, the derived R-module $X \in (\mathbf{D}_R)_{\Sigma^{n-1}N//M}$ has the property that its structure maps $\Sigma^{n-1}N \xrightarrow{f} X \xrightarrow{g} M$ respectively induce isomorphisms on H_{n-1} and H_0. On the other hand, as we have seen, this does *not* characterize X as an object of $(\mathbf{D}_R)_{\Sigma^{n-1}N//M}$: equivalences $X \to X'$ therein correspond to homotopies $k \Rightarrow k'$ in $\mathsf{hom}_{\mathbf{D}_R}(M, \Sigma^n N)$, which are obstructed by $\mathsf{H}_0(\mathsf{hom}_{\mathbf{D}_R}(M, \Sigma^n N)) \cong \mathsf{Ext}^n_R(M, N)$. Indeed, the equivalence class of the object $X \in (\mathbf{D}_R)_{\Sigma^{n-1}N//M}$ is *equivalent data* to the morphism $M \xrightarrow{k} \Sigma^n N$, which is called the $(n-1)^{st}$ ***k-invariant*** of X.

More generally, we can now make precise the assertion of §2.4.4 that a derived R-module is specified by the data of its homology groups and its k-invariants. To explain this, let us write $\mathbf{D}_R^{\leq n} \subseteq \mathbf{D}_R$ for the full subcategory of *n-truncated* derived R-modules, i.e. those with vanishing homology above dimension n. Its inclusion admits a left adjoint

$$\mathbf{D}_R \underset{\hookleftarrow}{\overset{\tau_{\leq n}}{\underset{\perp}{\dashrightarrow}}} \mathbf{D}_R^{\leq n} ,$$ [90]

and from here it should be plausible that we have an equivalence

$$\mathsf{id}_{\mathbf{D}_R} \xrightarrow{\sim} \lim \left(\cdots \longrightarrow \tau_{\leq (n+1)} \longrightarrow \tau_{\leq n} \longrightarrow \tau_{\leq (n-1)} \longrightarrow \cdots \right)$$

in $\mathsf{Fun}(\mathbf{D}_R, \mathbf{D}_R)$:[91] every derived R-module $Y \in \mathbf{D}_R$ is canonically equivalent to the limit of its ***Postnikov tower***

$$\cdots \longrightarrow \tau_{\leq (n+1)} Y \longrightarrow \tau_{\leq n} Y \longrightarrow \tau_{\leq (n-1)} Y \longrightarrow \cdots .$$

Moreover, we see from the iterated co/kernel diagram

$$\begin{array}{ccccc} \Sigma^n \mathsf{H}_n(Y) & \longrightarrow & \tau_{\leq n} Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tau_{\leq (n-1)} Y & \xrightarrow[k_n(Y)]{} & \Sigma^{n+1} \mathsf{H}_n(Y) \end{array}$$

in $\mathbf{D}_R$ (akin to diagram (17)) that given $\tau_{\leq (n-1)} Y$, the data of $\tau_{\leq n} Y$ (equipped with its canonical n-truncation map to $\tau_{\leq (n-1)} Y$) is equivalent to the data of the n^{th} ***k-invariant***

[90] A dual adjunction (from which this one may be immediately deduced) is essentially established in Exercise 8.7.

[91] This assertion is slightly subtle, because homology does *not* commute with (∞-categorical) inverse limits in $\mathbf{D}_R$ in general. However, it *does* commute with the limits of inverse systems whose structure maps are surjective on homology.

of Y, namely the morphism

$$k_n(Y) \in \mathsf{hom}_{\mathbf{D}_R}(\tau_{\leq(n-1)}Y, \Sigma^{n+1}\mathsf{H}_n(Y)) \ .$$

7.4. **Group co/homology.**

7.4.1. Let G be a discrete group. Then, a ($\Bbbk$-linear) ***G-module*** is a $\Bbbk$-module equipped with a (right) G-action. It is easy to see that these are equivalent to modules over the group algebra $\Bbbk[G]$.

Given any G-module $M \in \mathsf{Mod}_{\Bbbk[G]}$, its ***G-invariants*** and ***G-coinvariants*** are the $\Bbbk$-modules

$$M^G := \{m \in M : m \cdot g = m \text{ for all } g \in G\} \qquad \text{and} \qquad M_G := M/\{m - m \cdot g\}_{m \in M, g \in G} \ .$$

These are respectively the maximal $\Bbbk$-submodule and minimal quotient $\Bbbk$-module on which G act trivially.

Let us equip $\Bbbk$ itself with the trivial G-bimodule structure: $g \cdot \lambda := \lambda =: \lambda \cdot g$ for all $g \in G$ and for all $\lambda \in \Bbbk$. Then, it is clear that we have isomorphisms

$$M^G \cong \mathsf{hom}_{\mathsf{Mod}_{\Bbbk[G]}}(\Bbbk, M) \qquad \text{and} \qquad M_G \cong M \underset{\Bbbk[G]}{\otimes} \Bbbk \ .$$ [92]

This makes it clear how to derive these functors, and we define the ***homotopy G-invariants*** and ***homotopy G-coinvariants*** of M to be the derived $\Bbbk$-modules

$$M^{\mathsf{h}G} := \mathbb{R}\mathsf{hom}_{\mathsf{Mod}_{\Bbbk[G]}}(\Bbbk, M) \qquad \text{and} \qquad M_{\mathsf{h}G} := M \overset{\mathbb{L}}{\underset{\Bbbk[G]}{\otimes}} \Bbbk \ .$$

Said differently, we have an evident *augmentation* homomorphism $\Bbbk[G] \to \Bbbk$ given by the formula $g \mapsto 1$ for all $g \in G$, and these derived functors can then be interpreted as defining adjoints

(18)
$$\mathbf{D}_{\Bbbk[G]} \ \underset{(-)^{\mathsf{h}G}\ \bot}{\overset{(-)_{\mathsf{h}G}\ \bot}{\xleftarrow{\ \ \mathsf{fgt}\ \ }}} \ \mathbf{D}_{\Bbbk}$$

(as in Exercise 5.13(a)). Thereafter, we recover the ***cohomology*** and ***homology*** of G with coefficients in M as

$$\mathsf{H}^n(G; M) := \mathsf{H}_{-n}(M^{\mathsf{h}G}) =: \mathsf{Ext}^n_{\Bbbk[G]}(\Bbbk, M) \qquad \text{and} \qquad \mathsf{H}_n(G; M) := \mathsf{H}_n(M_{\mathsf{h}G}) =: \mathsf{Tor}_n^{\Bbbk[G]}(M, \Bbbk) \ .$$ [93]

[92] The first uses the right G-action on $\Bbbk$, while the second uses the left G-action on $\Bbbk$. So these both carry residual right G-actions, but of course these actions are trivial.

[93] As we will see, the G-module M defines a local system on the space $\mathsf{B}G$, and these are precisely its co/homology groups.

7.4.2. In view of Exercise 5.3(c)(d), to compute the co/homology of G it suffices to choose a projective resolution of $\Bbbk$ as a left or right $\Bbbk[G]$-module; this is certainly easier than injectively resolving M in order to compute cohomology, and if we do it once and for all then we need not projectively resolve M in order to compute homology. For this, it is standard to use the one-sidedly projective (in fact free) resolutions

$$\mathsf{Bar}(\Bbbk[G], \Bbbk[G], \Bbbk) \in \left({}_{\Bbbk[G]}\mathbf{P} \underset{{}_{\Bbbk[G]}\mathbf{K}}{\times} {}_{\Bbbk[G]}\mathbf{K}_{\Bbbk[G]} \right) \qquad \text{and} \qquad \mathsf{Bar}(\Bbbk, \Bbbk[G], \Bbbk[G]) \in \left(\mathbf{P}_{\Bbbk[G]} \underset{\mathbf{K}_{\Bbbk[G]}}{\times} {}_{\Bbbk[G]}\mathbf{K}_{\Bbbk[G]} \right) .$$

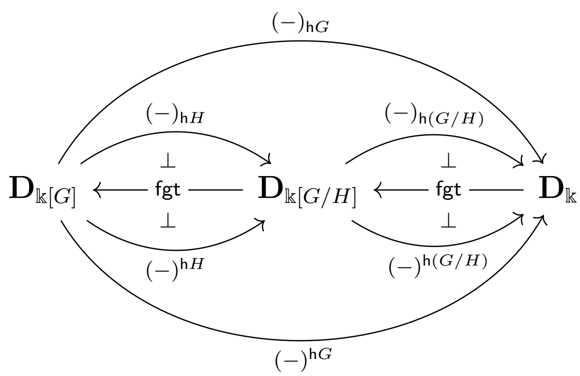
Exercise 7.7 (8 points)**.** Use bar resolutions to compute the homology and cohomology of the cyclic group $\mathsf{C}_n := \mathbb{Z}/n$ with coefficients in the trivial module $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}[\mathsf{C}_n]}$.

In solving Exercise 7.7, one finds that group homology and group cohomology exhibit "mostly periodic" behavior. The corresponding complexes can be combined into a single periodic complex that computes *Tate cohomology*, a fascinating construction originating in the study of class field theory.

7.4.3. Homotopy co/invariants compose nicely. For instance, for any subgroup $H \subseteq G$, taking (homotopy) H-coinvariants is implemented by taking the (resp. derived) tensor product with $\Bbbk[G/H] \in {}_{\Bbbk[G]}\mathsf{Mod}$. When H is normal this carries a commuting right action of G/H,[94] and we find that

$$(-)_{\mathsf{h}(G/H)} \circ (-)_{\mathsf{h}H} := (-) \otimes^{\mathbb{L}}_{\Bbbk[G]} \Bbbk[G/H] \otimes^{\mathbb{L}}_{\Bbbk[G/H]} \Bbbk \simeq (-) \otimes^{\mathbb{L}}_{\Bbbk[G]} \Bbbk =: (-)_{\mathsf{h}G} .$$[95]

Of course, an analogous relation holds for homotopy invariants. From the perspective of the diagram (18) describing homotopy co/invariants as adjoint functors, we can also view these relations as arising from a diagram



in which the upper and lower (hyperbolic) triangles commute by the the uniqueness of adjoints.

Homotopy co/invariants admit other subtler relations.

[94]More generally, this carries an action of the *Weyl group* of H in G, the quotient $\mathsf{W}_G(H) := \mathsf{N}_G(H)/H$ by it of its normalizer. (In fact, $\mathsf{W}_G(H)$ is precisely the monoid of automorphisms of G/H as a left G-set.)

[95]Here we use the facts that the derived relative tensor product is associative and the equivalence $\Bbbk[G/H] \otimes^{\mathbb{L}}_{\Bbbk[G/H]} \Bbbk \xrightarrow{\sim} \Bbbk$ of derived left $\Bbbk[G]$-modules.

Exercise 7.8 (12 points)**.** Let $G * H$ denote the coproduct (a.k.a. free product) of groups G and H. For any derived $(G*H)$-module $M \in \mathbf{D}_{\Bbbk[G*H]}$, construct exact squares

$$\begin{array}{ccc} M & \longrightarrow & M_{\mathsf{h}G} \\ \downarrow & & \downarrow \\ M_{\mathsf{h}H} & \longrightarrow & M_{\mathsf{h}(G*H)} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \longleftarrow & M^{\mathsf{h}G} \\ \uparrow & & \uparrow \\ M^{\mathsf{h}H} & \longleftarrow & M^{\mathsf{h}(G*H)} \end{array}$$

in $\mathbf{D}_{\Bbbk}$.[96]

7.5. Decategorification.

7.5.1. The term ***decategorification*** refers informally to the process of taking a mathematical object and extracting an invariant of it that lies at a lower categorical level – e.g. extracting a vector space from a (linear) category, or extracting a number from a vector space. Indeed, a first example – which appeared in §1 – is given by taking the dimension of a (finite-dimensional) vector space. This is generalized by taking the ***Euler characteristic*** of a (perfect) derived module. We explain this in the two relevant examples introduced in §1.4, both of which take place over the field $\mathbb{R}$, and then discuss Euler characteristics over more general rings.

7.5.2. Let us briefly recall the setup of §1. We wished to compute algebraic intersection numbers of subvarieties of complementary codimensions. That is, we wished to compute the "expected" number of intersection points that would be literally obtained by perturbing the intersection to be transverse, without actually making such a perturbation. For a transverse intersection (and in somewhat more generality (recall §1.4.2)), we saw that computing the dimension of a corresponding tensor product gave the correct answer. It was asserted that derived tensor products would give the correct answer unconditionally.

7.5.3. As we saw in §1.4.3, the first instance where ordinary tensor products fail to give the correct answer is for the self-intersection of a single point $a \in \mathbb{R}^1$. In this case, the corresponding ***derived intersection*** is computed by the derived tensor product

$$\mathbb{R}[x]/(x-a) \overset{\mathbb{L}}{\underset{\mathbb{R}[x]}{\otimes}} \mathbb{R}[x]/(x-a) \ . \tag{19}$$

That is, we can enhance the ordinary intersection by passing to the world of ***derived algebraic geometry***: the derived intersection has an underlying variety given by the ordinary intersection (which is just the point $a \in \mathbb{R}^1$), but then it has a *derived* commutative $\mathbb{R}$-algebra of functions, namely this derived relative tensor product.[97]

[96] This may be seen as a Mayer–Vietoris decomposition for (twisted) co/homology via the wedge sum decomposition $\mathsf{B}(G*H) \simeq \mathsf{B}G \vee \mathsf{B}H$, as we will see below.

[97] Recall from §6.3.5 that projective resolution carries commutative algebras to commutative algebras.

In order to compute this derived relative tensor product, we form the projective resolution

$$P := \left(\mathbb{R}[x] \xrightarrow{(x-a)} \underset{\sim\sim\sim}{\mathbb{R}[x]} \right) \xrightarrow{\approx} \mathbb{R}[x]/(x-a)$$

(as a complex of $\mathbb{R}[x]$-modules). Then, the ordinary relative tensor product

$$P \underset{\mathbb{R}[x]}{\otimes} \mathbb{R}[x]/(x-a) \cong \left(\mathbb{R} \xrightarrow{0} \underset{\sim}{\mathbb{R}} \right)$$

in $\mathbf{K}_{\mathbb{R}}$ represents the derived relative tensor product (19) in $\mathbf{D}_{\mathbb{R}}$.[98]

The relevant numerical invariant of this derived $\mathbb{R}$-module is defined on the full subcategory $\mathbf{D}_{\mathbb{R}}^{\mathsf{perf}} \subseteq \mathbf{D}_{\mathbb{R}}$ of ***perfect*** (derived) $\mathbb{R}$-modules, i.e. those with finite-dimensional total homology (equivalently, those with finitely many nonzero homology groups that are each finite-dimensional $\mathbb{R}$-modules).[99] Namely, given a perfect $\mathbb{R}$-module $M \in \mathbf{D}_{\mathbb{R}}^{\mathsf{perf}}$, its ***Euler characteristic*** is the alternating sum

$$\chi(M) := \sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim_{\mathbb{R}}(\mathsf{H}_n(M)) \ .$$

Here we find that

$$\chi \left(\mathbb{R}[x]/(x-a) \overset{\mathbb{L}}{\underset{\mathbb{R}[x]}{\otimes}} \mathbb{R}[x]/(x-a) \right) = \dim_{\mathbb{R}}(\mathbb{R}) - \dim_{\mathbb{R}}(\mathbb{R}) = 0 \ ,$$

as desired: generically, the intersection of two points in the line is empty.

7.5.4. We now explain how to compute the derived self-intersection of a projective line in $\mathbb{RP}^2$ (as asserted in §1.4.5), using some basic facts about the cohomology of line bundles on projective space (see e.g. [Har77, §III.5]).[100]

Given a curve $Z \subseteq X := \mathbb{RP}^2$, let us write $\mathcal{O}_Z$ for the corresponding ideal sheaf on X. Then, the ***derived self-intersection*** of Z with itself in X is the derived scheme

$$Y := Z \overset{\mathbb{R}}{\underset{X}{\times}} Z := \left(|Z|, \mathcal{O}_Z \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{O}_Z \right)$$

(expressed as a topological space equipped with a sheaf of derived commutative $\mathbb{R}$-algebras, and omitting the pullback functor from the notation). Now, because the underlying variety of Y is no longer just a point, in order to extract a numerical invariant we must take the Euler characteristic of Y: that is, the Euler characteristic of the derived global sections (a.k.a. (hyper)cohomology) of its structure sheaf.[101]

[98] It is not hard to see that the commutative algebra structure is the evident one that exists at the point-set level on this particular chain complex representative.

[99] In general, the subcategory $\mathbf{D}_R^{\mathsf{perf}} \subseteq \mathbf{D}_R$ of perfect (derived) R-modules can be defined as the smallest full subcategory containing the object $R \in \mathbf{D}_R$ that is closed under co/kernels, de/suspensions, and retracts.

[100] In fact, essentially the same computation applies to the derived intersection of any pair of (possibly equal) algebraic curves in $\mathbb{RP}^2$, yielding the product of their degrees as the algebraic intersection number.

[101] The fact that Y arose as the intersection of two subvarieties of complementary codimension is reflected in its ***virtual dimension***, which can be read off from its cotangent complex (as described below).

Exercise 7.9 (8 points)**.** For definiteness, let us take $Z \subseteq X := \mathbb{RP}^2$ to be the closure of the x-axis, i.e. the locus $\{[x : y : z] \in \mathbb{RP}^2 : y = 0\}$.

(a) Find a line bundle L over X and a section s of L such that $Z = s^{-1}(0)$.

This gives a short exact sequence $0 \to L^{-1} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_Z \to 0$, which we may view as defining a quasi-isomorphism $M := (L^{-1} \xrightarrow{s} \underset{\sim\sim}{\mathcal{O}_X}) \xrightarrow{\approx} \mathcal{O}_Z$. Thereafter, we may compute the derived tensor product as $\mathcal{O}_Z \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_Z \simeq M \otimes_{\mathcal{O}_X} \mathcal{O}_Z$.[102]

(b) Compute the Euler characteristic of Y, i.e. the Euler characteristic of the derived global sections (a.k.a. hypercohomology) of its structure sheaf.[103]

7.5.5. The Euler characteristic of derived modules over more general rings is not a priori well-defined. This is rectified by a universal construction known as ***algebraic K-theory***, as we now explain.

We begin with the following motivating observation: given a co/kernel sequence of derived $\mathbb{R}$-modules $L \to M \to N$, by the long exact sequence in homology we have an equality $\chi(M) = \chi(L) + \chi(N)$. This is referred to as the ***additivity*** of Euler characteristic.[104]

We define the full subcategory $\mathbf{D}_R^{\mathsf{perf}} \subseteq \mathbf{D}_R$ of ***perfect*** (derived) R-modules to be the smallest subcategory containing R and closed under retracts, co/kernels, and de/suspensions.[105] (Alternatively, these are the derived R-modules that can be presented by bounded complexes of finite-rank projective R-modules.) Then, the 0^{th} ***algebraic K-group*** of $\mathbf{D}_R^{\mathsf{perf}}$ is the abelian group

$$\mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}})$$

defined as follows: it has a generator $[M]$ for every perfect R-module $M \in \mathbf{D}_R^{\mathsf{perf}}$, and for every co/kernel sequence $L \to M \to N$ it has a relation $[M] = [L] + [N]$.[106] (In particular, if $M \simeq M'$ then $[M] = [M']$, because the co/kernel of an equivalence is zero.) By construction,

[102]That this indeed computes the derived tensor product follows from the fact that this is a *flat* resolution of $\mathcal{O}_X$-modules. (An $\mathcal{O}_X$-module is flat iff it is locally flat.) Note that there are no nonzero projective objects in the abelian category of $\mathcal{O}_X$-modules, so our previous considerations do not immediately apply here.

[103]Equivalently, one can compute the alternating sum of the Euler characteristics of a specific chain complex representative of the structure sheaf. This is an enhancement of the fact that given a bounded complex $M \in \mathsf{Ch}_{\Bbbk}$ of modules over a field $\Bbbk$, we have the alternative formula

$$\chi(M) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim_{\Bbbk}(M_n)$$

for the Euler characteristic of its underlying derived $\Bbbk$-module.

[104]The Euler characteristic for (suitably finite) spaces is likewise additive for cofiber sequences.

[105]This finiteness restriction is to avoid the so-called "Eilenberg swindle": for instance, the short exact sequence

$$0 \longrightarrow R \xrightarrow{r \longmapsto (r,0,0,\ldots)} R^{\oplus \mathbb{N}} \xrightarrow{(r_1,r_2,\ldots) \longmapsto (r_2,r_3,\ldots)} R^{\oplus \mathbb{N}} \longrightarrow 0$$

of R-modules would yield the relation that $[R] = 0$.

[106]The German word for "class" (as in "equivalence class") begins with the letter "K".

we have a "generalized Euler characteristic" function

$$\{\text{objects of } \mathbf{D}_R^{\mathsf{perf}}\} \xrightarrow{M \longmapsto [M]} \mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}}) ,$$

which is additive for co/kernel sequences.

In fact, it is not so hard to compute $\mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}})$. Let us define an abelian group $\mathsf{K}_0(\mathsf{Mod}_R^{\mathsf{f.g.,proj}})$ as follows: it has a generator $[M]$ for each finitely-generated projective R-module $M \in \mathsf{Mod}_R^{\mathsf{f.g.,proj}}$, and for every short exact sequence $0 \to L \to M \to N \to 0$ among such it has a relation $[M] = [L] + [N]$. Observe that there is an the evident homomorphism

$$\mathsf{K}_0(\mathsf{Mod}_R^{\mathsf{f.g.,proj}}) \longrightarrow \mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}}) . \tag{20}$$

Exercise 7.10 (10 points)**.** Prove that there is a well-defined homomorphism

$$\begin{array}{ccc} \mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}}) & \longrightarrow & \mathsf{K}_0(\mathsf{Mod}_R^{\mathsf{f.g.,proj}}) \\ \Downarrow & & \Downarrow \\ [M] & \longmapsto & \displaystyle\sum_{n \in \mathbb{Z}} (-1)^n \cdot [\mathsf{H}_n(M)] \end{array}$$

that defines an inverse to the homomorphism (20).

In particular, we see that K-theory classes are insensitive to k-invariants.

As the notation indicates, it is possible to define higher (and in fact also lower) K-groups. As with the discussion of §7.1.1, these constructions are motivated by a desire to repair certain failures of exactness of the functor K_0, although now the notion of "exactness" becomes substantially more subtle. To a first approximation, the functor K takes values in derived $\mathbb{Z}$-modules, and we recover the n^{th} algebraic K-group as $\mathsf{K}_n := \mathsf{H}_n \circ \mathsf{K}$. We refer the reader to [Wei13] for a comprehensive introduction to algebraic K-theory.

Part II. Higher category theory

In this part, we give a rapid introduction to ∞-category theory. We take the point of view that it is easiest to learn this theory through examples; those that we choose to highlight are provided by sheaf theory, which is the subject of **??**. So, the present part will contain relatively few examples, in the interest of reaching sheaf theory as quickly as possible. In particular, the present account of ∞-category theory is very far from exhaustive. Rather, it is intended to simultaneously explain

(1) how ∞-categories are actually used in practice, and
(2) how this usage connects with the rigorous definitions.

We address these goals in reverse order: the foundations of ∞-category theory are based in the theory of model categories, and so we begin with a rapid summary of the latter.[107]

[107] Also, some indication of how ∞-categories are used in practice already appeared in Part I.

The character of our exposition will change fairly dramatically here: we will introduce many results here without any semblance of proof, especially after §8 (although we will give references where relevant). Based on these black boxes, we will return to proving most of our assertions in **??**.

8. Model categories

8.1. Model categories, Quillen adjunctions, and Quillen equivalences.

8.1.1. A ***relative category*** is a pair $(\mathcal{C}, \mathbf{W})$ of a category $\mathcal{C}$ equipped with a subcategory $\mathbf{W} \subseteq \mathcal{C}$, called the subcategory of ***weak equivalences***. We generally denote weak equivalences by $\xrightarrow{\approx}$. Given a relative category $(\mathcal{C}, \mathbf{W})$, its ***localization*** (also often called its ***homotopy category***) is the category $\mathcal{C}[\mathbf{W}^{-1}]$ obtained from $\mathcal{C}$ by freely inverting the morphisms in $\mathbf{W}$.[108]

A ***model structure*** on a relative category $(\mathcal{C}, \mathbf{W})$ consists of additional structure which makes it feasible to perform computations in the localization $\mathcal{C}[\mathbf{W}^{-1}]$. In particular, this gives a means of obtaining adjunctions and equivalences among localizations of relative categories.

We quickly give the relevant definitions, and then illustrate them through a number of examples. We note here that the examples will be much more important for us than the specific details of the definitions.

8.1.2. A ***model structure*** on a relative category $(\mathcal{C}, \mathbf{W})$ consists of subcategories $\mathbf{C}, \mathbf{F} \subseteq \mathcal{C}$, whose morphisms are respectively called ***cofibrations*** and ***fibrations*** and are respectively denoted by $\rightarrowtail$ and $\twoheadrightarrow$; morphisms in $\mathbf{W} \cap \mathbf{C}$ (resp. $\mathbf{W} \cap \mathbf{F}$) are respectively called ***acyclic co/fibrations***. These data are required to satisfy the following axioms.

(1) The category $\mathcal{C}$ has all finite limits and colimits.
(2) The subcategory $\mathbf{W} \subseteq \mathcal{C}$ satisfies the two-out-of-three property: given any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, if any two of the morphisms f, g, and gf lie in $\mathbf{W}$ then so does the third.
(3) Every morphism $X \to Y$ admits factorizations

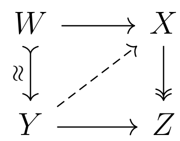
$$\left(X \overset{\approx}{\rightarrowtail} X' \twoheadrightarrow Y \right) \quad \text{and} \quad \left(X \rightarrowtail Y' \overset{\approx}{\twoheadrightarrow} Y \right) .$$

(4) We have equalities $(\mathbf{W} \cap \mathbf{C}) = \mathsf{llp}(\mathbf{F})$ (or equivalently $\mathbf{F} = \mathsf{rlp}(\mathbf{W} \cap \mathbf{C})$) and $\mathbf{C} = \mathsf{llp}(\mathbf{W} \cap \mathbf{F})$ (or equivalently $(\mathbf{W} \cap \mathbf{F}) = \mathsf{rlp}(\mathbf{C})$).[109]

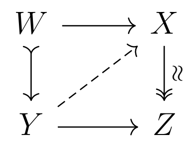
[108]Beware that we previously wrote $\mathcal{C}[\mathbf{W}^{-1}]$ to denote ∞-categorical localizations.

[109]Conditions 3 and 4 are sometimes expressed by saying that the pairs $(\mathbf{W} \cap \mathbf{C}, \mathbf{F})$ and $(\mathbf{C}, \mathbf{W} \cap \mathbf{F})$ form *weak factorization systems* on $\mathcal{C}$.

In particular, axiom 4 implies that given solid commutative squares



and

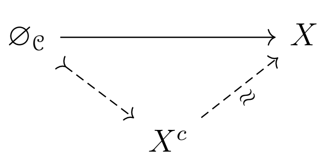


there always exist dashed lifts making the diagrams commute. It additionally requires that these various classes of morphisms are *characterized* by these lifting conditions.[110]

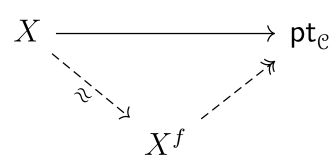
[changed 2/11] In general, we will notate a model category and its various attendant data using a subscript that indicates the name of the model structure. However, we will notate the weak equivalences with a subscript that indicates the more standard term for them when one exists; for instance, we will always write $\mathbf{W}_{\mathsf{q.i.}} \subseteq \mathsf{Ch}_R$ for the subcategory of quasi-isomorphisms.

Fix a model category $\mathcal{C}$. We say that an object $X \in \mathcal{C}$ is ***cofibrant*** if the unique map $\varnothing_{\mathcal{C}} \to X$ from the initial object is a cofibration, ***fibrant*** if the unique map $X \to \mathsf{pt}_{\mathcal{C}}$ to the terminal object is a fibration, and ***bifibrant*** if it is both cofibrant and fibrant. We respectively write $\mathcal{C}^c, \mathcal{C}^f, \mathcal{C}^{cf} \subseteq \mathcal{C}$ for the full subcategories on the cofibrant, fibrant, and bifibrant objects.

Note that every object is weakly equivalent to both a cofibrant object and a fibrant object. Indeed, we define a ***cofibrant resolution*** and a ***fibrant resolution*** of an object $X \in \mathcal{C}$ to respectively be factorizations



and



;

these are not unique, but they are guaranteed to exist by axiom 3. Even further, axiom 3 can be used to construct a bifibrant object that is weakly equivalent to X.

8.1.3. The first purpose of co/fibrant objects is that they are "good for mapping from/to", as we now explain.[111]

[110] There are a number of slight variations on the axioms: notably, some authors strengthen axiom 1 to require that $\mathcal{C}$ have all (not necessarily finite) limits and colimits, and some authors strengthen axiom 3 to require that the factorizations can be made to be functorial. These stronger axioms hold in essentially all examples of interest.

[111] More generally, a cofibration $W \rightarrowtail X$ makes X "good for mapping from in $\mathcal{C}_{W/}$", while a fibration $Y \twoheadrightarrow Z$ makes Y "good for mapping to in $\mathcal{C}_{/Z}$". Indeed, a model structure on $\mathcal{C}$ determines a model structure on $\mathcal{C}_{W//Z}$ in an evident way.

Fix a model category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$. We define a ***cylinder object*** for X and a ***path object*** for Y to respectively be factorizations

$$\begin{array}{ccc} X \amalg X & \xrightarrow{(\mathsf{id}_X, \mathsf{id}_X)} & X \\ & {\scriptstyle (i_0, i_1)} \searrow \quad \nearrow {\scriptstyle \approx} & \\ & \mathsf{cyl}(X) & \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{(\mathsf{id}_Y, \mathsf{id}_Y)} & Y \times Y \\ & {\scriptstyle \approx} \searrow \quad \nearrow {\scriptstyle (p_0, p_1)} & \\ & \mathsf{path}(Y) & \end{array} \quad ;$$

these are not unique, but they are guaranteed to exist by axiom 3. Then, for any pair of morphisms $f, g \in \mathsf{hom}_{\mathcal{C}}(X, Y)$, we define a ***left homotopy*** and a ***right homotopy*** from f to g (with respect to these choices) to respectively be dashed morphisms

$$\begin{array}{ccc} X & & \\ {\scriptstyle i_0} \downarrow & \searrow^{f} & \\ \mathsf{cyl}(X) & \dashrightarrow & Y \\ {\scriptstyle i_1} \uparrow & \nearrow_{g} & \\ X & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & Y \\ & \nearrow^{f} & \uparrow {\scriptstyle p_0} \\ X & \dashrightarrow & \mathsf{path}(Y) \\ & \searrow_{g} & \downarrow {\scriptstyle p_1} \\ & & Y \end{array}$$

making the diagrams commute. We respectively denote the existence of such homotopies by $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$. Straightforward considerations then lead to the ***fundamental theorem of model categories***: if X is cofibrant and Y is fibrant, then the map

$$\mathsf{hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathsf{hom}_{\mathcal{C}[\mathbf{W}^{-1}]}(X, Y)$$

is surjective, with the equivalence relation implementing it given by either left homotopy or right homotopy (i.e. these relations are both equivalence relations and moreover they coincide). We refer the reader to [Hov99, §1.2] for a proof; we include the following to give a representative sample of the arguments involved.

Exercise 8.1 (8 points)**.** Prove the following statements.

(a) Given a diagram

$$W \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

in $\mathcal{C}$, $f \overset{l}{\sim} g$ implies $hf \overset{l}{\sim} hg$, and if Y is fibrant then $f \overset{l}{\sim} g$ implies $fe \overset{l}{\sim} ge$.

(b) If X is cofibrant then $\overset{l}{\sim}$ defines an equivalence relation on $\mathsf{hom}_{\mathcal{C}}(X, Y)$.

(c) If $Y \xrightarrow{h} Z$ is either an acyclic fibration or a weak equivalence between fibrant objects, then postcomposition with h induces an isomorphism

$$\begin{array}{ccc} \mathsf{hom}_{\mathcal{C}}(X,Y) & \xrightarrow{h\circ(-)} & \mathsf{hom}_{\mathcal{C}}(X,Z) \\ \downarrow & & \downarrow \\ \mathsf{hom}_{\mathcal{C}}(X,Y)/\overset{l}{\sim} & \overset{\cong}{\dashrightarrow} & \mathsf{hom}_{\mathcal{C}}(X,Z)/\overset{l}{\sim} \end{array} .$$

(d) Suppose X is cofibrant. Then, for any $f, g \in \mathsf{hom}_{\mathcal{C}}(X,Y)$, $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Moreover, the relation of right homotopy is realized by any fixed path object for Y.

We emphasize that the fundamental theorem of model categories should be seen as quite striking: a priori, morphisms in $\mathcal{C}[\mathbf{W}^{-1}]$ are given by zigzags (or arbitrary length) in $\mathcal{C}$ in which the backwards maps are weak equivalences, whereas it implies that for any objects $X, Y \in \mathcal{C}$, every morphism in $\mathsf{hom}_{\mathcal{C}[\mathbf{W}^{-1}]}(X,Y)$ is presented by a zigzag

$$X \xleftarrow{\approx} X^c \longrightarrow Y^f \xleftarrow{\approx} Y$$

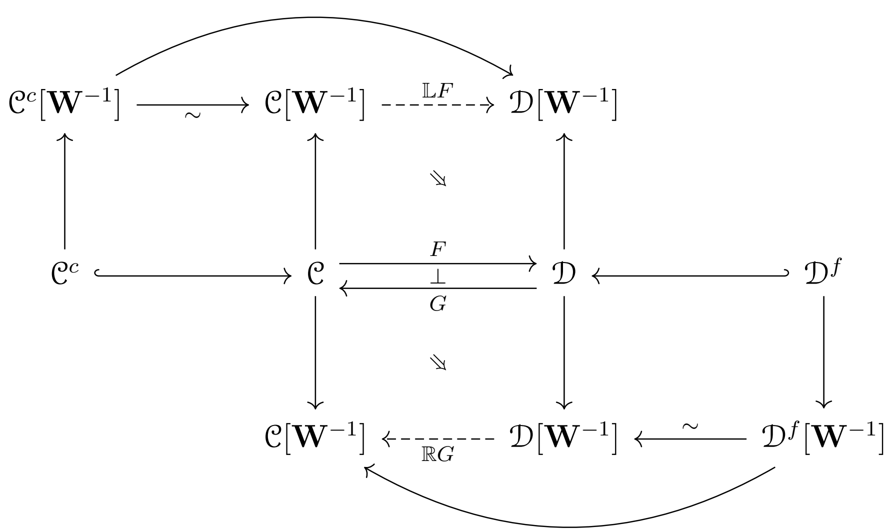
involving arbitrary but fixed co/fibrant resolutions of X and Y.

8.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A ***Quillen adjunction*** is an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

such that F preserves cofibrations and acyclic cofibrations; by axiom 4, these conditions are respectively equivalent to requiring that G preserves acyclic fibrations and fibrations.

As we now explain, such a Quillen adjunction determines a diagram



,

where we simply write $\mathbf{W}$ for all relevant subcategories of weak equivalences. First of all, the curved arrows arise from the easy consequence that F (resp. G) preserves weak equivalences between cofibrant (resp. fibrant) objects [Hov99, Lemma 1.1.12]. Moreover, the indicated horizontal equivalences follow easily from the fundamental theorem of model categories. Hence, we may define the dashed arrows so that the curved regions commute; we respectively call $\mathbb{L}F$ and $\mathbb{R}G$ the ***left derived functor*** of F and the ***right derived functor*** of G.[112] So by definition, for any objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we respectively have isomorphisms

$$(\mathbb{L}F)(X) \cong F(X^c) \qquad \text{and} \qquad (\mathbb{R}G)(Y) \cong G(Y^f)$$

in $\mathcal{D}[\mathbf{W}^{-1}]$ and $\mathcal{C}[\mathbf{W}^{-1}]$ for any co/fibrant resolutions $X^c \xrightarrow{\approx} X$ and $Y \xrightarrow{\approx} Y^f$. The natural transformations arise from the resolution maps: their components are

$$(\mathbb{L}F)(X) \cong F(X^c) \longrightarrow F(X) \qquad \text{and} \qquad G(Y) \longrightarrow G(Y^f) \cong (\mathbb{R}G)(Y)\ .$$[113]

(Of course, implicit in all of these assertions is the fact that they are independent of the choices of co/fibrant resolutions.)

The main point of such a Quillen adjunction is the following result.

Exercise 8.2 (8 points)**.** Prove that the derived functors participate in a canonical ***derived adjunction***

$$\mathcal{C}[\mathbf{W}^{-1}] \underset{\mathbb{R}G}{\overset{\mathbb{L}F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}[\mathbf{W}^{-1}] \quad .$$

The Quillen adjunction is called a ***Quillen equivalence*** if for every $X \in \mathcal{C}^c$ and every $Y \in \mathcal{D}^f$, a morphism $X \to G(Y)$ in $\mathcal{C}$ is a weak equivalence if and only if the corresponding morphism $F(X) \to Y$ in $\mathcal{D}$ is a weak equivalence. It is clear from (the solution of) Exercise 8.2 that the derived adjunction is an equivalence if and only if the Quillen adjunction is a Quillen equivalence.

[changed 2/11] We say that a Quillen functor is ***automatically derived*** if it carries *all* weak equivalences to weak equivalences. For instance, a left (resp.) right Quillen functor is automatically derived if in its source model category all objects are cofibrant (resp. fibrant). We simply write $F := \mathbb{L}F$ for the left derived functor of a left Quillen functor F that is automatically derived, and similarly we simply write $G := \mathbb{R}G$ for the right derived functor of a right Quillen functor G that is automatically derived.

8.2. **Model categories of derived modules.**

8.2.1. The category Ch_R is a relative category via the subcategory $\mathbf{W}_{\mathsf{q.i.}} \subseteq \mathsf{Ch}_R$ of quasi-isomorphisms. Its localization is the derived category of R: $\mathsf{Ch}_R[\mathbf{W}^{-1}] \simeq \mathsf{H}_0(\mathbf{D}_R)$.

[112] As the notation and terminology suggest, these are closely related to the derived functors introduced in §7.1.2.

[113] Note that F and G do *not* generally preserve weak equivalences between arbitrary objects.

8.2.2. The relative category $(\mathsf{Ch}_R, \mathbf{W}_{\mathsf{q.i.}})$ admits a ***projective model structure***, which is characterized by the fact that the fibrations are the levelwise surjective chain maps. So, all objects are fibrant, and the cofibrant objects are precisely the projective complexes. Hence, all weak equivalences are fibrant resolutions, and cofibrant resolutions are simply projective resolutions.

We now discuss axiom 3. For this, consider the sets

$$I := \{S^n \hookrightarrow D^{n+1}\}_{n\in\mathbb{Z}} \qquad \text{and} \qquad J := \{0 \overset{\approx}{\hookrightarrow} D^n\}_{n\in\mathbb{Z}}$$

of morphisms in Ch_R, where I is the set introduced in §5.5. Now, the factorizations factorizations $\overset{\approx}{\rightarrowtail}\twoheadrightarrow$ were constructed in §5.6 using the small object argument applied to the set I. It is easy to see that the small object argument applied to the set J yields the factorizations $\rightarrowtail\overset{\approx}{\twoheadrightarrow}$.[114] We summarize this situation by saying that the projective model structure is ***cofibrantly generated*** by the sets I and J, which are respectively called the set of *generating cofibrations* and the set of *generating acyclic cofibrations*.

Let us say that a morphism is a ***relative I-cell complex*** if it can be expressed as a transfinite composition of pushouts of elements of I (such as the morphism $c^{(\infty)}$ constructed in §5.6).[115] Then, the relative I-cell complexes are cofibrations, and in fact every cofibration is a retract of a relative I-cell complex. Similarly, every acyclic cofibration is a retract of a relative J-cell complex. We also have $\mathbf{F} = \mathsf{rlp}(J)$ and $(\mathbf{W} \cap \mathbf{F}) = \mathsf{rlp}(I)$: to detect (resp. acyclic) fibrations, it suffices to check the right lifting property merely against the set J (resp. I). Of course, these statements hold in any cofibrantly generated model category.

8.2.3. The relative category $(\mathsf{Ch}_R, \mathbf{W}_{\mathsf{q.i.}})$ also admits an ***injective model structure***, which is characterized by the fact that the cofibrations are the levelwise injective chain maps. So, all objects are cofibrant, and the fibrant objects are precisely the injective complexes. Hence, all weak equivalences are cofibrant resolutions, and fibrant resolutions are simply injective resolutions.[116]

8.2.4. The identity adjunction defines a Quillen equivalence

$$(\mathsf{Ch}_R)_{\mathsf{proj}} \underset{\mathsf{id}_{\mathsf{Ch}_R}}{\overset{\mathsf{id}_{\mathsf{Ch}_R}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathsf{Ch}_R)_{\mathsf{inj}}$$

in which both adjoints are automatically derived, whose derived equivalence is the identity functor on the derived category $\mathsf{H}_0(\mathbf{D}_R) := \mathsf{Ch}_R[\mathbf{W}_{\mathsf{q.i.}}^{-1}]$.

[114] To apply the small object argument to the set I, we needed to know that the source objects $S^n \in \mathsf{Ch}_R$ were compact (Exercise 5.18). It is trivial to verify that the object $0 \in \mathsf{Ch}_R$ is compact, so that we may apply the small object argument to the set J.

[115] To be precise, in §5.6 we constructed transfinite compositions of pushouts of *coproducts* of elements of I, but it is not hard to see that these are also I-cell complexes.

[116] This model structure is also cofibrantly generated, but the generating sets are quite inexplicit; see [Hov99, Theorem 2.3.13].

8.2.5. The model category $(\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}}$ is a ***symmetric monoidal model category***. In particular, this means that the bifunctor

$$(\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}} \times (\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}} \xrightarrow{(-)\otimes_{\Bbbk}(-)} (\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}}$$

is a *left Quillen bifunctor*, which in particular means that for any cofibrant object $P \in (\mathsf{Ch}_{\Bbbk})^{c}_{\mathsf{proj}} = \mathbf{P}_{\Bbbk}$, the functors $P\otimes_{\Bbbk}(-)$ and $(-)\otimes_{\Bbbk}P$ are left Quillen functors (i.e. they preserve cofibrations and acyclic cofibrations). This also means that the internal hom bifunctor

$$(\mathsf{Ch}_{\Bbbk})^{\mathsf{op}}_{\mathsf{proj}} \times (\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}} \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} (\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}}$$

is also suitably compatible with the projective model structure.

8.2.6. Similarly, the bifunctor

$$(\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}} \times (\mathsf{Ch}_{R})_{\mathsf{proj}} \xrightarrow{(-)\otimes_{\Bbbk}(-)} (\mathsf{Ch}_{R})_{\mathsf{proj}}$$

is a left Quillen bifunctor. This also means that its two-variable adjoints

$$(\mathsf{Ch}_{R})^{\mathsf{op}}_{\mathsf{proj}} \times (\mathsf{Ch}_{R})_{\mathsf{proj}} \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{R}}(-,-)} (\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}} \qquad \text{and} (\mathsf{Ch}_{\Bbbk})^{\mathsf{op}}_{\mathsf{proj}} \times (\mathsf{Ch}_{R})_{\mathsf{proj}} \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} (\mathsf{Ch}_{R})_{\mathsf{proj}}$$

are also suitably compatible with the projective model structures.

8.3. Model categories of spaces.

8.3.1. The category Top is a relative category via the subcategory $\mathbf{W}_{\mathsf{w.h.e.}} \subset \mathsf{Top}$ of weak homotopy equivalences. We refer to objects of the localization $\mathsf{Top}[\mathbf{W}^{-1}_{\mathsf{w.h.e.}}]$ as ***spaces***.[117] So by definition, a space is a weak homotopy equivalence class of topological spaces. For reasons that will become clear later, we write $\mathsf{ho}(\mathcal{S}) := \mathsf{Top}[\mathbf{W}^{-1}_{\mathsf{w.h.e.}}]$ and refer to this as ***the homotopy category of spaces***.

8.3.2. The relative category $(\mathsf{Top}, \mathbf{W}_{\mathsf{w.h.e.}})$ admits a model structure known as the ***Quillen–Serre model structure***, which we denote by $\mathsf{Top}_{\mathsf{QS}}$. This model structure is cofibrantly generated by the sets

$$I_{\mathsf{QS}} := \{S^{n-1} \hookrightarrow D^n\}_{n\geq 0} \qquad \text{and} \qquad J_{\mathsf{QS}} := \{D^n \cong D^n \times \{0\} \xhookrightarrow{\approx} D^n \times [0,1]\}_{n\geq 0}\ .$$

So, the cofibrations are the retracts of relative cell complexes (in the standard sense), and the fibrations are precisely the Serre fibrations. In particular, the cofibrant objects are the cell complexes and their retracts, and all objects are fibrant.

In any model category $\mathcal{C}$, if X is fibrant and Y is cofibrant, then any weak equivalence $X \xrightarrow{\approx} Y$ admits a "weak inverse", i.e. a weak equivalence $Y \xrightarrow{\approx} X$ that becomes an inverse in $\mathcal{C}[\mathbf{W}^{-1}]$. Applied to $\mathsf{Top}_{\mathsf{QS}}$, this recovers (a slight strengthening of) ***Whitehead's theorem***.

[117] We will later also refer to spaces as ∞-***groupoids***.

8.3.3. Recall the category $\mathbf{\Delta}$ of finite nonempty totally ordered sets and order-preserving functions, and recall the object $[n] := \{0 < 1 < \cdots < n\} \in \mathbf{\Delta}$ for every $n \geq 0$. Evidently, every object of $\mathbf{\Delta}$ is isomorphic to $[n]$ for some $n \geq 0$.

The category of ***cosimplicial objects*** in a category $\mathcal{C}$ is the category $\mathsf{c}\mathcal{C} := \mathsf{Fun}(\mathbf{\Delta}, \mathcal{C})$. In general, given a cosimplicial object $X \in \mathsf{c}\mathcal{C}$, we generally write $X^n := X([n]) \in \mathcal{C}$ for its value at the object $[n] \in \mathbf{\Delta}$; correspondingly, we may write $X^\bullet := X$ for emphasis.

Dually, the category of ***simplicial objects*** in a category $\mathcal{C}$ is the category $\mathsf{s}\mathcal{C} := \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{C})$. Given a simplicial object $X \in \mathsf{s}\mathcal{C}$, we generally write $X_n := X([n]^\circ) \in \mathcal{C}$ for its value at the object $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$; correspondingly, we may write $X_\bullet := X$ for emphasis.

Simplicial sets can be used as models for spaces. That is, the category sSet admits a model structure known as the ***Kan–Quillen model structure***, which we denote by $\mathsf{sSet}_{\mathsf{KQ}}$, which participates in a Quillen equivalence

$$\mathsf{sSet}_{\mathsf{KQ}} \underset{\mathsf{Sing}}{\overset{|-|}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Top}_{\mathsf{QS}} \quad , \tag{21}$$

as we explain shortly. Of course, it will follow that we have an equivalence $\mathsf{sSet}[\mathbf{W}_{\mathsf{KQ}}^{-1}] \simeq \mathsf{ho}(\mathcal{S})$, and in particular that a space can *also* be defined as a weak equivalence class of simplicial set.

8.3.4. We begin with some background on simplicial sets.

We write $\Delta^n := \mathsf{hom}_{\mathbf{\Delta}}(-, [n])$ for the (***combinatorial***) ***n-simplex***. These assemble into a functor $\mathbf{\Delta} \overset{\Delta^\bullet}{\hookrightarrow} \mathsf{sSet}$, i.e. a cosimplicial object in sSet, which is simply the Yoneda embedding.

Given a simplicial set $X \in \mathsf{sSet}$, we refer to the set $X_n := X([n]) \cong \mathsf{hom}_{\mathsf{sSet}}(\Delta^n, X)$ as its set of ***n-simplices***. An n-simplex of X is called ***degenerate*** if it arises as a composition $\Delta^n \to \Delta^i \to X$ for some $i < n$, and ***nondegenerate*** otherwise. The simplices of X assemble into a category

$$\mathbf{\Delta}_{/X} := \mathbf{\Delta} \underset{\mathsf{sSet}}{\times} \mathsf{sSet}_{/X} \ ,$$

which is called the ***category of simplices*** of X.

As in any presheaf category, limits and colimits in sSet are computed pointwise.[118]

Exercise 8.3 (4 points)**.** For every $n \geq 0$, determine the number of degenerate and nondegenerate n-simplices of $\Delta^1 \times \Delta^1$.

Exercise 8.4 (6 points)**.** Prove that any simplicial set $X \in \mathsf{sSet}$ is the colimit of its simplices, i.e. that the canonical morphism

$$\underset{(\Delta^n \downarrow X) \in \mathbf{\Delta}_{/X}}{\mathsf{colim}} \Delta^n \longrightarrow X$$

[118]More generally, in any functor category $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$, if $\mathcal{C}$ admits co/limits indexed over $\mathcal{J}$, then so does $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$ and these co/limits are computed pointwise. However, beware that there may exist co/limits in $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$ that are *not* computed pointwise (necessarily arising in the case that $\mathcal{C}$ does *not* admit such co/limits).

is an isomorphism.

We can rephrase Exercise 8.4 as saying that the left Kan extension

$$\begin{array}{ccc} \mathbf{\Delta} & \overset{\Delta^\bullet}{\hookrightarrow} & \mathsf{sSet} \\ {\scriptstyle \Delta^\bullet}\big\downarrow & \nearrow & \\ \mathsf{sSet} & & \end{array}$$

is the identity functor.[119] Indeed, in general the left Kan extension

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle \varphi}\big\downarrow & \Downarrow \nearrow_{\varphi_! F} & \\ \mathcal{J} & & \end{array}$$

(when it exists) is given by the formula

$$j \longmapsto (\varphi_! F)(j) \cong \mathsf{colim}\left(\mathcal{I} \underset{\mathcal{J}}{\times} \mathcal{J}_{/j} \xrightarrow{\mathsf{fgt}} \mathcal{I} \xrightarrow{F} \mathcal{C} \right) \ .$$

Exercise 8.5 (6 points)**.** Prove that any simplicial set $X \in \mathsf{sSet}$ is also the colimit of its *nondegenerate* simplices. That is, writing $\mathbf{\Delta}_{\mathsf{nondeg}/X} \subseteq \mathbf{\Delta}_{/X}$ for the full subcategory on the nondegenerate simplices, prove that the canonical morphism

$$\underset{(\Delta^n \downarrow X) \in \mathbf{\Delta}_{\mathsf{nondeg}/X}}{\mathsf{colim}} \Delta^n \longrightarrow X$$

is an isomorphism.

In fact, Exercise 8.5 follows from the more general claim that *any* colimit over $\mathbf{\Delta}_{/X}$ is isomorphic to the colimit of the restriction to $\mathbf{\Delta}_{\mathsf{nondeg}/X}$, as we will see below.

8.3.5. We now return to the Quillen equivalence (21).

For any $n \geq 0$, the ***topological n-simplex*** is the topological space

$$\Delta^n_{\mathsf{top}} := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=0}^n x_i = 1 \text{ and } x_i \geq 0 \right\} \in \mathsf{Top} \ .$$

These assemble into a cosimplicial topological space $\mathbf{\Delta} \xrightarrow{\Delta^\bullet_{\mathsf{top}}} \mathsf{Top}$.[120] We can now define the right adjoint Sing: it is simply the restricted Yoneda functor. That is, for any topological

[119] Note that Kan extensions along a fully faithful functor do not change the values on the full subcategory, so the natural transformation that should appear here is a natural isomorphism.

[120] The functoriality of $\Delta^\bullet_{\mathsf{top}}$ is uniquely specified by the following requirements: all maps in its image are linear (in the evident sense), and the function $\mathsf{fgt}([n]) \to \mathsf{fgt}(\Delta^n_{\mathsf{top}})$ on underlying sets taking $i \in [n]$ to the i^{th} unit basis vector assembles into a natural transformation

$$\begin{array}{ccc} \mathbf{\Delta} & \xrightarrow{\Delta^\bullet_{\mathsf{top}}} & \mathsf{Top} \\ & {\scriptstyle\mathsf{fgt}}\searrow \ \Uparrow & \big\downarrow {\scriptstyle \mathsf{fgt}} \\ & & \mathsf{Set} \end{array} \ .$$

space $Y \in \mathsf{Top}$, its ***singular simplicial set*** is the functor

$$\mathsf{Sing}(Y)_\bullet : \boldsymbol{\Delta}^{\mathsf{op}} \xrightarrow{\hom_{\mathsf{Top}}(\Delta^\bullet_{\mathsf{top}}, Y)} \mathsf{Set} \ .$$

We refer to the set $\mathsf{Sing}(Y)_n := \hom_{\mathsf{Top}}(\Delta^n_{\mathsf{top}}, Y)$ as the set of (***singular***) n***-simplices*** of Y. Its left adjoint functor $|-|$ is called ***geometric realization***. The most efficient definition is that it is the left Kan extension

$$\begin{array}{ccc} \boldsymbol{\Delta} & \xrightarrow{\Delta^\bullet_{\mathsf{top}}} & \mathsf{Top} \\ {\scriptstyle \Delta^\bullet} \big\downarrow & \nearrow_{|-|} & \\ \mathsf{sSet} & & \end{array}$$

of $\Delta^\bullet_{\mathsf{top}}$ along the Yoneda embedding $\Delta^\bullet$. So by definition, for any simplicial set $X \in \mathsf{sSet}$, we have

$$|X| := \mathsf{colim}\left(\boldsymbol{\Delta}_{/X} \xrightarrow{\mathsf{fgt}} \boldsymbol{\Delta} \xrightarrow{\Delta^\bullet_{\mathsf{top}}} \mathsf{Top} \right) \ .$$

So, the topological space $|X| \in \mathsf{Top}$ is built by gluing together topological n-simplices, one for each combinatorial n-simplex of X.

Exercise 8.6 (4 points)**.** Verify the adjunction (21).

Now, we define the subcategory $\mathbf{W}_{\mathsf{w.h.e.}} \subset \mathsf{sSet}$ of ***weak homotopy equivalences*** simply by pullback along $\mathsf{sSet} \xrightarrow{|-|} \mathsf{Top}$. These are the weak equivalences of the Kan–Quillen model structure. The cofibrations are the monomorphisms (i.e. the levelwise injections), so that every object is cofibrant. To describe the fibrations, we introduce some notation. First of all, for any $n \geq 0$, we write $\partial\Delta^n \subseteq \Delta^n$ for the largest simplicial subset not containing the nondegenerate n-simplex. For any $0 \leq i \leq n$, the i^{th} ***face*** of Δ^n is the nondegenerate $(n-1)$-simplex given by the morphism $[n-1] \xrightarrow{d^i_n} [n]$ in $\boldsymbol{\Delta}$ defined by

$$d^i_n(j) = \left\{ \begin{array}{cc} j \ , & 0 \leq j < i \\ j+1 \ , & i \leq j \leq n \end{array} \right. \ .$$

Then, the i^{th} ***horn*** of Δ^n is the largest simplicial subset $\Lambda^n_i \subseteq \Delta^n$ not containing the i^{th} face. So for instance, we may depict Λ^2_0 as

$$\begin{array}{ccc} & 1 & \\ \nearrow & & \\ 0 & \longrightarrow & 2 \end{array} \ .$$

Finally, the Kan–Quillen model structure is cofibrantly generated by the sets

$$I := \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \geq 0} \qquad \text{and} \qquad J := \{\Lambda^n_i \overset{\approx}{\hookrightarrow} \Delta^n\}_{0 \leq i \leq n \geq 1} \ .$$

In particular, an object is fibrant iff it has the extension property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$. A fibrant object of $\mathsf{sSet}_{\mathsf{KQ}}$ is called a ***Kan complex***.[121]

Now, it is immediate from the definitions that the left adjoint $\mathsf{sSet}_{\mathsf{KQ}} \xrightarrow{|-|} \mathsf{Top}_{\mathsf{QS}}$ is a left Quillen functor; indeed, it even carries the generating (resp. acyclic) cofibrations in $\mathsf{sSet}_{\mathsf{KQ}}$ to the generating (resp. acyclic) cofibrations in $\mathsf{Top}_{\mathsf{QS}}$ (up to certain straightforward homeomorphisms). To see that the Quillen adjunction (21) is in fact a Quillen equivalence, for any $X \in \mathsf{sSet}^c = \mathsf{sSet}$ and any $Y \in \mathsf{Top}^f = \mathsf{Top}$ we must have that a morphism $|X| \xrightarrow{f} Y$ lies in $\mathbf{W}_{\mathsf{QS}}$ iff its adjunct morphism $X \xrightarrow{f^\sharp} \mathsf{Sing}(Y)$ lies in $\mathbf{W}_{\mathsf{KQ}}$. By definition, the latter holds iff the morphism $|X| \xrightarrow{|f^\sharp|} |\mathsf{Sing}(Y)|$ lies in $\mathbf{W}_{\mathsf{QS}}$. Via the commutative triangle

$$\begin{array}{ccc} & & |\mathsf{Sing}(Y)| \\ & \overset{|f^\sharp|}{\nearrow} & \downarrow \varepsilon \\ |X| & \underset{f}{\searrow} & Y \end{array}$$

and the two-out-of-three property for $\mathbf{W}_{\mathsf{QS}} \subset \mathsf{Top}$, it is equivalent to verify that the counit morphism $|\mathsf{Sing}(Y)| \xrightarrow{\varepsilon} Y$ lies in $\mathbf{W}_{\mathsf{QS}}$. This is a nontrivial fact, but it should be plausible; indeed, this morphism is a cellular approximation (i.e. a weak equivalence from a cell complex).

We note that both adjoints of the Quillen equivalence (21) are automatically derived, simply because $\mathsf{sSet}^c_{\mathsf{KQ}} = \mathsf{sSet}$ and $\mathsf{Top}^f_{\mathsf{QS}} = \mathsf{Top}$. Also, $\mathsf{sSet}_{\mathsf{KQ}}$ is a symmetric monoidal model categories with respect to cartesian product, and $\mathsf{Top}_{\mathsf{QS}}$ is as well once restricting to a "convenient" subcategory (so that cartesian product admits a right adjoint).

8.4. Homology.

8.4.1. As we now explain, homology of spaces essentially amounts to ***derived linearization***. Beyond its intrinsic interest, this will be relevant for us in understanding the relationship between ∞-categories and $\Bbbk$-linear ∞-categories.

8.4.2. We begin with a brief digression.

We write $\mathsf{Ch}^{\geq 0}_R \subseteq \mathsf{Ch}_R$ for the full subcategory on the ***nonnegatively-graded complexes*** of R-modules, i.e. those $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n < 0$. We write

$$\mathsf{H}_0(\mathbf{D}^{\geq 0}_R) := (\mathsf{Ch}^{\geq 0}_R)[\mathbf{W}^{-1}_{\mathsf{q.i.}}]$$

for its localization, and refer to it as the category of ***nonnegatively-graded derived R-modules***, as justified by Exercise 8.7.

[121]It is worth noting that Kan complexes are necessarily very large. In fact, a Kan complex with any nondegenerate positive-dimensional simplices will necessarily have infinitely many simplices in all dimensions. In a strong sense, this largeness reflects the fundamental incalculability of unstable homotopy theory. This may be compared e.g. with the "small simplicial circle", namely $\Delta^1/\partial\Delta^1 \in \mathsf{sSet}$: it only has one nondegenerate edge, whereas a fibrant replacement must have at least as many edges as there are homotopy classes of maps $S^1 \to S^1$.

Exercise 8.7 (6 points).

(a) Prove that the projective model structure on Ch_R restricts to a ("projective") model structure on $\mathsf{Ch}_R^{\geq 0} \subseteq \mathsf{Ch}_R$.

(b) Construct a Quillen coreflective localization adjunction

$$(\mathsf{Ch}_R^{\geq 0})_{\mathsf{proj}} \underset{\tau_{\geq 0}}{\overset{i_{\geq 0}}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\bot}}}} (\mathsf{Ch}_R)_{\mathsf{proj}} \quad ,$$

and verify that both adjoints are automatically derived.

(c) Prove that the resulting derived adjunction

$$\mathsf{H}_0(\mathbf{D}_R^{\geq 0}) := \mathsf{Ch}_R^{\geq 0}[\mathbf{W}_{\mathsf{q.i.}}^{-1}] \underset{\tau_{\geq 0}}{\overset{i_{\geq 0}}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\bot}}}} \mathsf{Ch}_R[\mathbf{W}_{\mathsf{q.i.}}^{-1}] =: \mathsf{H}_0(\mathbf{D}_R)$$

is also a coreflective localization adjunction, and prove that the image of the derived left adjoint $i_{\geq 0} = \mathbb{L}(i_{\geq 0})$ consists of precisely those derived R-modules $M \in \mathsf{H}_0(\mathbf{D}_R)$ such that $\mathsf{H}_n(M) = 0$ for all $n < 0$.[122]

8.4.3. We now study certain intermediate categories that are relevant in the definition of homology.

Given a category $\mathcal{C}$, we write $\mathsf{Mod}_R(\mathcal{C})$ for the category of right R-module objects in $\mathcal{C}$. Note that $\mathsf{Mod}_R(\mathsf{sSet}) \simeq \mathsf{sMod}_R(\mathsf{Set}) =: \mathsf{sMod}_R$.[123] As a special case, we have the category $\mathsf{Ab}(\mathcal{C}) := \mathsf{Mod}_{\mathbb{Z}}(\mathcal{C})$ of abelian group objects, and no intuition is lost by restricting to this special case.

It will be convenient to refer to certain *lifted* model structures. Namely, we have Quillen adjunctions

$$\mathsf{sSet}_{\mathsf{KQ}} \underset{\mathsf{fgt}}{\overset{R\{-\}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathsf{sMod}_R)_{\mathsf{KQ}} \qquad \text{and} \qquad \mathsf{Top}_{\mathsf{QS}} \underset{\mathsf{fgt}}{\overset{R\{-\}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Mod}_R(\mathsf{Top})_{\mathsf{QS}} \quad .$$[124]

The model structures on categories of R-module objects are "created by the right adjoints", in the sense that a morphism is a (resp. acyclic) fibration iff its image under fgt is a (resp. acyclic) fibration. Because every object of $\mathsf{Top}_{\mathsf{QS}}$ is fibrant, so is every object of $\mathsf{Mod}_R(\mathsf{Top})_{\mathsf{QS}}$.

Exercise 8.8 (2 points). Show that every object of $(\mathsf{sMod}_R)_{\mathsf{KQ}}$ is fibrant.

[122] In other words, a derived R-module with nonnegatively-graded homology can be presented by a nonnegatively-graded complex of R-modules.

[123] This equivalence follows from the fact that products in sSet are computed pointwise.

[124] The free simplicial R-module on a simplicial set $X \in \mathsf{sSet}$ has $R\{X\}_n := R\{X_n\}$, i.e. it is given by applying the free R-module functor $\mathsf{Set} \xrightarrow{R\{-\}} \mathsf{Mod}_R$ levelwise. The free topological R-module on a topological space is the free R-module on its underlying set equipped with a suitable topology: its elements are given by finite unordered configurations of points in X labeled by elements of R, where nearby configurations with identical labels are nearby, and with addition and right R-action given pointwise.

8.4.4. There is a close relationship between the categories sMod_R and $\mathsf{Ch}_R^{\geq 0}$, which is known as the ***Dold–Kan correspondence***.

Namely, there are two functors $\overline{\mathsf{C}}_\bullet, \mathsf{C}_\bullet \in \mathsf{Fun}(\mathsf{sMod}_R, \mathsf{Ch}_R^{\geq 0})$, the *normalized* and *unnormalized chains* on a simplicial R-module. These are related by a commutative diagram

$$\begin{array}{ccc} \overline{\mathsf{C}}_\bullet & \overset{\approx}{\hookrightarrow} & \mathsf{C}_\bullet \\ & \searrow^{\mathrm{id}_{\overline{\mathsf{C}}_\bullet}} & \downarrow\!\!\downarrow \wr \\ & & \overline{\mathsf{C}}_\bullet \end{array} \tag{22}$$

of natural quasi-isomorphisms, and moreover $\overline{\mathsf{C}}_\bullet$ is an equivalence of categories [Wei94, §8.3].

We do not discuss the diagram (22), but we at least define the functors that it involves. First of all, for a simplicial R-module $X \in \mathsf{sMod}_R$, we define $\mathsf{C}_\bullet(X) \in \mathsf{Ch}_R^{\geq 0}$ by setting $\mathsf{C}_n(X) = X_n$ and defining the differentials $\mathsf{C}_n(X) \xrightarrow{d_n} \mathsf{C}_{n-1}(X)$ to be

$$\mathsf{C}_n(X) := X_n \xrightarrow{\sum_{i=0}^n (-1)^i \cdot X(d_n^i)} X_{n-1} =: \mathsf{C}_{n-1}(X) \ .$$

Then, $\overline{\mathsf{C}}_\bullet(X) \subseteq \mathsf{C}_\bullet(X)$ is the subcomplex defined levelwise by

$$\overline{\mathsf{C}}_n(X) := \bigcap_{i=0}^{n-1} \mathsf{ker}\left(X_n \xrightarrow{X(d_n^i)} X_{n-1} \right) \ ,$$

so that its differential is simply $\overline{\mathsf{C}}_n(X) \xrightarrow{(-1)^n \cdot X(d_n^n)} \overline{\mathsf{C}}_{n-1}(X)$.

Exercise 8.9 (8 points)**.** For any simplicial R-module $M \in \mathsf{sMod}_R$ and any $n \geq 0$, establish a commutative diagram

$$\begin{array}{ccc} \mathsf{hom}_{\mathsf{sSet}_*}(\Delta^n/\partial\Delta^n, \mathsf{fgt}(M)) & \xrightarrow{\cong} & \mathsf{Z}_n(\overline{\mathsf{C}}_\bullet(M)) \\ \uparrow & & \uparrow d_{n+1} \\ \mathsf{hom}_{\mathsf{sSet}_*}(\Delta^{n+1}/\Lambda^{n+1}_{n+1}, \mathsf{fgt}(M)) & \xrightarrow{\cong} & \overline{\mathsf{C}}_{n+1}(M) \end{array}$$

in Mod_R (where we consider $\mathsf{fgt}(M) := \mathsf{fgt}(0 \to M) \in \mathsf{sSet}_*$), and use this to deduce an isomorphism

$$\pi_n(\mathsf{fgt}(M)) \cong \mathsf{H}_n(\mathsf{C}_\bullet(M))$$

in Mod_R.

It follows from Exercise 8.9 and the above discussion that the functors $\overline{\mathsf{C}}_\bullet$ and $\mathsf{C}_\bullet$ induce canonically equivalent equivalences

$$\mathsf{sMod}_R[\mathbf{W}^{-1}_{\mathsf{w.h.e.}}] \xrightarrow{\sim} \mathsf{Ch}_R^{\geq 0}[\mathbf{W}^{-1}_{\mathsf{q.i.}}]$$

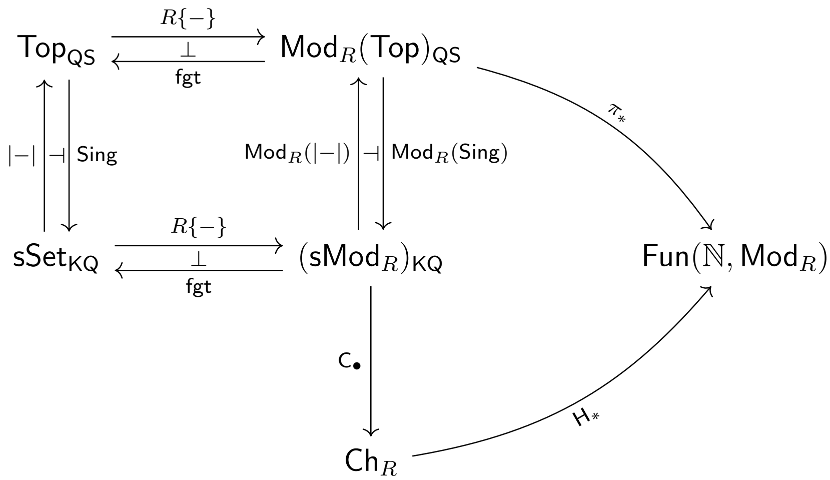
on localizations.[125] So, the model category $(\mathsf{sMod}_R)_{\mathsf{KQ}}$ is a presentation of $\mathsf{H}_0(\mathbf{D}_R^{\geq 0})$.

[125] In fact, it is not hard to see that the equivalence of categories

$$(\mathsf{sMod}_R)_{\mathsf{KQ}} \xrightarrow{\overline{\mathsf{C}}_\bullet} (\mathsf{Ch}_R^{\geq 0})_{\mathsf{proj}}$$

8.4.5. Now, both singular and simplicial homology may be located in the diagram

(23)



as we now explain. First of all, ***simplicial homology*** is the composite functor

$$\mathsf{H}_*^\Delta(-;R) : \mathsf{sSet} \xrightarrow{R\{-\}} \mathsf{sMod}_R \xrightarrow{\mathsf{C}_\bullet} \mathsf{Ch}_R \xrightarrow{\mathsf{H}_*} \mathsf{Fun}(\mathbb{N}, \mathsf{Mod}_R) ,$$

and ***singular homology*** is the composite functor

$$\mathsf{H}_*^{\mathsf{sing}}(-;R) : \mathsf{Top} \xrightarrow{\mathsf{Sing}} \mathsf{sSet} \xrightarrow{\mathsf{H}_*^\Delta(-;R)} \mathsf{Fun}(\mathbb{N}, \mathsf{Mod}_R)$$

(both implicitly with coefficients in R).[126] The adjunction $\mathsf{Mod}_R(|-| \dashv \mathsf{Sing})$ arises from the fact that both functors in the adjunction $|-| \dashv \mathsf{Sing}$ preserve finite products, and it is clear that the square in diagram (23) commutes after omitting all left adjoints or all right adjoints.

Exercise 8.10 (4 points)**.** Prove that the adjunction $\mathsf{Mod}_R(|-| \dashv \mathsf{Sing})$ is a Quillen equivalence.

Combining Exercise 8.10 with the discussion of § 8.4.4, we see that the model category $\mathsf{Mod}_R(\mathsf{Top})_{\mathsf{QS}}$ is also a presentation of $\mathsf{H}_0(\mathbf{D}_R^{\geq 0})$ and that the triangle in diagram (23) commutes.

From diagram (23), we also see that homology with coefficients in R is indeed derived R-linearization; note that both composites

$$\mathsf{C}_\bullet^\Delta(-;R) : \mathsf{sSet} \xrightarrow{R\{-\}} \mathsf{sMod}_R \xrightarrow{\mathsf{C}_\bullet} \mathsf{Ch}_R$$

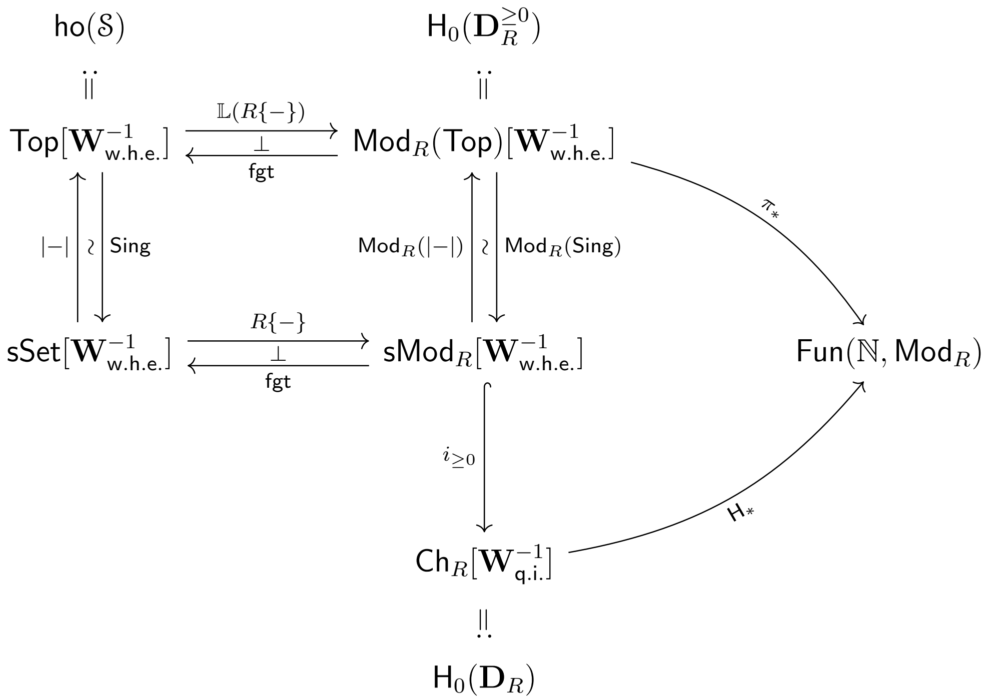
and

$$\mathsf{C}_\bullet^{\mathsf{sing}}(-;R) : \mathsf{Top} \xrightarrow{\mathsf{Sing}} \mathsf{sSet} \xrightarrow{\mathsf{C}_\bullet^\Delta(-;R)} \mathsf{Ch}_R$$

is both a left and right Quillen equivalence (using the notation of Exercise 8.7).

[126]More generally, homology with coefficients in a (possibly derived) R-module is obtained by taking the (resp. derived) tensor product before taking homology of complexes.

are automatically derived (in the evident sense).[127] Indeed, diagram (23) yields the commutative diagram



on localizations.[128] From this, we immediately deduce the ***Dold–Thom theorem***: for any cofibrant object $X \in \mathsf{Top}_{\mathsf{QS}}$ (i.e. a cell complex or retract thereof), we have a canonical isomorphism

$$\pi_*(R\{X\}) \cong \mathsf{H}_*^{\mathsf{sing}}(X; R) \ .$$

Said differently, the singular homology of X with coefficients in R is computed by the homotopy groups of the free topological R-module on X. We also deduce a sort of free/forget adjunction

$$\mathsf{ho}(\mathcal{S}) := \mathsf{Top}[\mathbf{W}_{\mathsf{w.h.e.}}^{-1}] \underset{\mathsf{fgt} \circ \tau_{\geq 0}}{\overset{\mathsf{C}_\bullet^{\mathsf{sing}}(-;R)}{\rightleftarrows}} \mathsf{Ch}_R[\mathbf{W}_{\mathsf{q.i.}}^{-1}] =: \mathsf{H}_0(\mathbf{D}_R)$$

between the homotopy category of spaces and the category of derived R-modules.[129]

[127]As always, we prefer to interpret these "homology" functors without actually passing to homology groups of complexes.

[128]That is, the two adjunctions are identified via the equivalences, so that in particular the diagram commutes upon omitting both left adjoints or both right adjoints.

[129]Note that this adjunction restricts to the free/forget adjunction

$$\mathsf{Set} \underset{\mathsf{fgt}}{\overset{R\{-\}}{\rightleftarrows}} \mathsf{Mod}_R$$

between sets and R-modules. This analogy will be amplified below.

References

[EH16] David Eisenbud and Joe Harris, 3264 and all that—a second course in algebraic geometry, Cambridge University Press, Cambridge, 2016, available here.

[GH15] David Gepner and Rune Haugseng, Enriched ∞-categories via non-symmetric ∞-operads, Adv. Math. **279** (2015), 575–716.

[GP74] Victor Guillemin and Alan Pollack, Differential topology, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.

[Har77] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[Hov99] Mark Hovey, Model categories, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[Lur] Jacob Lurie, Higher algebra, available from the author's website (version dated 9/18/2017).

[Lur09] ———, Higher topos theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Mac71] Saunders MacLane, Categories for the working mathematician, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 5.

[MG] Aaron Mazel-Gee, The zen of ∞-categories, available here.

[Saf] Pavel Safronov, Category theory [lecture notes], available here.

[Spa88] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Math. **65** (1988), no. 2, 121–154.

[Wei94] Charles A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wei13] ———, The K-book, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory.