SYLLABUS
HOMOLOGICAL ALGEBRA (MATH 128)
WINTER 2021

Contents

1. General information

1.1. Instructor.

name: Aaron Mazel-Gee
email: mazelgee@caltech.edu

office hours: by appointment
pronouns: he/him

1.2. Course logistics.

All interactions will be conducted via Zoom. Links will be made available in Canvas.
lectures: TuTh at 2:30-4pm
course website: etale.site/teaching/w21/
textbook: none required; possible supplementary texts include Weibel's *Homological algebra*, Loday's *Cyclic homology*, Kashiwara & Schapira's *Sheaves on manifolds*, and Lurie's *Higher algebra* – all should eventually be available in Canvas (under Course Reserves)
prerequisites: basic algebra (e.g. rings and modules) and basic category theory (e.g. co/limits and adjunctions); familiarity with basic algebraic topology (e.g. the fundamental group and homology) will also be helpful
Canvas: yes, barely (see §4.1)

1.3. **Homework, participation, and grades.** Homework is described in §1.3.1, and participation (which may be synchronous or asynchronous) is described in §1.3.2. Your final grade in the course will be obtained from your homework grade by subtracting N letters from your homework grade, where N is determined by your participation according to the following table.

last updated: January 3, 2021

minimum participation	75%	50%	25%	0%
N	0	1	2	3

If you are taking the course for a pass/fail grade, you will receive a "pass" as long as you earn a C- or better.

1.3.1. *Homework.* Homework problems will be interwoven through the lecture notes. They will have various purposes: to check your understanding, to illustrate an example or technique, or to offer a further avenue of exploration.

Each homework problem will be assigned a point value.[1] At the same time, I will keep a running tally of the number of points necessary for a grade of 100% on homework. This will be *far* less than the total number of points available: it won't be necessary to do anywhere close to all the homework problems in order to get 100% on homework.

Your homework grade will translate into a letter grade according to the following cutoffs.

percentage cutoff	95%	85%	80%	75%	65%	60%	55%	45%	40%
letter grade	A+	A	A-	B+	B	B-	C+	C	C-

Homework will be due at the end of the quarter. However, you are strongly encouraged to turn in problems early. The main reason for this is that working on homework problems as they appear is the best way of staying on top of the material. But moreover, I will grade the problems as you submit them, and you will be free to correct your solutions as necessary in order to receive full credit.

Depending on interest and on our progress through the material, I may also make available the possibility of giving a presentation on an advanced topic towards the end of the quarter as an alternate way of accumulating homework points.

Additionally, you will receive homework points for informing me of any typos or mistakes in the lecture notes (with the point value depending on the severity of the issue).

Please read the brief document Guidelines for Good Mathematical Writing, and adhere to its suggestions in your homework problem submissions to the greatest extent possible.

You are encouraged to write your homework in LaTeX (pronounced "LAH-teck"). This is a versatile and easy-to-use typesetting system, which is the industry standard in mathematics and adjacent fields. Some resources to help you get started writing in LaTeX are outlined in §4.4.

You will submit your homework simply by emailing it to me. If you write your homework in LaTeX, you may simply submit the pdf. Alternatively, you may write your homework out by hand and submit photos of it, assembled as a single pdf (e.g. using an online photo-to-pdf converter). At times, your solution to a problem may involve a picture or diagram; if you are not up for creating this via LaTeX, you may either embed a photo into your LaTeX file or simply attach a separate photo to your email.

[1]Please note that I may change the point-values slightly from time to time, especially early on in the quarter as I am getting a sense of things.

1.3.2. *Participation.* In my own experience, mathematics is most exciting and most rewarding when it is pursued as an interpersonal endeavor: as an interesting and unique way to interact with other human beings. Moreover, I have learned through experience that it is extremely discouraging for me to lecture to an empty Zoom room. And of course, active participation is an important prerequisite for engaging with the material. For these reasons, your participation will be factored into your final grade.

I would very much appreciate your synchronous attendance during lectures.[2] However, I also understand that this may be impossible or extremely inconvenient (e.g. due to time zone differences). In view of these considerations, your participation will be graded as follows. If you attend lecture synchronously and participate when prompted, this will fulfill your participation requirement. If you do not attend lecture synchronously, you can fulfill your participation requirement by taking notes on the recorded lecture and writing down any questions you might have as they arise, and then sharing these notes with me by email and engaging in further discussion as appropriate.[3] The deadline for such asynchronous participation for each week will be the end of the following weekend.[4]

2. Course description

Consider the two-element set $\{x, y\}$. Suppose that we impose the relation "$x = y$". This gives us a singleton, i.e. a set with one element. Moreover, if we impose the relation "$x = y$" again, we get the same result: the singleton admits no nontrivial quotients. Such an outcome is often undesirable: experience teaches us that it is often helpful to remember not simply "the fact that x and y are equal", but instead "the ways in which x and y are equal".

In this simple situation, we can record "the ways in which x and y are equal" through the following general construction. Suppose we are given a set S and a set R of (possibly apparently redundant) relations, i.e. a set R equipped with a function

$$R \xrightarrow{f=(f_0,f_1)} S \times S$$

[2] Indeed, I would very much appreciate you leaving your video on during lectures. This will help me gauge your comprehension and attention. You will also likely find it useful as a way of keeping yourself accountable, so that you are less likely to distract yourself (and therefore more likely to ask a question!) if and when you get lost. More broadly, I would hope for this to help us foster a sense of connection and shared experience.

[3] Your notes do not need to be extremely detailed; indeed, you may find it beneficial to make an "executive summary" of each lecture for your own mental organization. However, I would expect your notes to be at least half a page per lecture.

[4] So for instance, you should send me your notes and questions on the lectures of Tuesday 1/5 and Thursday 1/7 by the end of the day on Sunday 1/10.

to the cartesian product of S with itself.[5] Then, rather than form the quotient set S/R, we may form a graph $S/\!/R$ whose vertices are the elements of S and whose edges are the elements of R, such that the endpoints of the edge corresponding to an element $r \in R$ are the vertices $f_0(r)$ and $f_1(r)$. Note that the path components of $S/\!/R$ are precisely the elements of S/R, i.e. the equivalence classes of S under (the equivalence relation generated by) R. However, the graph $S/\!/R$ has not forgotten about the multiplicity of ways in which the elements of S are related by R. For instance, in the above example we obtain the graphs $x \;\text{——}\; y$ and $x \overset{\frown}{\underset{\smile}{\quad}} y$, which are respectively shaped like a closed interval and a circle.[6]

Broadly speaking, ***homological algebra*** is the field of mathematics that arises from applying this same line of reasoning in the context of algebra (e.g. involving groups, rings, and modules, instead of just sets). For instance, given an abelian group A and a pair of elements $a, b \in A$, if we impose the relation "$a = b$" twice in the context of homological algebra, we will end up with "a circle's worth of elements" in the resulting equivalence class. This turns out to be a remarkably fruitful perspective, whose influence has only grown in the half-century or so since it first appeared.

In this course, we will study homological algebra and its applications, with a focus on examples and computations. Possible topics to be covered, roughly order from most likely to least likely, include:

- abelian categories and derived functors (e.g. Tor and Ext), filtrations and spectral sequences;
- sheaves, the six functor formalism, Verdier duality;
- model categories and nonabelian derived functors (e.g. co/homology, the cotangent complex), the Dold–Kan correspondence and the Dold–Thom theorem;
- dg-categories and stable ∞-categories, spectra and generalized co/homology theories;
- $\mathbb{E}_n$-algebras, Koszul duality;
- Hochschild homology, cyclic homology, and algebraic K-theory;
- Goodwillie's calculus of functors.

3. Resources

Collaboration with your fellow students can be a very effective way of learning, and is highly encouraged. Often, we may think we understand something very well, but in trying to explain it to somebody else we find the gaps in our understanding.[7] However, *everyone*

[5] These relations are irredundant if and only if the function f is injective. In this case, f may be considered as the inclusion of a subset, which reduces us to the more familiar definition of a(n equivalence) relation on a set.

[6] The graph $S/\!/R$ may be referred to as the *homotopy quotient* of S by R.

[7] There was some research that came out about this recently. Subjects were asked to rate their understanding of some everyday mechanism (e.g. a ballpoint pen or the flushing mechanism of a toilet). Then, they were asked to explain it. Then, they were asked to rate their understanding again. As you might

must write up their own solutions separately: you may not copy each other's work. Moreover, you must indicate any help that you received on each problem.[8]

As a guideline for this collaboration policy, keep in mind that **you should be able to reproduce any solution you hand in without assistance**.

Here is a list of allowed and disallowed resources. This is intended to be reasonably comprehensive, but the fact that something is not on this list does *not* mean that it is permitted. If you are ever in doubt about whether something is allowed, it is **your responsibility** to ask.

you may consult:	
textbooks	yes
solution manuals (official or unofficial)	no
internet	yes
notes from lecture (yours or others')	yes
course handouts	yes
communications (e.g. emails) from instructor	yes
you may:	
discuss problems with others	yes
look at communal materials (e.g. (e-)whiteboard) when writing up solutions	yes
look at written work of others	no
post about problems online (e.g. at math.stackexchange)	yes

4. Miscellanea

4.1. **Canvas.** Much of the relevant information for this course will be posted on the course website. However, we will use Canvas for a few purposes, e.g. for any material that may be considered as sensitive or unpolished – lecture video recordings, Zoom links, chat, discussion boards, grades, etc.

4.2. **Students with documented disabilities.** If you need an academic accommodation based on the impact of a disability, you must initiate the request with Caltech Accessibility Services for Students (CASS). Professional staff will evaluate your request with required documentation, recommend reasonable accommodations, and prepare an Accommodation

imagine, their own estimation of their understanding dropped significantly from the first round of rating to the second.

[8]For instance, as a footnote to Problem N you might write "I worked on this problem with Alice and Bob." or "I found a hint/solution to this problem at `http://mathhelp4u.blogspot.com`.".

Letter for the quarter in which the request is being made. In this case, you should contact CASS as soon as possible, since timely notice is needed to coordinate accommodations. Undergraduate students should contact Dr. Lesley Nye, Associate Dean of Undergraduate Students (administrative contact: Beth Larranaga) and graduate students should contact Dr. Kate McAnulty, Associate Dean of Graduate Studies (administrative contact: Jacob Dalton).

4.3. **Academic integrity.** You must abide by Caltech's honor code: *No member of the Caltech community shall take unfair advantage of any other member of the Caltech community.* General guidelines for how this applies to plagiarism are given here. Specific guidelines for this course are described in §3.

4.4. **LaTeX resources.** LaTeX code is stored as plain text, and therefore can be written and saved with the filename extension `txt` (e.g. as `my-LaTeX-code.txt`). This raw code can then be *compiled* into a `pdf` file (such as the one you are reading now). You can compile LaTeX code directly on your computer after a somewhat lengthy download/installation process. Alternatively, you can write and compile LaTeX code online through the website Overleaf. I would recommend the latter option, especially if you are having difficulty with the former option (which is somewhat nontrivial).

A simple template that you can use to get started on your homework is available here; when compiled, that LaTeX code produces this file. Of course, you are also welcome to start from scratch. Along with occasional online searches, that template should suffice to get you started.[9] However, if you would like a further introduction to LaTeX you can check out some of the instructions provided by Overleaf: this shorter one or this longer one.

Of course, there are many online LaTeX resources. In particular, there are many "crib sheets", see e.g. here and here. Another great resource is Detexify: if you draw a mathematical symbol, it will attempt to suggest how to write it in LaTeX.

4.5. **Revisions to this syllabus.** This syllabus is subject to minor changes throughout the semester, as needed. In the interest of transparency, all versions will remain available on the course website.

[9] The diagram is drawn using the `tikz-cd` package, more information about which can be found here.