Goerss–Hopkins obstruction theory via model ∞-categories

by

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Abstract

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We develop a theory of *model ∞-categories* – that is, of model structures on ∞-categories – which provides a robust theory of resolutions entirely native to the ∞-categorical context. Using model ∞-categories, we generalize *Goerss–Hopkins obstruction theory* from spectra to an arbitrary (presentably symmetric monoidal stable) ∞-category. We give a sample application of this generalized obstruction theory in the setting of motivic homotopy theory, where we construct $\mathbb{E}_\infty$ structures on the motivic Morava E-theories and compute their automorphism spaces (as $\mathbb{E}_\infty$ algebras).

This thesis is dedicated to my grandparents.

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Chapter 0

Introduction

The bulk of this introductory chapter is split into three sections.

In § 0.1, we provide an expository overview of abstract homotopy theory. In the interest of accessibility to a broad mathematical audience, we begin with the motivation provided by abelian categories. In order to set the stage for the following section, we place particular emphasis on model categories and ∞-categories.

Next, in §0.2, we describe the theory of *model ∞-categories* which is introduced in this thesis. This provides a robust theory of resolutions entirely native to the world of ∞-categories. The context provided by § 0.1 makes this a fairly straightforward endeavor. We also describe a number of auxiliary results that we establish in ∞-category theory that serve as input to the theory of model ∞-categories.

Then, in §0.3 we describe our generalization of *Goerss–Hopkins obstruction theory* from spectra (in the sense of stable homotopy theory) to an arbitrary sufficiently nice ∞-category. This is a powerful tool for constructing "highly structured" objects (e.g. $\mathbb{E}_\infty$ algebras) out of purely algebraic data.

The original obstruction theory is based in a point-set model category of spectra satisfying a host of technical assumptions, which makes its direct generalization rather difficult. Thus, our generalization relieves the original construction of unnecessary point-set technicalities. However, as it turns out, relieving the construction of point-set technicalities is not the same thing as relieving it of model structures: as we will see, the obstruction theory relies crucially on the notion of a resolution, and so our generalization necessitates the use of the full strength of the theory of model ∞-categories.

This section begins with an introduction to spectra (as the "nonabelian derived ∞-category of sets") and to stable ∞-categories more generally. It then proceeds to give an impressionistic survey of derived algebraic geometry and chromatic homotopy theory, which provide context for some of the most important applications

of Goerss–Hopkins obstruction theory to date, most notably the construction of the cohomology theory *tmf* of topological modular forms. After describing the obstruction theory itself (in two passes), it closes by describing our sample application. This comes from motivic homotopy theory, which is a homotopical context for studying algebraic varieties and their various cohomology theories. Our application concerns the *motivic Morava E-theories*, which are certain "higher chromatic analogs" of (motivic) algebraic K-theory.

Finally, in §0.4 we say a few words regarding our conventions (which are spelled out in full detail in §A), and in §0.5 we express our acknowledgments.

0.1 A brief history of derived categories, nonabelian derived categories, and abstract homotopy theory

In this expository section, we provide a broad overview of abstract homotopy theory. In the interest of accessibility to a wide mathematical audience, we center our discussion around the theme of (derived) functors between abelian categories. We place particular emphasis on the theories of model categories and of ∞-categories, since the intuition surrounding them will play a prominent role in the remainder of this introduction (especially in §0.2).

0.1.1 Derived categories, derived functors, and resolutions

In studying abelian categories, one immediately encounters the inescapable fact that not every functor

$$F : \mathcal{A} \to \mathcal{B}$$

among them is exact: some are only left-exact (i.e. preserve kernels), some are only right-exact (i.e. preserve cokernels), and some are neither left- nor right-exact. For example, if we take $\mathcal{A} = \mathcal{B} = \mathrm{Mod}_R$ for a commutative ring R, then for an arbitrary R-module M the functor

$$M \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_R$$

will always be right-exact but will not generally be left-exact.

In his groundbreaking "Tôhoku paper" [Gro57], Grothendieck introduced an organizational framework for understanding and quantifying these failures of exactness, based on the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. This category provides a home

for *resolutions* of objects of $\mathcal{A}$: these are objects which are "weakly equivalent" to our original objects of $\mathcal{A}$, but which are better behaved with respect to our given functor of interest (in a sense to be described shortly). One would now like to define the *derived functor* of F to be the value of the induced functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$$

on an appropriately chosen resolution.

However, such resolutions – and thence their values under the functor $\mathrm{Ch}(F)$ – are generally only well-defined up to weak equivalence (a/k/a "quasi-isomorphism"). There are two ways of remedying this situation.

- One may take homology of these values in $\mathrm{Ch}(\mathcal{B})$ to obtain well-defined objects of $\mathcal{B}$. For example, this technique leads to the definition of $\mathrm{Tor}^R_*(M, -)$ as the derived functor of $M \otimes_R -$.

- Alternatively, writing $\mathbf{W}_{\text{q.i.}} \subset \mathrm{Ch}(\mathcal{B})$ for the subcategory of quasi-isomorphisms, one can consider the derived functor of F as taking values in the *derived category* of $\mathcal{B}$, i.e. the localization $\mathscr{D}(\mathcal{B}) = \mathrm{Ch}(\mathcal{B})[\mathbf{W}^{-1}_{\text{q.i.}}]$.

In fact, the first approach can always be recovered from the second: by the definition of quasi-isomorphism, homology descends along the canonical localization functor $\mathrm{Ch}(\mathcal{B}) \to \mathscr{D}(\mathcal{B})$.

Of course, a derived functor should in particular be a functor, but it is not immediately obvious that the process we have described defines one. In fact, our desired functoriality will be a consequence of our definition of "resolution". The appropriate notion will vary from one application to another, but in any case the crucial property will be that the restriction

$$\mathrm{Ch}(\mathcal{A})^{\mathrm{res}} \hookrightarrow \mathrm{Ch}(\mathcal{A}) \xrightarrow{\mathrm{Ch}(F)} \mathrm{Ch}(\mathcal{B})$$

to the full subcategory of "resolutions" preserves weak equivalences. For example, given any R-module N, any weak equivalence $P_\bullet \xrightarrow{\approx} Q_\bullet$ between projective resolutions of N induces a weak equivalence

$$M \otimes_R P_\bullet \xrightarrow{\approx} M \otimes_R Q_\bullet$$

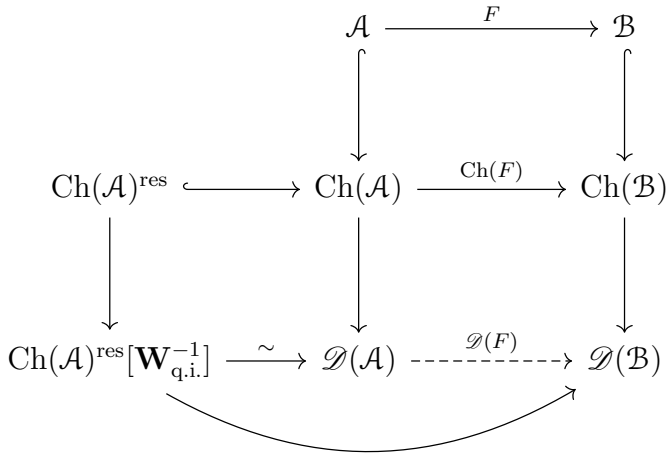
upon tensoring with M.[1] Moreover, every object should admit a resolution: indeed, in many cases (such as with model categories, as we will see in §0.1.2), the inclusion

[1] On the other hand, these objects are *not* generally weakly equivalent to $M \otimes_R N$: this is the entire point of resolving N in the first place.

$\mathrm{Ch}(\mathcal{A})^{\mathrm{res}} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ even induces an equivalence

$$\mathrm{Ch}(\mathcal{A})^{\mathrm{res}}[\mathbf{W}_{\mathrm{q.i.}}^{-1}] \xrightarrow{\sim} \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\mathrm{q.i.}}^{-1}] = \mathscr{D}(A)$$

on localizations. In such a situation, we then obtain the derived functor $\mathscr{D}(F)$ of the original functor F as an extension in the commutative diagram



of categories (which is well-defined up to natural isomorphism). The resulting composite

$$\mathcal{A} \to \mathscr{D}(\mathcal{A}) \xrightarrow{\mathscr{D}(F)} \mathscr{D}(\mathcal{B})$$

is sometimes referred to as the *total derived functor* of F (recovering as it does the "i^{th} derived functor" of F upon postcomposition with the functor $H_i : \mathscr{D}(\mathcal{B}) \to \mathcal{B}$).

0.1.2 Model categories

By definition, the derived category $\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\mathrm{q.i.}}^{-1}]$ of an abelian category $\mathcal{A}$ is the universal recipient of homological invariants. For example, the derived category $\mathscr{D}_R = \mathscr{D}(\mathrm{Mod}_R)$ is the target of the derived functor

$$\mathrm{Mod}_R \to \mathscr{D}_R \xrightarrow{\mathscr{D}(M \otimes_R -)} \mathscr{D}_R$$

of the functor

$$M \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

Correspondingly, the derived category enjoys a universal property *as a category*. However, it tends to be quite difficult to make computations *within* the derived

category. In effect, this is because its universal property takes place "one category-level higher" than do its actual objects and morphisms themselves.

In order to discuss this phenomenon, it is convenient to introduce the notion of a *relative category*: this is a (strict) category $\mathcal{R}$ equipped with a distinguished subcategory $\mathbf{W} \subset \mathcal{R}$ of "weak equivalences" which is required to contain the subcategory $\mathcal{R}^{\cong} \subset \mathcal{R}$ of isomorphisms. The category $\mathfrak{relcat}$ of relative categories admits a *localization* functor

$$\mathfrak{relcat} \to \mathfrak{cat}$$

to the category $\mathfrak{cat}$ of (strict) categories (which we have already referred to in §0.1.1), which is by definition left adjoint to the "minimal relative category structure" functor $\mathcal{C} \mapsto (\mathcal{C}, \mathcal{C}^{\cong})$. Given a relative category $(\mathcal{R}, \mathbf{W})$, its localization $\mathcal{R}[\mathbf{W}^{-1}]$ – which is also in some contexts called its "homotopy category" – is therefore equipped with a canonical localization functor

$$\mathcal{R} \to \mathcal{R}[\mathbf{W}^{-1}]$$

with the universal property that for any category $\mathcal{C} \in \mathfrak{cat}$, the restriction map

$$\hom_{\mathfrak{cat}}(\mathcal{R}[\mathbf{W}^{-1}], \mathcal{C}) \to \hom_{\mathfrak{cat}}(\mathcal{R}, \mathcal{C})$$

defines an isomorphism onto the set of functors $\mathcal{R} \to \mathcal{C}$ which take the subcategory $\mathbf{W} \subset \mathcal{R}$ of weak equivalences into the subcategory $\mathcal{C}^{\cong} \subset \mathcal{C}$ of isomorphisms.[2] At one extreme, the localization $\mathcal{R}[(\mathcal{R}^{\cong})^{-1}]$ of the minimal relative category structure is therefore simply $\mathcal{R}$ itself, while at the other extreme, the localization $\mathcal{R}[\mathcal{R}^{-1}]$ of the "maximal" relative category structure recovers the *groupoid completion* of the category $\mathcal{R}$.

Using this language, we can now illustrate the difficulty of making computations within the localization of a relative category, such as the derived category $\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\mathrm{q.i.}}^{-1}]$ of an abelian category $\mathcal{A}$.

We begin with the smallest possible example. Recall that a category with a single object is completely specified by the monoid of endomorphisms of its object; given a monoid G_0, we write $\mathfrak{B}G_0$ for the corresponding one-object category. Under this correspondence, the group completion G of the monoid G_0 corresponds to the groupoid completion of $\mathfrak{B}G_0$: that is, there is a canonical isomorphism

$$\mathfrak{B}G \cong \mathfrak{B}G_0[(\mathfrak{B}G_0)^{-1}]$$

in $\mathfrak{cat}$. But while the groupoid $\mathfrak{B}G$ is easy to characterize by means of its universal property, it is hopelessly difficult to describe in concrete terms. Indeed, understanding composition in $\mathfrak{B}G$ amounts to understanding the multiplication law of G, but

[2] The term "localization functor" is certainly overloaded, but it should always be clear what is meant in any given situation.

this is an intractable (in fact, computationally undecidable) task, closely related to the so-called "word problem" for generators and relations in abstract algebra.

More generally, given a relative category $(\mathcal{R}, \mathbf{W})$ and any two objects $x, y \in \mathcal{R}$, morphisms from x to y in the localization $\mathcal{R}[\mathbf{W}^{-1}]$ will be represented by equivalence classes of "zigzags"

$$x \xleftarrow{\approx} \bullet \to \bullet \xleftarrow{\approx} \cdots \xleftarrow{\approx} \bullet \to \bullet \xleftarrow{\approx} y$$

in $(\mathcal{R}, \mathbf{W})$ from x to y.[3,4] In particular, note that elements of $\hom_{\mathcal{R}[\mathbf{W}^{-1}]}(x, y)$ will generally fail drastically to be represented by elements of $\hom_{\mathcal{R}}(x, y)$.

It was against this backdrop that Quillen introduced the general theory of *model categories* in his seminal work [Qui67]. A model category $\mathcal{M}$ consists of a relative category equipped with certain additional data that are collectively called a *model structure*, which in particular specify full subcategories

$$\mathcal{M}^c \hookrightarrow \mathcal{M} \hookleftarrow \mathcal{M}^f$$

of *cofibrant* objects and of *fibrant* objects. Moreover, the axioms dictate that every object of $\mathcal{M}$ is weakly equivalent to a cofibrant object and is also weakly equivalent to a fibrant object. Thus, the following ***fundamental theorem of model categories*** provides a direct and computable method of accessing the hom-sets in the localization $\mathcal{M}[\mathbf{W}^{-1}]$.

Theorem 0.1.1 (Quillen)**.** *Let $\mathcal{M}$ be a model category, and suppose that $x \in \mathcal{M}$ is cofibrant and that $y \in \mathcal{M}$ is fibrant. Then the canonical map*

$$\hom_{\mathcal{M}}(x, y) \to \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x, y)$$

is a surjection, which moreover becomes an isomorphism after applying either equivalence relation of "left homotopy" or "right homotopy" to the source.

Thus, cofibrant objects should be thought of as being "good for mapping out of", while fibrant objects should be thought of as being "good for mapping into".[5]

[3] Strictly speaking, we should really be referring to the *images* of x and y under the localization functor $\mathcal{R} \to \mathcal{R}[\mathbf{W}^{-1}]$, but (since we are speaking strictly) this induces an isomorphism on sets of objects and so there is no real ambiguity.

[4] As an example of the equivalence relation on zigzags, if one of the backwards-pointing weak equivalences happens to be an isomorphism, then the displayed zigzag must be declared equivalent to the one obtained by replacing this weak equivalence with its (forwards-pointing) inverse and then composing with any adjacent forwards-pointing arrows.

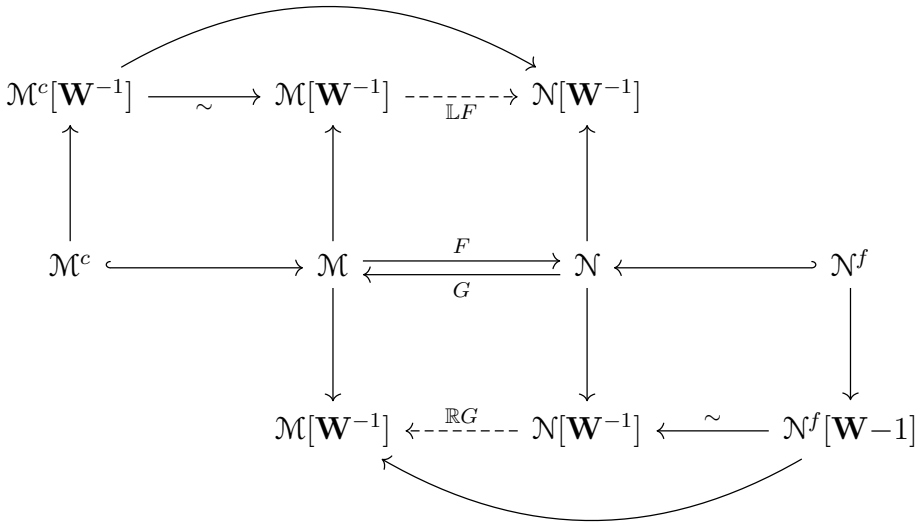
[5] For example, the relative category $(\mathrm{Ch}_R, \mathbf{W}_{\text{q.i.}})$ admits a model structure in which bounded-below complexes of projective R-modules are cofibrant and all objects are fibrant, and the "ho-

Beyond providing direct access to computations in derived categories, the theory of model categories moreover bears directly on the construction of derived functors. Given two model categories $\mathcal{M}$ and $\mathcal{N}$, a *Quillen adjunction* between them is an adjunction

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

satisfying certain conditions related to their respective model structures, and a *Quillen equivalence* is a Quillen adjunction satisfying a further condition. These notions are immensely useful for both constructing and computing derived functors, as a consequence of the following result.

Theorem 0.1.2 (Quillen)**.** *Given a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$, the left adjoint F preserves weak equivalences between cofibrant objects of $\mathcal{M}$ and the right adjoint G preserves weak equivalences between fibrant objects of $\mathcal{N}$. These induce derived functors via the commutative diagram*



of categories, which moreover participate in a canonical ***derived adjunction***

$$\mathbb{L}F : \mathcal{M}[\mathbf{W}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}^{-1}] : \mathbb{R}G$$

motopy" relations can be computed via the usual notion of chain homotopy. In fact, this same relative category admits *another* model structure, in which bounded-above complexes of injectives are fibrant and all objects are cofibrant. The existence of these two distinct model structures is responsible e.g. for the fact that we can compute $\mathrm{Ext}^*_R(M,N)$ either by applying the functor $\hom_R(-,N)$ to a projective resolution $P_\bullet \xrightarrow{\approx} M$ or by applying the functor $\hom_R(M,-)$ to an injective resolution $N \xrightarrow{\approx} I^\bullet$.

on localizations. If this Quillen adjunction is moreover a Quillen equivalence, then the derived adjunction is an adjoint equivalence of categories.

Thus, more generally, cofibrant objects should be thought of as *left resolutions*, while fibrant objects should be thought of as *right resolutions*.

0.1.3 Nonabelian derived categories

Let us return to our discussion of functors $F : \mathcal{A} \to \mathcal{B}$ between abelian categories. Recall that in good cases, the derived functor of F can be computed by passing to the induced functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$$

on categories of chain complexes and then restricting to a subcategory of "resolutions" (whose precise nature depends on the situation at hand).

Let us restrict our attention for a moment to the subcategory $\mathrm{Ch}_{\geq 0}(\mathcal{A}) \subset \mathrm{Ch}(\mathcal{A})$ of nonnegatively-graded chain complexes. Then, there is an equivalence

$$\mathrm{Ch}_{\geq 0}(\mathcal{A}) \simeq s\mathcal{A}$$

with the category of *simplicial objects* in $\mathcal{A}$, i.e. the category $s\mathcal{A} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ of $\mathcal{A}$-valued presheaves on the category $\mathbf{\Delta}$ of finite nonempty totally-ordered sets.

This leads to an enormously fruitful idea: if we are interested in resolving objects of a *nonabelian* category $\mathcal{C}$, then the category $s\mathcal{C} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ provides a reasonable substitute for the (nonexistent) category of "nonnegatively-graded chain complexes in $\mathcal{C}$". Moreover, the category $s\mathcal{C}$ still comes equipped with a subcategory $\mathbf{W} \subset s\mathcal{C}$ of weak equivalences (which reduces to that of quasi-isomorphisms in the abelian case), allowing us to form the *nonnegatively-graded nonabelian derived category* of $\mathcal{C}$ as the localization

$$\mathscr{D}_{\geq 0}(\mathcal{C}) = s\mathcal{C}[\mathbf{W}^{-1}].$$

(We will explain how to recover the full nonabelian derived category $\mathscr{D}(\mathcal{C})$ in §0.3.1.)[6]

As a first example, let us take $\mathcal{C} = \mathcal{S}\mathrm{et}$ to be the category of sets. Now, the category $s\mathcal{S}\mathrm{et}$ of simplicial sets admits a *geometric realization* functor

$$|-| : s\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$$

to the category of topological spaces: this uses a simplicial set as a recipe for assembling a simplicial complex, with the structure maps between the various constituent

[6] Actually, this is all a very slight simplification, which we will elide for now but return to in §0.3.4.

sets specifying the gluing data between topological simplices. In this case, the subcategory $\mathbf{W} \subset s\mathcal{S}\mathrm{et}$ is pulled back from the subcategory $\mathbf{W}_{\mathrm{w.h.e.}} \subset \mathcal{T}\mathrm{op}$ of weak homotopy equivalences, and moreover the geometric realization functor induces an equivalence

$$\mathscr{D}_{\geq 0}(\mathcal{S}\mathrm{et}) = s\mathcal{S}\mathrm{et}[\mathbf{W}^{-1}] \xrightarrow{\sim} \mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}].$$

In other words, the nonnegatively-graded nonabelian derived category of sets is nothing other than the classical homotopy category of topological spaces! In this sense, the category $s\mathcal{S}\mathrm{et}$ of simplicial sets can be seen as a combinatorial presentation of the homotopy category $\mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]$ of topological spaces, and simplicial sets themselves can be seen as combinatorial presentations of homotopy types.

In fact, the geometric realization functor participates in an adjunction

$$|-| : s\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}(-)_\bullet$$

with the *singular simplicial set* functor: given a topological space $X \in \mathcal{T}\mathrm{op}$, the corresponding simplicial set $\mathrm{Sing}(X)_\bullet \in s\mathcal{S}\mathrm{et} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}\mathrm{et})$ is given by taking the object $[n] = \{0, \ldots, n\} \in \mathbf{\Delta}^{op}$ to the set

$$\mathrm{Sing}_n(X) = \hom_{\mathcal{T}\mathrm{op}}(\Delta^n_{\mathrm{top}}, X)$$

of continuous maps into X from the standard topological n-simplex Δ^n_{top}.[7] Moreover, there exist model structures on these two relative categories – the *Kan–Quillen model structure* on $s\mathcal{S}\mathrm{et}$ and the *Quillen–Serre model structure* on $\mathcal{T}\mathrm{op}$ – making this adjunction into a Quillen equivalence.[8] In particular, the category $s\mathcal{S}\mathrm{et}$ is not merely a combinatorial presentation of the homotopy category $\mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]$: its Kan–Quillen model structure moreover allows for extremely efficient computations therein.

0.1.4 The homotopy theory of homotopy theories

In their radical and innovative paper [DK80c], Dwyer–Kan turned the lens of abstract homotopy theory onto itself, introducing a *derived functor* of the localization functor

[7] The functoriality of $\mathrm{Sing}(X)_\bullet : \mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ arises from pulling back along certain continuous functions between the various topological simplices, which are defined by mimicking the behavior of the corresponding morphisms in $\mathbf{\Delta}$ on vertices and then extending linearly. In fact, these assemble into a *cosimplicial* object $\Delta^\bullet_{\mathrm{top}} : \mathbf{\Delta} \to \mathcal{T}\mathrm{op}$, and we can consider the functor

$$\mathrm{Sing}(-)_\bullet = \hom^{\mathrm{lw}}_{\mathcal{T}\mathrm{op}}(\Delta^\bullet_{\mathrm{top}}, -)$$

as arising from taking "levelwise maps" out of this cosimplicial topological space.

[8] The derived left adjoint of this Quillen equivalence recovers the equivalence described above: all objects of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ are cofibrant, so the restriction to the subcategory $s\mathcal{S}\mathrm{et}^c_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ (as in the statement of Theorem 0.1.2) is already implicit.

$\mathfrak{Rel}\mathfrak{Cat} \to \mathfrak{Cat}$: this is a functor

$$\mathfrak{Rel}\mathfrak{Cat} \to \mathfrak{Cat}_{s\mathcal{S}\mathrm{et}}$$

landing in the category of *simplicially-enriched categories*, i.e. in the category of categories enriched in the category $s\mathcal{S}\mathrm{et}$ of simplicial sets.[9,10] As simplicial sets can be considered as presentations of homotopy types, objects of $\mathfrak{Cat}_{s\mathcal{S}\mathrm{et}}$ can be considered as presentations of "categories enriched in homotopy types".[11]

Of course, such a viewpoint immediately suggests a notion of "weak equivalence" among simplicially-enriched categories; weak equivalence classes of objects of $\mathfrak{Cat}_{s\mathcal{S}\mathrm{et}}$ came to be known colloquially as *homotopy theories*, and the corresponding localization

$$\mathfrak{Cat}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$$

came to be known as *the homotopy theory of homotopy theories*.

As we have seen, such a definition is of rather limited use in and of itself: it is generally extremely difficult to make computations in a localization. However, in [Ber07], Bergner drastically improved the state of affairs by constructing a model structure on $\mathfrak{Cat}_{s\mathcal{S}\mathrm{et}}$ extending this relative category structure, providing the first *model category* presenting the homotopy theory of homotopy theories.

Given a relative category $(\mathcal{R}, \mathbf{W})$, we denote its derived localization – also known as its *underlying homotopy theory* – by

$$\mathcal{R}[\![\mathbf{W}^{-1}]\!] \in \mathfrak{Cat}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}].$$

This power series notation is meant to indicate that the derived localization contains "higher-order" information than does the ordinary localization $\mathcal{R}[\mathbf{W}^{-1}] \in \mathfrak{Cat}$. Indeed, the "homotopy category" functor

$$\mathfrak{Cat}_{s\mathcal{S}\mathrm{et}} \to \mathfrak{Cat}$$

[9] Simplicially-enriched categories are not quite the same thing as simplicial objects in $\mathfrak{Cat}$: rather, $\mathfrak{Cat}_{s\mathcal{S}\mathrm{et}} \subset s(\mathfrak{Cat})$ defines a full subcategory on those objects whose "simplicial set of objects" is constant.

[10] Their technique falls squarely in line with the "simplicial objects as resolutions" paradigm described in §0.1.3: the derived localization functor is defined as the composite

$$\mathfrak{Rel}\mathfrak{Cat} \to s(\mathfrak{Rel}\mathfrak{Cat}) \to \mathfrak{Cat}_{s\mathcal{S}\mathrm{et}}$$

of a "free simplicial resolution" functor followed by a levelwise application of the ordinary localization functor $\mathfrak{Rel}\mathfrak{Cat} \to \mathfrak{Cat}$.

[11] Actually, this is not quite correct: a simplicially-enriched category also contains "homotopy-coherence data" for its composition (in a sense to be described in §0.1.5) which are not present in a category enriched in homotopy types.

(which takes each hom-simplicial set (considered as a homotopy type) to its set of path components) takes the subcategory $\mathbf{W} \subset \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ of weak equivalences into the subcategory $\mathbf{W} \subset \mathcal{C}\mathrm{at}$ of *equivalences* of categories, and the induced diagram

$$\begin{array}{ccc} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} & \xrightarrow{(\mathcal{R},\mathbf{W})\mapsto \mathcal{R}[\mathbf{W}^{-1}]} & \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}] \\ & \searrow^{(\mathcal{R},\mathbf{W})\mapsto \mathcal{R}[\mathbf{W}^{-1}]} & \downarrow \mathrm{ho} \\ & & \mathcal{C}\mathrm{at}[\mathbf{W}^{-1}] \end{array}$$

commutes (up to natural isomorphism).

0.1.5 The zen of ∞-categories

Since the work of Bergner, there has been a proliferation of model categories which are Quillen equivalent to $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ and thus likewise present the homotopy theory of homotopy theories (by virtue of Theorem 0.1.2). Purely as a matter of terminology, objects of *any* of these model categories – or more precisely, their weak equivalence classes – have come to be referred to as ∞-*categories*.

In fact, some of these other model categories of ∞-categories enjoy better technical properties than does $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ (or does its close cousin $(\mathcal{C}\mathrm{at}_{\mathcal{T}\mathrm{op}})_{\mathrm{Bergner}}$), making them far more useful in practice.[12] However, in addition to these technical advantages, certain of these other model categories admit *philosophical* advantages. In essence, the idea is that ∞-categories should not really be thought of as being *strictly* enriched – in topological spaces, or simplicial sets, or anything else: rather, they should be thought of as being enriched *in the* ∞-*category of spaces*, namely the equivalence class

$$\mathcal{S} \in \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$$

of either equivalent derived localization

$$\mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}] \simeq s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{KQ}}^{-1}]$$

[12] The issue is that the model category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ behaves poorly with respect to products: the product of two cofibrant objects will not generally be cofibrant. This is a major issue, for it obstructs a clean construction of a "homotopically correct" internal hom-object. At the level of the homotopy category $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$, this should be an object $\underline{\mathrm{hom}}(\mathcal{C},\mathcal{D})$ with represented functor given by

$$\mathcal{E} \mapsto \mathrm{hom}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]}(\mathcal{E}, \underline{\mathrm{hom}}(\mathcal{C},\mathcal{D})) \cong \mathrm{hom}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]}(\mathcal{E}\times\mathcal{C},\mathcal{D}).$$

But since it's not straightforward to obtain a cofibrant representative of the product $\mathcal{E}\times\mathcal{C}$ at the level of the model category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$, it becomes difficult to naturally construct an object at that level that descends through the localization $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$ to represent the functor $\mathrm{hom}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]}((-)\times\mathcal{C},\mathcal{D})$.

of a relative category.[13] In other words, the ∞-category $\mathcal{S}$ of spaces plays an analogous role in ∞-category theory to the one played by the category $\mathcal{S}\mathrm{et}$ in 1-category theory. In order to illustrate this idea, we briefly survey the theory of *quasicategories*.

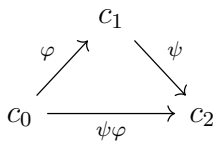
We begin by recalling the *nerve* construction, which is a functor

$$\mathrm{N}(-)_\bullet : \mathbb{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}.$$

By definition, the category $\mathbf{\Delta}$ is a category of posets, which are particular examples of categories; thus there is an inclusion functor $\mathbf{\Delta} \hookrightarrow \mathbb{C}\mathrm{at}$. Then, the nerve functor is given by the restricted Yoneda embedding: for any $\mathcal{C} \in \mathbb{C}\mathrm{at}$ and any $[n] \in \mathbf{\Delta}$, we define

$$\mathrm{N}(\mathcal{C})_n = \hom_{\mathbb{C}\mathrm{at}}([n], \mathcal{C}).$$

So the set $\mathrm{N}(\mathcal{C})_n$ of n-simplices is the set of sequences of n composable morphisms in $\mathcal{C}$ (with $\mathrm{N}(\mathcal{C})_0$ simply the set of objects), and for instance the morphism $\{0,1\} \to \{0,1,2\}$ in $\mathbf{\Delta}$ given by $0 \mapsto 0$ and $1 \mapsto 2$ determines a function $\mathrm{N}(\mathcal{C})_2 \to \mathrm{N}(\mathcal{C})_1$ which takes a pair of composable morphisms $(c_0 \xrightarrow{\varphi} c_1, c_1 \xrightarrow{\psi} c_2)$ to its composite $(c_0 \xrightarrow{\psi\varphi} c_2)$. Thus, a 2-simplex of $\mathrm{N}(\mathcal{C})_\bullet$ may be thought of as encoding a commutative triangle



in $\mathcal{C}$, and we may therefore think of it as a "witness" to the fact that the morphism $\psi\varphi$ is the composite of the morphisms φ and ψ. As composition in the category $\mathcal{C}$ is uniquely defined, it follows that for any two "composable 1-simplices" of $\mathrm{N}(\mathcal{C})_\bullet$ (such as φ and ψ as above), there exists a *unique* 2-simplex extending them (such as the 2-simplex above).

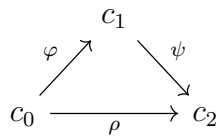
Now, in the setting of simplicially-enriched categories, the nerve functor can be enhanced to the *homotopy-coherent nerve* functor, denoted

$$\mathrm{N}^{\mathrm{hc}}(-)_\bullet : \mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to s\mathcal{S}\mathrm{et}.$$

Rather than describe this in full, we will simply indicate its values in the bottom few dimensions. For a simplicially-enriched category $\mathcal{C} \in \mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$, we once again have that the set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_0$ of 0-simplices is given by the set of objects of $\mathcal{C}$, and that the

[13] In fact, the ∞-category of spaces admits various universal characterizations which make no reference whatsoever to topological spaces or to simplicial sets: for instance, it is the *free cocompletion* of the terminal ∞-category.

set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_1$ of 1-simplices is given by the set of morphisms of $\mathcal{C}$ (i.e. the 0-simplices of its various hom-simplicial sets, or equivalently the morphisms in its underlying unenriched category). However, the set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_2$ of 2-simplices is more interesting: for any three morphisms $c_0 \xrightarrow{\varphi} c_1$, $c_1 \xrightarrow{\psi} c_2$, and $c_0 \xrightarrow{\rho} c_2$ in $\mathcal{C}$, a 2-simplex



is determined by a *1-simplex* in the simplicial set $\underline{\hom}_{\mathcal{C}}(c_0, c_2)$ connecting the 0-simplices $\psi\varphi$ and ρ. Thinking of such a 1-simplex as a "path" in this "hom-space", we may therefore think of such a 2-simplex as a witness to the *homotopy commutativity* of this triangle.

Of course, one such 2-simplex of $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$ can be obtained simply by taking $\rho = \psi\varphi$ (and by taking the "path" to be the constant one). Thus, any two "composable 1-simplices" of $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$ admit *some* 2-simplex extending them. However, in the abstract simplicial set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$, it is no longer possible to tell which 2-simplices arose from "strict composition" and which 2-simplices arose from "homotopy-coherent composition". And indeed, this is the entire point: *any* of the possible 2-simplex extensions of our two composable 1-simplices should be considered to be "just as good" as any other. In other words, the strict composition of composable 1-simplices in this simplicial set is *not even well-defined*.

The homotopy-coherent nerve $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$ is the canonical example of a *quasicategory*. This is nothing other than a simplicial set $\mathtt{C}$ in which, for all $n \geq 2$, any string of n composable 1-simplices admits *some* extension to an n-simplex. If this string is selected by a morphism

$$\left(\Delta^{\{0,1\}} \underset{\Delta^{\{1\}}}{\amalg} \cdots \underset{\Delta^{\{n-1\}}}{\amalg} \Delta^{\{n-1,n\}}\right) \to \mathtt{C}$$

of simplicial sets, then such an n-simplex $\Delta^n \to \mathtt{C}$ may be thought of as a witness to the fact that its 1-subsimplex

$$\Delta^{\{0,n\}} \to \Delta^n \to \mathtt{C}$$

is a composite of the string. Of course, in general such an extension will not be unique: indeed, all such extensions (for all strings of all lengths) will be unique precisely when $\mathtt{C}$ is the nerve of an ordinary category. Nevertheless, there is a strong sense in which such an extension is "essentially unique": the set of extensions of a string naturally extends to a *simplicial* set, which will always be contractible *when*

considered as a space (i.e. as an object of the ∞-category $\mathcal{S} \simeq s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{KQ}}^{-1}]$). Quasicategories are the fibrant objects of the *Joyal model structure* on the category $s\mathcal{S}\mathrm{et}$, to which the homotopy-coherent nerve functor defines a right Quillen equivalence

$$s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \leftarrow (\mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}} : \mathrm{N}^{\mathrm{hc}}(-)_{\bullet}.$$

Of course, the ∞-category of spaces is a rather abstract object. By contrast, its objects can be presented by topological spaces or by simplicial sets, both of which notions are quite concrete. For instance, one can speak of the "underlying set" of a topological space, whereas a *space* admits no such notion: a weak equivalence between topological spaces will not generally respect their underlying sets.

It would therefore appear to afford much more control to work directly with topologically- or simplicially-enriched categories, rather than considering them only as being enriched in the ∞-category of spaces (e.g. so that one can speak of the "underlying set" of a hom-space). Thus, the idea that an ∞-category should only be considered as being enriched over spaces runs directly against intuition, and against deeply-ingrained human urges for control.

However, ***the sheer power of this idea is impossible to overstate***.

To illustrate this striking phenomenon, we give two examples. For concreteness, both will concern the relationship between the 1-category $\mathcal{T}\mathrm{op}$ of topological spaces and the ∞-category $\mathcal{S}$ of spaces. For the present purposes, it will be convenient to consider the ∞-category of spaces as being presented by the homotopy-coherent nerve of the *topologically*-enriched category of CW complexes (although we only really consider it as a quasicategory at all to emphasize the non-strictness of its composition).[14,15]

[14] Both functors in the adjunction $|-| : s\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}(-)_{\bullet}$ preserve finite products; applying them "locally" (i.e. to each hom-object individually) therefore defines an adjunction $\mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \rightleftarrows \mathbb{C}\mathrm{at}_{\mathcal{T}\mathrm{op}}$, and one can define the homotopy-coherent nerve of a topologically-enriched category simply by precomposing with its right adjoint.

[15] There is a notion of a model category being *compatibly enriched* over a given monoidal model category (the definition of which itself requires certain compatibilities between the model structure and the monoidal structure); for instance, both $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$ and $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ are compatibly self-enriched. Given a model category $\underline{\mathcal{M}}$ which is compatibly enriched over either $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$ or $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ and writing $\mathcal{M}$ for its underlying unenriched model category, the underlying ∞-category $\mathcal{M}[\mathbf{W}^{-1}]$ is presented by the $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$- or $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$-enriched category $\underline{\mathcal{M}}^{cf}$ of *bifibrant* (i.e. cofibrant and fibrant) objects. In particular, if either $x \in \mathcal{M}$ is not cofibrant or $y \in \mathcal{M}$ is not fibrant, then the enriched hom-object $\underline{\hom}_{\mathcal{M}}(x, y)$ will not generally have the "correct" weak equivalence class. In $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$, all objects are fibrant and CW complexes are cofibrant. In fact, they are not *all* of the cofibrant objects (these are "cell complexes and retracts thereof"), but their full inclusion into the topologically-enriched category $\underline{\mathcal{T}\mathrm{op}}^{cf}_{\mathrm{QS}}$ is a weak equivalence (and hence presents an equivalence of ∞-categories), so we've just restricted to them for simplicity of terminology.

Our first example of the power of ∞-categorical thinking illustrates the following paradigm: *working ∞-categorically, it's impossible to say the wrong thing.*

Given a based CW complex X, its *suspension* is defined to be the pushout

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

with itself of the inclusion of X into the cone $CX = (X \times [0,1])/(X \times \{1\})$ (as the subspace $X \times \{0\}$). This is an extremely useful construction in homotopy theory: for instance, it participates in a *suspension isomorphism*

$$\tilde{H}_i(X) \cong \tilde{H}_{i+1}(\Sigma X)$$

in (reduced) homology.

However, this definition itself is clearly not the "true" thing. After all, the cone CX is contractible, and indeed any other two contractible CW complexes into which X maps as closed inclusions would function just as well: more precisely, the resulting pushout would be weakly equivalent to the suspension ΣX. One gets the distinct sense that this "wants to be" the pushout

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \Sigma X \end{array}$$

along the unique terminal maps into (what end up being) the two cone points of ΣX – the only problem being that this diagram of topological spaces simply doesn't commute, let alone define a pushout.

On the other hand, this *canonically* defines a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{pt} \\ \downarrow & \overset{\simeq}{\searrow} & \downarrow \\ \mathrm{pt} & \longrightarrow & \Sigma X \end{array}$$

in the ∞-category of spaces! First of all, the map $X \to \Sigma X$ is given by the equatorial inclusion. Then, the homotopy-commutativity of each of the two triangles is selected by the canonical homotopy

$$X \times [0,1] \to CX$$

given by the formula

$$(x, t) \mapsto (x, t).$$

That is, this postcomposes to define homotopies

$$X \times [0, 1] \to CX \to \Sigma X,$$

which select canonical paths in the hom-topological space $\underline{\hom}_{\mathcal{T}\mathrm{op}}(X, \Sigma X)$ between the equatorial inclusion and the inclusion of one or the other cone point.

Even better, this commutative square is a pushout in the ∞-categorical sense. Working ∞-categorically, the universal property of a pushout

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \coprod_A C \end{array}$$

has no choice but to read: "an object which, when mapped into a test object Y, corepresents the data of a map $B \to Y$, a map $C \to Y$, and a *path* witnessing the agreement of the two composites $A \to B \to Y$ and $A \to C \to Y$". Returning to our original example, we see that this is precisely the functor that the suspension ΣX was born to corepresent all along.

Finally, we reach our "impossible to be wrong" paradigm: for *any* contractible CW complexes B and C and *any* pair of maps $X \to B$ and $X \to C$, the ∞-categorical pushout

$$\begin{array}{ccc} X & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \coprod_X C \end{array}$$

is canonically equivalent (in the ∞-category of spaces) to the suspension ΣX. This is in stark contrast to the situation in the 1-category $\mathcal{T}\mathrm{op}$ of topological spaces, where one must demand that the maps $X \to B$ and $X \to C$ be closed inclusions. From this point of view, we learn the additional lesson that *working 1-categorically makes us want to force something which is* naturally *homotopy-coherent to be* unnaturally *strict*.

A pushout among CW complexes in which the two maps are closed inclusions is an example of a *homotopy pushout* in the model category $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$, which is in turn a particular example of a *homotopy colimit*. The theory of homotopy colimits in general model categories is well-studied, but it is fairly subtle and unreasonably technical:

for instance, a homotopy colimit in a model category $\mathcal{M}$ over an indexing category $\mathcal{I}$ should be the left derived functor of the colimit functor

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{I}, \mathcal{M}) \to \mathcal{M},$$

but the requisite model structure needed to actually obtain this (i.e. a model structure on $\mathrm{Fun}(\mathcal{I}, \mathcal{M})$ for which this is a left Quillen functor) need not even exist. But more importantly, even in the extremely simple case of homotopy pushouts, these point-set considerations obscure the true and essential ∞-categorical meaning of the suspension construction $X \mapsto \Sigma X$, which – tying everything together – actually gives a conceptual explanation for the suspension isomorphism in the first place.

Our second example of the power of ∞-categorical thinking illustrates the following paradigm: *homotopy-coherence appears everywhere, and working ∞-categorically sweeps homotopy-coherence into the ambient machinery.*

Given a based topological space $X = (X, x)$, its *based loopspace* is the topological space

$$\Omega X = \{\gamma : [0,1] \to X \ : \ \gamma(0) = \gamma(1) = x\},$$

or equivalently the topological space $\underline{\hom}_{\mathcal{T}\mathrm{op}_*}(S^1, X)$ of based maps from the circle

$$S^1 = [0,1]/(0 \sim 1)$$

into X.[16] By adjunction, there is a natural isomorphism

$$\pi_0(\Omega X) \cong \pi_1(X)$$

between the set of path components of ΩX and the fundamental group of X. Moreover, the group structure on the fundamental group $\pi_1(X)$ comes from concatenation of (homotopy classes of) based loops. For instance, given two based loops $\gamma_1, \gamma_2 \in \Omega X$, we can define a new based loop $(\gamma_1 * \gamma_2) \in \Omega X$ to be given by the formula

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

However, this formula is just the most straightforward option: any choice of "pinch map"

$$S^1 \xrightarrow{\Delta} S^1 \vee S^1$$

gives rise to a concatenation operation

$$\Omega X \times \Omega X \xrightarrow{\mu} \Omega X$$

[16] In fact, this is a completely dual object to the suspension ΣX: it's the ∞-*categorical pullback* of the diagram $\{x\} \to X \leftarrow \{x\}$.

which defines the same group structure on the set $\pi_0(\Omega X)$ of path components.

These considerations strongly suggest that the based loopspace ΩX should *itself* be some manner of "group", in which the multiplication law is given by concatenation of loops. However, a moment's reflection reveals that it is impossible to make this concatenation operation strictly associative, no matter which pinch map we choose.

On the other hand, in a sense, this failure of associativity is not so severe. Suppose that we fix a pinch map Δ on S^1 inducing a multiplication map μ on ΩX. Then, the associativity diagram

$$\begin{array}{ccc} (\Omega X)^{\times 3} & \xrightarrow{\mathrm{id}_{\Omega X} \times \mu} & (\Omega X)^{\times 2} \\ {\scriptstyle \mu \times \mathrm{id}_{\Omega X}}\downarrow & & \downarrow{\scriptstyle \mu} \\ (\Omega X)^{\times 2} & \xrightarrow[\mu]{} & \Omega X \end{array}$$

does not strictly commute, but it commutes *up to homotopy*: in order to specify such a homotopy, it suffices to choose once and for all a homotopy witnessing the homotopy-commutativity of the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\Delta} & S^1 \vee S^1 \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{S^1} \vee \Delta} \\ S^1 \vee S^1 & \xrightarrow[\Delta \vee \mathrm{id}_{S^1}]{} & S^1 \vee S^1 \vee S^1, \end{array}$$

and this can be done straightforwardly (in essentially the same manner that one proves that the fundamental group is associative – it can be slightly easier to visualize the analogous picture with intervals instead of wedges of circles).

This is concordant with the ***core philosophy of higher category theory***: rather than merely positing the *existence* of a homotopy witnessing the homotopy-commutativity of the associativity diagram, we should instead keep track of such a homotopy as *additional data*.

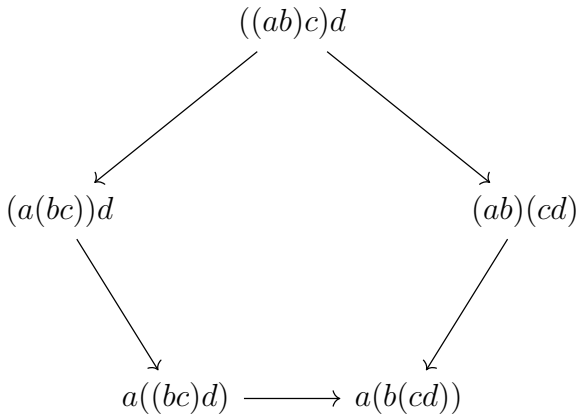
These observations are sufficient for producing the group structure on $\pi_0(\Omega X)$, but they do not yet allow us treat ΩX as a "group" itself. For instance, suppose that we would like to concatenate *four* loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Omega X$. So far, we have only chosen a multiplication

$$\mu : \mathrm{pt} \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 2}, \Omega X)$$

along with an "associator", i.e. a path

$$\mu_3 : [0,1] \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 3}, \Omega X)$$

between the two resulting composites $\mu \circ (\mathrm{id}_{\Omega X} \times \mu)$ and $\mu \circ (\mu \times \mathrm{id}_{\Omega X})$. As it turns out, μ determines five iterated multiplication maps $(\Omega X)^{\times 4} \to \Omega X$, which are in turn related by application of the associator at various stages: these can be schematically organized into the famous "Mac Lane pentagon"



in which the arrows indicate "front-to-back" associations $(xy)z \to x(yz)$.[17] This can in turn be organized as a map from the boundary of a pentagon (thought of as a 1-dimensional simplicial complex) into the enriched hom-topological space

$$\underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 4}, \Omega X).$$

As this may a priori select a nontrivial loop, it is clear that we cannot yet declare our multiplication μ to be "unambiguously associative up to homotopy".

In his thesis [Sta61], Stasheff uncovered a strong sense in which the multiplication μ on ΩX is indeed "unambiguously associative up to homotopy", using what are now called the *Stasheff associahedra*. This is a sequence of convex polytopes $\{(\mathbb{A}_\infty)_n\}_{n \geq 2}$: it begins with $(\mathbb{A}_\infty)_2 = \mathrm{pt}$ and $(\mathbb{A}_\infty)_3 = [0, 1]$, and so far we have observed maps

$$\mu_2 : (\mathbb{A}_\infty)_2 \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 2}, X)$$

and

$$\mu_3 : (\mathbb{A}_\infty)_3 \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 3}, X),$$

where the value of μ_3 on the boundary of $(\mathbb{A}_\infty)_3$ is determined by μ_2. Moreover, $(\mathbb{A}_\infty)_4$ is precisely the (filled-in) pentagon we have seen above, and we can similarly choose a map

$$\mu_4 : (\mathbb{A}_\infty)_4 \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 4}, X)$$

[17] This diagram appeared in Mac Lane's foundational study of monoidal categories [ML63], hence the name.

(which can likewise be determined "universally" by studying the pinch map Δ and its iterates) which extends the map on the boundary of $(\mathbb{A}_\infty)_4$ determined by μ_2 and μ_3. As $(\mathbb{A}_\infty)_4$ is convex and in particular contractible, this gives a precise sense in which *all four-fold multiplications are equivalent, up to contractible ambiguity.* Of course, this pattern continues: the maps $\mu_2, \dots, \mu_{n-1}$ determine a map from the boundary of $(\mathbb{A}_\infty)_n$ into $\underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times n}, \Omega X)$, and it is possible to (universally) choose an extension over this contractible topological space. The relationships between these polytopes which inductively determine the maps on their boundaries assemble into certain structure maps which makes them into an *operad* (namely the $\mathbb{A}_\infty$ operad), and the compatible sequence of maps

$$\{\mu_n : (\mathbb{A}_\infty)_n \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times n}, \Omega X)\}$$

makes the topological space ΩX into an *algebra* over this operad.[18,19]

In fact, not only does the based loopspace of a based topological space carry the structure of an $\mathbb{A}_\infty$ algebra, but in a sense this structure *characterizes* based loopspaces: after we restrict to connected based spaces for obvious reasons, the based loopspace functor

$$\Omega : \mathcal{T}\mathrm{op}_*^{\geq 1} \to \mathcal{T}\mathrm{op}_*$$

defines an equivalence

$$\Omega : \mathcal{T}\mathrm{op}_*^{\geq 1}[\mathbf{W}_{\text{w.h.e.}}^{-1}] \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{A}_\infty}^{\mathrm{gp}}(\mathcal{T}\mathrm{op}_*)[\mathbf{W}_{\text{w.h.e.}}^{-1}]$$

onto the homotopy category of *grouplike* $\mathbb{A}_\infty$ topological spaces, i.e. those $\mathbb{A}_\infty$ algebras $Y \in \mathcal{T}\mathrm{op}_*$ for which the induced multiplication on $\pi_0(Y)$ makes it into a group (instead of just a monoid). In other words, a grouplike $\mathbb{A}_\infty$ structure on a topological space allows us to construct a *delooping* of that topological space (up to weak homotopy equivalence). The analogous result is false for "grouplike h-spaces", i.e. group objects in $\mathcal{T}\mathrm{op}[\mathbf{W}_{\text{w.h.e.}}^{-1}]$, for which we are only assured the *existence* of a homotopy making the associativity diagram commute (in the homotopy category). Thus, it is indeed only by *keeping track* of the homotopies making the (higher) associativity diagrams commute that we can construct a delooping.

[18] Actually, we have only parametrized n-fold multiplications for $n \geq 2$, whereas operads begin in degree 0. To extend this to a true $\mathbb{A}_\infty$ algebra structure, we should additionally specify the map $(\mathbb{A}_\infty)_0 = \mathrm{pt} \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 0}, \Omega X) \cong \Omega X$ selecting the basepoint (which functions as the "identity element" for the multiplication) as well as the map $(\mathbb{A}_\infty)_1 = \mathrm{pt} \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 1}, \Omega X)$ selecting the identity map on ΩX (the "1-fold multiplication").

[19] Operads were introduced by May in his landmark work [May72], in which he characterized *all* iterated loopspaces. As we will see presently, the $\mathbb{A}_\infty$ operad completely governs 1-fold loopspaces; this is also called the $\mathbb{E}_1$ operad, and more generally the $\mathbb{E}_n$ operad completely governs n-fold loopspaces for all n (including $n = \infty$, in a suitable sense).

Now, the $\mathbb{A}_\infty$ operad is an example of a ("non-symmetric") operad in topological spaces. Another example of an object in this category is the *associative operad*, denoted $\mathrm{Ass} \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$. This object is much simpler than $\mathbb{A}_\infty$: for all $n \geq 0$, we simply have $\mathrm{Ass}_n = \mathrm{pt}$. In other words, Ass parametrizes *strictly associative* multiplications.

In fact, the category $\mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$ of these *itself* admits a model structure, in which the weak equivalences are determined "level by level" (in $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$). Moreover, Ass is the terminal object of $\mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$, and the unique map $\mathbb{A}_\infty \to \mathrm{Ass}$ is a cofibrant replacement (and in particular, a weak equivalence).

This is relevant for the following reason. First of all, any object $Y \in \mathcal{T}\mathrm{op}$ determines an *endomorphism operad*

$$\mathcal{E}\mathrm{nd}^{\mathrm{ns}}(Y) \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$$

given by

$$\mathcal{E}\mathrm{nd}^{\mathrm{ns}}(Y)_n = \underline{\hom}_{\mathcal{T}\mathrm{op}}(Y^{\times n}, Y).$$

Moreover, as suggested by the above discussion, for an arbitrary operad $\mathcal{O} \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$, one can *define* an $\mathcal{O}$-algebra structure on Y to be a morphism

$$\mathcal{O} \to \mathcal{E}\mathrm{nd}^{\mathrm{ns}}(Y)$$

of operads. Thus, to say that $\mathbb{A}_\infty$ is "good for mapping out of" (in a way that Ass is not) is to say that certain topological spaces (e.g. and i.e. based loopspaces) "want" to be associative algebras, but are in fact only $\mathbb{A}_\infty$ algebras.

In fact, the term "$\mathbb{A}_\infty$ operad" has come to refer to *any* cofibrant replacement of the associative operad. Moreover, a weak equivalence between cofibrant operads induces a Quillen equivalence between their model categories of algebras. Thus, we see that the point-set $\mathbb{A}_\infty$ operad in topological spaces is not the "true" thing: the homotopy category of based loopspaces can be organized as the homotopy category of grouplike algebras over *any* cofibrant replacement of the associative operad.

By now, the punch line should be clear: based loopspaces *are* associative algebras, but only when considered in the ∞-*category* of spaces! Moreover, the above equivalence of homotopy categories lifts to an equivalence

$$\Omega : \mathcal{S}_*^{\geq 1} \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathrm{Ass}}(\mathcal{S}) = \mathcal{G}\mathrm{rp}(\mathcal{S})$$

of ∞-categories. Thus, as advertised, the homotopy-coherence inherent in the very foundations of ∞-categories turns a complicated and un-"true" assertion about not-even-canonical point-set operads into the simple, canonical, and compelling state-

ment that we were really after all along: based loopspaces of pointed spaces determine group objects in spaces.[20]

On the other hand, there is another approach to studying the homotopy category of based loopspaces: in fact, it turns out that the canonical map $\mathbb{A}_\infty \to \mathrm{Ass}$ *also* induces an equivalence

$$\mathrm{Alg}^{\mathrm{gp}}_{\mathrm{Ass}}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}] \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{A}_\infty}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}]$$

of homotopy categories (even though $\mathrm{Ass} \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$ is not cofibrant). In particular, any grouplike $\mathbb{A}_\infty$ algebra in $\mathcal{T}\mathrm{op}$ is weakly equivalent (as an $\mathbb{A}_\infty$ algebra) to a *topological group*. Thus, one can also study the homotopy category of based loopspaces by studying the homotopy category of topological groups.

However, it is only due to the simplicity of the $\mathbb{A}_\infty$ operad that such strictification is possible. For instance, a cofibrant replacement of the *commutative operad*

$$\mathrm{Comm} \in \mathrm{Op}(\mathcal{T}\mathrm{op})$$

(which can only be defined as a "symmetric" operad – in fact, it is likewise the terminal object of $\mathrm{Op}(\mathcal{T}\mathrm{op})$) is called an "$\mathbb{E}_\infty$ operad", and restriction along the canonical map $\mathbb{E}_\infty \to \mathrm{Comm}$ determines a functor

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{T}\mathrm{op}) \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{T}\mathrm{op})$$

which does *not* induce an equivalence on homotopy categories. In particular, the induced functor

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}] \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}]$$

on homotopy categories is not essentially surjective: not every topological space equipped with a homotopy-coherently commutative and associative multiplication can be rigidified to one with a strictly commutative and associative multiplication.

From a more philosophical perspective, we posit that it should feel *morally reprehensible* to attempt to force a based loopspace to be something which it is not: it is truly and essentially a homotopy-coherent object, and its strictifiability is ultimately just an intriguing coincidence.

In fact, recall from §0.1.2 that a one-object category is completely specified by the monoid of endomorphisms of its unique object. In an identical fashion, a one-object topologically-enriched category is completely specified by the topological monoid of

[20] The fact that this induces an equivalence when we restrict to connected based spaces is a homotopical form of *Koszul duality*, which features prominently in the study of deformation theory.

endomorphisms of its unique object. Thus, the coincidence that based loopspaces can be rectified to topological groups is (up to questions of grouplikeness) nothing other than a "one-object" version of the coincidence that topologically-enriched categories present ∞-categories! We may therefore view this connection as justifying our philosophical assertion that *we never should have considered ∞-categories as having strictly associative composition in the first place.*[21]

Of course, these two examples of the power of ∞-categorical thinking are merely toys, which we chose in order to highlight the differences between working strictly and working homotopy-coherently. The real fun begins when one actually starts to *use* ∞-category theory, at which point the world becomes a magical place: one's power to make new definitions is limited only by one's imagination, and one's ability to prove new theorems is limited only by the clarity of one's understanding (at least as far as the purely formal aspects are concerned). The many fussy details that arise when one attempts to use point-set techniques to work homotopy-coherently simply melt away: they were in fact irrelevant all along to the true and underlying mathematics, and their disappearance into the ambient machinery brings with it a harmony that is only possible when intuition and language are once again aligned. Thus, paradoxically, by *discarding* such emotional crutches as underlying sets and strict composition and by *embracing* the apparent chaos and uncontrol of homotopy-coherence, we acquire a measure of power of which previous generations of mathematicians could barely have dreamed.

0.1.6 The praxis of ∞-categories

In case it was not evident from the discussion of §0.1.5, we now make an explicit clarification: in reality, a large number of users of ∞-categories throughout mathematics

[21] Recall that before extolling the philosophical advantages of homotopy-coherent models for ∞-categories (over strict ones), we actually began this subsection by mentioning certain technical advantages that they also enjoy. In fact, it turns out that these technical advantages can *themselves* be seen as arising from the fact ∞-categories fundamentally "want" to be homotopy-coherent objects. Thus, these technical and philosophical advantages are actually two sides of the same coin.

In order to see this, recall that the technical disadvantages e.g. of simplicially-enriched categories are ultimately due to the failure of the cartesian product of two cofibrant objects to again be cofibrant. Indeed, this failure is in turn due to the fact that the "correct" hom-set must encode all *homotopy-coherent* functors. If the target object already accounts for this homotopy-coherence (as does e.g. a quasicategory), then the source object doesn't need to (and indeed, all objects of $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ are cofibrant). But if the target object is forced to be strict (as is e.g. a simplicially-enriched category), then to get the correct hom-set we need to account for our desired homotopy-coherence in the source. As taking a product generally introduces new composites that weren't present in either factor individually (e.g. consider the product $[1] \times [1]$), it should come as no surprise that products of cofibrant simplicially-enriched categories don't generally remain cofibrant.

do not actually choose *any* particular model category of them, instead working in a purely formal manner and only making reference to universal constructions (such as limits, colimits, adjoint functors, etc.).

Most pragmatically, this (absence of) choice can be justified by declaring that such manipulations are "secretly" taking place among quasicategories. Indeed, although quasicategories are in the end nothing more than certain simplicial sets, they collectively assemble into a quasicategory *of quasicategories*, in which e.g. it is only possible to speak of homotopy-coherent composition of functors between them. Moreover, the theory of quasicategories has been developed extensively, most notably by Joyal and Lurie. As a result, nearly any 1-categorical maneuver one might wish to imitate (e.g. an appeal to the adjoint functor theorem) can be rigorously performed in the quasicategorical setting.[22]

The "underlying ∞-category" of this quasicategory – or indeed, of e.g. either relative category $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ or $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ – is denoted $\mathcal{C}\mathrm{at}_\infty$ and is referred to as *the ∞-category of ∞-categories*.

0.1.7 Model categories and ∞-categories

A technically advantageous model category of ∞-categories is absolutely essential for the full and rigorous development of the theory of ∞-categories. Thus, the theory of ∞-categories rests firmly on the theory of model categories.

However, both can be used as frameworks for abstract homotopy theory. On the one hand, a model structure on a relative category $(\mathcal{M}, \mathbf{W}) \in \mathscr{R}\mathcal{C}\mathrm{at}$ provides an efficient method of making computations not just in its localization

$$\mathcal{M}[\mathbf{W}^{-1}] \in \mathcal{C}\mathrm{at}$$

but in its *derived localization*

$$\mathcal{M}[\![\mathbf{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_\infty$$

(which is indeed its localization when considered as a *relative ∞-category*). On the other hand, essentially every ∞-category of lasting interest can be presented by a model category $\mathcal{M}$ in this sense. It is therefore often analogized that model categories are to ∞-categories as atlases are to manifolds: a model category is a

[22] For a beautiful and compelling introduction to quasicategories, we refer the interested reader to [Lur09b, Chapter 1].

convenient presentation of an ∞-category, but not every operation that one might like to perform in an ∞-category can be presented within a given model category.[23]

By no means does the theory of ∞-categories render the theory of model categories obsolete, even beyond the obvious issue of logical reliance. To wit, model categories are still an indispensable component of the homotopical toolkit because *it is essentially impossible to perform any non-formal computations using ∞-category theory alone.*

To give an example of this, we return to the original thread with which our story began. Given an abelian category $\mathcal{A}$, the relative category $(\mathrm{Ch}(\mathcal{A}), \mathbf{W}_{\text{q.i.}})$ is the natural home of "resolutions" of objects of $\mathcal{A}$. Out of this, we can form the *derived ∞-category* of $\mathcal{A}$, namely the ∞-categorical localization

$$\mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\text{q.i.}}^{-1}]\!] \in \mathcal{C}\mathrm{at}_\infty$$

of this relative category. This admits a canonical functor

$$\mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\text{q.i.}}^{-1}]\!] \to \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\text{q.i.}}^{-1}],$$

which witnesses the ordinary localization $\mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\text{q.i.}}]$ as the *homotopy category* of the ∞-categorical localization (obtained by applying the functor $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$ "locally" (i.e. to each hom-space individually)); the derived ∞-category of $\mathcal{A}$ is thus a *refinement* of the ordinary derived category, and we will henceforth reappropriate the notation

$$\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\text{q.i.}}^{-1}]\!]$$

accordingly.

Now, suppose we are given two objects $M, N \in \mathcal{A}$, and suppose we would like to understand the hom-space

$$\hom_{\mathscr{D}(\mathcal{A})}(M, N).$$

Though it arises from a modern construction, this space is often of classical interest: for instance, if $\mathcal{A} = \mathrm{Mod}_R$, then its homotopy groups are precisely the Ext groups $\mathrm{Ext}_R^*(M, N)$. However, we are once again faced with precisely the same issue that we confronted in §0.1.2: the derived ∞-category admits a universal characterization *as an ∞-category*, but this abstract characterization takes place at the wrong "category-level" for direct computation within it to be even remotely possible. Rather, it remains as necessary as ever to take *resolutions*, i.e. to make use of a model structure

[23] However, the analogy breaks down quickly: for example, the existence of a model category presenting an ∞-category implies the existence of all limits and colimits in the latter (or at least the finite ones, depending on which variant of the definition "model category" one chooses). As a result, not every ∞-category can be presented by a model category.

on the relative category $(\mathrm{Ch}(\mathcal{A}), \mathbf{W}_{\mathrm{q.i.}})$. For instance, if $\mathcal{A} = \mathrm{Mod}_R$, then it is necessary to take either a projective resolution of M or an injective resolution of N.[24]

On the other hand, ∞-categories make possible a number of obviously desirable maneuvers which model categories do not accommodate (or do not easily accommodate). The consideration of *functors* is surely the most important example.

Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, it is utterly straightforward to define the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ (whose morphisms are natural transformations). For example, if $\mathtt{C}$ and $\mathtt{D}$ are quasicategories which respectively present $\mathcal{C}$ and $\mathcal{D}$, then the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is presented by the internal hom-object $\underline{\hom}_{s\mathcal{S}\mathrm{et}}(\mathtt{C}, \mathtt{D})$ in simplicial sets. As an ∞-category, this represents the desired functor

$$\mathcal{E} \mapsto \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{E}, \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{E} \times \mathcal{C}, \mathcal{D});$$

there's nothing more to it.

By contrast, almost without exception the only meaningful "morphisms" between model categories are given by Quillen adjunctions. Moreover, a Quillen adjunction

$$\mathcal{M} \rightleftarrows \mathcal{N}$$

between model categories induces not just a derived adjunction $\mathcal{M}[\mathbf{W}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}^{-1}]$ (as described in Theorem 0.1.2) but an ∞-categorical adjunction

$$\mathcal{M}[\![\mathbf{W}^{-1}]\!] \rightleftarrows \mathcal{N}[\![\mathbf{W}^{-1}]\!]$$

between underlying ∞-categories (as a special case of Theorem 0.2.3 below). Thus, model categories provide an acutely restrictive framework if one is interested in non-adjoint functors between underlying ∞-categories.[25]

However, given a diagram category $\mathcal{I}$ and a model category $\mathcal{M}$, it is sometimes possible to endow the functor category $\mathrm{Fun}(\mathcal{I}, \mathcal{M})$ with a "pointwise" model structure (i.e. one whose weak equivalences are precisely those natural transformations whose components are all weak equivalences in $\mathcal{M}$). For example, under certain (often-satisfied) restrictions on $\mathcal{M}$, there exists a *projective model structure* $\mathrm{Fun}(\mathcal{I}, \mathcal{M})_{\mathrm{proj}}$,

[24] On the other hand, it must also be said that many operations in the literature which happen to be performed within model categories are actually essentially formal and hence could be done equally well – or perhaps better, in the interest of conceptual clarity – in their underlying ∞-categories.

[25] Actually, the key to Goerss–Hopkins obstruction theory is the homotopical control of a certain functor between model (∞-)categories which is *not* a Quillen adjoint (see §0.3.4). But this actually *also* presents a left adjoint functor between ∞-categories, and in any case it is an extremely rare exception.

while under certain (still often-satisfied) further restrictions on $\mathcal{M}$ there also exists an *injective model structure* $\mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{inj}}$. When they exist, these model structures can be used to compute homotopy co/limits, as they participate in Quillen adjunctions

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathcal{M} : \mathrm{const}$$

and

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{inj}} : \lim .$$

Alternatively, if $\mathcal{I}$ is a *Reedy* category (which condition is quite restrictive but is satisfied for a reasonably large class of examples of practical interest, including e.g. the categories $\mathbf{\Delta}$ and $\mathbf{\Delta}^{op}$), then for *any* model category $\mathcal{M}$ there exists a *Reedy model structure* $\mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{Reedy}}$. However, in general the Reedy model structure need not be compatible with either the colimit functor or the limit functor in the sense described above.

As should be clear from the complexity of this discussion, in practice these pointwise model structures can be a nuisance. For instance, there does not generally exist such a model structure on $\mathrm{Fun}(\mathcal{I},\mathcal{M})$ which is compatible with *both* the colimit functor and the limit functor, and so as a result one must pass through the Quillen equivalence

$$\mathrm{id} : \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{inj}} : \mathrm{id}$$

to mediate between the opposite "handedness" of these two model structures. Moreover, this entire discussion only allows $\mathcal{I}$ to be a diagram *1-category*: it is extremely difficult to work with diagrams in a model category which are meant to present diagrams indexed by a more general ∞-category.

0.2 Model ∞-categories

In this thesis, we take the novel perspective that the apparent dichotomy between model categories and ∞-categories of §0.1.7 is actually ill-founded. More precisely, we posit that the notion of a ***model structure*** remains one of fundamental importance within the context of ∞***-categories***. This is due to the more primitive fact that *resolutions* remain a pertinent and effective technique.

We never actually completely defined model categories in §0.1, and correspondingly we will not completely define model ∞-categories here.[26] However, having just

[26]The eager reader is welcome to jump directly to Definition 1.1.1, but it is safe to say that the definition should be completely unsurprising to anyone who is familiar both with model categories and with ∞-categories.

thoroughly contextualized model categories, we can straightforwardly explain the essential features of model ∞-categories and indicate the computations and maneuvers that they make possible, providing references throughout to the precise results that are proved in the main body of this thesis.

We will discuss our primary motivation for the introduction of this theory in §0.3: suffice it to say that this application rests crucially on the existence of a good theory of resolutions in ∞-categories.

0.2.1 Main results

A *relative ∞-category* is an ∞-category $\mathcal{R}$ equipped with a subcategory $\mathbf{W} \subset \mathcal{R}$ which contains the subcategory $\mathcal{R}^{\simeq} \subset \mathcal{R}$ of equivalences (which are the direct ∞-categorical analog of "isomorphisms" in an ordinary category). Then, a ***model ∞-category*** is a relative ∞-category $(\mathcal{M}, \mathbf{W})$ equipped with certain additional data that are collectively called a *model structure*, which in particular specify full subcategories

$$\mathcal{M}^c \hookrightarrow \mathcal{M} \hookleftarrow \mathcal{M}^f$$

of *cofibrant* objects and of *fibrant* objects. The axioms dictate that every object of $\mathcal{M}$ is weakly equivalent to a cofibrant object and is also weakly equivalent to a fibrant object. Thus, the following ***fundamental theorem of model ∞-categories*** – an analog (and generalization) of Theorem 0.1.1 – provides a direct and computable method of accessing the hom-spaces in the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$.

Theorem 0.2.1 (6.1.9)**.** *Let $\mathcal{M}$ be a model ∞-category, and suppose that $x \in \mathcal{M}$ is cofibrant and that $y \in \mathcal{M}$ is fibrant. Then the canonical map*

$$\hom_{\mathcal{M}}(x, y) \to \hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x, y)$$

is an "∞-categorical surjection", which moreover becomes an equivalence after applying either "∞-categorical equivalence relation" of "left homotopy" or "right homotopy" to the source.

In the ∞-categorical setting, the appropriate notion of "applying an equivalence relation" is taking the *geometric realization* – that is, the ∞-categorical colimit – of a simplicial object. In order to show why this might be reasonable, we first recall that the category $\mathbf{\Delta}$ admits a "generators and relations" presentation as depicted in the diagram

$$[0] \;\; \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} \;\; [1] \;\; \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} \;\; \cdots ,$$

in which each "upwards" map is a section of any adjacent "downwards" maps. Moreover, in the ∞-category of spaces, a "surjection" is simply a map which is a surjection on π_0. Thus, a simplicial space $X_\bullet : \mathbf{\Delta}^{op} \to \mathcal{S}$ in particular determines a pair of surjections

$$X_1 \rightrightarrows X_0$$

of spaces. Under the two maps, a point $x \in X_1$ will be sent to points $d_0(x), d_1(x) \in X_0$. Thus, taking the colimit

$$|X_\bullet| = \mathrm{colim}_{\mathbf{\Delta}^{op}} X_\bullet = \mathrm{colim} \left(\cdots \; \substack{\longrightarrow \\ \longleftarrow \\ \longleftarrow \\ \longrightarrow} \; X_1 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X_0 \right)$$

forces those two points to become equivalent under the canonical map $X_0 \to |X_\bullet|$.[27] From here, we see that the space X_2 should be thought of as encoding "equivalences between equivalences", that the space X_3 should be thought of as encoding "equivalences between equivalences between equivalences", and so on.

So, the ∞-categorical equivalence relations appearing in Theorem 0.2.1 are really objects of the ∞-category $s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$ of simplicial spaces, and to apply these equivalence relations amounts to applying the colimit functor

$$|-| : s\mathcal{S} \to \mathcal{S}$$

down to the ∞-category of spaces. Thus, in order to prove this theorem, it is necessary to have a good handle on the ∞-category $s\mathcal{S}$ as it relates to $\mathcal{S}$ via this functor. In other words, if we write

$$\mathbf{W}_{|-|} \subset s\mathcal{S}$$

for the subcategory of morphisms between simplicial spaces which become equivalences of spaces upon geometric realization, then it is necessary to have a good handle on the *relative* ∞-category $(s\mathcal{S}, \mathbf{W}_{|-|})$.

In fact, a completely analogous problem already appears in the setting of ordinary model categories: these have hom-*sets* instead of hom-*spaces*, and so one must correspondingly have a good handle on the category $s\mathcal{S}\mathrm{et}$ of simplicial sets as it relates via the composite functor

$$s\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S} \xrightarrow{|-|} \mathcal{S}$$

[27] In contrast with ordinary equivalence relations on sets, in this higher-categorical setting, a point $x \in X_1$ is not redundant if its two images $d_0(x), d_1(x) \in X_0$ are already equivalent. Indeed, in the ∞-groupoid X_0, one can only speak of a path between two points which *witnesses* their equivalence. Moreover, even if there already exists an equivalence $d_0(x) \simeq d_1(x)$ in X_0, taking the geometric realization will freely adjoin a *new* equivalence (i.e. path) between them.

to the ∞-category of spaces.[28] Moreover, this problem has already been solved: the resulting subcategory $\mathbf{W}_{|-|} \subset s\mathcal{S}\mathrm{et}$ of weak equivalences is precisely that of the *Kan–Quillen model structure* introduced in §0.1.3; in other words, the above composite is an ∞-categorical localization

$$s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{|-|}^{-1}]\!] \simeq \mathcal{S},$$

and moreover this ∞-categorical localization is controlled by the Kan–Quillen model structure.

And beautifully so: the Kan–Quillen model structure enjoys excellent technical properties, and is hence extremely convenient to work with. Notably, it is *proper*, and it is *cofibrantly generated* by the sets

$$I_{\mathrm{KQ}} = \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}$$

of "boundary inclusions" and

$$J_{\mathrm{KQ}} = \{\Lambda^n_i \to \Delta^n\}_{0\leq i\leq n\geq 1}$$

of "horn inclusions".[29] In our ∞-categorical setting, pursuing this analogy (and a proof of Theorem 0.2.1), we construct a ***Kan–Quillen model structure on the ∞-category of simplicial spaces***, which likewise presents the ∞-category of spaces.

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[28] This explanation is ahistorical but accurate: this new geometric realization functor is compatible with the one

$$|-| : s\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$$

defined in §0.1.3 in the sense that they participate in a commutative diagram

$$\begin{array}{ccc} s\mathcal{S}\mathrm{et} & \xrightarrow{|-|} & \mathcal{T}\mathrm{op} \\ \downarrow & & \downarrow \\ s\mathcal{S} & \xrightarrow[|-|]{} & \mathcal{S} \end{array}$$

of ∞-categories (in which the right vertical map is the ∞-categorical localization functor $\mathcal{T}\mathrm{op} \to \mathcal{T}\mathrm{op}[\![\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]\!] \simeq \mathcal{S}$). (This is because our original "geometric realization" functor is actually a *homotopy colimit*, in the sense that it takes a simplicial set $\mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ to a homotopy colimit of the composite $\mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et} \hookrightarrow \mathcal{T}\mathrm{op}_{\mathrm{QS}}$.) In particular, this abuse of notation is actually quite slight, and should introduce no real confusion if for no other reason than because we will essentially never again mean to refer to the one landing in $\mathcal{T}\mathrm{op}$.

[29] The reader unfamiliar with this terminology should feel free to ignore the specific details.

Theorem 0.2.2 (1.4.4)**.** *There is a* proper *model structure on the relative* ∞*-category* $(s\mathcal{S}, \mathbf{W}_{|-|})$ *presenting the* ∞*-category* $\mathcal{S}$*, which is* cofibrantly generated *by the same sets* I_{KQ} *and* J_{KQ} *of morphisms in* $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ *(i.e. considered as discrete simplicial spaces).*

Via geometric realization, simplicial spaces have been used pervasively throughout algebraic topology as resolutions of spaces (notably in algebraic K-theory and in loopspace theory). Theorem 0.2.2 represents a dramatic improvement over comparable preexisting results in the literature, many of which can in retrospect be seen as pale shadows of its full strength. For more on this point, we refer the reader to §1.0.2.

Intriguingly, it does not appear that Theorem 0.2.2 can be proved completely formally. Our own proof is quite technical, and ultimately relies on rather involved manipulations within the model category $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ (which presents the ∞-category $s\mathcal{S}$).

We also mention in passing that in §1.6.3, we define an Ex^∞ fibrant replacement functor for $s\mathcal{S}_{\mathrm{KQ}}$ in analogy with Kan's classic Ex^∞ fibrant replacement functor for $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, and we establish that it enjoys various corresponding convenient properties.

Now, recall that model categories do not exist in isolation, but can be related by *Quillen adjunctions* and *Quillen equivalences*. Model ∞-categories can be related in completely identical ways, and we prove the following analog (and generalization) of Theorem 0.1.2.

Theorem 0.2.3 (5.1.1 and 5.1.3)**.** *A Quillen adjunction* $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ *between model* ∞*-categories induces a canonical* ***derived adjunction***

$$\mathbb{L}F : \mathcal{M}[\![\mathbf{W}^{-1}]\!] \rightleftarrows \mathcal{N}[\![\mathbf{W}^{-1}]\!] : \mathbb{R}G$$

on localizations. If this Quillen adjunction is moreover a Quillen equivalence, then the derived adjunction is an adjoint equivalence of ∞*-categories.*

We established the restriction of Theorem 0.2.3 to model 1-categories (but still applying to their ∞-categorical localizations) in [MG16]. This had been previously been known under various (restrictive but often satisfied) hypotheses.

We now list the more specialized results that we establish surrounding model ∞-categories. These concern straightforward generalizations of various concepts from model 1-categories; thus, we expect them to be of interest mainly to readers already familiar with their model 1-categorical counterparts. We will therefore content ourselves with simply stating the results, without defining (or even introducing) the terms involved. We also note here that a number of these results are new even when

restricted to model 1-categories (see Remark 5.0.2 for a precise account). Indeed, the classical statements (involving 1-categorical localizations) are fairly straightforward to prove, but the more refined statements (involving ∞-categorical localizations) are decidedly not – at least, not as far as we are aware. Of course, the distinction lies in the crucial difference between ignorance and incorporation of homotopy-coherence.

Theorem 0.2.4 (5.4.6)**.** *A* ***two-variable Quillen adjunction*** *between model ∞-categories induces a canonical two-variable derived adjunction on their localizations.*

Theorem 0.2.5 (5.5.4 and 5.5.6)**.** *The localization of a (resp.* ***symmetric****)* ***monoidal model ∞-category*** *is canonically a closed (resp. symmetric) monoidal ∞-category.*

Theorem 0.2.6 (5.6.7)**.** *If $\mathcal{M}$ is an* ***enriched model ∞-category*** *over a monoidal model ∞-category $\mathcal{V}$, then its localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$ is canonically enriched and bitensored over $\mathcal{V}[\![\mathbf{W}^{-1}]\!]$.*

Our key example of a (symmetric) monoidal model ∞-category is $s\mathcal{S}_{\mathrm{KQ}}$, equipped with the cartesian symmetric monoidal structure. Via Theorem 0.2.5, this presents the symmetric monoidal ∞-category $\mathcal{S}$ of spaces, equipped with the cartesian symmetric monoidal structure: geometric realization of simplicial spaces commutes with finite products.

In turn, via Theorem 0.2.6, a compatible enrichment of a model ∞-category $\mathcal{M}$ over $s\mathcal{S}_{\mathrm{KQ}}$ necessarily presents the tautological enrichment of its localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$ over the localization $s\mathcal{S}[\![\mathbf{W}_{\mathrm{KQ}}^{-1}]\!] \simeq \mathcal{S}$. This can be extremely convenient in practice, as it provides an even more direct way of computing hom-spaces in the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$ than does Theorem 0.2.1: as soon as x is cofibrant and y is fibrant, we obtain a canonical equivalence

$$|\underline{\hom}_{\mathcal{M}}(x,y)| \xrightarrow{\sim} \hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y).$$

Of course, this is a model ∞-categorical counterpart to the classical theory of *simplicial model categories* (i.e. model categories that are compatibly enriched over $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$), which history has shown to be likewise extremely convenient. In fact, the results that we describe in §0.3 all turn on the *resolution model structure* (on the ∞-category $s\mathcal{C}$ of simplicial objects in a suitable ∞-category $\mathcal{C}$), and use in an essential way the fact that it is compatibly enriched over the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$.

We end this subsection by mentioning that we also establish in §5.1 the basics of the theory of *homotopy co/limits* in model ∞-categories (including analogs of the standard "pointwise" model structures mentioned in §0.1.7).

0.2.2 Auxiliary results

In this subsection, we highlight some auxiliary results contained in this thesis that are used to prove the various foundational theorems regarding model ∞-categories described in §0.2.1. Once again, we refer the reader to the main body of the thesis for precise definitions and statements.

As the theory of model ∞-categories ultimately concerns localizations of relative ∞-categories, it is necessary to have a good handle on the localization functor. To this end, we prove the following general result.

Theorem 0.2.7 (2.3.8 and 2.3.9)**.** *For any relative ∞-category $(\mathcal{R}, \mathbf{W})$, its ∞-categorical Rezk nerve*

$$\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W}) \in s\mathcal{S}$$

is taken to the localization

$$\mathcal{R}[\![\mathbf{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_{\infty}$$

under the composite functor

$$s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathrm{cSS}}} \mathcal{CSS} \xrightarrow[\sim]{\mathrm{N}^{-1}_{\infty}} \mathcal{C}\mathrm{at}_{\infty},$$

where

$$\mathrm{N}_{\infty}(-)_{\bullet} = \hom^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}_{\infty}}([\bullet], -) : \mathcal{C}\mathrm{at}_{\infty} \xrightarrow{\sim} \mathcal{CSS}$$

denotes the equivalence from the ∞-category of ∞-categories to the ∞-category of complete Segal spaces. Moreover, this behaves well in families: the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty} \xrightarrow{\mathrm{N}^{\mathrm{R}}_{\infty}} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathrm{cSS}}} \mathcal{CSS} \xrightarrow[\sim]{\mathrm{N}^{-1}_{\infty}} \mathcal{C}\mathrm{at}_{\infty}$$

is canonically equivalent to the localization functor on relative ∞-categories.

A complete Segal space is essentially the homotopical analog of the nerve of a 1-category; indeed, complete Segal spaces provide a model for the ∞-category of ∞-categories (internal to the theory of ∞-categories). As the Rezk nerve functor $\mathrm{N}^{\mathrm{R}}_{\infty}$ is relatively explicit, it should not really be expected that the Rezk nerve of an arbitrary relative ∞-category would already be a complete Segal space: localization is a fundamentally difficult and complicated procedure.

Much like Theorem 0.2.2, it does not appear to be possible to prove Theorem 0.2.7 in a completely formal manner; its proof likewise relies on delicate manipulations involving bisimplicial sets, this time also using the model structure $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ (which presents the ∞-category $s(\mathcal{C}\mathrm{at}_{\infty})$).

We also require a number of results concerning various colimit and limit operations; these are contained in Chapter 3. We mention here three results contained there.

We begin with a result which encodes, for a cocomplete ∞-category $\mathcal{C}$, the simultaneous and interwoven functoriality of colimits

- for *natural transformations* – that is, for maps in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$, where $\mathcal{D}$ is any diagram ∞-category – and
- for *pullbacks along maps of diagram ∞-categories* – that is, for maps in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$.

Theorem 0.2.8 (3.2.12)**.** *If $\mathcal{C}$ is a cocomplete ∞-category, then there exists a global colimit functor*

$$\mathcal{L}\mathrm{ax}(\mathcal{C}) \to \mathcal{C}$$

from its lax overcategory. Dually, if $\mathcal{C}$ is complete, there exists a global limit functor

$$\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})^{op} \to \mathcal{C}$$

from the opposite of its oplax overcategory.

We then have an ∞-categorical analog of the classical Bousfield–Kan formula for homotopy colimits in model categories.

Theorem 0.2.9 (3.5.8)**.** *Let $\mathcal{C}$ be a cocomplete ∞-category, and let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be a diagram. Then there is a canonical equivalence*

$$\mathrm{colim}_{\mathcal{D}} F \simeq |\mathrm{srep}(F)_\bullet|$$

in $\mathcal{C}$ between the colimit of the diagram F and the geometric realization of its simplicial replacement

$$\mathrm{srep}(F)_\bullet : \mathbf{\Delta}^{op} \to \mathcal{C}.$$

Thirdly, we have the following generalization of certain 1-categorical results of Barwick–Kan and Dwyer–Kan–Smith, which are themselves vast generalizations of Quillen's Theorem B.

Theorem 0.2.10 (3.4.23 and 3.4.26)**.** *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be ∞-categories. If a functor $\mathcal{D} \to \mathcal{C}$ has property B_n, then for any functor $\mathcal{E} \to \mathcal{C}$ the (not generally commutative) square*

$$\begin{array}{ccc} (F(\mathcal{D}) \downarrow_n G(\mathcal{E})) & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{C} \end{array}$$

of ∞-categories induces a (commutative) pullback square

$$\begin{array}{ccc} (F(\mathcal{D}) \downarrow_n G(\mathcal{E}))^{\mathrm{gpd}} & \longrightarrow & \mathcal{D}^{\mathrm{gpd}} \\ \downarrow & & \downarrow \\ \mathcal{E}^{\mathrm{gpd}} & \longrightarrow & \mathcal{C}^{\mathrm{gpd}} \end{array}$$

of spaces upon groupoid completion. Moreover, if $\mathcal{C}$ has property C_n, then any functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has property B_n.

We end this subsection by mentioning two results which go into the proof of the fundamental theorem of model ∞-categories (0.2.1). Both rely on the notion of a relative ∞-category admitting a homotopical three-arrow calculus, which is a slight variation on a classical definition of Dwyer–Kan (for relative 1-categories).

The first of these is in fact a generalization of the main theorem regarding this notion, the 1-categorical version of which is due to Dwyer–Kan.

Theorem 0.2.11 (4.3.4)**.** *Given a relative ∞-category $(\mathcal{R}, \mathbf{W})$ admitting a homotopical three-arrow calculus, the hom-spaces in the underlying ∞-category of its hammock localization admit a canonical equivalence*

$$\underline{\mathbf{3}}(x, y)^{\mathrm{gpd}} \xrightarrow{\sim} \left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x, y)\right|$$

from the groupoid completion of the ∞-category of three-arrow zigzags $x \xleftarrow{\approx} \bullet \to \bullet \xleftarrow{\approx} y$ in $(\mathcal{R}, \mathbf{W})$.

The simplicial space $\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x, y)$ is a refinement of the "quotient of zigzags" procedure described in § 0.1.2. As these zigzags can be arbitrarily long and the equivalence relations that must be imposed on them are generally difficult to control, Theorem 0.2.11 provides a substantial simplification of the hammock localization of a relative ∞-category.

The second result, which generalizes our joint work with Low [LMG15], provides sufficient conditions on a relative ∞-category for its Rezk nerve to be a (complete) Segal space.

Theorem 0.2.12 (4.5.1)**.** *Given a relative ∞-category $(\mathcal{R}, \mathbf{W})$, its Rezk nerve*

$$\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W}) \in s\mathcal{S}$$

- *is a Segal space if $(\mathcal{R}, \mathbf{W})$ admits a homotopical three-arrow calculus, and*

- *is moreover a complete Segal space if moreover* $(\mathcal{R}, \mathbf{W})$ *is saturated and satisfies the two-out-of-three property.*

It is a direct procedure to extract hom-spaces from a Segal space – that is, to extract hom-spaces corresponding to the ∞-category that it presents (via completion to a complete Segal space). As any model ∞-category $\mathcal{M}$ admits a homotopical three-arrow calculus, the combination of Theorems 0.2.7 and 0.2.12 is ultimately what allows us to compute the hom-spaces in its localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$, and Theorem 0.2.11 provides a key intermediate step.

In fact, it follows a posteriori from Theorem 0.2.1 that any model ∞-category is saturated. As the two-out-of-three property is part of the definition of a model ∞-category, Theorem 0.2.12 then implies that the Rezk nerve of a model ∞-category is not just a Segal space but is in fact a complete Segal space.

0.3 Goerss–Hopkins obstruction theory

In this section, we outline and contextualize our primary application of the theory of model ∞-categories discussed in §0.2, namely *Goerss–Hopkins obstruction theory*.

First, in §0.3.1, we begin by giving some background on spectra and their corresponding extraordinary co/homology theories.

Next, in §0.3.2, we briefly survey the (intimately related) fields of derived algebraic geometry and chromatic homotopy theory, which are the ambient context for some of the most important and compelling applications of Goerss–Hopkins obstruction theory to date.

Then, in §0.3.3, we proceed to describe Goerss–Hopkins obstruction theory itself. Here we describe it purely as a black box, in terms of its input and output.

As a follow-up, in §0.3.4, we explain the internal workings of Goerss–Hopkins obstruction theory, albeit still in somewhat broad strokes. This explanation reveals the motivation for a good theory of resolutions internal to the world of ∞-categories. Even though it still only provides an overview, this subsection is quite technical and may be safely skipped (or merely skimmed).

Finally, in §0.3.5, we briefly introduce motivic homotopy theory and describe our sample application of our generalized Goerss–Hopkins obstruction theory.

0.3.1 Stable ∞-categories, spectra, and co/homology theories

Let us begin by recalling our discussion of *nonabelian derived categories* from §0.1.3.

Given an abelian category $\mathcal{A}$, the category $\mathrm{Ch}(\mathcal{A})$ is the home of resolutions of objects of $\mathcal{A}$; these are well-defined up to quasi-isomorphism, and the localization $\mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\text{q.i.}}^{-1}]$ is called the *derived category* of $\mathcal{A}$. Moreover, we have an equivalence

$$\mathrm{Ch}_{\geq 0}(\mathcal{A}) \simeq s\mathcal{A}$$

between the categories of nonnegatively-graded chain complexes and of simplicial objects in $\mathcal{A}$; this leads us to define, for *any* category $\mathcal{C}$, the *nonnegatively-graded nonabelian derived category* of $\mathcal{C}$ to be the localization

$$s\mathcal{C}[\mathbf{W}^{-1}],$$

where the definition of the subcategory $\mathbf{W} \subset s\mathcal{C}$ is a natural extension of that of the subcategory

$$\mathbf{W}_{\text{q.i.}} \subset \mathrm{Ch}_{\geq 0}(\mathcal{A}) \simeq s\mathcal{A}$$

of quasi-isomorphisms.

Of course, we will be working with ∞-categorical (rather than 1-categorical) localizations, in which setting we obtain the *derived ∞-category*

$$\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\text{q.i.}}^{-1}]\!]$$

of $\mathcal{A}$ and the *nonnegatively-graded nonabelian derived ∞-category*

$$\mathscr{D}_{\geq 0}(\mathcal{C}) = s\mathcal{C}[\![\mathbf{W}^{-1}]\!]$$

of $\mathcal{C}$. (Again, these ∞-categories admit universal characterizations making no reference to either chain complexes or simplicial objects.) In the case that $\mathcal{C} = \mathcal{S}\mathrm{et}$, the subcategory $\mathbf{W} \subset s\mathcal{S}\mathrm{et}$ of weak equivalences is precisely that of the Kan–Quillen model structure, which immediately furnishes an equivalence

$$\mathscr{D}_{\geq 0}(\mathcal{S}\mathrm{et}) \simeq \mathcal{S}$$

between the nonnegatively-graded nonabelian derived ∞-category of $\mathcal{S}\mathrm{et}$ and the ∞-category of spaces.

Now, the full derived ∞-category $\mathscr{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ enjoys certain properties not shared by its nonnegatively-graded variant $\mathscr{D}_{\geq 0}(\mathcal{A})$. The key difference between these is that the former is an example of a *stable* ∞-category, while the latter is not. In fact, stable ∞-categories are themselves a sort of ∞-categorical analog of abelian categories. Their definition is so simple that we cannot help but give it in full: an ∞-category $\mathcal{C}$ is called ***stable*** if

- it admits a zero object $0 \in \mathcal{C}$ (i.e. an object which is both initial and terminal),

- a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow{\scriptstyle \psi} \\ 0 & \longrightarrow & Z \end{array}$$

  is a pushout square (i.e. ψ is the *cofiber* of φ) if and only if it is a pullback square (i.e. φ is the *fiber* of ψ), and

- every morphism in $\mathcal{C}$ admits both a fiber and a cofiber.[30]

Crucially, if $\mathcal{C}$ is a stable ∞-category, then the suspension/loop adjunction

$$\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega$$

is an adjoint *equivalence*. For this reason, when working in a stable ∞-category it is common to write $\Sigma^{-1} = \Omega$ for the loop functor. In the case of the derived ∞-category $\mathscr{D}(\mathcal{A})$, these adjoints are simply the "shift" functors, often denoted $Y \mapsto$

[30]In fact, stable ∞-categories are a robust enhancement of a more classical notion, namely that of a *triangulated category*. These were introduced by Verdier in his thesis [Ver96] (which was written in 1963 but only published in 1996), appropriately entitled *Des catégories derivées des catégories abéliennes* ("On derived categories of abelian categories"): the canonical example of a triangulated category is the derived category $\mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\text{q.i.}}^{-1}]$ of an abelian category $\mathcal{A}$. From our original example $\mathscr{D}(\mathcal{A})$ of a stable ∞-category, we recover this derived category as its *homotopy category*: the canonical functor

$$\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\text{q.i.}}^{-1}]\!] \to \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\text{q.i.}}^{-1}]$$

from the derived ∞-category of $\mathcal{A}$ is obtained simply by collapsing the hom-spaces in $\mathscr{D}(A)$ onto their sets of path components, which are the corresponding hom-sets in the derived category. In fact, the homotopy category of any stable ∞-category is canonically triangulated (with the "distinguished triangles" given equivalently by the cofiber sequences or the fiber sequences), and indeed essentially every triangulated category of interest is (in retrospect) *defined* as the homotopy category of a stable ∞-category.

It has long been held that triangulated categories are "not the true thing"; in particular, it has long been observed as a flaw in their definition that cofibers (or fibers) are not unique up to *unique* isomorphism. In fact, this flaw is completely repaired by the use of stable ∞-categories, in which there exists a *contractible space* of cofibers (or fibers) of a given map. Moreover, the so-called "octahedral axiom" for triangulated categories is both complicated and rather difficult to motivate, but from the perspective of stable ∞-categories it becomes a straightforward consequence of the fact that pushout squares can be "composed" (a/k/a "pasted") into another pushout square. (This observation also indicates that the octahedral axiom should in fact be just the first in an infinite hierarchy of such axioms, illustrating the further suboptimality of the definition of a triangulated category.)

$Y[\pm 1]$ in the literature.[31] This illustrates why the nonnegatively-graded derived ∞-category $\mathscr{D}_{\geq 0}(\mathcal{A})$ cannot be stable. Namely, the inclusion $\mathscr{D}_{\geq 0}(\mathcal{A}) \hookrightarrow \mathscr{D}(\mathcal{A})$ admits a right adjoint $\tau_{\geq 0}$, called the *connective cover* or 0^{th} *cotruncation* functor, and the suspension/loop adjunction

$$\Sigma : \mathscr{D}_{\geq 0}(\mathcal{A}) \rightleftarrows \mathscr{D}_{\geq 0}(\mathcal{A}) : \Omega$$

has its left adjoint computed in $\mathscr{D}(\mathcal{A})$ but its right adjoint computed by the composite

$$\mathscr{D}_{\geq 0}(\mathcal{A}) \xleftarrow{\tau_{\geq 0}} \mathscr{D}(\mathcal{A}) \xleftarrow{\Omega} \mathscr{D}(\mathcal{A}) \hookleftarrow \mathscr{D}_{\geq 0}(\mathcal{A}).$$

That is, the suspension functor in $\mathscr{D}_{\geq 0}(\mathcal{A})$ is simply given by "shifting up", but the loop functor in $\mathscr{D}_{\geq 0}(\mathcal{A})$ is given by "shifting down and then cotruncating away any negative-dimensional homology".

We can now proceed to indicate how to recover the *full* nonabelian derived ∞-category $\mathscr{D}(\mathcal{C})$ of an arbitrary category $\mathcal{C}$. For this, we will draw on intuition coming from the relationship between $\mathscr{D}(\mathcal{A})$ and $\mathscr{D}_{\geq 0}(\mathcal{A})$ in the abelian case. Namely, we will ponder the question: how can we describe an object $X \in \mathscr{D}(\mathcal{A})$ while making reference only to objects of $\mathscr{D}_{\geq 0}(\mathcal{A})$?

In answering this question, the first thing we will do is extract the 0^{th} contruncation

$$X_0 = \tau_{\geq 0} X \in \mathscr{D}_{\geq 0}(\mathcal{A}),$$

though clearly this will not generally suffice to recover X itself.[32] On the other hand, to do slightly better we can instead extract the -1^{st} contruncation $\tau_{\geq -1} X$, but then suspend it once to obtain another object

$$X_1 = \Sigma(\tau_{\geq -1} X) \in \mathscr{D}_{\geq 0}(\mathcal{A}).$$

Of course, these two objects of $\mathscr{D}_{\geq 0}(\mathcal{A})$ will be compatible in a certain sense: the map

$$\Sigma X_0 \to X_1$$

will be adjoint to an *equivalence*

$$X_0 \xrightarrow{\sim} \Omega X_1$$

[31] We prefer the topologically motivated notation $\Sigma \dashv \Omega$ since it is more descriptive, and because the underlying mathematical meaning is independent of whether one chooses to use homological grading conventions or cohomological ones.

[32] This subscript notation is motivated by the aim of the discussion. Although perhaps mildly confusing, it is at least not technically abusive: it is impossible to speak of the "object of $\mathcal{A}$ in homological degree 0" of the object $X \in \mathscr{D}(\mathcal{A})$, just as it is impossible to speak of the "underlying set" of an object of $\mathcal{S}$.

in $\mathscr{D}_{\geq 0}(\mathcal{A})$. In other words, the -1^{st} cotruncation $\tau_{\geq -1}X$ "remembers one more layer" of X than does the 0^{th} cotruncation $\tau_{\geq 0}X$, and when we strip off that bottommost layer then they become equivalent. From here, the general pattern is clear: the object $X \in \mathscr{D}(\mathcal{A})$ is entirely recovered by the sequences of objects

$$X_n = \Sigma^n(\tau_{\geq -n}X)$$

and of equivalences

$$X_n \xrightarrow{\sim} \Omega X_{n+1}$$

in $\mathscr{D}_{\geq 0}(\mathcal{A})$ (indexed over $n \geq 0$). In fact, this assembles into an equivalence

$$\mathscr{D}(\mathcal{A}) \xrightarrow{\sim} \lim\left(\cdots \xrightarrow{\Omega} \mathscr{D}_{\geq 0}(\mathcal{A}) \xrightarrow{\Omega} \mathscr{D}_{\geq 0}(\mathcal{A}) \xrightarrow{\Omega} \mathscr{D}_{\geq 0}(\mathcal{A})\right)$$

of ∞-categories: the functor from $\mathscr{D}(\mathcal{A})$ to the n^{th}-from-rightmost copy of $\mathscr{D}_{\geq 0}(\mathcal{A})$ is given by the composite

$$\mathscr{D}(\mathcal{A}) \xrightarrow{\tau_{\geq -n}} \mathscr{D}_{\geq -n}(\mathcal{A}) \xrightarrow[\sim]{\Sigma^n} \mathscr{D}_{\geq 0}(\mathcal{A}).$$

Motivated by this, for *any* ∞-category $\mathcal{D}$ admitting a terminal object, we define the ∞-category of ***spectrum objects*** in $\mathcal{D}$ to be the limit

$$\mathcal{S}\mathrm{p}(\mathcal{D}) = \lim\left(\cdots \xrightarrow{\Omega} \mathcal{D}_* \xrightarrow{\Omega} \mathcal{D}_* \xrightarrow{\Omega} \mathcal{D}_*\right)$$

of ∞-categories.[33] Going a step further back, for any (not necessarily abelian) category $\mathcal{C}$, we define its (***full***) ***nonabelian derived ∞-category*** to be the ∞-category

$$\mathscr{D}(\mathcal{C}) = \mathcal{S}\mathrm{p}(\mathscr{D}_{\geq 0}(\mathcal{C}))$$

of spectrum objects in its nonnegatively-graded nonabelian derived ∞-category $\mathscr{D}_{\geq 0}(\mathcal{C})$. As a special case, the ∞-category of ***spectra*** is by definition

$$\mathcal{S}\mathrm{p} = \mathcal{S}\mathrm{p}(\mathcal{S}) \simeq \mathcal{S}\mathrm{p}(\mathscr{D}_{\geq 0}(\mathcal{S}\mathrm{et})) = \mathscr{D}(\mathcal{S}\mathrm{et}),$$

the full nonabelian derived ∞-category of the category of sets. This ∞-category is the setting of the field of ***stable homotopy theory***.

[33] We work with pointed objects in $\mathcal{D}$ in order to obtain a sensible "loop" functor; if the terminal object of $\mathcal{D}$ is already also initial (and hence is a zero object), then the forgetful functor determines an equivalence $\mathcal{D}_* \xrightarrow{\sim} \mathcal{D}$.

Whereas stable ∞-categories are a homotopical analog of abelian *categories*, spectra are a homotopical analog of abelian *groups*. Indeed, there is a fully faithful embedding

$$H : \mathrm{Ab} \to \mathcal{S}\mathrm{p},$$

the *Eilenberg–Mac Lane spectrum* functor: this takes an abelian group A to the spectrum HA determined by the Eilenberg–Mac Lane spaces $\{K(A,n)\}_{n\geq 0}$ and the canonical equivalences $K(A,n) \xrightarrow{\sim} K(A,n+1)$. Moreover, the ∞-category $\mathcal{S}\mathrm{p}$ admits a symmetric monoidal structure – alternately called the *smash product* (and denoted $\wedge$) or the *tensor product* (and denoted $\otimes$), depending on the desired emphasis – and with respect to this the Eilenberg–Mac Lane spectrum functor H is lax symmetric monoidal. It follows that an associative (resp. commutative) ring R gives rise to an *associative* (resp. *commutative*) *ring spectrum* HR.

Just as sets give rise to abelian groups via the free/forget adjunction

$$\mathrm{F}_{\mathrm{Ab}} : \mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Ab} : \mathrm{U}_{\mathrm{Ab}},$$

so do *spaces* give rise to *spectra*: there is an adjunction

$$\Sigma^\infty : \mathcal{S}_* \rightleftarrows \mathcal{S}\mathrm{p} : \Omega^\infty,$$

which can be extended to unbased spaces simply by precomposing with the free/forget adjunction

$$(-)_+ : \mathcal{S} \rightleftarrows \mathcal{S}_* : \mathrm{U}_+.$$

The left adjoint (either on $\mathcal{S}_*$ or restricted to $\mathcal{S}$) is called the *suspension spectrum* functor, and should be thought of as a homotopical analog of the "free abelian group" functor (with the basepoint serving as the "identity element").

Of central importance among suspension spectra is the *sphere spectrum*

$$\mathbb{S} = \Sigma^\infty S^0 \simeq \Sigma^\infty_+ \mathrm{pt},$$

which is the unit of the symmetric monoidal ∞-category $(\mathcal{S}\mathrm{p}, \otimes)$. For any $n \in \mathbb{Z}$, we also write $\mathbb{S}^n = \Sigma^n \mathbb{S}$, and refer to these objects collectively as the *sphere spectra*; for $n \geq 0$ we have $\mathbb{S}^n \simeq \Sigma^\infty S^n$, but these are no longer suspension spectra for $n < 0$. Then, for any spectrum E and any $n \in \mathbb{Z}$, we define the n^{th} *homotopy group* of E to be the abelian group

$$\pi_n E = [\mathbb{S}^n, E]_{\mathcal{S}\mathrm{p}} = \hom_{\mathrm{ho}(\mathcal{S}\mathrm{p})}(\mathbb{S}^n, E)$$

of homotopy classes of maps from the n-fold suspension $\Sigma^n \mathbb{S}$. For a based space $X \in \mathcal{S}_*$, the homotopy groups $\pi_*(\Sigma^\infty X)$ are called the *stable homotopy groups* of X.

Homotopy groups create the equivalences among spectra: that is, a map of spectra is an equivalence if and only if it induces isomorphisms on all homotopy groups.

It is no accident that the Eilenberg–Mac Lane spaces $\{K(A,n)\}_{n\geq 0}$ simultaneously assemble into a spectrum HA and represent the singular cohomology theory $H^*(-;A)$. In fact, *any* spectrum $E \in \mathcal{S}\mathrm{p}$ represents a(n ***extraordinary***) ***cohomology theory***, given by defining

$$E^n X = [\Sigma_+^{\infty -n} X, E]_{\mathcal{S}\mathrm{p}} \cong [\mathbb{S}^{-n}, \underline{\hom}_{\mathcal{S}\mathrm{p}}(\Sigma_+^\infty X, E)] = \pi_{-n}\left(\underline{\hom}_{\mathcal{S}\mathrm{p}}(\Sigma_+^\infty X, E)\right)$$

for any $n \in \mathbb{Z}$. Moreover, E also defines a(n ***extraordinary***) ***homology theory*** as well, given by setting

$$E_n X = \pi_n(E \otimes \Sigma_+^\infty X) = [\mathbb{S}^n, E \otimes \Sigma_+^\infty X]_{\mathcal{S}\mathrm{p}}$$

for any $n \in \mathbb{Z}$.[34] In a precise sense, these formulas define *all* co/homology theories: this is the *Brown representability theorem*, introduced in its original form in [Bro62]. We observe that these formulas continue to make sense if we replace the suspension spectrum $\Sigma_+^\infty X$ by an arbitrary spectrum Y: that is, we can define its E-cohomology groups to be

$$E^n Y = [\Sigma^{-n} Y, E]_{\mathcal{S}\mathrm{p}} \cong \pi_{-n}\left(\underline{\hom}_{\mathcal{S}\mathrm{p}}(Y, E)\right)$$

and its E-homology groups to be

$$E_n Y = \pi_n(E \otimes Y).$$

If the spectrum E carries additional structure, this endows the homology theory E_* with corresponding additional structure. For instance, if E carries an associative (resp. commutative) algebra structure in the homotopy category $\mathrm{ho}(\mathcal{S}\mathrm{p})$, this endows the $\mathbb{Z}$-graded abelian group $E_* = \pi_* E$ with the structure of a $\mathbb{Z}$-graded associative (resp. commutative) ring, and moreover we obtain a lift of the homology theory E_* to the category Mod_{E_*} of E_*-modules.

In fact, if we have $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$ and moreover $E_* E$ is *flat* as an E_*-module, then the pair $(E_*, E_* E)$ becomes a *Hopf algebroid* (i.e. $E_* E$ becomes a Hopf algebra over E_*), and moreover we obtain a canonical lift

$$\begin{array}{ccc} & & \mathrm{Comod}_{(E_*, E_* E)} \\ & \overset{\nearrow}{\cdots} & \downarrow \\ \mathcal{S}\mathrm{p} & \xrightarrow[E_*]{} & \mathrm{Mod}_{E_*} \end{array}$$

[34] The term "extraordinary" is meant to emphasize the fact that they satisfy all of the Eilenberg–Steenrod axioms characterizing ordinary (i.e. singular) co/homology, except for possibly the dimension axiom.

of the E-homology functor to the category of comodules over this Hopf algebroid.[35] In this case, we may simply refer to the object $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$ as "flat". We may also refer to objects of $\mathrm{Comod}_{(E_*, E_*E)}$ as "E_*E-comodules", or even simply as "comodules" if the choice of E is clear.

Now, just as we have seen in §0.1.5, there is a world of difference between carrying multiplicative structure in the ∞-category $\mathcal{S}\mathrm{p}$ and carrying multiplicative structure in its homotopy category $\mathrm{ho}(\mathcal{S}\mathrm{p})$: the latter only requires the *existence* of homotopies making various structure diagrams commute, whereas the former demands *coherent choices of witnesses* to the homotopy-commutativity of an infinite hierarchy of higher structure diagrams.

Despite the fact that we are working ∞-categorically, we will sometimes refer to associative (resp. commutative) algebra objects in the ∞-category $\mathcal{S}\mathrm{p}$ as "$\mathbb{A}_\infty$ ring spectra" (resp. "$\mathbb{E}_\infty$ ring spectra").[36] This is partly for historical reasons, since these objects were studied long before ∞-categories existed (so that they could only be defined as algebras over such a cofibrant operad in a model category of spectra (i.e. a model category presenting the ∞-category $\mathcal{S}\mathrm{p}$)). However, this will also be for emphasis: for instance, if E is an $\mathbb{E}_\infty$ ring spectrum, we obtain a truly vast amount of additional structure on its corresponding cohomology theory E^*, collectively referred to as *power operations*.[37] A prototypical example is the family of *Steenrod operations* on mod-p singular cohomology, which arise from the $\mathbb{E}_\infty$ ring structure on the Eilenberg–Mac Lane spectrum $H\mathbb{F}_p$. Conversely, for emphasis we may refer to an object of $\mathrm{Alg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$ (resp. $\mathrm{CAlg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$) as a *homotopy* associative (resp. commutative) ring spectrum.

Power operations are an extremely valuable tool in the project of algebraic topology, which is after all precisely the study of topological objects via algebraic invariants. If we refine the target of a topology-to-algebra functor, this imposes further

[35] In the language of algebraic geometry, a Hopf algebroid is precisely a groupoid object in the category of affine schemes. This represents a groupoid-valued functor, and comodules over the Hopf algebroid can be profitably thought of as quasicoherent sheaves on its associated stack. This perspective on flat extraordinary homology theories (as taking values in sheaves on stacks) plays a fundamental role in *chromatic homotopy theory*, which beautiful research area we will briefly describe in §0.3.2.

[36] However, we will nevertheless write $\mathrm{Alg}(\mathcal{S}\mathrm{p})$ (resp. $\mathrm{CAlg}(\mathcal{S}\mathrm{p})$) for the ∞-category of $\mathbb{A}_\infty$ (resp. $\mathbb{E}_\infty$) ring spectra.

[37] Power operations are a rare example of a homotopy-invariant phenomenon which is perhaps strictly easier to see from a model-categorical point of view than from an ∞-categorical one. For example, given an $\mathbb{E}_\infty$ ring spectrum E and a space X, there is a power operation $E^0X \to E^0(B\Sigma_n \times X)$ for each $n \geq 0$, which effectively arises from the point-set structure of the $\mathbb{E}_\infty$ operad (say in $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$), which in level n is a contractible topological space $E\Sigma_n$ equipped with a free Σ_n-action (so that its quotient recovers a classifying space $B\Sigma_n$). For a comprehensive treatment of power operations, we refer the interested reader to [BMMS86].

conditions on morphisms in the algebraic category; this generally implies the existence of fewer morphisms there, which then can be used to draw conclusions back upstairs in the world of topology.

0.3.2 Derived algebraic geometry and chromatic homotopy theory

Let us return our attention to the Eilenberg–Mac Lane spectrum functor

$$H : \mathrm{Ab} \to \mathcal{S}\mathrm{p}.$$

Recall that its lax symmetric monoidality implies that an associative (resp. commutative) ring R gives rise to an associative (resp. commutative) ring spectrum HR. In fact, more is true: it implies that we obtain a lift

$$\begin{array}{ccc} \mathrm{Mod}_R(\mathrm{Ab}) & \overset{H}{\dashrightarrow} & \mathrm{Mod}_{HR}(\mathcal{S}\mathrm{p}) \\ {\scriptstyle \mathrm{U}_R}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{HR}} \\ \mathrm{Ab} & \xrightarrow[H]{} & \mathcal{S}\mathrm{p}, \end{array}$$

i.e. the Eilenberg–Mac Lane spectra of R-modules are naturally HR-module spectra.[38] Even better, there exists a canonical equivalence

$$\mathscr{D}_R \simeq \mathrm{Mod}_{HR}(\mathcal{S}\mathrm{p})$$

between the derived ∞-category of R and the ∞-category of HR-module spectra (which is symmetric monoidal if R is commutative). Thus, for instance, for any R-modules M and N, we obtain canonical identifications

$$\mathrm{Tor}_n^R(M, N) \cong \pi_n(HM \otimes_{HR} HN)$$

and

$$\mathrm{Ext}^n_R(M, N) = \pi_{-n}\left(\underline{\hom}_{HR}(HM, HN)\right).$$

In summary, *embedding ordinary algebra into spectral algebra naturally brings its derived aspects to the fore.* This observation lies at the heart of the field of ***derived algebraic geometry***.

In order to illustrate this philosophy and the power it affords, let us draw an example from *intersection theory*.

[38] Everything here must be suitably interpreted (in the evident way) in the associative case.

Suppose that Z is a smooth variety over a field k, and that $X, Y \subset Z$ are smooth subvarieties of complementary dimensions. It has long been recognized that if one counts intersection points in the naivest possible way, it is only when X and Y intersect transversely that one obtains the "correct" number (say, the number corresponding to the cup product in cohomology).

On the other hand, by passing from varieties to schemes, it is often possible to directly obtain the correct intersection *as a geometric object*. This is much better than simply knowing how to attach the correct multiplicities to intersection points: it allows us to *continue doing geometry* with their correct intersection.

For instance, suppose that we take

$$Z = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$$

to be the affine plane,

$$X = \operatorname{Spec} k[x, y]/(y) \cong \operatorname{Spec} k[x]$$

to be the x-axis, and

$$Y = \operatorname{Spec} k[x, y]/(y - x^2)$$

to be the standard parabola. Then, the naive set-theoretic intersection of X and Y is just a single point, namely the origin of Z. This is the wrong number: generically, a line and a parabola intersect in two points. On the other hand, the *scheme-theoretic* intersection is given by the intersection

$$X \underset{Z}{\times} Y = \operatorname{Spec} \left(k[x, y]/(y) \underset{k[x,y]}{\otimes} k[x, y]/(y - x^2) \right) \cong \operatorname{Spec} k[x]/(x^2).$$

Indeed, this is one of the major selling points of scheme theory: this geometric object intrinsically keeps track of the fact that this intersection point has multiplicity 2.

However, even the scheme-theoretic intersection does not always give the correct answer. For simplicity, let us assume that set-theoretically, X and Y only intersect at a single point $p \in Z$. Then, the *Serre intersection formula*, introduced in [Ser65], asserts that the correct intersection multiplicity is given by the alternating sum

$$\sum_i (-1)^i \dim \left(\operatorname{Tor}_i^{\mathcal{O}_{Z,p}}(\mathcal{O}_{X,p}, \mathcal{O}_{Y,p}) \right).$$

However, as we have just seen, this is nothing other than the Euler characteristic of the *derived* tensor product

$$\mathcal{O}_{X,p} \overset{\mathbb{L}}{\underset{\mathcal{O}_{Z,p}}{\otimes}} \mathcal{O}_{Y,p},$$

i.e. the tensor product *in the derived* ∞*-category*

$$\mathscr{D}_{\mathcal{O}_{Z,p}} \simeq \mathrm{Mod}_{\mathcal{O}_{Z,p}}(\mathcal{S}\mathrm{p}).$$

Thus, by passing further from schemes to ***derived schemes***, we can *always* obtain the correct intersection *as a geometric object*, namely as the *derived* intersection

$$X \overset{\mathbb{R}}{\underset{Z}{\times}} Y = \mathrm{Spec}\left(\mathcal{O}_{X,p} \overset{\mathbb{L}}{\underset{\mathcal{O}_{Z,p}}{\otimes}} \mathcal{O}_{Y,p}\right).$$

Now, much of the work done in derived algebraic geometry takes place over a field k (i.e. within the derived ∞-category $\mathscr{D}_k \simeq \mathrm{Mod}_{Hk}(\mathcal{S}\mathrm{p})$), often even under the assumption that k has characteristic 0. This is not without reason: such assumptions drastically simplify matters, but are nevertheless sufficiently general to encompass many of the desired applications.

However, the observation that the derived ∞-category of a ring R can be found "inside" of $\mathcal{S}\mathrm{p}$ (that is, is monadic over it) suggests an extremely tantalizing direction of investigation: by passing to spectra, it is possible to work *under* $\mathrm{Spec}\,\mathbb{Z}$.[39]

Now, the unit object – namely, the sphere spectrum $\mathbb{S}$ – is the initial object of the ∞-category $\mathrm{CAlg}(\mathcal{S}\mathrm{p})$ of commutative ring spectra. Thus, at least within *spectral* algebraic geometry (i.e. derived algebraic geometry, but emphatically not necessarily over an ordinary commutative ring), $\mathrm{Spec}\,\mathbb{S}$ is the deepest possible base with respect to which one might work.

Spectral algebraic geometry is absolutely full of mysteries – not the least of which is that we're not even totally sure what the right definition of "$\mathrm{Spec}\,\mathbb{S}$" should be. As it turns out, direct generalizations of the classical definition of the Zariski spectrum (in terms of prime ideals) don't really give the desired answer.[40] Instead, what appears to work better is a mild generalization, which still recovers the usual Zariski spectrum in the classical case.

[39]Such a world has long been sought: for instance, it has been a continual source of inspiration that the Riemann hypothesis might admit a "geometric" proof, if only one could make rigorous sense of the expression "$\mathrm{Spec}\,\mathbb{Z} \times_{\mathrm{Spec}\,\mathbb{F}_1} \mathrm{Spec}\,\mathbb{Z}$" (in analogy with Weil's proof for the case of curves over finite fields). To be clear, we do not necessarily mean to assert that spectra provide a suitable context for attacking the Riemann hypothesis, though that would be mildly satisfying. On the other hand, let us recall that the reigning (but very possibly overly naive) sense is that a "finitely generated projective module over $\mathbb{F}_1$" should be nothing other than a finite pointed set. Via the discussion of §0.3.1, it would follow that the ∞-category $\mathcal{S}\mathrm{p}^{\omega}$ of compact spectra is a compelling candidate for "the ∞-category of perfect complexes on $\mathrm{Spec}\,\mathbb{F}_1$". Thus, if nothing else, one might consider the sphere spectrum $\mathbb{S}$ to be *Morita equivalent* to $\mathbb{F}_1$.

[40]In fact, "ideals" of ring spectra are themselves a poorly behaved notion. For instance, given a commutative ring spectrum R and an element $f \in \pi_0 R$, the quotient $R/(f) = \mathrm{cofib}(f : R \to R)$ is rarely another commutative ring spectrum.

To explain this, we must introduce some terminology. In essence, the idea will be to treat a ***presentably symmetric monoidal stable ∞-category*** as a *categorification* of a commutative ring. Let us first explain this terminology.

To begin, an ∞-category is called *presentable* if it is "generated by a small subcategory". More precisely, for a regular cardinal κ, the functor Ind_κ of *freely adjoining κ-filtered colimits* defines an equivalence

$$\mathrm{Ind}_\kappa : \left\{ \begin{array}{c} \text{small} \\ \kappa\text{-cocomplete} \\ \infty\text{-categories} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \kappa\text{-compactly generated} \\ \infty\text{-categories} \end{array} \right\},$$

and a presentable ∞-category is one which is κ-compactly generated for some regular cardinal κ. (It is then also κ'-compactly generated for any regular cardinal $\kappa' \geq \kappa$.) In these settings, if $\kappa = \omega$ is countable then one often omits the cardinal unless it is meant to be emphasized. So for example, the category $\mathcal{S}\mathrm{et}$ of sets is the Ind-completion of its subcategory of finite sets, and the category Ab of abelian groups is the Ind-completion of its subcategory of finitely generated abelian groups. In practice, most ∞-categories of lasting interest are presentable (often even ω-compactly generated).

Next, an ∞-category which is both presentable and symmetric monoidal is called *presentably symmetric monoidal* if its symmetric monoidal structure distributes over colimits separately in each variable. This may be thought of as a categorification of the usual distributivity axiom in a commutative ring. The above equivalence Ind_κ refines to an equivalence

$$\mathrm{Ind}_\kappa : \left\{ \begin{array}{c} \text{small} \\ \kappa\text{-cocompletely s.m.} \\ \infty\text{-categories} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \kappa\text{-compactly generated} \\ \text{presentably s.m.} \\ \infty\text{-categories} \end{array} \right\}.$$

In practice, most symmetric monoidal ∞-categories of lasting interest are presentably symmetric monoidal (again often even coming from a small ω-cocompletely symmetric monoidal ω-cocomplete ∞-category).

We can now describe the notions that will participate in our categorified notion of the Zariski spectrum.

First, suppose that $\mathcal{C}$ is a symmetric monoidal ∞-category. A full subcategory $\mathscr{I} \subset \mathcal{C}$ is called an *ideal* if it is "contagious" under the symmetric monoidal structure, i.e. if $X \otimes Y \in \mathscr{I}$ whenever $X \in \mathcal{C}$ and $Y \in \mathscr{I}$. Then, a subcategory $\mathscr{P} \subset \mathcal{C}$ is called a *prime ideal* if it is a proper ideal and moreover if $X \otimes Y \in \mathscr{P}$ then either $X \in \mathscr{P}$ or $Y \in \mathscr{P}$.

Next, suppose that $\mathcal{D}$ is a stable ∞-category. A full subcategory $\mathscr{K} \subset \mathcal{D}$ is called a *stable subcategory* if it is stable and its inclusion is exact (i.e. the co/fiber sequences

in $\mathscr{K}$ agree with those of $\mathcal{D}$). In this case, we can form the *Verdier quotient* of $\mathcal{D}$ by $\mathscr{K}$ as the pushout

$$\begin{array}{ccc} \mathscr{K} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{D}/\mathscr{K} \end{array}$$

in the ∞-category of stable ∞-categories (and exact functors between them). Under the composite

$$\mathscr{K} \to \mathcal{D} \to \mathcal{D}/\mathscr{K},$$

all of the objects of $\mathscr{K}$ are sent to the zero object of $\mathcal{D}/\mathscr{K}$. However, the objects of $\mathcal{D}$ that are sent to the zero object of $\mathcal{D}/\mathscr{K}$ are precisely the *retracts* of objects of the subcategory $\mathscr{K} \subset \mathcal{D}$. Motivated by this, we define a *thick subcategory* of $\mathcal{D}$ to be a stable subcategory which is closed under retracts. Any stable subcategory $\mathscr{K} \subset \mathcal{D}$ can be *idempotent-completed* to a thick subcategory $\mathscr{K}^{\wedge} \subset \mathcal{D}$ with the same Verdier quotient.

If our stable ∞-category $\mathcal{D}$ is presentable, then we can say more. For starters, the quotient functor $\mathcal{D} \to \mathcal{D}/\mathscr{K}$ must then admit a fully faithful right adjoint: in other words, it participates in a *reflective localization*

$$\mathrm{L}_{\mathscr{K}} : \mathcal{D} \rightleftarrows \mathcal{D}/\mathscr{K} : \mathrm{U}_{\mathscr{K}}.$$

Moreover, from this adjunction, we can recover the idempotent completion $\mathscr{K}^{\wedge} \subset \mathcal{D}$ of $\mathscr{K}$ as the full subcategory of *acyclic* objects for the reflective localization, i.e. those that are sent to the zero object of $\mathcal{D}/\mathscr{K}$. We thus obtain an order-reversing isomorphism

$$\{\text{thick subcategories of } \mathcal{D}\} \cong \{\text{reflective localizations of } \mathcal{D}\}^{op}$$

of posets.

Let us now suppose further that our presentable stable ∞-category $\mathcal{D}$ is in fact presentably symmetric monoidal (i.e. a "categorified commutative ring"). Then, a thick subcategory $\mathscr{K} \subset \mathcal{D}$ will be an ideal precisely if the corresponding reflective localization

$$\mathrm{L}_{\mathscr{K}} : \mathcal{D} \to \mathcal{D}/\mathscr{K}$$

is compatible with the symmetric monoidal structure of $\mathcal{D}$, in the sense that for all $X, Y \in \mathcal{D}$, the canonical dotted factorization

$$\begin{array}{ccccc} X \otimes Y & \longrightarrow & \mathrm{L}_{\mathscr{K}}(X) \otimes \mathrm{L}_{\mathscr{K}}(Y) & \longrightarrow & \mathrm{L}_{\mathscr{K}}\left(\mathrm{L}_{\mathscr{K}}(X) \otimes \mathrm{L}_{\mathscr{K}}(Y)\right) \\ & \searrow & & \nearrow & \\ & & \mathrm{L}_{\mathscr{K}}(X \otimes Y) & & \end{array}$$

is an equivalence. In this case, the presentable stable ∞-category $\mathcal{D}/\mathscr{K}$ inherits a *completed* symmetric monoidal structure defined by

$$X \widehat{\otimes} Y = \mathrm{L}_{\mathscr{K}}(X \otimes Y),$$

with respect to which it and the reflective localization become presentably symmetric monoidal. Thus, we obtain a restricted isomorphism

$$\{\text{thick ideal subcategories of } \mathcal{D}\} \cong \left\{ \begin{array}{c} \text{presentably s.m.} \\ \text{reflective localizations of } \mathcal{D} \end{array} \right\}^{op}$$

of posets, in direct analogy with the isomorphism

$$\{\text{ideals of } R\} \cong \{\text{surjective ring homomorphisms from } R\}^{op}$$

for a commutative ring R.

Finally, taking a cue from commutative algebra, let us say that a presentably symmetric monoidal stable ∞-category is *integral* if whenever $X \otimes Y \simeq 0$ then either $X \simeq 0$ or $Y \simeq 0$. It is immediate from this definition that our isomorphism restricts further to an isomorphism

$$\left\{ \begin{array}{c} \text{thick prime ideal} \\ \text{subcategories of } \mathcal{D} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{integral} \\ \text{presentably s.m.} \\ \text{reflective localizations of } \mathcal{D} \end{array} \right\}^{op}$$

of posets.

Forgetting the poset structure (which we only kept track of to indicate the order-reversal), we can endow the above set with a topology, with basis given by the subsets

$$U_X = \left\{ \begin{array}{c} \mathscr{P} \subset \mathcal{D} \text{ a thick prime} \\ \text{ideal subcategory} \end{array} \; : \; X \in \mathscr{P} \right\}$$

for $X \in \mathcal{D}$.[41] We denote the resulting topological space by

$$\operatorname{Spec} \mathcal{D},$$

and refer to it as the (***Zariski*** or ***prime***) ***spectrum*** of the presentably symmetric monoidal stable ∞-category $\mathcal{D}$.[42] This definition still makes sense if $\mathcal{D}$ is instead a

[41] The condition defining U_X is reversed from what might be expected from commutative algebra; this can be attributed to *Hochster duality* (see [Hoc69] as well as the recent [KP]).

[42] The analogous construction in the setting of tensor triangulated categories has been studied extensively; for an overview, see Balmer's prominent work [Bal10].

small κ-cocompletely symmetric monoidal stable ∞-category, and we double-book the notation accordingly.[43,44,45] Although the analogy with ordinary scheme theory is clearest in the setting of presentably symmetric monoidal ∞-categories, this latter version actually appears to be better behaved.[46]

Let us now describe how this recovers the usual Zariski spectrum. Suppose that X is a qcqs scheme. Let us write $\mathscr{D}_X$ for its derived ∞-category (of quasicoherent sheaves), and let us write $\mathscr{D}_X^\omega \subset \mathscr{D}_X$ for its subcategory of compact objects (i.e. perfect complexes). This is a small ω-cocompletely symmetric monoidal stable ∞-category. The fact to which we have been alluding is that this gives rise to a homeomorphism

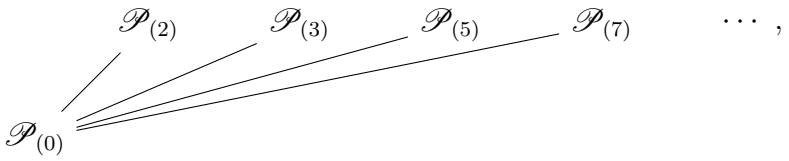
$$X \cong \operatorname{Spec} \mathscr{D}_X^\omega$$

of topological spaces.

For an explicit example, let us take $X = \operatorname{Spec} \mathbb{Z}$: its derived ∞-category is the presentably symmetric monoidal stable ∞-category

$$\mathscr{D}_X \simeq \mathscr{D}_{\mathbb{Z}} \simeq \operatorname{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p}).$$

Then, the subcategory $\operatorname{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p})^\omega$ of compact objects has its poset of thick prime ideal subcategories given by the usual diagram



where for any prime p we write

$$\mathscr{P}_{(p)} = \ker\left(\operatorname{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p})^\omega \xrightarrow{H\mathbb{F}_p \otimes -} \operatorname{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p})^\omega\right)$$

[43] In this latter version, it seems appropriate to require that $\mathcal{D}$ be idempotent-complete (which is actually already automatic unless $\kappa = \omega$).

[44] In this latter version, the Verdier quotients of $\mathcal{D}$ corresponding to the points of $\operatorname{Spec} \mathcal{D}$ will not generally be reflective localizations, since the right adjoint comes from the adjoint functor theorem (which requires presentability).

[45] In either version, it is possible to endow the topological space $\operatorname{Spec} \mathcal{D}$ with a structure sheaf, but we will not pursue that point here.

[46] If a presentable ∞-category $\mathcal{D}$ is κ-compactly generated, then ideals of its subcategory $\mathcal{D}^\kappa \subset \mathcal{D}$ of κ-compact objects are in bijective correspondence with κ-compactly generated ideals of $\mathcal{D}$ whose inclusions preserve κ-compact objects. So in general, the presentable version will have more points. Of course, one could also modify the definition in the presentable case along these lines.

for the thick prime ideal subcategory of p-torsion-free complexes, and we write

$$\mathscr{P}_{(0)} = \ker\left(\mathrm{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p})^{\omega} \xrightarrow{H\mathbb{Q}\otimes -} \mathrm{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p})^{\omega}\right)$$

for the thick prime ideal subcategory of torsion complexes. At the level of presentably symmetric monoidal stable ∞-categories, these respectively correspond to the integral presentably symmetric monoidal reflective localizations

$$\mathrm{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p}) \rightleftarrows \mathrm{Mod}_{H\mathbb{F}_p}(\mathcal{S}\mathrm{p})$$

and

$$\mathrm{Mod}_{H\mathbb{Z}}(\mathcal{S}\mathrm{p}) \rightleftarrows \mathrm{Mod}_{H\mathbb{Q}}(\mathcal{S}\mathrm{p}),$$

which at the level of algebraic geometry are precisely the sheaf-theoretic adjunctions corresponding to the inclusions of the respective subschemes $\operatorname{Spec}\mathbb{F}_p \subset \operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{Q} \subset \operatorname{Spec}\mathbb{Z}$.

We can now describe our candidate for $\operatorname{Spec}\mathbb{S}$: as we have $\mathrm{Mod}_{\mathbb{S}}(\mathcal{S}\mathrm{p}) \simeq \mathcal{S}\mathrm{p}$, we simply define

$$\operatorname{Spec}\mathbb{S} = \operatorname{Spec}\mathcal{S}\mathrm{p}^{\omega}$$

to be the Zariski spectrum of the small ω-cocompletely symmetric monoidal stable ∞-category of compact spectra. The beautiful *thick subcategory theorem* of Hopkins–Smith (see [HS98]) asserts that this takes the form

$$\operatorname{Spec}\mathbb{S} = \mathbb{N}_0^{\infty} \wedge \operatorname{Spec}\mathbb{Z},$$

the smash product of the poset

$$\mathbb{N}_0^{\infty} = \{0 < 1 < 2 < \cdots < \infty\}$$

(equipped with the basepoint 0) and the poset

$$\operatorname{Spec}\mathbb{Z}$$

(equipped with the basepoint (0)).[47] This is illustrated in Figure 0.1, where we write

$$\mathscr{P}_{n,(p)} = \ker\left(\mathcal{S}\mathrm{p}^{\omega} \xrightarrow{K(n,p)\otimes -} \mathcal{S}\mathrm{p}^{\omega}\right)$$

[47] The thick subcategory theorem is intimately related to the equally beautiful *nilpotence theorem* of Devinatz–Hopkins–Smith (see [DHS88]); together, these two works resolved most of the *Ravenel conjectures*, which were put forth in the deeply influential paper [Rav84].

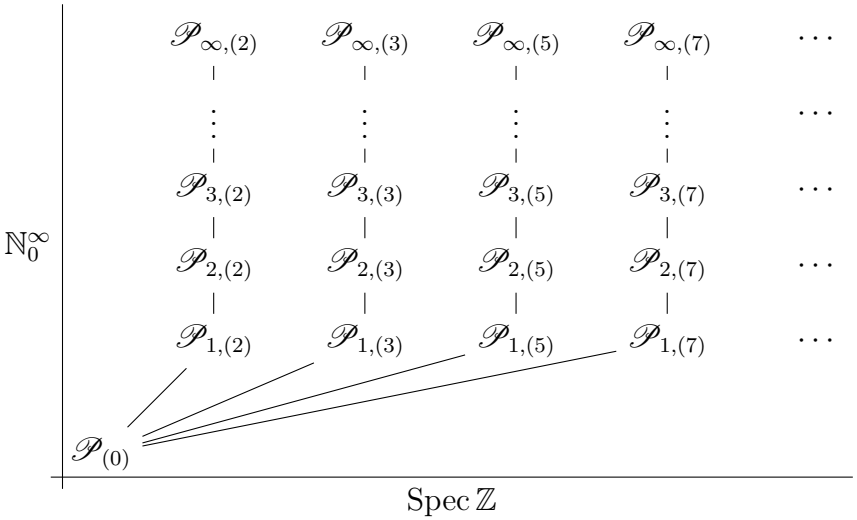


Figure 0.1: $\operatorname{Spec}\mathbb{S}$.

for the kernel of the n^{th} *Morava K-theory at the prime p*, and similarly to above we write

$$\mathscr{P}_{(0)} = \ker\left(\mathcal{S}\mathrm{p}^\omega \xrightarrow{H\mathbb{Q}\otimes -} \mathcal{S}\mathrm{p}^\omega\right);$$

it is often convenient to set $K(0,p) = H\mathbb{Q}$ for any prime p.[48] The elements of $\mathbb{N}_0^\infty$ are referred to as *chromatic heights*, and the study of stable homotopy theory through the lens of this stratification is the field of ***chromatic homotopy theory***.[49] In fact, we have $K(\infty,p) \simeq H\mathbb{F}_p$: in other words, the canonical map

$$\operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{S}$$

[48]It seems likely that this picture of $\operatorname{Spec}\mathbb{S}$ should also be recoverable from a *noncommutative* version of spectral algebraic geometry. The Morava K-theories are generally only associative ring spectra, but their homotopy groups nevertheless happen to form commutative rings, which makes their algebro-geometric manipulation substantially more tractable than that of arbitrary associative ring spectra. (This property is sometimes cheekily referred to as "$\mathbb{E}_{1\frac{1}{2}}$", a term attributed to Hopkins.)

[49]The field of chromatic homotopy theory, which finds its origins in Morava's celebrated paper [Mor85], is so named for its basis in certain periodicity phenomena in the ring $\pi_*\mathbb{S}$ of stable homotopy groups of spheres: the word "chromatic" is meant to evoke a connection with periods and wavelengths. This terminology inspires the names of the *redshift* and *blueshift* conjectures, which (in rough terms) respectively assert that algebraic K-theory "raises chromatic height" and that Tate cohomology "lowers chromatic height".

cuts out the locus at chromatic height ∞ (along with the generic point). Thus, most of $\operatorname{Spec}\mathbb{S}$ does indeed live "under $\operatorname{Spec}\mathbb{Z}$".

Now, for any prime p, the containments

$$\mathscr{P}_{\infty,(p)} \subset \cdots \subset \mathscr{P}_{n,(p)} \subset \mathscr{P}_{n-1,(p)} \subset \cdots \subset \mathscr{P}_{1,(p)} \subset \mathscr{P}_{(0)}$$

are tantamount to saying that for any compact spectrum $X \in \mathcal{S}\mathrm{p}^{\omega}$ and for any $n \geq 1$, if $K(n,p)\otimes X \simeq 0$ then $K(n-1,p)\otimes X \simeq 0$. However, this relationship need not hold for noncompact spectra. On the other hand, for any $n \geq 0$ and any prime p, there exists a spectrum $E_{n,p}$, the n^{th} *Morava E-theory at the prime* p, such that for *any* spectrum $Y \in \mathcal{S}\mathrm{p}$, we have $E_{n,p}\otimes Y \simeq 0$ if and only if $K(i,p)\otimes Y \simeq 0$ for all $0 \leq i \leq n$. (At $n = 0$, we once again simply set $E_{0,p} = H\mathbb{Q}$ (or perhaps a 2-periodification thereof).) Thus, the Morava E-theories govern arbitrary quasicoherent sheaves on $\operatorname{Spec}\mathbb{S}$, in a manner which may be referred to as ***chromatic globalization***.

This state of affairs naturally leads one to wonder about the possibility of *arithmetic* globalization. Of course, at chromatic height 0, the question is trivial. But in fact, there is an obvious candidate for such an "arithmetically global" theory at chromatic height 1 as well: the Morava E-theory $E_{1,p}$ is nothing other than the p-completed complex K-theory spectrum $KU_p^{\wedge}$. Clearly, these are all recovered from the *uncompleted* complex K-theory spectrum KU, which may therefore be deemed to be ***arithmetically global of chromatic height 1***.

In order to explain how this generalizes upwards, we briefly indicate the intimate connection between chromatic homotopy theory and the theory of formal groups (in the sense of formal algebraic geometry). A *complex orientable cohomology theory* is one which "admits a theory of Chern classes"; a *complex orientation* is a choice of "coordinate" for such a theory. Given a complex orientation, the formula for the first Chern class of a tensor product of line bundles then provides a formal group law, and changing the complex orientation preserves the underlying formal group. For example, the classical equation

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

implies that singular cohomology theories correspond to the *additive* formal group $\widehat{\mathbb{G}}_a$. In fact, the chromatic height of a complex orientable cohomology theory is by definition the height of its corresponding formal group.[50] In particular, the Morava K-theory $K(n,p)$ corresponds to the height-n *Honda formal group* at the prime p, while the Morava E-theory $E_{n,p}$ corresponds to its *universal deformation*, also called the height-n *Lubin–Tate formal group* at the prime p.

[50] Actually, the notion of the "height" of a formal group only makes sense in positive characteristic; for convenience, we simply declare the height of $(\widehat{\mathbb{G}}_a)|_{\mathbb{Q}}$ to be 0.

This perspective provides a more primordial explanation for why the height-1 Morava E-theories admit such a straightforward "integral model". Namely, at any prime p, the height-1 Lubin–Tate formal group can be more concretely identified as the *multiplicative* formal group $(\widehat{\mathbb{G}}_m)|_{\mathbb{Z}_p^\wedge}$ over the ring $\mathbb{Z}_p^\wedge$ of p-adic integers. These are all pulled back from the absolute formal multiplicative group $\widehat{\mathbb{G}}_m = (\widehat{\mathbb{G}}_m)|_{\mathbb{Z}}$, which in turn corresponds to KU. Recasting this situation in rough algebro-geometric terms, we can consider the scheme $\operatorname{Spec}\mathbb{Z}$ as being equipped with a sheaf $\widehat{\mathbb{G}}_m$ of formal groups and a compatible quasicoherent sheaf $KU^\sim$ of commutative ring spectra, such that their evaluations on the object

$$\operatorname{Spf}\mathbb{Z}_p^\wedge \to \operatorname{Spec}\mathbb{Z}$$

gives the corresponding Lubin–Tate formal group

$$(\widehat{\mathbb{G}}_m)|_{\mathbb{Z}_p^\wedge} \cong \widehat{\mathbb{G}}_m \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p^\wedge$$

and the Morava E-theory spectrum

$$E_{1,p} \simeq KU \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p^\wedge.$$

This, then, suggests our upwards generalization: we would like to find some algebro-geometric object equipped with a sheaf of formal groups which naturally contains the height-2 Lubin–Tate formal groups at all primes, and we would like to enhance this sheaf of formal groups to a compatible sheaf of commutative ring spectra.

In fact, at chromatic height 2, the *moduli stack of elliptic curves* $\mathscr{M}_{\mathrm{ell}}$ is just such an object. Among elliptic curves (over a field of positive characteristic), there are the *ordinary* ones whose formal groups have height 1, and there are the *supersingular* ones whose formal groups have height 2; moreover, the *Serre–Tate theorem* of [ST68] implies that the deformation theory of a supersingular elliptic curve is equivalent to that of its formal group. However, in this case, it is much more difficult to construct the relevant sheaf $\mathcal{O}^{\mathrm{der}}$ of commutative ring spectra: the moduli stack $\mathscr{M}_{\mathrm{ell}}$ is not affine, and so it is far from sufficient to merely specify a single commutative ring spectrum (as we merely specified KU previously at height 1). Nevertheless, it can be done.

Locally, the homotopy groups of the sheaf $\mathcal{O}^{\mathrm{der}}$ of commutative ring spectra encompass both the structure sheaf of $\mathscr{M}_{\mathrm{ell}}$ as well as the sheaf of invariant differentials and all of its tensor powers. Its commutative ring spectrum of global sections is therefore denoted

$$tmf = \mathcal{O}^{\mathrm{der}}(\mathscr{M}_{\mathrm{ell}})$$

and referred to as ***topological modular forms***: its homotopy groups may be thought of as something like "derived modular forms", and indeed they agree with the ring of ordinary modular forms as soon as we invert $6 \in \mathbb{Z}$.[51,52] Up to taking fixedpoints by stabilizer groups, the local sections of $\mathcal{O}^{\mathrm{der}}$ are given by the various Morava E-theories $E_{0,p}$, $E_{1,p}$, and $E_{2,p}$ at all primes p. Thus, *tmf* may be deemed to be ***arithmetically global of chromatic height 2***.[53]

0.3.3 Goerss–Hopkins obstruction theory: the black box

In broad terms, an *obstruction theory* is a machine which provides *algebraic* criteria for determining the possibility of some *topological* construction. For example, the obstructions to the triviality of a vector bundle are given by its characteristic classes.

Perhaps the most sophisticated obstruction theory in all of algebraic topology is ***Goerss–Hopkins obstruction theory***. Suppose we are given a flat homotopy commutative ring spectrum

$$E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$$

satisfying *Adams's condition*, which we will describe in §0.3.4; we will refer to the its corresponding homology theory E_* as our "detecting" homology theory. Suppose moreover that we are given a commutative algebra

$$A \in \mathrm{CAlg}(\mathrm{Comod}_{(E_*, E_*E)})$$

in comodules. Then, Goerss–Hopkins obstruction theory provides a method for computing the ***moduli space of*** (E***-local***) ***realizations*** of A as an $\mathbb{E}_\infty$ ring spectrum

[51] We are actually being slightly sloppy here, in a few different ways: regarding periodic versus connective spectra, and regarding compactifications of $\mathscr{M}_{\mathrm{ell}}$. However, it doesn't seem useful clarify this point here, and we will remain imprecise in this way throughout this introduction.

[52] In his forthcoming alternative construction of *tmf*, Lurie takes a different approach to the one we have described (which is admittedly rather ad hoc in comparison), viewing *tmf* as originating fundamentally in spectral algebraic geometry. Namely, rather than take the *ordinary* moduli stack of elliptic curves and endow it with a sheaf of commutative ring spectra, he defines a spectral stack of (oriented) spectral elliptic curves: its global sections are then *tmf*, and in fact its truncation recovers the "spectrally enhanced ordinary stack" that we have described. For an overview of this approach, we refer the reader to [Lur09a].

[53] At present, the question of arithmetic globalization at chromatic heights greater than 2 is wide open. In particular, there are not really even any candidates for analogous arithmetically global ring spectra at higher heights. An apparent primary obstruction is the (current) absence of a connection between higher-dimensional formal groups and chromatic homotopy theory. In what represents the current state of the art, Behrens–Lawson construct a spectrum of *topological automorphic forms* at a prime p in [BL10], which they accomplish by considering moduli stacks of higher-dimensional abelian varieties that are equipped with enough extra structure that their formal completions canonically split off 1-dimensional summands.

– the first question being whether it is nonempty.[54] Let us explain this all in more detail.

First of all, a *realization* of A is an $\mathbb{E}_\infty$ ring spectrum X for which there exists an isomorphism $E_*X \cong A$ (of algebras in comodules). These are our objects of interest. Note that we do *not* require the existence of a spectrum realizing the underlying comodule of A: that is, we start with *purely algebraic* data.

Next, an *E-equivalence* is a map $X \to Y$ of spectra that induces an isomorphism $E_*X \xrightarrow{\cong} E_*Y$ of E_*E-comodules (or equivalently of E_*-modules). In a universal way, we can invert the E-equivalences in the ∞-category of spectra to form the ∞-category $\mathrm{L}_E(\mathcal{S}\mathrm{p})$ of *E-local spectra*. The terminology stems from the fact that this localization actually participates in a reflective localization

$$\mathrm{L}_E : \mathcal{S}\mathrm{p} \rightleftarrows \mathrm{L}_E(\mathcal{S}\mathrm{p}) : \mathrm{U}_E,$$

i.e. an adjunction whose right adjoint is fully faithful; in particular, we can consider $\mathrm{L}_E(\mathcal{S}\mathrm{p}) \subset \mathcal{S}\mathrm{p}$ as a full subcategory.[55,56] In other words, E-local spectra are just particular sorts of spectra, but E-equivalences between them are necessarily equivalences.

Finally, the *moduli space of E-local realizations* of A is the full subgroupoid

$$\mathscr{M}_A \subset \mathrm{CAlg}(\mathrm{L}_E(\mathcal{S}\mathrm{p}))$$

on the E-local $\mathbb{E}_\infty$ ring spectra which are realizations of A; its morphisms are the E-equivalences (which are also equivalences) between them. As indicated above, we will generally leave the descriptor "E-local" implicit.

Of course, this necessarily only produces E-local spectra. Thus, if one is interested in obtaining an $\mathbb{E}_\infty$ ring structure on a particular spectrum $X \in \mathcal{S}\mathrm{p}$, one must choose a detecting homology theory E_* for which X is E-local. On the other hand, this locality is not so hard to satisfy in practice: crucially, any E-module is necessarily E-local. Note that this is a relatively weak (and in particular, unstructured) hypothesis: we have only assumed that E is a *homotopy* commutative ring spectrum, and thus by "module" we can only possibly mean an object $X \in \mathrm{Mod}_E(\mathrm{ho}(\mathcal{S}\mathrm{p}))$.

[54] In fact, the obstruction theory applies to algebras in spectra over *any* operad $\mathcal{O}$, though this changes the nature of the algebraic object in comodules that we must consider. For the purposes of this overview, we will focus on the $\mathbb{E}_\infty$ case, which is of central interest.

[55] This is in fact a *presentably symmetric monoidal* reflective localization of the presentably symmetric monoidal stable ∞-category $\mathcal{S}\mathrm{p}$ of spectra, as described in §0.3.2. Under the contravariant isomorphism of posets described there, this corresponds to the thick ideal subcategory of *E-acyclic* spectra, i.e. those objects $X \in \mathcal{S}\mathrm{p}$ such that $E \otimes X \simeq 0$.

[56] This is the underlying ∞-categorical content of the theory of *Bousfield localization* of spectra, as introduced in the classic paper [Bou79].

In particular, it follows that E is E-local. This implies the nearly unbelievable conclusion that if we would like to endow a homotopy commutative ring spectrum $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$ with an $\mathbb{E}_\infty$-structure, then E can *itself* serve as the detecting homology theory!

In fact, this was precisely the technique employed by Goerss–Hopkins as the very first application of their newly minted obstruction theory: they proved that the Morava E-theory spectra $E_{n,p}$ described in §0.3.2 admit unique $\mathbb{E}_\infty$ structures, and that their automorphism spaces (as such) are in fact discrete.

Even more spectacularly, Goerss–Hopkins obstruction theory is also the key ingredient of the construction of the sheaf $\mathcal{O}^{\mathrm{der}}$ over $\mathscr{M}_{\mathrm{ell}}$ whose global sections are *tmf*. This actually uses a *relative* version of the obstruction theory (which in our description we have left implicit): it is necessary to construct not just the local sections along with their actions of the automorphisms groups of elliptic curves, but also the more general restriction maps between them.

Let us now explain what we mean when we say that Goerss–Hopkins obstruction theory allows us to "compute" the moduli space $\mathscr{M}_A$ of realizations of our chosen object A.

First of all, given our commutative algebra A in comodules, one can speak of *modules* over A (in comodules); we mention now that for any $n \geq 1$ one can define a canonical A-module $\Omega^n A$, which will play a role in our story shortly. For any A-module M and any augmented commutative algebra

$$X \in \mathrm{CAlg}(\mathrm{Comod}_{(E_*, E_*E)})_{/A}$$

in comodules, we can define the corresponding *André–Quillen cohomology groups* $H^*(X; M)$. In fact, these are given by the homotopy groups of a certain spectrum

$$\mathscr{H}(X; M) = \{\mathscr{H}^n(X; M)\}_{n \geq 0},$$

in the sense that

$$H^n(X; M) = \pi_{-n}\mathscr{H}(X; M) \cong \pi_0\mathscr{H}^n(X; M) \cong \pi_1\mathscr{H}^{n+1}(X; M) \cong \cdots$$

for any $n \geq 0$ (or really for any $n \in \mathbb{Z}$: this spectrum has vanishing positive-dimensional homotopy groups, not unlike $\underline{\hom}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+ X, E)$ for any $X \in \mathcal{S}$ and any $E \in \mathcal{S}\mathrm{p}$). The group

$$\mathrm{Aut}(A, M)$$

of automorphisms of the pair (A, M) (whose elements are pairs of an isomorphism $\varphi : A \xrightarrow{\cong} A$ and an isomorphism $M \to \varphi^* M$) naturally acts on this spectrum. In particular, it acts on each constituent space $\mathscr{H}^n(X; M)$, and we write

$$\widehat{\mathscr{H}^n}(X; M) = (\mathscr{H}^n(X; M))_{\mathrm{Aut}(A,M)}$$

for the (homotopy) quotient. This action fixes the basepoint of $\mathscr{H}^n(X;M)$ (whose path component corresponds to the zero element $0 \in H^n(X;M)$), and so the inclusion of the basepoint is $\mathrm{Aut}(A,M)$-equivariant and hence determines a map

$$B\mathrm{Aut}(A,M) \to \widehat{\mathscr{H}^n}(X;M)$$

on quotients. We note for future reference that this map, whose source is connected, lands entirely in the path component selected by the composite

$$\mathrm{pt} \xrightarrow{0} \mathscr{H}^n(X;M) \to \widehat{\mathscr{H}^n}(X;M).$$

Now, as we will describe in more depth in §0.3.4, our understanding of the moduli space $\mathscr{M}_A$ actually comes from a sequence of moduli spaces $\mathscr{M}_n(A)$ of "n-stage approximations" to a realization of A. These moduli spaces are related by pullback squares

$$\begin{array}{ccc} \mathscr{M}_n(A) & \longrightarrow & B\mathrm{Aut}(A,\Omega^n A) \\ \downarrow & & \downarrow \\ \mathscr{M}_{n-1}(A) & \longrightarrow & \widehat{\mathscr{H}^{n+2}}(A;\Omega^n A) \end{array}$$

(for all $n \geq 1$), in which the left vertical map is induced by an "$(n-1)^{\mathrm{st}}$ Postnikov trunction" functor and the lower map is induced by an "n^{th} k-invariant" functor

$$\mathscr{M}_{n-1}(A) \xrightarrow{\chi_n} \mathscr{H}^{n+2}(A;\Omega^n A).$$

Moreover, we have a canonical identification

$$\mathscr{M}_A \xrightarrow{\sim} \lim\left(\cdots \to \mathscr{M}_2(A) \to \mathscr{M}_1(A) \to \mathscr{M}_0(A)\right)$$

of our moduli space of realizations as the limit of the resulting tower. Finally, as the base for our inductive understanding, we have an equivalence

$$\mathscr{M}_0(A) \simeq B\mathrm{Aut}(A).$$

We can now describe the sense in which we can "compute" the moduli space $\mathscr{M}_A$. Observe that the above pullback square implies that an $(n-1)$-stage X can be lifted to an n-stage if and only if the k-invariant

$$[\chi_n(X)] \in H^{n+2}(A;\Omega^n A)$$

vanishes: this is the only case in which there exists a nonempty fiber in the diagram

$$\begin{array}{ccccccc} & & & & & & B\mathrm{Aut}(A, \Omega^n A) \\ & & & & & & \downarrow \\ \mathrm{pt} & \xrightarrow[X]{} & \mathscr{M}_{n-1}(A) & \xrightarrow[\chi_n]{} & \mathscr{H}^{n+2}(A; \Omega^n A) & \longrightarrow & \widehat{\mathscr{H}}^{n+2}(A; \Omega^n A), \end{array}$$

which is necessary and sufficient for there to exist a nonempty fiber in the diagram

$$\begin{array}{ccc} & & \mathscr{M}_n(A) \\ & & \downarrow \\ \mathrm{pt} & \xrightarrow[X]{} & \mathscr{M}_{n-1}(A). \end{array}$$

Of course, this is most useful in the "étale" case, i.e. when the relevant André–Quillen cohomology groups all vanish. Under this assumption, the entire tower collapses to an equivalence

$$\mathscr{M}_A \xrightarrow{\sim} \mathscr{M}_0(A) \simeq B\mathrm{Aut}(A).$$

This is visibly the case with Goerss–Hopkins's original application to the Morava E-theories. In fact, after enough algebraic manipulation, it also becomes the case in the construction of the sheaf $\mathcal{O}^{\mathrm{der}}$ (but these manipulations are themselves not completely trivial).

In fact, this is also the case in another prominent application of Goerss–Hopkins obstruction theory as well. In his inspiring monograph [Rog08], Rognes develops the *Galois theory* of $\mathbb{E}_\infty$ ring spectra. This may be seen as the study of *covering spaces* among affine spectral schemes, and provides a remarkably effective framework for the organization of chromatic homotopy theory from the viewpoint of spectral algebraic geometry. Just as classical Galois theory, this is governed by a Galois correspondence, i.e. a contravariant equivalence of posets. In order to prove this fundamental theorem, Rognes uses Goerss–Hopkins obstruction theory to obtain the desired intermediate Galois extension from a subgroup of the Galois group.

In addition to the applications we have mentioned, Goerss–Hopkins obstruction theory finds frequent and crucial use throughout the literature on structured ring spectra.

0.3.4 Goerss–Hopkins obstruction theory: under the hood

In order to explain the inner workings of Goerss–Hopkins obstruction theory and our generalization thereof, we must first explain what exactly we mean by the expression "$\mathscr{D}_{\geq 0}(\mathcal{C})$" for a (1- or ∞-)category $\mathcal{C}$.

In fact, our notation has been very slightly misleading: this construction does not depend on the ∞-category $\mathcal{C}$ alone. Rather, we must first choose a full subcategory $\mathcal{G} \subset \mathcal{C}$ which is closed under finite coproducts, which should be thought of as a subcategory of "projective generators". Out of this, we define

$$\mathscr{D}_{\geq 0}(\mathcal{C}) = \mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}) = \mathcal{P}_\Sigma(\mathcal{G}) = \mathrm{Fun}^\times(\mathcal{G}^{op}, \mathcal{S})$$

to be the ∞-category of *product-preserving presheaves of spaces* on $\mathcal{G}$, i.e. the full subcategory of $\mathrm{Fun}(\mathcal{G}^{op}, \mathcal{S})$ on those contravariant functors that take finite coproducts in $\mathcal{G}$ to finite products in $\mathcal{S}$.[57]

The connection with the discussion of §0.1.3 is as follows. There, we considered the category $s\mathcal{C}$ of simplicial objects in a 1-category $\mathcal{C}$, and we asserted that it comes with a canonical subcategory $\mathbf{W} \subset s\mathcal{C}$ of weak equivalences. After choosing the subcategory $\mathcal{G} \subset \mathcal{C}$, this subcategory is pulled back from the equivalences under the functor

$$s\mathcal{C} \xrightarrow{X_\bullet \mapsto \left(Y \mapsto |\hom^{\mathrm{lw}}_{\mathcal{C}}(Y, X_\bullet)|\right)} \mathcal{P}_\Sigma(\mathcal{G}),$$

which can equivalently be thought of as the composite

$$s\mathcal{C} \xrightarrow{s(\text{ɟ})} s(\mathrm{Fun}(\mathcal{G}^{op}, \mathcal{S}\mathrm{et})) \simeq \mathrm{Fun}(\mathcal{G}^{op}, s\mathcal{S}\mathrm{et}) \xrightarrow{\mathrm{Fun}(\mathcal{G}^{op}, |-|)} \mathrm{Fun}(\mathcal{G}^{op}, \mathcal{S})$$

of the levelwise restricted Yoneda embedding followed by a pointwise application of the geometric realization functor

$$|-| : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}[\![\mathbf{W}^{-1}_{\mathrm{KQ}}]\!] \simeq \mathcal{S}.$$

In fact, suppose we are given a cocomplete 1-category $\mathcal{C}$ admitting a set of projective generators; assume without loss of generality that this set is closed under finite coproducts, and denote the full subcategory it determines by $\mathcal{G} \subset \mathcal{C}$. Then, in [Qui67, §II.4], Quillen defined a model structure on $s\mathcal{C}$ which (in hindsight) is precisely a presentation of the ∞-category $\mathcal{P}_\Sigma(\mathcal{G})$. If we take $\mathcal{C} = \mathcal{S}\mathrm{et}$ and $\mathcal{G} = \mathcal{F}\mathrm{in}$, this recovers the standard Kan–Quillen model structure $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ (which, finally, explains the example given in §0.1.3). In general, cofibrant replacements in these model structures may thus be thought of as ***nonabelian projective resolutions***.

[57] The secondary notation $\mathcal{P}_\Sigma(\mathcal{G})$ is commonly used in the literature, and stems from the notation $\mathcal{P}(\mathcal{G}) = \mathrm{Fun}(\mathcal{G}^{op}, \mathcal{S})$ for the ∞-category of *all* presheaves of spaces on $\mathcal{G}$ (the inclusion of $\mathcal{P}_\Sigma(\mathcal{G})$ into which admits a left adjoint). We will generally prefer the notation $\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G})$, which emphasizes the analogy with the classical setting as well as the fact that this ∞-category should be thought of as being that of "$\mathcal{G}$-projective resolutions of objects of $\mathcal{C}$", even though the larger ∞-category $\mathcal{C}$ actually plays no direct role in the definition. If the subcategory $\mathcal{G} \subset \mathcal{C}$ contains a set of generators, then the corresponding restricted Yoneda functor $\mathcal{C} \to \mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G})$ is (by definition) fully faithful.

In fact, this same idea has been carried further in homotopy theory. In [DKS93], Dwyer–Kan–Stover defined a *resolution model structure* on the category $s\mathcal{T}\mathrm{op}_*$ of simplicial pointed topological spaces based on the set of generators

$$\{S^n \in \mathcal{T}\mathrm{op}_*\}_{n\geq 1},$$

and in [Bou03] Bousfield generalized this to a general (pointed, right proper) model category equipped with a set of h-cogroup objects satisfying certain conditions. In both cases, the restriction to h-cogroup objects is motivated by the desire for spectral sequences converging to the "homotopy groups" (with respect to the generators and their finite coproducts) of the geometric realization of an object (in the model-categorical sense). The levelwise weak equivalences are weak equivalences in these model structures, but there are strictly more of the latter.

From the perspective of model ∞-categories, it is clear that these model 1-categories are fairly inefficient: it is wholly unnecessary to distinguish between objects which are levelwise weakly equivalent. On the other hand, the resolutions that these model structures afford *are* necessary – indeed, they are the entire point. Thus, one might expect to freely invert the levelwise weak equivalences while keeping track of the remaining resolution weak equivalences. To this end, we have the following theorem.

Theorem 0.3.1 (7.1.19 and 7.1.22)**.** *Let $\mathcal{C}$ be a presentable ∞-category, let $\{Z_\alpha \in \mathcal{C}\}$ be a set of compact objects, and write $\mathcal{G} \subset \mathcal{C}$ for the full subcategory generated by the objects Z_α and their finite coproducts. Then there exists a* ***resolution model structure*** *on the ∞-category $s\mathcal{C}$, denoted $s\mathcal{C}_{\mathrm{res}}$. This model structure is simplicial (i.e., it is compatibly enriched over $s\mathcal{S}_{\mathrm{KQ}}$). Moreover, it participates in a Quillen adjunction*

$$\mathrm{Fun}(\mathcal{G}, s\mathcal{S}_{\mathrm{KQ}})_{\mathrm{proj}} \rightleftarrows s\mathcal{C}_{\mathrm{res}},$$

whose derived adjunction (as guaranteed by Theorem 0.2.3) is precisely the canonical adjunction

$$\mathcal{P}(\mathcal{G}) \rightleftarrows \mathcal{P}_\Sigma(\mathcal{G}).$$

This is indeed much more efficient than its 1-categorical analogs. For example, every object in $s\mathcal{C}_{\mathrm{res}}$ is fibrant; by contrast, in the resolution model structures of Dwyer–Kan–Stover and Bousfield, the fibrant objects are precisely the Reedy fibrant objects. (This is by no means a decisive advantage, but it seems worth pointing out nonetheless.)

Let us now turn to the inner workings of Goerss–Hopkins obstruction theory; of course, what we have just discussed will become relevant shortly. Given everything that we have explained so far, we can in fact proceed to directly explain our generalization, which is one of the central goals of this thesis.

Thus, let us begin with a presentably symmetric monoidal stable ∞-category $\mathcal{C}$. This replaces a model 1-category of spectra, which in the original construction must be assumed to satisfy a long list of technical assumptions. We assume that $\mathcal{C}$ is equipped with a full subcategory $\mathcal{G} \subset \mathcal{C}$ of generators, which we assume to be sufficiently nice (e.g. its objects must all have inverses with respect to the symmetric monoidal structure – thereafter, our assumptions will imply that its objects are compact). This generalizes the set of sphere spectra. These generators define a "homotopy groups" functor π_{*}.

We now discuss our detecting homology theory, which we assume to be given by a flat homotopy commutative algebra $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{C}))$. We can now explain the all-important *Adams's condition* (which was mentioned in §0.3.3). This is the requirement that E be obtainable as a filtered colimit

$$\mathrm{colim}_{\mathcal{J}} E_\alpha \xrightarrow{\sim} E$$

of *dualizable* objects E_α, such that their duals $\mathrm{D}E_\alpha$ have projective E-homology. This condition allows us to treat E-homology as being given by "homotopy groups with respect to these duals". More precisely, our assumptions guarantee that for any generator $S^\beta \in \mathcal{G}$ we have a string of isomorphisms

$$\begin{aligned}\mathrm{colim}_{\alpha\in\mathcal{J}}[\Sigma^\beta \mathrm{D}E_\alpha, X]_{\mathcal{C}} &\cong \mathrm{colim}_{\alpha\in\mathcal{J}}[S^\beta, E_\alpha \otimes X]_{\mathcal{C}} \cong [S^\beta, \mathrm{colim}_{\alpha\in\mathcal{J}}(E_\alpha \otimes X)]_{\mathcal{C}} \\ &\cong [S^\beta, \mathrm{colim}_{\alpha\in\mathcal{J}}(E_\alpha) \otimes X]_{\mathcal{C}} \cong [S^\beta, E \otimes X]_{\mathcal{C}} = E_\beta X\end{aligned}$$

(where we suggestively write Σ^β for the functor $S^\beta \otimes -$). Therefore, if a map $X \to Y$ induces "$\mathrm{D}E_\alpha$-homotopy" isomorphisms

$$[\Sigma^\beta \mathrm{D}E_\alpha, X]_{\mathcal{C}} \xrightarrow{\cong} [\Sigma^\beta \mathrm{D}E_\alpha, Y]_{\mathcal{C}}$$

for all $S^\beta \in \mathcal{G}$ and all $\alpha \in \mathcal{J}$, then it induces an isomorphism on E-homology. On the other hand, the converse will not generally hold. This subtlety can be handled with a little bit of care (or with a lot of care, in the original model 1-categorical case), and we will return to it in due time.

Let us write $\mathcal{G}^E_{\mathcal{C}} \subset \mathcal{C}$ for the full subcategory generated by the subcategory $\mathcal{G}$ and the objects $\mathrm{D}E_\alpha$ under finite coproducts. Then, our resolutions will be based on the nonnegatively-graded nonabelian derived ∞-category

$$\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}^E_{\mathcal{C}}).$$

However, we will need to make computations using actual simplicial resolutions (i.e. objects of $s\mathcal{C}$) instead of their images under the functor

$$s\mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{G}^E_{\mathcal{C}}) = \mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}^E_{\mathcal{C}}),$$

and for this we will use the resolution model structure provided by Theorem 0.3.1.

As we will explain shortly, we will not actually be using this model ∞-category directly, but rather a generalization of it. However, even in this special case we can point out an essential feature of the story. Let us write $\tilde{\mathcal{A}}$ for the category of (E_*, E_*E)-comodules, and let us write $\mathcal{G}_{\tilde{A}} \subset \tilde{\mathcal{A}}$ for the full subcategory on objects of the form E_*S^ε for some $S^\varepsilon \in \mathcal{G}_{\mathcal{C}}^E$; by our assumptions, these will be projective as E_*-modules. As we have assumed that $\mathcal{C}$ is presentably symmetric monoidal, it follows that the induced functor

$$E_* : \mathcal{G}_{\mathcal{C}}^E \to \mathcal{G}_{\tilde{A}}$$

preserves finite coproducts. It follows formally that the induced functor

$$\mathcal{P}_\Sigma(E_*) : \mathcal{P}_\Sigma(\mathcal{G}_{\mathcal{C}}^E) \to \mathcal{P}_\Sigma(\mathcal{G}_{\tilde{A}})$$

preserves all colimits. Ultimately, this fact will be (a shadow of) the reason that our topological obstructions can be computed purely algebraically. At the level of model ∞-categories, this can be seen as resulting from the fact that the functor

$$E_*^{\mathrm{lw}} : s\mathcal{C}_{\mathrm{res}} \to s\tilde{\mathcal{A}}_{\mathrm{res}}$$

preserves cofibrations between cofibrant objects relative to an analogous resolution model structure on $s\tilde{\mathcal{A}}_{\mathrm{res}}$.

We now proceed to discuss multiplicative structures. We henceforth use the terminology "operad" to refer to a (single-colored) ∞-operad; the ∞-category Op of operads is presented by the relative category $\mathrm{Op}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$ (as well as by the relative category $\mathrm{Op}(\mathcal{T}\mathrm{op}_{\mathrm{QS}})$), whose weak equivalences are determined levelwise on underlying objects (i.e. ignoring the symmetric group actions). This relative category structure enhances to a *Boardman–Vogt* model structure, which (using a generalization of Theorem 0.3.1) we incidentally generalize to the ∞-category $\mathrm{Op}(s\mathcal{V})$ of internal operads (for a suitable symmetric monoidal ∞-category $\mathcal{V}$) as Proposition 7.3.23.

Now, our obstruction theory can be used to construct (E-local) $\mathcal{O}$-algebras in $\mathcal{C}$, for any operad $\mathcal{O} \in \mathrm{Op}$. Given a choice of $\mathcal{O}$, however, we must choose a monad Φ on $\tilde{\mathcal{A}}$ which will parametrize our "algebraic structures": in other words, we must have a lift

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) & \overset{E_*}{\dashrightarrow} & \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \\ {\scriptstyle \mathrm{U}_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\Phi}} \\ \mathcal{C} & \xrightarrow[E_*]{} & \tilde{\mathcal{A}} \end{array}$$

of our E-homology functor. For instance, in the special case where $\mathcal{O} = \mathrm{Comm} = \mathbb{E}_\infty$ that we described in §0.3.3, we also took $\Phi = \mathrm{Comm}$. However, even in the case that we take $\mathcal{O} = \mathrm{Comm}$, it can be useful – essential, even – to have this added generality.[58]

So of course, we will not be interested in resolving objects of $\mathcal{C}$, but rather objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. However, it will not suffice to simply resolve them by *simplicial* objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$: at no point will this allow us to gain control over their levelwise E-homology (in the model category $s\tilde{\mathcal{A}}_{\mathrm{res}}$).

On the other hand, there is a special case in which this does hold, namely when the operad $\mathcal{O}$ is π_0-$\mathfrak{S}$-*free*: by definition, this means that for every $n \geq 0$, the symmetric group $\mathfrak{S}_n$ acts freely on the set $\pi_0(\mathcal{O}(n))$ of path components of the n^{th} constituent space of $\mathcal{O}$. When this is the case, the "free $\mathcal{O}$-algebra" functor

$$X \mapsto \coprod_{n \geq 0} (\mathcal{O}(n) \odot X^{\otimes n})_{\mathfrak{S}_n}$$

simplifies dramatically. Even better, if we assume that $E_* X$ is projective – such as when $X = \mathrm{D}E_\alpha$ –, then the Künneth spectral sequence for the E-homology of this free $\mathcal{O}$-algebra (which is guaranteed by Adams's condition) immediately collapses!

Thus, a key insight of Goerss–Hopkins obstruction theory (over its predecessors) was, for a general operad $\mathcal{O}$, to take a simplicial resolution $T_\bullet \in s\mathrm{Op}$ by π_0-$\mathfrak{S}$-free operads. Amusingly, this can be achieved by choosing a cofibrant representative of $\mathcal{O}$ in the model category $\mathrm{Op}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{BV}}$ via the embedding

$$\mathrm{Op}(s\mathcal{S}\mathrm{et}) \simeq s\,(\mathrm{Op}(\mathcal{S}\mathrm{et})) \hookrightarrow s\mathrm{Op}.$$

A simplicial operad can be made to act on simplicial objects in $\mathcal{C}$, and from here we obtain (as Theorem 7.3.13) a lifted resolution model structure through the adjunction

$$\mathrm{F}_T : s\mathcal{C}_{\mathrm{res}} \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}} : \mathrm{U}_T$$

[58] The construction of *tmf* (via the sheaf $\mathcal{O}^{\mathrm{der}}$ on $\mathscr{M}_{\mathrm{ell}}$), which was spelled out in full detail by Behrens in [DFHH14], makes essential use of such generality. In order to construct the height-1 component of the sheaf (which is necessary in order to "interpolate" between the supersingular loci at distinct primes, and which is by far the most technical aspect of the construction), one must take the p-adic complex K-theory spectrum $KU_p^\wedge$ as the detecting homology theory, and one must enhance the nature of the algebraic input from a commutative algebra in comodules to what is called a *θ-algebra* (which structure is canonically present on the p-adic K-theory of an $\mathbb{E}_\infty$ ring spectrum).

This is the model ∞-category we have been seeking. On the one hand, its objects are resolutions of $\mathcal{O}$-algebras in $\mathcal{C}$: we have a canonical lift

$$\begin{array}{ccc} \mathrm{Alg}_T(s\mathcal{C}) & \overset{|-|}{\dashrightarrow} & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \\ {\scriptstyle \mathrm{U}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\mathcal{O}}} \\ s\mathcal{C} & \xrightarrow[|-|]{} & \mathcal{C} \end{array}$$

of the geometric realization functor. On the other hand, we will assume enough so that there is a monad $\tilde{T}_E$ on $s\tilde{\mathcal{A}}$ admitting a lift

$$\begin{array}{ccc} \mathrm{Alg}_T(s\mathcal{C}) & \overset{E^{\mathrm{lw}}_*}{\dashrightarrow} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \\ {\scriptstyle \mathrm{U}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\tilde{T}_E}} \\ s\mathcal{C} & \xrightarrow[E^{\mathrm{lw}}_*]{} & s\tilde{\mathcal{A}}. \end{array}$$

Just as our unstructured functor

$$E^{\mathrm{lw}}_* : s\mathcal{C}_{\mathrm{res}} \to s\tilde{\mathcal{A}}_{\mathrm{res}}$$

preserves cofibrations between cofibrant objects, so will this lifted functor E^{lw}_* (with respect to an analogously lifted resolution model structure $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\mathrm{res}}$), which crucially implies that its localization

$$E^{\mathrm{lw}}_* : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$$

preserves colimits. Although there will be one more small wrinkle that must be smoothed out, this fact is very nearly the true reason that our topological obstructions can be computed purely algebraically.

Given our algebraic object $A \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, we can now explain that our "n-stage approximations" to A will be objects of the ∞-category $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$, and our André–Quillen cohomology spaces will be certain mapping spaces extracted from the ∞-category $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$. However, these facts are technically true but slightly misleading.

To clarify both at once, let us recall for the sake of analogy that in the ∞-category $\mathcal{C}$, a map becoming an isomorphism under all of functors $[\Sigma^{\beta}\mathrm{D}E_{\alpha}, -]_{\mathcal{C}}$ implies that it also becomes an isomorphism under the functor E_*, but that the converse is generally false. Then, in the algebraic case, note that there exists a forgetful functor

$$\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_{\tilde{T}_E}} s\tilde{\mathcal{A}} \xrightarrow{s(\mathrm{U}_{\tilde{A}})} s\mathcal{S}\mathrm{et}_*,$$

which takes the subcategory $\mathbf{W}_{\mathrm{res}} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$ into the subcategory $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}_*$, but not only this subcategory; defining

$$\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$$

to be the pullback of $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}_*$, we obtain a reflective localization

$$\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \rightleftarrows \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!].$$

Similarly, in the topological case, the functor

$$E_*^{\mathrm{lw}} : \mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$$

takes the subcategory $\mathbf{W}_{\mathrm{res}} \subset \mathrm{Alg}_T(s\mathcal{C})$ into the subcategory $\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$, but not only this subcategory; defining

$$\mathbf{W}_{E_*^{\mathrm{lw}}} \subset \mathrm{Alg}_T(s\mathcal{C})$$

to be the pullback of $\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$, we obtain a reflective localization

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!].$$

Now, we can clarify that in that the moduli spaces of n-stages for A are naturally subgroupoids

$$\mathscr{M}_n(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$$

of the reflective localization, while the relevant André–Quillen cohomology spaces are computed by mapping in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ to an object of the reflective subcategory $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$. Moreover, these two reflective localization functors participate as the downwards arrows in a commutative square

$$\begin{array}{ccc} \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] & \xrightarrow{E_*^{\mathrm{lw}}} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \\ \downarrow & & \downarrow \\ \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] & \xrightarrow[E_*^{\mathrm{lw}}]{} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!], \end{array}$$

in which the dotted arrow exists by the universal property of localization and preserves colimits by an easy diagram chase. This, finally, is the *true* reason that our topological obstructions can be computed purely algebraically. However, in order to explain this, we must introduce the *spiral exact sequence*.

Given a simplicial object $X \in s\mathcal{C}$, there are two sorts of E-homology groups that one might extract: the *classical* E-homology groups

$$\pi_n E^{\mathrm{lw}}_\beta X = \pi_n [S^\beta, E \otimes X]^{\mathrm{lw}}_{\mathcal{C}}$$

and the *natural* E-homology groups

$$E^{\natural}_{n,\beta} X = \pi_n \left(\hom_{\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}^E_{\mathcal{C}})}(S^\beta, (E \otimes X)^{\mathrm{lw}}) \right).$$

These serve dual purposes.

On the one hand, the classical E-homology groups assemble into the E^2 page of a spectral sequence

$$\mathrm{E}^2 = \pi_n E^{\mathrm{lw}}_\beta X \Rightarrow \mathrm{E}^\infty = E_{\beta+n}|X|,$$

where we write $S^{\beta+n} = S^\beta \otimes S^n = \Sigma^n S^\beta$. Of course, this spectral sequence allows us to obtain control over the E-homology of the geometric realization $|X|$.

On the other hand, the natural E-homology groups are by their very definition much more directly related to the ∞-category

$$\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}^E_{\mathcal{C}}) \simeq s\mathcal{C}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!].$$

Thus, they participate in a "cells and disks" obstruction theory within this ∞-category. In order to explain this, we introduce the notation

$$D^n_{\mathbf{\Delta}} = \Delta^n / \Lambda^n_0 \in (s\mathcal{S}\mathrm{et}_*)_{\mathrm{KQ}}$$

and

$$S^n_{\mathbf{\Delta}} = \Delta^n / \partial\Delta^n \in (s\mathcal{S}\mathrm{et}_*)_{\mathrm{KQ}}.$$

There are evident cofibrations

$$S^n_{\mathbf{\Delta}} \rightarrowtail D^{n+1}_{\mathbf{\Delta}}$$

in $(s\mathcal{S}\mathrm{et}_*)_{\mathrm{KQ}}$, which present the maps

$$S^n \to D^{n+1} \simeq \mathrm{pt}$$

in $\mathcal{S}_*$. Moreover, for any $K \in s\mathcal{S}_*$ and any $X \in s\mathcal{C}$, there exists a "based tensor" object $K \overline{\odot} X \in s\mathcal{C}$, which is compatible with the canonical enrichment of $s\mathcal{C}$ over $s\mathcal{S}_*$ (where the basepoint is given by the zero morphism). Writing $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$ for an arbitrary object, the fact that the model ∞-category $s\mathcal{C}_{\mathrm{res}}$ is *simplicial* implies that the "cells" given by

$$S^n_{\mathbf{\Delta}} \overline{\odot} \mathrm{const}(S^\varepsilon) \in s\mathcal{C}_{\mathrm{res}}$$

and the "disks" given by

$$D^n_{\mathbf{\Delta}} \overline{\odot} \mathrm{const}(S^\varepsilon) \in s\mathcal{C}_{\mathrm{res}}$$

together control the theory of *Postnikov towers* in $s\mathcal{C}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$.[59]

Now, the ("***localized***") ***spiral exact sequence*** relates these two types of E-homology, running

$$\cdots \longrightarrow \pi_{i+1}E_\beta X \xrightarrow{\delta} E^\natural_{i-1,\beta+1}X \longrightarrow E^\natural_{i,\beta}X \longrightarrow \pi_i E_\beta X \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} E^\natural_{0,\beta+1}X \longrightarrow E^\natural_{1,\beta}X \longrightarrow \pi_1 E_\beta X \longrightarrow 0.$$

Note that it is two-thirds natural E-homology, and one-third classical E-homology.[60] Thus, via the spiral exact sequence, *by controlling the natural E-homology groups* (via "cells and disks") *we can also control the classical E-homology groups* (which assemble into the E^2 page of the spectral sequence).

We can now explain the connection with "n-stages" for our chosen object $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$ of which we are interested in realizations. First of all, an *∞-stage* for A is an object of $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ whose E^2 page is simply given by A, concentrated in the bottom row; these assemble into a moduli space

$$\mathscr{M}_\infty(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!].$$

We then have the following result, which cements the relationship between realizations of A and their (approximate) resolutions.

Theorem 0.3.2 (7.7.5)**.** *The geometric realization functor*

$$|-| : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))$$

induces an equivalence

$$\mathscr{M}_\infty(A) \xrightarrow{\sim} \mathscr{M}_A.$$

We emphasize that the moduli space $\mathscr{M}_\infty(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ will *not* generally contain all of the objects whose geometric realizations are realizations of A: rather,

[59] Examining the structure maps of the simplicial sets $D^n_{\mathbf{\Delta}}$ and $S^n_{\mathbf{\Delta}}$, one sees that they may be seen as corepresenting the *nonabelian n-cycles* and *nonabelian normalized n-chains* objects of an object $X \in s\mathcal{C}$ (via a "based cotensor" bifunctor $-\overline{\pitchfork}- : s\mathcal{S}_* \times s\mathcal{C} \to \mathcal{C}$ which we will not make precise here).

[60] In fact, these long exact sequences are what organize into the exact couple defining the above spectral sequence.

it only contains those whose geometric realizations are realizations of A "for obvious reasons" (namely that their spectral sequences collapse immediately).

Let us now move to the bottom of the tower. A 0-stage for A is an object $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]$ whose natural E-homology is given by

$$E_{i,*}X \cong \begin{cases} A, & i = 0 \\ 0, & i > 0. \end{cases}$$

As the natural E-homology groups govern cellular approximations in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]$, the following result should be plausible.

Theorem 0.3.3 (7.7.8)**.** *The moduli space of 0-stages for A admits a canonical equivalence*

$$\mathscr{M}_0(A) \simeq B\mathrm{Aut}(A).$$

Now, if $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]$ is a 0-stage for A, then its natural E-homology is extremely simple. On the other hand, as dictated by the spiral exact sequence, its classical E-homology – and hence its E^2 page – is not quite correct for it to be an ∞-stage: instead, we will have

$$\pi_i E^{\mathrm{lw}}_* X \cong \begin{cases} A, & i = 0 \\ \Omega A, & i = 2 \\ 0, & i \notin \{0, 2\}. \end{cases}$$

In fact, more generally, if X is an n-stage for A, then we will have

$$\pi_i E^{\mathrm{lw}}_* X \cong \begin{cases} A, & i = 0 \\ \Omega^{n+1} A, & i = n+2 \\ 0, & i \notin \{0, n+2\}. \end{cases}$$

Thus, to move upwards through the tower of moduli spaces is to push the failure of X to be an ∞-stage "further and further away".[61] However, we emphasize that the above identification of natural homotopy groups does not alone imply that X is an n-stage: it must also have the correct k-invariants (or equivalently, it must also have the correct natural E-homology).

We now explain why this iterative *topological* procedure is indeed governed by *algebraic* computations. (In fact, a somewhat simpler argument will also justify Theorem 0.3.3.) This is where we will use the cocontinuity of the functor

$$E^{\mathrm{lw}}_* : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!] \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$$

[61] In fact, the spectral sequence for an n-stage will collapse after the E^{n+2} page, directly after cancelling out the entire $(n+2)^{\mathrm{nd}}$ row with the corresponding entries of the 0^{th} row.

between presentable ∞-categories.[62]

Suppose that $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]$ is an $(n-1)$-stage for A. As we have just seen, its image

$$Y = E^{\mathrm{lw}}_* X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$$

will have its homotopy concentrated in degrees 0 and $n+1$: for brevity, we simply write

$$\pi_* Y \cong A \times (\Omega^n A)[n+1].$$

We are interested in modifying X to obtain an n-stage for A: this entails simultaneously peeling off this copy of $(\Omega^n A)[n+1]$ and replacing it with a copy of $(\Omega^{n+1} A)[n+2]$, all in a way that behaves correctly with respect to the natural E-homology groups.

In order to address this question, we first examine the levelwise E-homology object $Y = E^{\mathrm{lw}}_* X$. Now, in the ∞-category $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, homotopy groups alone do not characterize equivalence classes: just as with (based) spaces, one must also keep track of the k-invariants. In this case, since Y only has potentially nonvanishing homotopy in dimensions 0 and $(n+1)$, it participates in a uniquely determined pullback square

$$\begin{array}{ccc} Y & \longrightarrow & K_A \\ \downarrow & & \downarrow \\ A & \xrightarrow[\chi_n(Y)]{} & K_A(\Omega^n A, n+2) \end{array}$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, in which the objects on the right are ***algebraic Eilenberg–Mac Lane objects*** with $\pi_* K_A \cong A$ and $\pi_* K_A(\Omega^n A, n+2) \cong A \times (\Omega^n A)[n+2]$, the right vertical map between them is an isomorphism on π_0, and the map $\chi_n(Y)$ is the unique potentially nontrivial k-invariant of Y. This defines a class

$$[\chi_n(Y)] \in H^{n+2}(A; \Omega^n A)$$

in the indicated André–Quillen cohomology group, and taken over all $(n-1)$-stages $X \in \mathscr{M}_{n-1}(A)$ this defines a map

$$\mathscr{M}_{n-1}(A) \xrightarrow{\chi_n} \mathscr{H}^{n+2}(A; \Omega^n A)$$

to the indicated André–Quillen cohomology space.

[62] The adjoint functor theorem implies that this functor admits a right adjoint. However, it appears extremely unlikely that this lifts to the level of model ∞-categories. And even if it does, the functor E^{lw}_* will not generally be a left Quillen functor, since it generally only preserves weak equivalences between cofibrant objects (instead of all acyclic cofibrations between arbitrary objects).

Returning to topology, we now come to the crucial point: for any object $Z \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, the composite functor

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]^{op} \xrightarrow{E^{\mathrm{lw}}_*} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{op} \xrightarrow{\hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]}(-,Z)} \mathcal{S}$$

preserves limits (i.e. takes colimits in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]$ to limits in $\mathcal{S}$) and so must be representable (by presentability). When $Z = K_A$ or $Z = K_A(\Omega^n A, n+2)$, we obtain ***topological Eilenberg–Mac Lane objects***, which we respectively denote by B_A and $B_A(\Omega^n A, n+2)$.

Now, if there exists an n-stage $\widetilde{X}$ lifting X, then Postnikov theory in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]$ implies that it must fit into a pullback square

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & B_A \\ \downarrow & & \downarrow \\ X & \longrightarrow & B_A(\Omega^n A, n+2), \end{array}$$

in which the right vertical map classifies the standard map $K_A \to K_A(\Omega^n A, n+2)$. Conversely, if we define $\widetilde{X}$ to be such a pullback, then it will be an n-stage if and only if the lower map corresponds to an equivalence

$$E^{\mathrm{lw}}_* X = Y \xrightarrow{\sim} K_A(\Omega^n A, n+2).$$

As we have just seen, the equivalence class of Y is entirely classified by a k-invariant

$$[\chi_n(Y)] \in H^{n+2}(A; \Omega^n A),$$

and it is not hard to show that such an equivalence $Y \xrightarrow{\sim} K_A(\Omega^n A, n+2)$ exists if and only this k-invariant vanishes.

All in all, an expansion of this argument can be used to prove the following.

Theorem 0.3.4 (7.7.9)**.** *For any $n \geq 1$, there is a natural pullback square*

$$\begin{array}{ccc} \mathscr{M}_n(A) & \longrightarrow & B\mathrm{Aut}(A, \Omega^n A) \\ \downarrow & & \downarrow \\ \mathscr{M}_{n-1}(A) & \longrightarrow & \widehat{\mathscr{H}}^{n+2}(A; \Omega^n A). \end{array}$$

This is the final ingredient in our generalized Goerss–Hopkins obstruction theory, which allows us to compute the purely algebraic obstructions to the inductive passage up the tower of moduli spaces

$$\begin{array}{ccc}
\mathscr{M}_A & \xleftarrow{\sim} & \mathscr{M}_\infty(A) \\
 & & \downarrow{\scriptstyle \lim} \\
 & & \vdots \\
 & & \downarrow \\
 & & \mathscr{M}_n(A) \longrightarrow B\mathrm{Aut}(A, \Omega^n A) \\
 & & \downarrow \qquad\qquad \downarrow \\
 & & \mathscr{M}_{n-1}(A) \longrightarrow \widehat{\mathscr{H}}^{n+2}(A; \Omega^n A) \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
B\mathrm{Aut}(A) & \simeq & \mathscr{M}_0(A).
\end{array}$$

0.3.5 Motivic Morava E-theories

The field of ***motivic homotopy theory*** was originally introduced by Voevodsky in order to resolve the Milnor and Bloch–Kato conjectures (see [Voe03] and [Voe11]). In short, the ∞-category $\mathcal{S}^{\mathrm{mot}}$ of *motivic spaces* is the free (∞-categorical) cocompletion of the category of smooth schemes over a fixed base, subject to *Nisnevich descent* and to the $\mathbb{A}^1$-*homotopy invariance* condition that all projections $X \times \mathbb{A}^1 \to X$ become equivalences.[63]

From here, the ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{mot}}$ of *motivic spectra* is obtained by formally inverting the endofunctor

$$\mathbb{P}^1 \wedge - : \mathcal{S}_*^{\mathrm{mot}} \to \mathcal{S}_*^{\mathrm{mot}}$$

on pointed motivic spaces. The pushout square

$$\begin{array}{ccc}
\mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \longrightarrow & \mathbb{P}^1
\end{array}$$

[63] For an introduction to motivic homotopy theory, we refer the reader to [MV99].

in ordinary schemes remains a pushout square in motivic spaces (because the open inclusions $\mathbb{G}_m \to \mathbb{A}^1$ are cofibrations in the appropriate model structure), whereafter since $\mathbb{A}^1$ becomes contractible by fiat we obtain an equivalence

$$\Sigma\mathbb{G}_m \simeq \mathbb{P}^1.$$

Thus, to invert the smash product with $\mathbb{P}^1$ is to invert both the smash product with S^1 (considered as a constant sheaf) – i.e. to stabilize in the sense described in §0.3.1 – and the smash product with $\mathbb{G}_m$ (also considered as a sheaf). Thus, among the motivic spectra which are invertible for the smash product are not just the "topological" motivic sphere spectra $\Sigma^n\Sigma^\infty S^0$ for $n \in \mathbb{Z}$ (where S^0 now denotes two disjoint copies of the base scheme), but more generally the *bigraded motivic sphere spectra*

$$S^{i,j} = \Sigma^{i-j}\Sigma^\infty(\mathbb{G}_m)^{\wedge j}$$

for all $i, j \in \mathbb{Z}$.

In contrast with the ∞-category $\mathcal{S}\mathrm{p}$ of ordinary spectra, in the ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{mot}}$ of motivic spectra, one cannot generally detect the equivalences via bigraded homotopy groups (i.e. homotopy classes of maps out of bigraded spheres). However, there exists a full subcategory $\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}} \subset \mathcal{S}\mathrm{p}^{\mathrm{mot}}$ of *cellular* motivic spectra, originally introduced by Dugger–Isaksen in [DI05], which is by construction the largest subcategory on which bigraded homotopy groups *do* detect equivalences.

As it turns out, the chromatic approach to studying the ∞-category of spectra described in §0.3.2 lifts to the ∞-category of motivic spectra. Though the global picture is much more mysterious and much less well understood, certain broad features are known to persist. For example, the role of the complex cobordism spectrum MU (which is implicit in the discussion of §0.3.2) is played by the *algebraic cobordism* motivic spectrum MGL, while the role of the complex K-theory spectrum KU is played by the ($\mathbb{A}^1$-*homotopy invariant*) *algebraic* K*-theory* spectrum KGL.

More generally, the classical *Landweber exact functor theorem* of [Lan76], a key tool for constructing complex oriented cohomology theories, has been generalized by Naumann–Spitzweck–Østvær in [NSØ09b] to the motivic setting. Whereas the classical theorem builds these cohomology theories out of MU, the motivic version builds these motivic cohomology theories out of MGL. As the motivic spectrum MGL is cellular, by construction the motivic cohomology theories that this theorem produces are cellular as well.

In particular, the motivic Landweber exact functor theorem produces ***motivic Morava*** E***-theory*** spectra, denoted $E^{\mathrm{mot}}_{n,p}$, for all $n \geq 0$ and at all primes p. In analogy with the classical case, at chromatic height 1 these can be identified as

$$E^{\mathrm{mot}}_{1,p} \simeq KGL^\wedge_p,$$

the p-completion of algebraic K-theory. Thus, the motivic Morava E-theories are *higher chromatic analogs* of algebraic K-theory.

Now, the ordinary Landweber exact functor theorem only produces plain spectra, rather than spectra with additional structure such as that of an $\mathbb{E}_\infty$ ring spectrum. Thus it was necessary for Goerss–Hopkins to use their obstruction theory in order to endow the Morava E-theory spectrum $E_{n,p}$ with an $\mathbb{E}_\infty$ ring structure. As a bonus, this technique showed that this $\mathbb{E}_\infty$ structure is actually *unique* (subject to the compatibility requirement that it recover the natural commutative ring structure on homotopy groups), and that the automorphism group of the resulting $\mathbb{E}_\infty$ ring spectrum is *discrete*.

Analogously, the motivic Landweber exact functor theorem only produces plain motivic spectra. Thus, we use our generalized Goerss–Hopkins obstruction theory to prove the following.result.

Theorem 0.3.5 (8.1.1)**.** *The motivic Morava E-theory spectrum $E_{n,p}^{\mathrm{mot}}$ admits a unique $\mathbb{E}_\infty$ structure that is compatible with the canonical commutative ring structure on its bigraded homotopy groups. Moreover, the automorphism group of the resulting motivic $\mathbb{E}_\infty$ ring spectrum is discrete.*

As in the case of ordinary spectra, an $\mathbb{E}_\infty$ structure endows the corresponding motivic co/homology theory with a vast amount of additional algebraic structure.

0.4 Conventions on ∞-categories and model-independence

We take quasicategories as our preferred model for ∞-categories, and in general we adhere to the notation and terminology of [Lur09b] and [Lur14]. In fact, our references to these two works will be frequent enough that it will be convenient for us to use the "code names" T and A for them, respectively. Thus, for instance, to refer to [Lur09b, Theorem 4.1.3.1], we will simply write Theorem T.4.1.3.1. (Due to the differences in their numbering systems, no ambiguity will arise between references to [Lur14] and to §A here.)

However, we work invariantly to the greatest possible extent: that is, we primarily work *within the ∞-category of ∞-categories*. Thus, for instance, we omit all technical uses of the word "essential", e.g. we use the term *unique* in situations where one might otherwise say "essentially unique" (i.e. parametrized by a contractible space). For a full treatment of this philosophy as well as a complete elaboration of our conventions, we refer the interested reader to §A. The casual reader should feel free to skip this on

a first reading; on the other hand, the careful reader may find it useful to peruse that section before reading the main body of this thesis. For the reader's convenience, we also provide a complete index of the notation that is used throughout this thesis in §B.

0.5 Acknowledgments

Each of the chapters comprising this thesis was a project in and of itself, and correspondingly they each contain their own acknowledgments. However, there are a number of people who have influenced me deeply that I would like to thank here.

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Lastly, to Eric: thanks for dragging me up out of the mud.

Chapter 1

Model ∞-categories I: some pleasant properties of the ∞-category of simplicial spaces

Both simplicial sets and simplicial spaces are used pervasively in homotopy theory as presentations of spaces, where in both cases we extract the "underlying space" by taking geometric realization. We have a good handle on the category of simplicial sets in this capacity; this is due to the existence of a suitable model structure thereon, which is particularly convenient to work with since it enjoys the technical properties of being *proper* and of being *cofibrantly generated*. This chapter is devoted to showing that, if one is willing to work ∞-categorically, then one can manipulate simplicial spaces exactly as one manipulates simplicial sets. Precisely, this takes the form of a proper, cofibrantly generated model structure on the ∞-*category* of simplicial spaces, the definition of which we also introduce here.

1.0 Introduction

1.0.1 Simplicial spaces and model ∞-categories

A simplicial space can be thought of as a *resolution* of a space, namely its homotopy colimit; in nice cases this is computed by its geometric realization, and we will henceforth blur the distinction. The operation of taking a homotopy colimit, being homotopy invariant, descends to a functor $|-| : s\mathcal{S} \to \mathcal{S}$ of ∞-*categories*, from that of simplicial spaces to that of spaces. The primary purpose of the present chapter is to introduce a new framework for studying this functor. In particular, we give

∞-*categorical* criteria for determining

(a) when a map of simplicial spaces becomes an equivalence upon geometric realization, and

(b) when a homotopy pullback of simplicial spaces remains a homotopy pullback upon geometric realization,

which we have found to be much easier to verify than their existing 1-categorical counterparts. We hope that this encourages homotopy theorists grappling with simplicial spaces to work ∞-categorically: even if a map of simplicial spaces or a homotopy pullback of simplicial spaces began its life 1-categorically, these questions are homotopy invariant and hence inherently ∞-categorical, and thus should be approachable using the framework given here.

To set the stage, let us recall Quillen's theory of ***model categories***: given a category $\mathcal{C}$ equipped with a subcategory $\mathbf{W} \subset \mathcal{C}$, a *model structure* on the category $\mathcal{C}$ consists of additional data which provide an efficient and computable method of accessing its localization $\mathcal{C}[\mathbf{W}^{-1}]$. As a prime example, the ***Kan–Quillen model structure*** on the category $s\mathcal{S}$et of simplicial sets provides a combinatorial framework for studying the homotopy category $s\mathcal{S}\mathrm{et}[\mathbf{W}^{-1}] \simeq \mathcal{T}\mathrm{op}[\mathbf{W}^{-1}]$ of topological spaces.

Now, suppose that $\mathcal{C}$ is not merely a category but an ∞-*category*, again equipped with a subcategory $\mathbf{W} \subset \mathcal{C}$. We can analogously localize the ∞-category $\mathcal{C}$ at the subcategory $\mathbf{W}$, forming a new ∞-category $\mathcal{C}[\![\mathbf{W}^{-1}]\!]$ equipped with a functor $\mathcal{C} \to \mathcal{C}[\![\mathbf{W}^{-1}]\!]$, which is initial among those functors from $\mathcal{C}$ which invert all the maps in $\mathbf{W}$.[1] A key example for us is $s\mathcal{S}[\![\mathbf{W}^{-1}_{|-|}]\!] \simeq \mathcal{S}$, where we denote by $\mathbf{W}_{|-|} \subset s\mathcal{S}$ the subcategory spanned by those maps which become invertible upon geometric realization.[2]

With this background in place, we can now explain the two goals of this chapter.

(1) Inspired by Quillen, we introduce the notion of a ***model ∞-category***: given an ∞-category $\mathcal{C}$ equipped with a subcategory $\mathbf{W} \subset \mathcal{C}$, a *model structure* on the ∞-category $\mathcal{C}$ consists of additional data which provide an efficient and computable method of accessing its localization $\mathcal{C}[\![\mathbf{W}^{-1}]\!]$. This provides a theory of *resolutions* which is native to the ∞-categorical context. As indicated by the title of this chapter, the theory of model ∞-categories is developed in subsequent chapters.

[1] Note that even if $\mathcal{C}$ is just a 1-category, this will generally differ from the more crude and lossy 1-categorical localization $\mathcal{C}[\mathbf{W}^{-1}]$: the canonical map $\mathcal{C}[\![\mathbf{W}^{-1}]\!] \to \mathcal{C}[\mathbf{W}^{-1}]$ is precisely the projection to the homotopy category $\mathrm{ho}(\mathcal{C}[\![\mathbf{W}^{-1}]\!]) \simeq \mathcal{C}[\mathbf{W}^{-1}]$.

[2] For an explanation of this equivalence, see item (20) of §A.

(2) In precise analogy with the Kan–Quillen model structure on the category $s\mathcal{S}\mathrm{et}$ of simplicial sets, we endow the ∞-category $s\mathcal{S}$ of simplicial spaces with a ***Kan–Quillen model structure***, whose subcategory of weak equivalences is exactly $\mathbf{W}_{|-|} \subset s\mathcal{S}$, which allows us to access its ∞-categorical localization $s\mathcal{S}[\![\mathbf{W}_{|-|}^{-1}]\!] \simeq \mathcal{S}$. This model structure is likewise *proper*, and is *cofibrantly generated* by the sets $I_{\mathrm{KQ}}, J_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}$ of boundary inclusions $\partial\Delta^n \subset \Delta^n$ and of horn inclusions $\Lambda^n_i \subset \Delta^n$, considered as maps of simplicial spaces via the inclusion $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ of simplicial sets as the discrete simplicial spaces.

By way of illustrating some typical applications of the Kan–Quillen model structure, we now return to the criteria promised above.

(a) As the Kan–Quillen model structure on $s\mathcal{S}$ is cofibrantly generated, a map is an *acyclic fibration* if it has the right lifting property against the set I_{KQ} of generating cofibrations. In particular, $\mathrm{rlp}(I_{\mathrm{KQ}}) \subset \mathbf{W}_{|-|}$.

(b) As the Kan–Quillen model structure on $s\mathcal{S}$ is right proper, then a pullback in which at least one of the two maps is a fibration (i.e. has $\mathrm{rlp}(J_{\mathrm{KQ}})$) is a *homotopy pullback*: that is, it remains a pullback under geometric realization.

However, far from providing just a few assorted tricks, the Kan–Quillen model structure on $s\mathcal{S}$ gives a robust and extensive framework for understanding simplicial spaces vis-à-vis their geometric realizations, which is further augmented by the general theory of model ∞-categories. For instance, the theory of *homotopy co/limits* in model ∞-categories furnishes criteria for comparing the co/limit of an arbitrary diagram $\mathcal{I} \to s\mathcal{S}$ of simplicial spaces with the co/limit of the resulting diagram $\mathcal{I} \to s\mathcal{S} \to \mathcal{S}$ obtained by componentwise geometric realization.

1.0.2 Relation to existing literature

Simplicial spaces and their geometric realizations play a key role in a number of different areas of topology. Their study appears to have been initiated in loopspace theory (see [Seg68], [Seg73], [Seg74], and [May72]). Relatedly, they find much use in algebraic K-theory (see e.g. [Wal85] or [Wei13]).[3] More recently, some of the results in the present chapter provide key input to applications in algebraic L-theory (see [Lur11b]) and in derived differential geometry (see [BEBdBP]).

[3]Many deep results in algebraic K-theory rely crucially on commuting certain pullbacks of simplicial spaces with geometric realization. This commutation is generally "not purely formal", but is instead based in delicate simplicial manipulations. We are hopeful that our Kan–Quillen model structure will prove useful in enabling (or at least streamlining) such arguments.

In addition to the references already mentioned, hints of the Kan–Quillen model structure abound in the literature. For instance, there is already explicit mention of the "two homotopy theories" on the category of simplicial topological spaces – the "topological" one corresponding to the interval object $\mathrm{const}([0,1])$, and the "algebraic" one corresponding to the interval object Δ^1 – as early as [May72]. Moreover, various scattered results bear anywhere between passing and striking resemblances to aspects of the Kan–Quillen model structure. To illustrate this, we compare

- our criterion (a) with analogous criteria arising
  - from the "Moerdijk model structure" (of [Moe89]) in Remark 1.7.3, and
  - from the Cegarra–Remedios "$\overline{W}$ model structure" (of [CR07]) in Remark 1.7.4;
- our criterion (b) with analogous criteria arising
  - from Bousfield–Friedlander's "π_*-Kan condition" (of [BF78]) and
  - from Anderson's notion of a simplicial groupoid being "fully fibrant" (of [And78])

  in Remark 1.6.13; and
- our notion of a fibration
  - with Rezk's "realization fibrations" – which sorts of maps have also at times been called "sharp maps", "fibrillations", "right proper maps", "h-fibrations", and "Grothendieck W-fibrations", and which are closely related to the classical notion of "quasifibrations" – (of [Rez14]) in Remark 1.6.12, and
  - with both of Seymour's and Brown–Szczarba's related but distinct "continuous" notions of Kan fibrations (of [Sey80] and [BS89] resp.) in Remark 1.6.15.

Remark 1.0.1. These sundry results make abundantly clear the vast superiority of the ∞-categorical perspective for the study of simplicial spaces and their geometric realizations. The definition of the Kan–Quillen model structure on the ∞-category of simplicial spaces is both straightforward and economical. By contrast, to describe it in purely model-categorical terms would require some sort of notion of a "double model structure", in which an "external" model structure would present the underlying ∞-category, and then the actual model structure of interest thereon would be presented by an "internal" model structure (whose distinguished classes of maps would then have to be invariant under the homotopy relation generated by the

"external" model structure; whose lifting axioms would need to be formulated in a homotopy-coherent way relative to the "external" model structure; etc.). It seems unlikely that the full extent of the Kan–Quillen model structure – or for that matter, anything comparable to the general theory of model ∞-categories – would ever have been discovered in a purely model-categorical context.

1.0.3 Goerss–Hopkins obstruction theory for ∞-categories

As the present chapter is the first in this thesis, we spend a moment describing the original motivation for the theory of model ∞-categories.

The overarching goal of this project is to generalize ***Goerss–Hopkins obstruction theory*** ([GH04, GHb]), a powerful tool for obtaining existence and uniqueness results for $\mathbb{E}_\infty$ ring spectra via purely algebraic computations, to the equivariant and motivic settings. However, the original obstruction theory is based in a model category of spectra satisfying a number of technical conditions, making it relatively difficult to generalize directly. Relatedly, the foundations for its construction rely on various point-set considerations, which appear for the sake of simplification but play no real mathematical role.[4] Thus, as the obstruction theory ultimately lives on the *underlying ∞-category* of spectra anyways, we instead aim to generalize Goerss–Hopkins obstruction theory to an arbitrary (presentably symmetric monoidal stable) ∞-category $\mathcal{C}$; this will yield equivariant and motivic obstruction theories simply by specializing to the ∞-categories of equivariant and motivic spectra.

Now, Goerss–Hopkins obstruction theory is constructed in the *resolution model structure* (a/k/a the "E^2 model structure") on simplicial spectra, originally introduced in [DKS93], which provides a theory of *nonabelian projective resolutions*.[5] Correspondingly, suppose we are given a presentable ∞-category $\mathcal{C}$, along with a set $\mathcal{G}$ of generators which we assume (without real loss of generality) to be closed under finite coproducts. Then, Goerss–Hopkins obstruction theory for $\mathcal{C}$ will take place in the ***nonabelian derived ∞-category*** of $\mathcal{C}$, i.e. the ∞-category $\mathcal{P}_\Sigma(\mathcal{G}) = \mathrm{Fun}^\times(\mathcal{G}^{op}, \mathcal{S})$ of those presheaves on $\mathcal{G}$ that take finite coproducts in $\mathcal{G}$ to finite products in $\mathcal{S}$, originally introduced in §T.5.5.8. (If $\mathcal{C}$ is the underlying ∞-category of an appropriately chosen model category, then $\mathcal{P}_\Sigma(\mathcal{G})$ will be the underlying ∞-category of the resolution model structure on the category of simplicial objects therein.) This admits a natural functor

$$s\mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{G}),$$

[4] In fact, to those unfamiliar with the more nuanced techniques of model categories, these considerations might even appear to amount to something like black magic.

[5] More precisely, it is actually constructed in a certain model category which is monadic over simplicial spectra, whose model structure is lifted along the defining adjunction.

given by taking a simplicial object $Y_\bullet \in s\mathcal{C}$ to the functor

$$S^\beta \mapsto |\hom^{\mathrm{lw}}_{\mathcal{C}}(S^\beta, Y_\bullet)|$$

(for any generator $S^\beta \in \mathcal{G}$, and writing "lw" to denote "levelwise"). In fact, this functor is a localization: denoting by $\mathbf{W}_{\mathrm{res}} \subset s\mathcal{C}$ the subcategory spanned by those maps which it inverts, it induces an equivalence

$$s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \xrightarrow{\sim} \mathcal{P}_\Sigma(\mathcal{G}).$$

From here, we can explain our motivation for the theory of model ∞-categories: the definition of the ∞-category $\mathcal{P}_\Sigma(\mathcal{G})$ is extremely efficient, but its abstract universal characterizations alone are insufficient for making the actual computations *within* this ∞-category that are necessary to set up the obstruction theory. Rather, computations therein generally rely on choosing *simplicial resolutions* of objects, i.e. preimages under the functor $s\mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{G})$, and then working in $s\mathcal{C}$ to deduce results back down in $\mathcal{P}_\Sigma(\mathcal{G})$.[6] Thus, in order to organize these computations, we will provide a ***resolution model structure*** on the ∞-category $s\mathcal{C}$, giving an efficient and computable method of accessing the localization $s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \simeq \mathcal{P}_\Sigma(\mathcal{G})$.

Remark 1.0.2. In a sense, the ∞-categorical Goerss–Hopkins obstruction theory is actually *easier* to set up than its classical counterpart. As described above, in the latter, one must carefully choose an appropriate "ground floor" model category of spectra on which to build the relevant resolution model structure (which depends nontrivially on the ground floor model structure). By contrast, in the former, the relevant resolution model structure on $s\mathcal{C}$ is built on the *trivial* model structure on the ∞-category $\mathcal{C}$, in which every map is both a cofibration and a fibration and the weak equivalences are precisely the equivalences (see Example 1.2.2).

We view this state of affairs as falling squarely in line with the core philosophy of ∞-categories, namely that they are meant to isolate and dispense with exactly those manipulations which ought to be purely formal. The usage of ∞-categories (and model structures thereon) allows us to sidestep the point-set technicalities which were ultimately of no homotopical interest in the first place, and hence to address only the truly interesting aspect of the story: nonabelian projective resolutions.

Remark 1.0.3. As a sample application of the ∞-categorical Goerss–Hopkins obstruction theory, we prove the following result in Chapter 7. As background, recall that the first application of Goerss–Hopkins obstruction theory was to prove that the Morava E-theory spectra admit essentially unique $\mathbb{E}_\infty$ structures, and moreover

[6] An important example is the "spiral exact sequence", which is a key ingredient to setting up the obstruction theory (see e.g. [GHb, Lemma 3.1.2(2)]).

that their spaces of $\mathbb{E}_\infty$ automorphisms are essentially discrete and are given by the corresponding Morava stabilizer groups (see [GH04, §7]). Bootstrapping up their arguments, we use Goerss–Hopkins obstruction theory in the ∞-category of motivic spectra to prove

- that the *motivic* Morava E-theory spectra again admit essentially unique $\mathbb{E}_\infty$ structures,
- that again their spaces of $\mathbb{E}_\infty$ automorphisms are essentially discrete, but
- that they can admit "exotic" $\mathbb{E}_\infty$ automorphisms not seen in ordinary topology.

(More precisely, their groups of $\mathbb{E}_\infty$ automorphisms will generally contain the corresponding Morava stabilizer groups as *proper* subgroups.)

1.0.4 Outline

We now provide a more detailed outline of the contents of this chapter.

- In §1.1, we define *model ∞-categories*, and we define the notions of *Quillen adjunctions* and *Quillen equivalences* between them.
- In §1.2, we provide a host of examples of the objects introduced in §1.1. We also speculate on the existence of other examples – some of a foundational nature, some which would provide yet more models for the ∞-category of ∞-categories, and one related to $\mathbb{E}_n$ deformation theory – whose verifications lie beyond the scope of the current project.
- In §1.3, we define *cofibrantly generated model ∞-categories* and provide recognition and lifting theorems analogous to the classical ones.
- In §1.4, we assert the existence of the *Kan–Quillen model structure* on the ∞-category $s\mathcal{S}$ of simplicial spaces. The proof (which we only outline, leaving the real substance for §1.7) relies on the recognition theorem of §1.3. We also define what it means for a model ∞-category to be *proper*.
- In §1.5, we collect some auxiliary results regarding spaces and simplicial spaces. In particular, we state a particular result – Lemma 1.5.4 – which ultimately represents the key piece of not-totally-formal input that makes the entire story tick (but we defer its proof to §1.8).

- In §1.6, we prove some convenient properties enjoyed by the fibrant objects and the fibrations in the Kan–Quillen model structure on $s\mathcal{S}$, and we define a "fibrant replacement" endofunctor Ex^∞ analogous to the classical one.

- In §1.7, we prove the main result: that the data described in §1.4 do indeed define a proper, cofibrantly generated model structure on the ∞-category $s\mathcal{S}$ of simplicial spaces.

- In §1.8, we prove Lemma 1.5.4, using the classical theory of model categories and ultimately some rather delicate arguments regarding bisimplicial sets.

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1.1 Model ∞-categories: definitions

In this section, we define model ∞-categories, Quillen adjunctions, and Quillen equivalences. We will provide numerous examples of all of these concepts in §1.2.

1.1.1 The definition of a model ∞-category

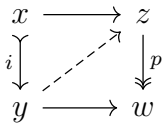
Definition 1.1.1. We say that three wide subcategories $\mathbf{W}, \mathbf{C}, \mathbf{F} \subset \mathcal{M}$ – called the subcategories of ***weak equivalences***, of ***cofibrations***, and of ***fibrations***, respectively, and with their morphisms denoted by the symbols $\xrightarrow{\sim}$, $\rightarrowtail$, and $\twoheadrightarrow$, respectively – make an ∞-category $\mathcal{M}$ into a ***model ∞-category*** if they satisfy the following evident ∞-categorical analogs of the usual axioms for a model category.

$\mathrm{M}_\infty 1$ *(limit)* $\mathcal{M}$ is finitely bicomplete.

$\mathrm{M}_\infty 2$ *(two-out-of-three)* $\mathbf{W}$ satisfies the two-out-of-three property.

$\mathrm{M}_\infty 3$ *(retract)* $\mathbf{W}$, $\mathbf{C}$, and $\mathbf{F}$ are all closed under retracts.

$\mathrm{M}_\infty 4$ *(lifting)* There exists a lift in any commutative square



in which (i is a cofibration, p is a fibration, and) either i or p is a weak equivalence.

$\mathrm{M}_\infty 5$ *(factorization)* Every map in $\mathcal{M}$ factors via both $\mathbf{F} \circ (\mathbf{W} \cap \mathbf{C})$ and $(\mathbf{W} \cap \mathbf{F}) \circ \mathbf{C}$.

To indicate that a morphism lies in one of these subcategories, we will use the symbols $\xrightarrow{\approx}$, $\rightarrowtail$, and $\twoheadrightarrow$, respectively. We call $\mathbf{W} \cap \mathbf{C} \subset \mathcal{M}$ the subcategory of ***acyclic cofibrations***, and we call $\mathbf{W} \cap \mathbf{F} \subset \mathcal{M}$ the subcategory of ***acyclic fibrations***. Morphisms lying in these subcategories are denoted by the symbols $\xrightarrow{\approx}\!\!\!\!\rightarrowtail$ and $\xrightarrow{\approx}\!\!\!\!\twoheadrightarrow$, respectively.

Notation 1.1.2. In order to disambiguate our notation associated to various model ∞-categories, we introduce the following conventions.

- We will always subscript the data associated to a "named" model ∞-category with (an abbreviation of) its name. For instance, we write $s\mathcal{S}_{\mathrm{KQ}}$ to denote the model ∞-category given by the Kan–Quillen model structure on the ∞-category $s\mathcal{S}$ of simplicial spaces (see Definition 1.4.5), and we write $\mathbf{F}_{\mathrm{KQ}} \subset s\mathcal{S}$ to denote its subcategory of fibrations.

- On the other hand, for an "unnamed" model ∞-category, we may subscript its associated data with the name of the underlying ∞-category. For instance, if $\mathcal{M}$ is a model ∞-category, we may write $\mathbf{C}_{\mathcal{M}} \subset \mathcal{M}$ to denote its subcategory of cofibrations.

- When two different ∞-categories have model structures with the same name, we may additionally superscript their associated data with the name of the ambient ∞-category. For instance, we may write $\mathbf{W}_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}} \subset s\mathcal{S}\mathrm{et}$ to denote the subcategory of weak equivalences in the classical Kan–Quillen model structure on the category $s\mathcal{S}\mathrm{et}$ of simplicial sets (see Definition 1.4.1). However, when no ambiguity should arise, we will generally omit this superscript.

Definition 1.1.3. An object of a model ∞-category $\mathcal{M}$ is called

- ***cofibrant*** if its unique map from the initial object $\varnothing_{\mathcal{M}}$ is a cofibration,
- ***fibrant*** if its unique map to the terminal object $\mathrm{pt}_{\mathcal{M}}$ is a fibration, and
- ***bifibrant*** if it is both cofibrant and fibrant.

We denote the full subcategories of these objects by $\mathcal{M}^c \subset \mathcal{M}$, $\mathcal{M}^f \subset \mathcal{M}$, and $\mathcal{M}^{cf} \subset \mathcal{M}$, respectively. More generally, we will use these same superscripts to denote the intersection of some other subcategory of $\mathcal{M}$ with the indicated subcategory just defined, so that e.g. $\mathbf{W}^{cf} = \mathbf{W} \cap \mathcal{M}^{cf} \subset \mathcal{M}$ denotes the subcategory of weak equivalences between bifibrant objects.

Remark 1.1.4. As in the classical case, one should think of cofibrant objects as being "good for mapping out of", and of fibrant objects as being "good for mapping into". Indeed, the *fundamental theorem of model ∞-categories* (6.1.9) asserts that if $x \in \mathcal{M}^c$ and $y \in \mathcal{M}^f$, then the natural map

$$\hom_{\mathcal{M}}(x, y) \to \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x, y)$$

is a surjection, and becomes an equivalence after applying either "∞-categorical equivalence relation" of *left homotopy* or of *right homotopy* to the source (in a sense made precise in §6.1).

Moreover, as in the classical case, factorization axiom $\mathrm{M}_\infty 5$ guarantees that any object of $\mathcal{M}$ admits both

- a cofibrant replacement by an acyclic fibration and
- a fibrant replacement by an acyclic cofibration.

Taken together, these imply that every object of $\mathcal{M}[\mathbf{W}^{-1}]$ can even be represented by a *bifibrant* object of the model ∞-category $\mathcal{M}$.

Remark 1.1.5. In view of Remark 1.1.4, we see that if we are only interested in computing hom-spaces in $\mathcal{M}[\mathbf{W}^{-1}]$, then it suffices to know only which objects of $\mathcal{M}$ are co/fibrant (as long as we also have some control over the left and right homotopy relations). However, there are many other constructions that one might perform in an ∞-category besides extracting hom-spaces, and from this point of view we can think of the subcategories $\mathbf{C}, \mathbf{F} \subset \mathcal{M}$ as telling us which objects are *relatively* co/fibrant, i.e. co/fibrant in some undercategory or overcategory (see Example 1.2.3). Nevertheless, we may view the data of the subcategories $\mathbf{W}, \mathcal{M}^c, \mathcal{M}^f \subset \mathcal{M}$ as a "first approximation" to the data of a model structure. As it can at times be quite difficult to prove the existence of a model structure, in some examples below we will just content ourselves to produce choices of co/fibrant objects corresponding to a given class of weak equivalences.

Remark 1.1.6. One of the most important constructions that one might perform in a (1- or ∞-)category is the extraction of co/limits. Classically, the theory of *homotopy co/limits* in a model 1-category gives a way of computing co/limits in its ∞-categorical localization (see e.g. Theorem T.4.2.4.1). (In contrast with the "first approximation" of Remark 1.1.5, these certainly require the full model structure!) We provide a theory of homotopy co/limits in model ∞-categories in §5.1.2.

Remark 1.1.7. The ∞-categorical lifting axiom $\mathrm{M}_\infty 4$ encodes a homotopically coherent version of the usual lifting axiom. It is strictly stronger than the usual lifting axiom for hom-sets in the homotopy category (see Example 1.2.11).

Remark 1.1.8. Factorization systems in ∞-categories (in which fillers are unique) are studied in §T.5.2.8. These are distinct from what we study here, which might be called *weak* factorization systems in ∞-categories.

Remark 1.1.9. When defining model categories, there are always the choices to be made

- of whether to require that the factorizations be functorial, and
- of whether to require bicompleteness or only finite bicompleteness.

Of course, in defining model ∞-categories, these choices persist. With regards to both of these, we have chosen the less restrictive option.

Remark 1.1.10. Since the opposite of a model ∞-category is canonically a model ∞-category, many of the statements that we make throughout this thesis have obvious duals. For conciseness, we will often just make whichever of the pair of dual statements is more convenient, and then we will simply refer to its dual if and when we require it.

Remark 1.1.11. There are many basic facts about model categories which follow easily and directly from the definitions; these generally remain true for model ∞-categories. For instance, we will repeatedly use the facts

- that $\mathbf{C} = \mathrm{llp}(\mathbf{W} \cap \mathbf{F})$,
- that $\mathbf{W} \cap \mathbf{C} = \mathrm{llp}(\mathbf{F})$,
- that $\mathbf{F} = \mathrm{rlp}(\mathbf{W} \cap \mathbf{C})$, and
- that $\mathbf{W} \cap \mathbf{F} = \mathrm{rlp}(\mathbf{C})$.

(The proofs remains the same, see [Hir03, Proposition 7.2.3].) We also note here that it follows from these characterizations that $\mathcal{M}^{\simeq} \subset \mathbf{W} \cap \mathbf{C} \cap \mathbf{F} \subset \mathcal{M}$, i.e. that the subcategory $\mathcal{M}^{\simeq} \subset \mathcal{M}$ of equivalences is contained in all three defining subcategories of a model ∞-category.

1.1.2 The definitions of Quillen adjunctions and Quillen equivalences

Model categories are useful not just in isolation, but in how they relate to one another. Following the classical situation, we make the following definition.

Definition 1.1.12. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are two model ∞-categories, and suppose that

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

is an adjunction between their underlying ∞-categories. We say that this adjunction is a ***Quillen adjunction*** if any of the following equivalent conditions is satisfied:

- F preserves cofibrations and acyclic cofibrations;
- G preserves fibrations and acyclic fibrations;
- F preserves cofibrations and G preserves fibrations;
- F preserves acyclic cofibrations and G preserves acyclic fibrations.

(That these conditions are indeed equivalent follows immediately from Remark 1.1.11.) In this situation, we call F a ***left Quillen functor*** and we call G a ***right Quillen functor***.

Remark 1.1.13. We prove as Theorem 5.1.1 that a Quillen adjunction

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

induces a canonical adjunction

$$\mathbb{L}F : \mathcal{M}[\mathbf{W}_{\mathcal{M}}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}_{\mathcal{N}}^{-1}] : \mathbb{R}G$$

on localizations, called its *derived adjunction* (see Definition 5.1.2).

Of course, we also have the following special case of Definition 1.1.12.

Definition 1.1.14. We say that a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a ***Quillen equivalence*** if for all $x \in \mathcal{M}^c$ and all $y \in \mathcal{N}^f$, the equivalence

$$\hom_{\mathcal{M}}(x, G(y)) \simeq \hom_{\mathcal{N}}(F(x), y)$$

induces an equivalence of subspaces

$$\hom_{\mathbf{W}_{\mathcal{M}}}(x, G(y)) \simeq \hom_{\mathbf{W}_{\mathcal{N}}}(F(x), y).$$

(Note that this condition can be checked on path components.)

Remark 1.1.15. Extending Remark 1.1.13, we prove as Corollary 5.1.3 that the derived adjunction of a Quillen equivalence induces an *equivalence* of localizations.

1.2 Model ∞-categories: examples

Having discussed the generalities of model ∞-categories, we now proceed to give a number of examples. In §1.2.1 we give some examples of model ∞-categories, in §1.2.2 we give some examples of Quillen adjunctions and Quillen equivalences between model ∞-categories, and in §1.2.3 we give some speculative examples whose verifications lie beyond the scope of the current project. This section may be safely omitted from a first reading.

1.2.1 Examples of model ∞-categories

In this subsection, we give a plethora of examples of model ∞-categories. They are organized into subsubsections based on their nature:

- in §1.2.1.1 we begin with some general examples of model ∞-categories,
- in §1.2.1.2 we list some more specific examples model ∞-categories,
- in §1.2.1.3 we give examples of model structures on ∞-categories of diagrams in a model ∞-category, and
- in §1.2.1.4 we explore some more exotic examples of model ∞-categories (which we have included primarily for their intrinsic interest).

1.2.1.1 General examples of model ∞-categories

Example 1.2.1. In the case that $\mathcal{M}$ is actually a 1-category considered as an ∞-category with discrete hom-spaces, then Definition 1.1.1 recovers the classical definition of a model category. Thus, any model 1-category gives an example of a model ∞-category.

Example 1.2.2. Any finitely bicomplete ∞-category $\mathcal{M}$ has a *trivial* model structure, in which we set $\mathbf{W} = \mathcal{M}^{\simeq}$ and $\mathbf{C} = \mathbf{F} = \mathcal{M}$. We denote this model ∞-category by $\mathcal{M}_{\mathrm{triv}}$.

Example 1.2.3. If $\mathcal{M}$ is a model ∞-category and $x \in \mathcal{M}$ is any object, then both the undercategory $\mathcal{M}_{x/}$ and the overcategory $\mathcal{M}_{/x}$ inherit the structure of a model ∞-category, where in both cases the three defining subcategories are created by the forgetful functor to $\mathcal{M}$. Iterating this observation, for any morphism $x \to y$ in $\mathcal{M}$ we obtain the structure of a model ∞-category on $\mathcal{M}_{x//y}$.

Example 1.2.4. The *recognition theorem* for cofibrantly generated model ∞-categories (1.3.11) gives general criteria for the existence of a cofibrantly generated model structure on an ∞-category with respect to given choices of weak equivalences, generating cofibrations, and generating acyclic cofibrations (see §1.3).

1.2.1.2 Specific examples of model ∞-categories

Example 1.2.5. The main purpose of this chapter is to define the *Kan–Quillen model structure* on $s\mathcal{S}$, which will be given as Definition 1.4.5, and which is denoted by $s\mathcal{S}_{\mathrm{KQ}}$.

Example 1.2.6. The ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories admits a *Thomason model structure*, described in Example 1.2.27, which is denoted by $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$.

Example 1.2.7. In future work, given a model ∞-category $\mathcal{M}$ (with a chosen set of generators), following [DKS93] and [Bou03] we will define a *resolution model structure* (a/k/a an "E^2 model structure") on the ∞-category $s\mathcal{M}$ which presents the corresponding *nonabelian derived ∞-category* of $\mathcal{M}$, which is denoted $s\mathcal{M}_{\mathrm{res}}$.

1.2.1.3 Examples of model structures on functor ∞-categories

Example 1.2.8. Given a model ∞-category $\mathcal{M}$ and an ∞-category $\mathcal{C}$, there is sometimes a *projective model structure* on the ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{M})$ (see §5.1.2), which is denoted by $\mathrm{Fun}(\mathcal{C},\mathcal{M})_{\mathrm{proj}}$.

Example 1.2.9. Given a model ∞-category $\mathcal{M}$ and an ∞-category $\mathcal{C}$, there is sometimes a *injective model structure* on the ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{M})$ (see §5.1.2), which is denoted by $\mathrm{Fun}(\mathcal{C},\mathcal{M})_{\mathrm{inj}}$.

Example 1.2.10. Given a model ∞-category $\mathcal{M}$ and a Reedy category $\mathcal{C}$ (see Definition 5.1.11), there is always a *Reedy model structure* on the ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{M})$ (see §5.1.3), which is denoted by $\mathrm{Fun}(\mathcal{C},\mathcal{M})_{\mathrm{Reedy}}$.

1.2.1.4 Exotic examples of model ∞-categories

Example 1.2.11. If $\mathcal{M}$ is a model ∞-category and $\mathrm{ho}(\mathcal{M})$ is finitely bicomplete, then the model structure on $\mathcal{M}$ descends to a model structure on $\mathrm{ho}(\mathcal{M})$. In fact, the two-out-of-three, retract, and factorization axioms in $\mathcal{M}$ are even verifiable in

$\mathrm{ho}(\mathcal{M})$. Moreover, the map

$$\lim \left(\begin{array}{cc} & \hom_{\mathcal{M}}(y,w) \\ & \downarrow{\scriptstyle i^*} \\ \hom_{\mathcal{M}}(x,z) & \xrightarrow[p_*]{} \hom_{\mathcal{M}}(x,w) \end{array} \right) \to \lim \left(\begin{array}{cc} & \hom_{\mathrm{ho}(\mathcal{M})}(y,w) \\ & \downarrow{\scriptstyle i^*} \\ \hom_{\mathrm{ho}(\mathcal{M})}(x,z) & \xrightarrow[p_*]{} \hom_{\mathrm{ho}(\mathcal{M})}(x,w) \end{array} \right)$$

is a surjection, so if the lifting axiom holds in $\mathcal{M}$ then it must hold in $\mathrm{ho}(\mathcal{M})$ as well. Of course, this generalizes Example 1.2.1: if $\mathcal{M}$ is a 1-category, then the projection $\mathcal{M} \xrightarrow{\sim} \mathrm{ho}(\mathcal{M})$ is an equivalence. On the other hand, since for an arbitrary ∞-category $\mathcal{M}$ there is usually a complicated interplay between co/limits in $\mathcal{M}$ and co/limits in $\mathrm{ho}(\mathcal{M})$, it does not appear possible to draw any general conclusions about the relationship between these two model ∞-categories, or about the induced map

$$\mathcal{M}[\![\mathbf{W}^{-1}]\!] \to \mathrm{ho}(\mathcal{M})[\![\mathrm{ho}(\mathbf{W})^{-1}]\!]$$

on localizations. (Note that the functor $\mathcal{M} \to \mathrm{ho}(\mathcal{M})$ is the unit of the left localization $\mathrm{ho} : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{C}\mathrm{at} : \mathrm{U}_{\mathcal{C}\mathrm{at}}$, but it is not generally itself an adjoint.)

As we will elaborate upon in Remark 1.2.18, the following example illustrates a particularly "one-sided" sort of model ∞-category.

Example 1.2.12. Suppose that $\mathcal{M}$ is a finitely bicomplete ∞-category and that $\mathrm{L} : \mathcal{M} \rightleftarrows \mathrm{L}\mathcal{M} : \mathrm{U}$ is a left localization. This means that we may consider the right adjoint $\mathrm{U} : \mathrm{L}\mathcal{M} \hookrightarrow \mathcal{M}$ as the inclusion of the reflective subcategory of "local" objects, and that for any $x \in \mathcal{M}$ and any $y \in \mathrm{L}\mathcal{M} \subset \mathcal{M}$, the localization map $x \to \mathrm{L}x$ induces an equivalence

$$\hom_{\mathcal{M}}(\mathrm{L}x, y) \xrightarrow{\sim} \hom_{\mathcal{M}}(x, y).$$

This fits squarely into the "first approximation" rubric of Remark 1.1.5: we should consider $\mathcal{M}^c = \mathcal{M}$ and $\mathcal{M}^f = \mathrm{U}(\mathrm{L}\mathcal{M}) \subset \mathcal{M}$ (with trivial left/right homotopy relations). Hence, it is natural to guess that there might be a *left localization* model structure $\mathcal{M}_{\mathrm{L}}$ on $\mathcal{M}$, in which

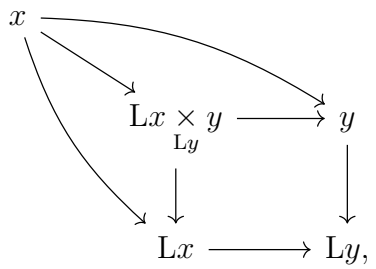
- $\mathbf{W}_{\mathrm{L}} \subset \mathcal{M}$ is the subcategory of morphisms that become equivalences in $\mathrm{L}\mathcal{M}$,
- $\mathbf{C}_{\mathrm{L}} = \mathcal{M}$, and
- $\mathbf{F}_{\mathrm{L}} \subset \mathcal{M}$ is determined by the lifting condition $\mathbf{F}_{\mathrm{L}} = \mathrm{rlp}((\mathbf{W} \cap \mathbf{C})_{\mathrm{L}}) = \mathrm{rlp}(\mathbf{W}_{\mathrm{L}})$;

in this case, the left adjoint will coincide with the localization $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}_{\mathrm{L}}^{-1}]\!] \simeq \mathrm{L}\mathcal{M}$ (and moreover, the left homotopy relation will be visibly trivial (and hence the right homotopy relation will be trivial as well)). What remains, then, is to check

- the half of the lifting axiom asserting that $(\mathbf{W} \cap \mathbf{F})_{\mathrm{L}} \subset \mathrm{rlp}(\mathbf{C}_{\mathrm{L}}) = \mathrm{rlp}(\mathcal{M})$, and
- the half of the factorization axiom regarding $\mathbf{F}_{\mathrm{L}} \circ (\mathbf{W} \cap \mathbf{C})_{\mathrm{L}} = \mathbf{F}_{\mathrm{L}} \circ \mathbf{W}_{\mathrm{L}}$.

Now, for a map to have $\mathrm{rlp}(\mathcal{M})$ in particular means that it has the right lifting property against the identity map from itself, and this implies that the map is an equivalence (with inverse given by the guaranteed lift). Thus, $\mathrm{rlp}(\mathcal{M}) \subset \mathcal{M}^{\simeq}$. On the other hand, clearly $\mathcal{M}^{\simeq} \subset \mathrm{rlp}(\mathcal{M})$, so $\mathrm{rlp}(\mathcal{M}) = \mathcal{M}^{\simeq}$, and hence to satisfy the lifting axiom it is both necessary and sufficient to have that $(\mathbf{W} \cap \mathbf{F})_{\mathrm{L}} \subset \mathcal{M}^{\simeq}$. (Of course, the reverse containment follows from the definitions, so it is also necessary and sufficient to have that $(\mathbf{W} \cap \mathbf{F})_{\mathrm{L}} = \mathcal{M}^{\simeq}$.)

The factorization axiom here is more subtle, and there does not appear to be a general criterion for when this might hold. One possibility is that we might try to use the factorization of an arbitrary map $x \to y$ via the construction



i.e. via the composite

$$x \to \mathrm{L}x \underset{\mathrm{L}y}{\times} y \to y.$$

In the case that the left localization L is additionally *left exact*, i.e. in the case that it commutes with finite limits (see Remark T.5.3.2.3), then this does indeed produce a factorization as $\mathbf{F}_{\mathrm{L}} \circ \mathbf{W}_{\mathrm{L}}$:

- the first map is in $\mathbf{W}_{\mathrm{L}}$ by the left exactness of L, and
- the second map is in $\mathbf{F}_{\mathrm{L}}$ since it is a pullback of the map $\mathrm{L}x \to \mathrm{L}y$, which is directly seen to have $\mathrm{rlp}(\mathbf{W}_{\mathrm{L}})$.

In fact, this does not require the full strength of left exactness, but is only using the weaker notion of L being a *locally cartesian* localization (see [GK, 1.2]).[7]

In a different direction, the proof of [Sal, Proposition 2.9] (which actually only uses *finite* bicompleteness) appears to generalize directly to show that whenever

[7] There are a number of results in the literature surrounding factorizations in 1-categories which, if suitably generalized to ∞-categories, would give a much wider class of left localizations in which the factorization axiom holds; a notable example is [CHK85, Corollary 3.4]. (Somewhat relatedly, factorization systems in stable ∞-categories are explored in [FL].)

the subcategory $\mathrm{L}\mathcal{M} \subset \mathcal{M}$ is additionally "strictly saturated" (in the sense of [Sal, Definition 2.7]), then the model ∞-category $\mathcal{M}_{\mathrm{L}}$ does indeed exist, and moreover

- the subcategory $\mathbf{F}_{\mathrm{L}} \subset \mathcal{M}$ consists of precisely those maps which are pullbacks of maps in the subcategory $\mathrm{L}\mathcal{M} \subset \mathcal{M}$, while
- $(\mathbf{W} \cap \mathbf{F})_{\mathrm{L}} \simeq \mathcal{M}^{\simeq}$.

More broadly, [Sal] makes a detailed study of these left localization model category structures and their evident duals (see Example 1.2.17), which should presumably generalize naturally to model ∞-categories.

Example 1.2.13. A particular case where Example 1.2.12 does indeed give a model structure is the left localization $\pi_0 : \mathcal{S} \rightleftarrows \mathcal{S}\mathrm{et} : \mathrm{disc}$. Here, we have that $\mathbf{F}_{\pi_0} = \mathrm{rlp}(\mathbf{W}_{\pi_0})$ consists of precisely the *étale* maps of spaces, i.e. those maps which induce $\pi_{\geq 1}$-isomorphisms for every basepoint of the source. (Note that this condition is independent from that of being 0-connected; in particular, such a map may be neither injective nor surjective on π_0.) It follows that $(\mathbf{W} \cap \mathbf{F})_{\pi_0} = \mathcal{S}^{\simeq}$, and so we have the required lifting condition. Moreover, the proposed factorization of Example 1.2.12 is in this case precisely the standard epi-mono factorization, and this gives the required factorization $\mathbf{F}_{\pi_0} \circ \mathbf{W}_{\pi_0}$. As $\mathcal{S}$ is certainly finitely bicomplete, we obtain a model ∞-category $\mathcal{S}_{\pi_0}$ with localization $\mathcal{S} \to \mathcal{S}[\![\mathbf{W}_{\pi_0}^{-1}]\!] \simeq \mathcal{S}\mathrm{et}$.

Example 1.2.14. In fact, Example 1.2.13 generalizes to the left localizations $\tau_{\leq n} : \mathcal{S} \rightleftarrows \mathcal{S}^{\leq n} : \mathrm{U}_{\leq n}$. Using the notation of Example 1.3.10 below, we have that

$$\mathbf{F}_{\tau_{\leq n}} = \mathrm{rlp}((I^{\mathcal{S}}_{\mathrm{triv}})_{\geq n+2}) \cap \mathrm{rlp}'(\{S^n \to \mathrm{pt}_{\mathcal{S}}\})$$

consists of precisely those maps which induce $\pi_{\geq n+1}$-isomorphisms for every basepoint in the source. So again $(\mathbf{W} \cap \mathbf{F})_{\tau_{\leq n}} = \mathcal{S}^{\simeq}$, and it is easy to check using the long exact sequence in homotopy for a pullback square that again the suggestion in Example 1.2.12 yields the factorization $\mathbf{F}_{\tau_{\leq n}} \circ \mathbf{W}_{\tau_{\leq n}}$. (Alternatively, an easy spheres-and-disks construction yields this factorization as well.) Hence, we obtain a model ∞-category $\mathcal{S}_{\tau_{\leq n}}$ with localization $\mathcal{S} \to \mathcal{S}[\![\mathbf{W}_{\tau_{\leq n}}^{-1}]\!] \simeq \mathcal{S}^{\leq n}$.

Remark 1.2.15. It is clear that Example 1.2.14 relies on the fact that there is a good theory of cellular approximation in $\mathcal{S}$ (e.g., the fact that attaching an $(n+2)$-cell to a space doesn't change its n-truncation). Thus, it does not appear to immediately generalize to an arbitrary ∞-topos: it is important that the generators be suitably compatible with the truncation functors.

Remark 1.2.16. The Kan–Quillen model structure on $s\mathcal{S}$ of Definition 1.4.5 has its weak equivalences created by the left adjoint $|-| : s\mathcal{S} \to \mathcal{S}$, but it is *not* a left

localization model structure (in the sense of Example 1.2.12). Indeed, this left adjoint is certainly not left exact: the question of when limits commute with geometric realizations is generally very difficult to answer. (However, in the case of pullbacks, this question is addressed to some extent by Corollary 1.6.7 below; see also the surrounding Remarks 1.6.6, 1.6.12, 1.6.13, and 1.6.14. The case of more general limits will also be addressed in §5.1.2.)

Example 1.2.17. Given a right localization $\mathrm{U} : \mathrm{R}\mathcal{M} \rightleftarrows \mathcal{M} : \mathrm{R}$ with $\mathcal{M}$ finitely bicomplete, we obtain a dual story to that of Example 1.2.12. Now we should think of every object as *fibrant*, and of the left adjoint as the inclusion of the coreflective subcategory of *cofibrant* objects. In this case, the guess at a *right localization* model structure $\mathcal{M}_{\mathrm{R}}$ on $\mathcal{M}$ has that

- $\mathbf{W}_{\mathrm{R}} \subset \mathcal{M}$ is the subcategory of morphisms that become equivalences in $\mathrm{R}\mathcal{M}$,
- $\mathbf{F}_{\mathrm{R}} = \mathcal{M}$, and
- $\mathbf{C}_{\mathrm{R}} \subset \mathcal{M}$ is determined by the lifting condition $\mathbf{C}_{\mathrm{R}} = \mathrm{llp}(\mathbf{W}_{\mathrm{R}})$.

From here, it remains to check

- the lifting condition $(\mathbf{W} \cap \mathbf{C})_{\mathrm{R}} \subset \mathrm{llp}(\mathbf{F}_{\mathrm{R}}) = \mathcal{M}^{\simeq}$, and
- the factorization condition for $(\mathbf{W} \cap \mathbf{F})_{\mathrm{R}} \circ \mathbf{C}_{\mathrm{R}} = \mathbf{W}_{\mathrm{R}} \circ \mathbf{C}_{\mathrm{R}}$.

Now, the factorization condition will be implied by the *right exactness* of the right localization R (or more generally, if the right localization is "locally cocartesian").

Remark 1.2.18. Example 1.2.12 reinterprets a left localization as describing a model ∞-category in which all objects are *cofibrant*; dually, Example 1.2.17 reinterprets a right localization as describing a model ∞-category in which all objects are *fibrant*. Since in an arbitrary model ∞-category, not all the objects will be cofibrant and not all the objects will be fibrant, we might think of the notion of a model ∞-category as a sort of *simultaneous generalization* of the notions of left and right localizations.[8]

Example 1.2.19. As a simple case of Remark 1.2.18, we can obtain a "first approximation" (as in Remark 1.1.5) to a model structure on the ∞-category $\mathcal{S}\mathrm{p}$ of spectra which would present the ∞-category $\mathcal{S}\mathrm{p}^{[m,n]}$ of spectra that only have nontrivial homotopy groups in some interval $[m, n] \subset \mathbb{Z}$: the cofibrant objects would be $\mathcal{S}\mathrm{p}^{\geq m} \subset \mathcal{S}\mathrm{p}$, while the fibrant objects would be $\mathcal{S}\mathrm{p}^{\leq n} \subset \mathcal{S}\mathrm{p}$. This example, though illustrative, is somewhat degenerate, since any weak equivalence between bifibrant objects is already an equivalence. Indeed, the "homotopy relations" would all be

[8] However, this is not to say that a model ∞-category structure in which all objects are cofibrant must determine a left localization (or dually); for instance, the localization $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} \to s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{\mathrm{KQ}}^{-1}]\!] \simeq \mathcal{S}$ provides a counterexample, as it is obviously not a left adjoint.

trivial, corresponding to the fact that we have an inclusion $\mathcal{S}\mathrm{p}^{[m,n]} \subset \mathcal{S}\mathrm{p}$ which is a section to the projection $\tau_{\geq m} \circ \tau_{\leq n} \simeq \tau_{\leq n} \circ \tau_{\geq m} : \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}^{[m,n]}$.

Remark 1.2.20. Analogously to Example 1.2.14, we might try to obtain a model structure as in Example 1.2.17 from the right localization $\mathrm{U}_{\geq n} : \mathcal{S}_*^{\geq n} \rightleftarrows \mathcal{S}_*^{\geq 1} : \tau_{\geq n}$. However, the existence of such a model structure is much less clear.

We do still have the lifting condition. Indeed, suppose that $x \to y$ is in $(\mathbf{W} \cap \mathbf{C})_{\tau_{\geq n}}$. Since $x \to y$ is in $\mathbf{W}_{\tau_{\geq n}}$, it induces an isomorphism on $\pi_{\geq n}$. On the other hand, obtaining a lift in the commutative square

$$\begin{array}{ccc} x & \longrightarrow & \tau_{\leq n-1} x \\ \downarrow & & \downarrow \\ y & \longrightarrow & \tau_{\leq n-1} y \end{array}$$

(in which $\tau_{\geq n}$ applied to the right map yields $\mathrm{id}_{\mathrm{pt}_{\mathcal{S}}}$), we see that the map also induces isomorphisms on $\pi_{\leq n-1}$.

But the factorization condition is trickier. Given a map $x \to y$ in $\mathcal{S}_*^{\geq 1}$, the map

$$\tau_{\geq n} y \to \tau_{\geq n} y \amalg_{\tau_{\geq n} x} x$$

will not necessarily be a $\pi_{\geq n}$-isomorphism. For instance, when $n \geq 2$ and x is a 1-type, then this map will just be the inclusion $\tau_{\geq n} y \to \tau_{\geq n} y \vee x$ of the first wedge summand, which will not generally induce a $\pi_{\geq n}$-isomorphism. (Of course, this observation does not preclude the existence of suitable factorizations.)

Remark 1.2.21. While not exactly an example of a model ∞-category, it seems worth observing that given a model ∞-category $\mathcal{M}$, the pair $(\mathcal{M}^c, \mathbf{W}^c)$ gives an example of the evident ∞-categorical analog of Waldhausen's notion of a "category with cofibrations and weak equivalences" (see [Wal85, §1.2]); moreover, a left Quillen functor $\mathcal{M} \to \mathcal{N}$ of model ∞-categories then gives rise to an "exact functor" of such. Consequently, there is an evident definition of the *algebraic K-theory* of these objects, visibly functorial for left Quillen functors (see [Wal85, §1.3]).

Note that this does *not* coincide with Barwick's notion of a "Waldhausen ∞-category" given as [Bara, Definition 2.7]. Rather, one recovers this latter notion as a special case of an "∞-category with cofibrations and weak equivalences" in which the weak equivalences are just the equivalences. On the other hand, it seems likely that the algebraic K-theory of $(\mathcal{M}^c, \mathbf{W}^c)$ in the above sense would simply compute the algebraic K-theory of the localization $\mathcal{M}^c[\![(\mathbf{W}^c)^{-1}]\!] \simeq \mathcal{M}[\![\mathbf{W}^{-1}]\!]$ with respect to its *maximal* pair structure in Barwick's sense (as this is true for model 1-categories, see [Bara, Proposition 9.15 and Corollary 10.10.3]).[9]

[9] This equivalence is given by (the dual of) Corollary 5.3.4.

1.2.2 Examples of Quillen adjunctions and Quillen equivalences

Example 1.2.22. As will be immediate from Definitions 1.4.1 and 1.4.5, the adjunction $\pi_0 : s\mathcal{S}_{\mathrm{KQ}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} : \mathrm{disc}$ (see Notation 1.4.3) between the Kan–Quillen model ∞-category structures on $s\mathcal{S}$ and $s\mathcal{S}\mathrm{et}$ is a Quillen equivalence.

Example 1.2.23. In Remark 1.6.23, we will see that the "subdivision" and "extension" endofunctors of [Kan57], suitably extended from $s\mathcal{S}\mathrm{et}$ to $s\mathcal{S}$ (see §1.6.3), define a Quillen equivalence $\mathrm{sd} : s\mathcal{S}_{\mathrm{KQ}} \rightleftarrows s\mathcal{S}_{\mathrm{KQ}} : \mathrm{Ex}$.

Example 1.2.24. The *lifting theorem* for cofibrantly generated model ∞-categories (1.3.12) gives general general criteria for constructing Quillen adjunctions by lifting a cofibrantly generated model structure along a left adjoint (see §1.3).

Example 1.2.25. If a left localization $\mathrm{L} : \mathcal{M} \rightleftarrows \mathrm{L}\mathcal{M} : \mathrm{U}$ induces a left localization model structure $\mathcal{M}_{\mathrm{L}}$ as in Example 1.2.12, then we obtain a Quillen adjunction $\mathrm{id}_{\mathcal{M}} : \mathcal{M}_{\mathrm{triv}} \rightleftarrows \mathcal{M}_{\mathrm{L}} : \mathrm{id}_{\mathcal{M}}$, whose derived adjunction is precisely the original left localization $\mathrm{L} : \mathcal{M} \rightleftarrows \mathrm{L}\mathcal{M} : \mathrm{U}$. This is of course closely related to the theory of *left Bousfield localizations* of model categories (see e.g. [Hir03, §3.3]). Dual statements apply to the right localization model structures of Example 1.2.17.

Example 1.2.26. As a particular case of Example 1.2.25, the model ∞-category $\mathcal{S}_{\tau_{\leq n}}$ of Example 1.2.14 participates in a Quillen adjunction $\mathrm{id}_{\mathcal{S}} : \mathcal{S}_{\mathrm{triv}} \rightleftarrows \mathcal{S}_{\tau_{\leq n}} : \mathrm{id}_{\mathcal{S}}$, whose derived adjunction is $\tau_{\leq n} : \mathcal{S} \rightleftarrows \mathcal{S}^{\leq n} : \mathrm{U}_{\leq n}$.

Example 1.2.27. Recall that the ∞-category $\mathcal{CSS}$ of *complete Segal spaces* sits as a left localization $s\mathcal{S} \rightleftarrows \mathcal{CSS}$, and moreover admits a natural equivalence $\mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty$ (see §2.2). We show as Theorem 3.6.1 that, as a particular case of Example 1.2.24, we can transfer the Kan–Quillen model structure of Definition 1.4.5 along the adjunction $s\mathcal{S} \rightleftarrows \mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty$ to obtain the *Thomason model structure* on the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories. This Quillen adjunction is in fact a Quillen equivalence, and hence Remark 1.1.15 implies that the Thomason model structure on $\mathcal{C}\mathrm{at}_\infty$ once again presents the ∞-category $\mathcal{S}$. As explained in Remark 3.6.6, this model structure resolves some of the less satisfying aspects of the classical Thomason model structure on $\mathfrak{C}\mathrm{at}$ (which also presents $\mathcal{S}$).

Example 1.2.28. Given a model ∞-category $\mathcal{M}$ and an ∞-category $\mathcal{C}$, if both the projective and injective model structures on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ exist (see Examples 1.2.8 and 1.2.9), then the identity adjunction defines a Quillen equivalence

$$\mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})} : \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{inj}} : \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}$$

between them (see Remark 5.1.20).

Example 1.2.29. Given a model ∞-category $\mathcal{M}$ and a Reedy category $\mathcal{C}$, if the projective model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ exists (see Example 1.2.8), then the identity adjunction defines a Quillen equivalence

$$\mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})} : \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}} : \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}$$

(see Remark 5.1.20).

Example 1.2.30. Given a model ∞-category $\mathcal{M}$ and a Reedy category $\mathcal{C}$, if the injective model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ exists (see Example 1.2.9), then the identity adjunction defines a Quillen equivalence

$$\mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})} : \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{inj}} : \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}$$

(see Remark 5.1.20).

Example 1.2.31. Given a model ∞-category $\mathcal{M}$ and an ∞-category $\mathcal{C}$ such that $\mathcal{M}$ admits $\mathcal{C}$-shaped colimits, the adjunction

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \rightleftarrows \mathcal{M} : \mathrm{const}$$

is

- always a Quillen adjunction with respect to the projective model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ if it exists (see Remark 5.1.9)
- sometimes (but not always) a Quillen adjunction if $\mathcal{C}$ is additionally a Reedy category and we equip $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ with the Reedy model structure, e.g. in the case that $\mathcal{C}$ has *fibrant constants* (see Definition 5.1.22).

Example 1.2.32. Given a model ∞-category $\mathcal{M}$ and an ∞-category $\mathcal{C}$ such that $\mathcal{M}$ admits $\mathcal{C}$-shaped limits, the adjunction

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{M}) : \mathrm{lim}$$

is

- always a Quillen adjunction with respect to the injective model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ if it exists (see Remark 5.1.9)
- sometimes (but not always) a Quillen adjunction if $\mathcal{C}$ is additionally a Reedy category and we equip $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ with the Reedy model structure, e.g. in the case that $\mathcal{C}$ has *cofibrant constants* (see Definition 5.1.22).

1.2.3 Speculative examples

Speculation 1.2.33. Let us temporarily refer to the model structure of Definition 1.4.5 as the "strong" Kan–Quillen model structure on $s\mathcal{S}$: its subcategory $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ of weak equivalences is created by the geometric realization functor $|-| : s\mathcal{S} \to \mathcal{S}$, and it is cofibrantly generated by the sets

$$I_{\mathrm{KQ}_{\mathrm{strong}}} = \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}$$

and

$$J_{\mathrm{KQ}_{\mathrm{strong}}} = \{\Lambda^n_i \to \Delta^n\}_{0\leq i\leq n\geq 1}.$$

Then, there should exist other Kan–Quillen model structures on $s\mathcal{S}$: these would have the same subcategory of weak equivalences, but would have more cofibrations. For instance, we might define "medium" and "weak" Kan–Quillen model structures by extending the set of generating cofibrations to be given by

$$I_{\mathrm{KQ}_{\mathrm{medium}}} = I_{\mathrm{KQ}_{\mathrm{strong}}} \cup \{S^i \odot \partial\Delta^n \to S^i \odot \Delta^n\}_{i\geq 1, n\geq 0}$$

and

$$I_{\mathrm{KQ}_{\mathrm{weak}}} = I_{\mathrm{KQ}_{\mathrm{medium}}} \cup \{S^i \odot \Delta^n \to \mathrm{pt}_{\mathcal{S}} \odot \Delta^n\}_{i\geq 1, n\geq 0},$$

with sets of generating acyclic cofibrations extended to match. There would then exist Quillen equivalences

$$s\mathcal{S}_{\mathrm{KQ}_{\mathrm{strong}}} \rightleftarrows s\mathcal{S}_{\mathrm{KQ}_{\mathrm{medium}}} \rightleftarrows s\mathcal{S}_{\mathrm{KQ}_{\mathrm{weak}}}$$

(in which all underlying functors are $\mathrm{id}_{s\mathcal{S}}$): moving to the right, more and more maps become cofibrations, while moving to the left, more and more maps become fibrations. This explains the terminology: the geometric realization functor (being a colimit) already plays well with colimits, and hence it does not seem to be particularly useful to identify more maps as cofibrations. On the other hand, these variants would enjoy certain features not shared by $s\mathcal{S}_{\mathrm{KQ}_{\mathrm{strong}}}$.

- The model ∞-category $s\mathcal{S}_{\mathrm{KQ}_{\mathrm{medium}}}$ would be obtained by closing up the generating sets under the tensoring, i.e. by performing an *enriched* small object argument (see Remark 1.3.7), which would easily provide *functorial* factorizations.

- The model ∞-category $s\mathcal{S}_{\mathrm{KQ}_{\mathrm{weak}}}$ would have all objects cofibrant, just as $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. However, it seems that the primary importance of this fact is that it implies left properness, so this may not be much of an advantage, since $s\mathcal{S}_{\mathrm{KQ}_{\mathrm{strong}}}$ is already left proper (for essentially trivial reasons).

The existence of these alternate Kan–Quillen model structures almost follows easily from the recognition theorem for cofibrantly generated model ∞-categories (1.3.11) and our proof of Theorem 1.4.4 (the main theorem of this chapter, which asserts the existence of $s\mathcal{S}_{\mathrm{KQ}_{\mathrm{strong}}}$); more precisely, using the results presented here, it is straightforward to verify all the conditions given in Theorem 1.3.11 except for condition (3). On the other hand, it seems eminently plausible that Smith's recognition theorem for combinatorial model categories (see Proposition T.A.2.6.8), especially its simpler special case given by Lurie (see Proposition T.A.2.6.13), would admit a straightforward generalization to the model ∞-categorical setting. From here, a version of Proposition T.A.2.6.13 would guarantee that *any* set of maps $I \subset s\mathcal{S}$ containing $I_{\mathrm{KQ}_{\mathrm{strong}}}$ would constitute a set of generating cofibrations for a model structure on $s\mathcal{S}$ with subcategory of weak equivalences given by $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$. (Condition (1) would be satisfied by a combination of variants of Example T.A.2.6.11 and Corollary T.A.2.6.12, condition (2) would be true (even without the assumption that $I \supset I_{\mathrm{KQ}_{\mathrm{strong}}}$) because geometric realization (being a colimit) commutes with pushouts, and condition (3) would follow from the fact that $\mathrm{rlp}(I) \subset \mathrm{rlp}(I_{\mathrm{KQ}_{\mathrm{strong}}}) \subset \mathbf{W}_{\mathrm{KQ}}$.)

Remark 1.2.34. It seems that the model ∞-category $s\mathcal{S}_{\mathrm{KQ}_{\mathrm{weak}}}$ of Speculation 1.2.33 would be closely related to the Moerdijk model structure on $ss\mathcal{S}\mathrm{et}$ described in Remark 1.7.3. Moreover, a putative set of generating acyclic cofibrations

$$J_{\mathrm{KQ}_{\mathrm{strong}}} \cup \{S^j \odot \Lambda^n_i \to S^j \odot \Delta^n\}_{j \geq 1, 0 \leq i \leq n \geq 1}$$

seems closely related to our comparison with the π_*-Kan condition in Remark 1.6.13.

Speculation 1.2.35. There should exist a Kan–Quillen model structure on the ∞-category $s\mathcal{S}\mathrm{p}$ of simplicial spectra. However, the "levelwise infinite loopspace" functor $s\Omega^\infty : s\mathcal{S}\mathrm{p} \to s\mathcal{S}_*$ isn't conservative, and so it wouldn't make much sense to lift this from a Kan–Quillen model structure on $s\mathcal{S}_*$.[10] (By contrast, the usual model structure on simplicial abelian groups is lifted from $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.) This should give rise to model structures e.g. on simplicial module spectra over a (simplicial) ring spectrum, and in other stable contexts. Alternatively, the putative applications of such model structures might all be handled sufficiently by resolution model structures.

Speculation 1.2.36. Given a simplicial model category $\mathcal{M}_\bullet$, we can consider its underlying $s\mathcal{S}\mathrm{et}$-enriched category as an ∞-category $\mathcal{M}_\bullet \in (\mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ in its own right. Closing up the defining simplicial subcategories $\mathbf{W}_\bullet, \mathbf{C}_\bullet, \mathbf{F}_\bullet \subset \mathcal{M}_\bullet$ to subcategories in the ∞-categorical sense should then determine a model ∞-category

[10] On the other hand, this issue would vanish if we were to restrict to *connective* spectra.

structure (though there is some subtlety in ensuring that the model ∞-category axioms continue to hold in the ∞-categorical sense).

As an illustrative example of this apparent phenomenon, let use consider the case of the simplicial model category $(s\mathcal{S}\mathrm{et}_{\bullet})_{\mathrm{KQ}}$. Let us write

$$\mathbf{W}_{\mathrm{h.e.}} \subset \mathbf{W}_{\mathrm{w.h.e.}} \subset s\mathcal{S}\mathrm{et}$$

for the subcategories of *homotopy equivalences* and *weak homotopy equivalences*, respectively (with respect to the Kan–Quillen model structure (so that by definition, $\mathbf{W}_{\mathrm{w.h.e.}} = \mathbf{W}_{\mathrm{KQ}}$)). Then, it is not hard to see that the canonical functor

$$s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{h.e.}}^{-1}] \to s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}] \simeq \mathrm{ho}(\mathcal{S})$$

on homotopy categories is a left localization. Moreover, every acyclic fibration in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is actually a homotopy equivalence, so that *every* map is simplicially homotopic to a cofibration. Taking these two facts together, it seems reasonable to guess that this procedure yields a left localization model structure (in the sense of Example 1.2.12) corresponding to a left localization adjunction

$$s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{\mathrm{h.e.}}^{-1}]\!] \rightleftarrows s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]\!] \simeq \mathcal{S}.$$

Speculation 1.2.37. There should exist a model structure on simplicial objects in algebras over an operad which accounts for *free resolutions*. For example, this would recover as a special case the model structure on simplicial commutative rings alluded to in [Qui, §2] (and laid out explicitly in [Sch97, §3.1]), and would provide a framework organizing the "prove it for a free simplicial resolution, then prove that it commutes with (sifted) colimits" arguments involving "B-structured n-disk algebras" (e.g. $\mathbb{E}_n$ algebras) that appear throughout [AF].

Speculation 1.2.38. There should exist a *Joyal model structure* on $s\mathcal{S}$, whose fibrant objects are the "homotopical quasicategories", namely those $Y \in s\mathcal{S}$ such that for all $n \geq 0$ and all $0 < i < n$, the inner horn inclusion $\Lambda^n_i \to \Delta^n$ induces a surjection

$$Y_n \simeq \mathrm{M}_{\Delta^n}(Y) \to \mathrm{M}_{\Lambda^n_i}(Y)$$

in $\mathcal{S}$. This should moreover participate in a left Bousfield localization $s\mathcal{S}_{\mathrm{Joyal}} \rightleftarrows s\mathcal{S}_{\mathrm{KQ}_{\mathrm{weak}}}$ with the "weak" Kan–Quillen model structure of Speculation 1.2.33.

The proof of the Joyal model structure on $s\mathcal{S}$ would presumably follow that of the one on $s\mathcal{S}\mathrm{et}$ fairly closely; a short and streamlined exposition of the latter is given in [DS11, Appendix C] (in contrast with the one given in §T.2.2.5, which proceeds by using the model category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$.)

There should similarly exist other model ∞-categories which present $\mathcal{C}\mathrm{at}_\infty$ based on model 1-categories that do, for instance

- a Barwick–Kan model structure on the ∞-category of relative ∞-categories (see Definitions 2.1.1 and 2.1.16),
- a Bergner model structure on the ∞-category of $s\mathcal{S}$-enriched ∞-categories (see Definition 4.1.12), and
- a Bergner model structure on "Segal pre-categories" in $s\mathcal{S}$, i.e. those simplicial spaces whose 0^{th} space is discrete.

Speculation 1.2.39. The central theorem regarding *formal moduli problems* for $\mathbb{E}_n$ algebras, described in [Lur10] (and made precise in [Lur11a]), posits an equivalence between the ∞-categories of formal $\mathbb{E}_n$ moduli problems and of augmented $\mathbb{E}_n$ algebras (see [Lur10, Theorem 6.20]). This equivalence takes an augmented $\mathbb{E}_n$ algebra to its associated *Maurer–Cartan functor*.

The inverse equivalence is somewhat trickier to describe. When the formal $\mathbb{E}_n$ moduli problem is *affine* or *pro-affine* (i.e. corepresented by a small $\mathbb{E}_n$ algebra or by a pro-object in such), then this inverse equivalence is implemented by *Koszul duality* (see [Lur10, Example 8.5 and Remark 8.9]). However, more generally one must take a resolution of the given formal $\mathbb{E}_n$ moduli problem by a *smooth hypercovering* consisting of pro-affine ones, apply Koszul duality levelwise to this simplicial object, and then take the colimit. Thus, in general this inverse equivalence is implemented by *the derived functor of Koszul duality* (in analogy with e.g. the statement that the cotangent complex is the derived functor of derivations).

Hence, there should then exist a model structure on the ∞-category of simplicial objects in formal $\mathbb{E}_n$ moduli problems (presumably related to the model structure of Speculation 1.2.37): appropriate levelwise pro-affine objects would be cofibrant, smooth hypercoverings would be acyclic fibrations, and then e.g. for an arbitrary formal $\mathbb{E}_n$ moduli problem, [Lur10, Proposition 8.19] would provide a cofibrant replacement by an acyclic fibration.

1.3 Cofibrantly generated model ∞-categories

In this section we discuss *cofibrantly generated* model ∞-categories. As in the classical situation, these are model structures which are determined by a relatively small amount of data, namely by a set of *generating cofibrations* and a set of *generating acyclic cofibrations*, which simultaneously

- generate the subcategories $\mathbf{C}$ and $\mathbf{W} \cap \mathbf{C}$, respectively, in a suitable sense (as their names suggest),

- detect the subcategories $\mathbf{W} \cap \mathbf{F}$ and $\mathbf{F}$, respectively, in accordance with Remark 1.1.11, and
- are suited for obtaining the factorizations required by Definition 1.1.1.

In §1.4, we will use this setup to define the Kan–Quillen model structure on the ∞-category $s\mathcal{S}$ of simplicial spaces.

We begin with a sequence of definitions. They are all direct generalizations of their 1-categorical counterparts (after replacing a set of maps with a set of homotopy classes of maps), and it is routine to verify that they enjoy completely analogous properties (see e.g. [Hir03, §10.4-5]). We also point out once and for all that these definitions do not depend on choices of representatives for the elements of the given set I of homotopy classes of maps.

Definition 1.3.1. Given a set I of homotopy classes of maps in $\mathcal{C}$, the subcategory I-inj $\subset \mathcal{C}$ of ***I-injectives*** is the subcategory of maps with $\mathrm{rlp}(I)$.

Definition 1.3.2. Given a set I of homotopy classes of maps in $\mathcal{C}$, the subcategory I-cof $\subset \mathcal{C}$ of ***I-cofibrations*** is the subcategory of maps with $\mathrm{llp}(\mathrm{rlp}(I))$.

Definition 1.3.3. Assume that $\mathcal{C}$ admits pushouts and sequential colimits. Given a set I of homotopy classes of maps in $\mathcal{C}$, the subcategory I-cell $\subset \mathcal{C}$ of ***relative I-cell complexes*** is the subcategory of maps that can be constructed as transfinite compositions of pushouts of elements of I. An object is called an ***I-cell complex*** if its unique map from $\varnothing_{\mathcal{C}}$ is a relative I-cell complex. Note that (I-cell)-inj $= I$-inj and that I-cell $\subset I$-cof.

Definition 1.3.4. Given a cardinal κ, an object $x \in \mathcal{C}$ is called ***κ-small relative to I*** if for every regular cardinal $\lambda \geq \kappa$ and every λ-sequence $\{y_\beta\}_{\beta<\lambda}$ of relative I-cell complexes, $\mathrm{colim}_{\beta<\lambda} \hom_{\mathcal{C}}(x, y_\beta) \xrightarrow{\sim} \hom_{\mathcal{C}}(x, \mathrm{colim}_{\beta<\lambda} y_\beta)$. An object of $\mathcal{C}$ is called ***small relative to I*** if it is κ-small relative to I for some cardinal κ. An object of $\mathcal{C}$ is called ***κ-small*** if it is κ-small relative to $\mathcal{C}$, and is called ***small*** if it is κ-small for some cardinal κ.

Definition 1.3.5. We say that a set I of homotopy classes of maps in $\mathcal{C}$ ***permits the small object argument*** if the sources of its elements are small relative to I.

We now come to the key result on which the theory of cofibrantly generated model ∞-categories rests, which we refer to as the ***small object argument***.

Proposition 1.3.6. *Suppose that $\mathcal{C}$ is an ∞-category that admits pushouts and sequential colimits, and suppose that I is a set of homotopy classes of maps in $\mathcal{C}$ which*

permits the small object argument. Then every map in $\mathcal{C}$ admits a factorization into a relative I-cell complex followed by an I-injective.

Proof. The proof runs identically to that of [Hir03, Proposition 10.5.16], except that we take a coproduct over *homotopy classes* of commutative squares and we choose arbitrary representatives for these classes when forming the pushout. (See also [Lur11a, Proposition 1.4.7].) □

Remark 1.3.7. Although the above definitions are analogous to the classical ones, there is one wrinkle that appears in the ∞-categorical case. Namely, the classical small object argument is visibly functorial: one simply takes a coproduct over the *set* of commutative squares (and no choices of representatives of homotopy classes is necessary). In an ∞-category, however, we instead have a *space* of commutative squares. Thus, it would be more natural in some respects for us to instead carry out an "$\mathcal{S}$-enriched small object argument", which would then be similarly functorial. However, this has two drawbacks for us.

First of all, to do so would shrink the right class of the associated weak factorization system. (Indeed, in enriched category theory, passing from an unenriched lifting condition to an enriched lifting condition is equivalent to closing up the given set of maps under tensors with the enriching category.) We will be making much use of fibrations and acyclic fibrations (for instance in the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$ of Definition 1.4.5), and so it is in our best interest to keep these classes as large as possible; given that the stronger statement holds (i.e. that in our cases of interest, it suffices to check an unenriched lifting condition), it seems reasonable to incorporate it into the theory.

More crucially, however, a key feature of the resolution model structure is that, given a model ∞-category $\mathcal{M}$ with chosen set of generators $\mathcal{G}$, we will obtain a cofibrant replacement of an object $X \in \mathcal{M}$ by an object $Y_\bullet \in s\mathcal{M}_{\mathrm{res}}^c$ which is given in each level as a *coproduct* of elements of $\mathcal{G}$ (see [DKS93, 3.3 and 5.5], Lemma T.5.5.8.13, and Proposition T.5.5.8.10(5)), as opposed to some more elaborate construction involving tensorings with spaces. This is useful because, denoting by $F : \mathcal{M} \to \mathcal{A}$ our topology-to-algebra functor of interest (for instance, a homology theory E_*), it affords us an approximation of $F(X) \in \mathcal{A}$ by $F^{\mathrm{lw}}(Y_\bullet) \in s\mathcal{A}$, which is only useful inasmuch as it is algebraically accessible (for instance, a levelwise-projective simplicial E_*-module). If we were to use an $\mathcal{S}$-enriched small object argument to construct the cofibrantly generated resolution model structure, we would cease to have any control whatsoever over the value of the functor $F^{\mathrm{lw}} : s\mathcal{M}_{\mathrm{res}} \to s\mathcal{A}$ on these cofibrant replacements.

We now come to the main definition of this section.

Definition 1.3.8. A ***cofibrantly generated model ∞-category*** is a model ∞-category $\mathcal{M}$ such that there exist sets of homotopy classes of maps I and J, respectively called the ***generating cofibrations*** and the ***generating acyclic cofibrations***, both permitting the small object argument, such that $\mathbf{W} \cap \mathbf{F} = \mathrm{rlp}(I)$ and $\mathbf{F} = \mathrm{rlp}(J)$. (It follows from Remark 1.1.11 that also $\mathbf{C} = I\text{-cof}$ and $\mathbf{W} \cap \mathbf{C} = J\text{-cof}$.)

Example 1.3.9. Let $\mathcal{S}_{\mathrm{triv}}$ denote the trivial model structure of Example 1.2.2 on the ∞-category $\mathcal{S}$ of spaces. Combined with a few basic observations coming from the theory of CW complexes, Lemma 1.5.1 shows that $\mathcal{S}_{\mathrm{triv}}$ is cofibrantly generated by the sets $I^{\mathcal{S}}_{\mathrm{triv}} = \{S^{n-1} \to \mathrm{pt}_{\mathcal{S}}\}_{n \geq 0}$ and $J^{\mathcal{S}}_{\mathrm{triv}} = \varnothing_{\mathcal{S}\mathrm{et}}$.

Example 1.3.10. Recall the model structure $\mathcal{S}_{\pi_0}$ on the ∞-category $\mathcal{S}$ of Example 1.2.13: $\mathbf{W}_{\pi_0}$ is created by $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$, $\mathbf{C}_{\pi_0} = \mathcal{S}$, and $\mathbf{F}_{\pi_0} = \mathrm{rlp}(\mathbf{W}_{\pi_0})$. This model structure is not cofibrantly generated (or at least, not obviously so), but we nevertheless have that

$$\mathbf{F}_{\pi_0} = \mathrm{rlp}((I^{\mathcal{S}}_{\mathrm{triv}})_{\geq 2}) \cap \mathrm{rlp}'(\{S^0 \to \mathrm{pt}_{\mathcal{S}}\}),$$

where

- we define

$$(I^{\mathcal{S}}_{\mathrm{triv}})_{\geq 2} = \{S^{n-1} \to \mathrm{pt}_{\mathcal{S}}\}_{n \geq 2} = \{S^1 \to \mathrm{pt}_{\mathcal{S}}, S^2 \to \mathrm{pt}_{\mathcal{S}}, \ldots\},$$

  and

- by $\mathrm{rlp}'(\{S^0 \to \mathrm{pt}_{\mathcal{S}}\})$, we mean to only require a lift in those commutative squares for which the upper map factors through the terminal map $S^0 \to \mathrm{pt}_{\mathcal{S}}$.

This observation generalizes to the model ∞-category $\mathcal{S}_{\tau_{\leq n}}$ of Example 1.2.14, as indicated there.

We have the following ***recognition theorem*** for cofibrantly generated model ∞-categories.

Theorem 1.3.11. *Let $\mathcal{M}$ be an ∞-category which is both cocomplete and finitely complete, and let $\mathbf{W} \subset \mathcal{M}$ be a subcategory which is closed under retracts and satisfies the two-out-of-three property. Suppose I and J are sets of homotopy classes of maps in $\mathcal{M}$, both permitting the small object argument, such that*

(1) $J\text{-cof} \subset (I\text{-cof} \cap \mathbf{W})$,

(2) $I\text{-inj} \subset (J\text{-inj} \cap \mathbf{W})$, *and*

(3) either

(a) $(I\text{-cof} \cap \mathbf{W}) \subset J\text{-cof}$ *or*

(b) $(J\text{-inj} \cap \mathbf{W}) \subset I\text{-inj}$.

Then the sets I and J define a cofibrantly generated model structure on $\mathcal{M}$ whose weak equivalences are $\mathbf{W}$.

Proof. With the small object argument in hand, the proof runs identically to that of [Hir03, Theorem 11.3.1]. □

We also have the following ***lifting theorem*** for cofibrantly generated model ∞-categories.

Theorem 1.3.12. *Let $\mathcal{M}$ be a cofibrantly generated model ∞-category with generating cofibrations I and generating acyclic cofibrations J, and let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjunction with $\mathcal{N}$ finitely bicomplete. If FI and FJ both permit the small object argument and G takes relative FJ-cell complexes into $\mathbf{W}_{\mathcal{M}}$, then FI and FJ define a cofibrantly generated model structure on $\mathcal{N}$ in which $\mathbf{W}_{\mathcal{N}}$ is created by G. Moreover, with respect to this lifted model structure, the adjunction $F \dashv G$ becomes a Quillen adjunction.*

Proof. The proof runs identically to that of [Hir03, Theorem 11.3.2]. □

1.4 The definition of the Kan–Quillen model structure

We are now in a position to state the main result of this chapter (Theorem 1.4.4), which gives a systematic way of manipulating simplicial spaces in their capacity as "presentations of spaces" via the geometric realization functor. This sits in precise analogy with the 1-category of simplicial sets, and so we begin with the following recollection.

Definition 1.4.1. The ***Kan–Quillen model structure*** on $s\mathcal{S}\mathrm{et}$, denoted $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, is the proper model structure which is cofibrantly generated by the sets $I_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}} = \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}$ and $J_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}} = \{\Lambda_i^n \to \Delta^n\}_{0\leq i\leq n\geq 1}$ (see e.g. [Hir03, Example 11.1.6 and Theorem 13.1.13]).

In order to be precise regarding model ∞-categories (and in case the reader has forgotten the classical definition), we also give the following.

Definition 1.4.2. A model ∞-category is called ***left proper*** if its weak equivalences are preserved under pushout along cofibrations, and dually is called ***right proper*** if its weak equivalences are preserved under pullback along fibrations. A model ∞-category is called ***proper*** if it is both left proper and right proper.

The categories of simplicial sets and simplicial spaces are related in the following way.

Notation 1.4.3. Recall the adjunction $\pi_0 : \mathcal{S} \rightleftarrows \mathcal{S}\mathrm{et} : \mathrm{disc}$. Applying $\mathrm{Fun}(\mathbf{\Delta}^{op}, -)$, this induces an adjunction which we again denote by $\pi_0 : s\mathcal{S} \rightleftarrows s\mathcal{S}\mathrm{et} : \mathrm{disc}$. We will generally omit this right adjoint from the notation unless we mean to emphasize it.

Using this terminology and notation, we can now state the main result of this chapter.

Theorem 1.4.4. *The sets* $I^{s\mathcal{S}}_{\mathrm{KQ}} = \mathrm{disc}(I^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}})$ *and* $J^{s\mathcal{S}}_{\mathrm{KQ}} = \mathrm{disc}(J^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}})$ *define a proper, cofibrantly generated model structure on* $s\mathcal{S}$*, in which the weak equivalences are created by the geometric realization functor* $|-| : s\mathcal{S} \to \mathcal{S}$*.*

Proof. We appeal to 1.3.11, verifying

- condition (1) in Proposition 1.7.1,
- condition (2) in Proposition 1.7.2,
- condition (3)(b) in Proposition 1.7.9, and
- that $I^{s\mathcal{S}}_{\mathrm{KQ}}$ and $J^{s\mathcal{S}}_{\mathrm{KQ}}$ both permit the small object argument in Corollary 1.5.3.

The weak equivalences are closed under retracts and satisfy the two-out-of-three property because they are pulled back from a class of equivalences. Lastly, left properness is immediate since the weak equivalences are created by a left adjoint (which commutes with pushouts), and right properness is proved as Corollary 1.6.8. □

Definition 1.4.5. In analogy with Definition 1.4.1, we also refer to the model structure on $s\mathcal{S}$ defined by Theorem 1.4.4 as the ***Kan–Quillen model structure***, denoted $s\mathcal{S}_{\mathrm{KQ}}$.

Remark 1.4.6. In the theory of model 1-categories, one of the most useful consequences of right properness is that a pullback in which just one of the maps is a fibration is already a homotopy pullback (and dually for left properness). This remains true in the theory of model ∞-categories; with the theory of homotopy co/limits in model ∞-categories in hand (see §5.1.2), the proof runs essentially identically (see e.g. [Hir03, §13.3]). However, for the sake of self-containment, we will also directly

prove this consequence of the right properness of $s\mathcal{S}_{\mathrm{KQ}}$ as Corollary 1.6.7. (In fact, we use this as an *input* to the proof of right properness in Corollary 1.6.8.) On the other hand, the dual corollary of the left properness of $s\mathcal{S}_{\mathrm{KQ}}$ is as trivial to verify as the left properness of $s\mathcal{S}_{\mathrm{KQ}}$ itself.

Remark 1.4.7. In the theory of model 1-categories, there are many other adjectives that one might attach to a model structure: combinatorial, cellular, tractable, etc. The model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$ enjoys completely analogous properties to those enjoyed by $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. However, we won't need these observations for now, so we just leave them here as a remark. (The model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$ will also be a *simplicial model ∞-category*, the analogous notion to that of a simplicial model 1-category (see Definition 5.6.2). (This is simply to say that it is a *symmetric monoidal model ∞-category* (see Definition 5.5.5 and Example 5.5.3).))

Remark 1.4.8. Note that if we apply the small object argument for $I^{s\mathcal{S}}_{\mathrm{KQ}}$ or $J^{s\mathcal{S}}_{\mathrm{KQ}}$ to a map in $s\mathcal{S}$ whose source is in $s\mathcal{S}\mathrm{et}$, then the intermediate object will also be in $s\mathcal{S}\mathrm{et}$. In particular, by the usual transfinite induction arguments, $\mathrm{rlp}(I^{s\mathcal{S}}_{\mathrm{KQ}}) = \mathrm{rlp}(\mathrm{disc}(\mathbf{C}^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}}))$ and $\mathrm{rlp}(J^{s\mathcal{S}}_{\mathrm{KQ}}) = \mathrm{rlp}(\mathrm{disc}((\mathbf{W}\cap\mathbf{C})^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}}))$. We will use these facts without further comment.

Remark 1.4.9. The adjunction $\pi_0 : s\mathcal{S} \rightleftarrows s\mathcal{S}\mathrm{et} : \mathrm{disc}$ could be used to lift the cofibrantly generated Kan–Quillen model structure on $s\mathcal{S}$ to the one on $s\mathcal{S}\mathrm{et}$ via the lifting theorem (1.3.12), except of course that that would be totally circular: the construction of $s\mathcal{S}_{\mathrm{KQ}}$ takes the existence of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ as input. As observed in Example 1.2.22, this adjunction is even a Quillen equivalence, whose derived adjunction is the identity adjunction on their common localization $\mathcal{S}$.

Remark 1.4.10. In fact, extending Remark 1.4.9, note that all three defining subcategories of the model category $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ are pulled back from the corresponding defining subcategories of the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$ along the inclusion $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$.

To codify the observation of Remark 1.4.10, we introduce the following.

Definition 1.4.11. Let $\mathcal{M}$ be a model ∞-category. We say that a model structure on a subcategory $\mathcal{N} \subset \mathcal{M}$ defines a ***model subcategory*** of $\mathcal{M}$ if we have

- $\mathbf{W}_{\mathcal{N}} = \mathcal{N} \cap \mathbf{W}_{\mathcal{M}} \subset \mathcal{M}$,
- $\mathbf{C}_{\mathcal{N}} = \mathcal{N} \cap \mathbf{C}_{\mathcal{M}} \subset \mathcal{M}$, and
- $\mathbf{F}_{\mathcal{N}} = \mathcal{N} \cap \mathbf{F}_{\mathcal{M}} \subset \mathcal{M}$.

Notation 1.4.12. As $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is a model subcategory of $s\mathcal{S}_{\mathrm{KQ}}$ by Remark 1.4.10, it will usually be unambiguous to omit the superscripts $s\mathcal{S}\mathrm{et}$ and $s\mathcal{S}$ from their defining subcategories: we will usually just write $\mathbf{W}_{\mathrm{KQ}}$, $\mathbf{C}_{\mathrm{KQ}}$, or $\mathbf{F}_{\mathrm{KQ}}$, leaving the ambient

∞-category implicit unless we mean to draw specific attention to it, as promised in Notation 1.1.2. (Of course, since the purpose of this chapter is to construct the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$, one would be safest to assume that we mean to refer to the subcategories of $s\mathcal{S}$ instead of those of $s\mathcal{S}\mathrm{et}$ when no superscript is included.) Similarly, we will henceforth generally simply denote

- both $I^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}}$ and $I^{s\mathcal{S}}_{\mathrm{KQ}}$ by I_{KQ}, and
- both $J^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}}$ and $J^{s\mathcal{S}}_{\mathrm{KQ}}$ by J_{KQ}.

Notation 1.4.13. In the course of proving Theorem 1.4.4, it will be important to take care in distinguishing which facts have been proved and which facts have not. Otherwise, our arguments might appear to be circular (for example as discussed in Remark 1.7.7). Thus, in order to be totally clear about this distinction, we take the following conventions regarding maps of simplicial sets and of simplicial spaces.

- We will only decorate our arrows if the corresponding property is relevant to the argument.
- We will only use the decoration $\xrightarrow{\approx}$ if we've actually proved that the map becomes a weak equivalence upon geometric realization.
- When working in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} \subset s\mathcal{S}_{\mathrm{KQ}}$, we will use all standard results and notation.
- By the nature of the arguments, the only cofibrations in $s\mathcal{S}_{\mathrm{KQ}}$ that appear will actually lie in the subcategory $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. On the other hand, rather than write $\twoheadrightarrow$ or $\xrightarrow[\approx]{}$ for the indicated maps in $s\mathcal{S}_{\mathrm{KQ}}$, we will instead label the arrows with their lifting properties (so $\mathrm{rlp}(J_{\mathrm{KQ}})$ or $\mathrm{rlp}(I_{\mathrm{KQ}})$, respectively).
- For convenience, we will still write $s\mathcal{S}^{f}_{\mathrm{KQ}}$ for those objects whose terminal map has $\mathrm{rlp}(J_{\mathrm{KQ}})$. (On the other hand, we will simply have $s\mathcal{S}^{c}_{\mathrm{KQ}} = s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$.)

1.5 Auxiliary results on spaces and simplicial spaces

In this section, we collect some auxiliary results regarding spaces and simplicial spaces which will be necessary for our proof of Theorem 1.4.4 (the bulk of which will be given in §1.7).

We begin with an easy folkloric result, which gives a criterion for a map of spaces to be an equivalence.

Lemma 1.5.1. *Let $Y \xrightarrow{\varphi} Z$ be a map in $\mathcal{S}$. Then φ is n-connected iff it has* $\mathrm{rlp}(\{S^{i-1} \to \mathrm{pt}_{\mathcal{S}}\}_{0\leq i\leq n})$. *In particular, φ is an equivalence iff it has* $\mathrm{rlp}(\{S^{n-1} \to \mathrm{pt}_{\mathcal{S}}\}_{n\geq 0})$.

Proof. First, note that if $Y \to Z$ has $\mathrm{rlp}(\{S^{i-1} \to \mathrm{pt}_{\mathcal{S}}\})$, then the map $[S^{i-1}, Y]_{\mathcal{S}} \to [S^{i-1}, Z]_{\mathcal{S}}$ is an inclusion. Since a map off of S^{i-1} is basedly nullhomotopic iff it's freely so, this means that for any basepoint $y \in Y$, the map $\pi_{i-1}(Y, y) \to \pi_{i-1}(Z, \varphi(y))$ is an inclusion.

On the other hand, considering the map $S^{i-1} \to \mathrm{pt}_{\mathcal{S}}$ as the standard inclusion $S^{i-1} \to D^i$, we see that if we begin with the constant map $S^{i-1} \to Y$ at some point $y \in Y$, then an extension of the composite $S^{i-1} \to Y \to Z$ over $S^{i-1} \to D^i$ really just selects a map $S^i \to Z$ which is based at $\varphi(y)$. So, if $Y \to Z$ has $\mathrm{rlp}(\{S^{i-1} \to \mathrm{pt}_{\mathcal{S}}\})$, then also $\pi_i(Y, y) \to \pi_i(Z, \varphi(y))$ is a surjection for any basepoint $y \in Y$.

Combining these two consequences of $Y \to Z$ having $\mathrm{rlp}(\{S^{i-1} \to \mathrm{pt}_{\mathcal{S}}\})$ proves both claims. □

We now turn to simplicial spaces. We begin with the necessary smallness results.

Lemma 1.5.2. *An object of $s\mathcal{S}$et with finitely many nondegenerate simplices is ω-small as an object of $s\mathcal{S}$.*

Proof. This follows from the fact that finite sets are small in $\mathcal{S}$ and from the inductive definition of a map of simplicial objects. □

Corollary 1.5.3. *The sets I_{KQ} and J_{KQ} permit the small object argument.*

Proof. This follows from Lemma 1.5.2. □

Many of the remaining results of this chapter will rely critically on the following one. Its proof is rather involved, and so we postpone it to §1.8. Roughly speaking, this result concerns the inductive construction of a presentation of a map in $\mathcal{S}$ by a map in $s\mathcal{S}$ whose source lies in $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$.

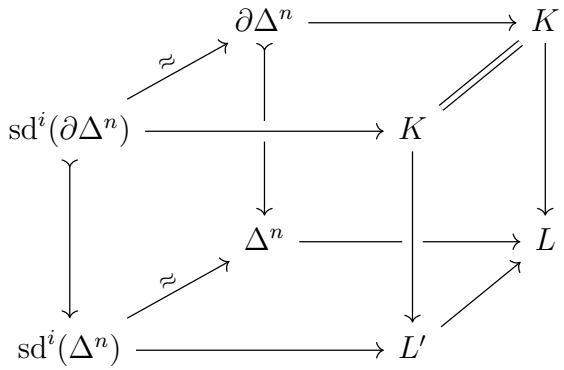
Lemma 1.5.4. *Suppose we are given any $W \in s\mathcal{S}$, any $K \in s\mathcal{S}$et, and any pushout*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & K \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & L \end{array}$$

in $s\mathcal{S}\mathrm{et}$. *Suppose further that we are given any point of the pullback*

$$\lim \left(\begin{array}{ccc} & & \hom_{s\mathcal{S}}(K,W) \\ & & \downarrow \\ \hom_{\mathcal{S}}(|L|,|W|) & \longrightarrow & \hom_{\mathcal{S}}(|K|,|W|) \end{array} \right).$$

Then there exists some $i \geq 0$ *such that if the front square in the cube*



is also a pushout in $s\mathcal{S}\mathrm{et}$, *then* $L' \xrightarrow{\approx} L$ *in* $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ *and the map*

$$\hom_{s\mathcal{S}}(L',W) \to \lim \left(\begin{array}{ccccc} & & & & \hom_{s\mathcal{S}}(K,W) \\ & & & & \downarrow \\ \hom_{\mathcal{S}}(|L'|,|W|) & \xleftarrow{\sim} & \hom_{\mathcal{S}}(|L|,|W|) & \longrightarrow & \hom_{\mathcal{S}}(|K|,|W|) \end{array} \right)$$

in $\mathcal{S}$ *is surjective onto the chosen point.*

Remark 1.5.5. Lemma 1.5.4 is not as strong as one might hope. First of all, it would be nice if the last map in its statement were actually a surjection, but to deduce this we would need to be able to bound the number i as we run through the path components of the pullback, which does not appear to be possible. But even more seriously, if instead we have a cofibration $K \rightarrowtail L$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ which can only be obtained through multiple pushouts of maps in I_{KQ}, then Lemma 1.5.4 cannot be made to guarantee the existence of an extension $L' \to W$ in $s\mathcal{S}$ (for some $L' \xrightarrow{\approx} L$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$) modeling the chosen extension $|L| \to |W|$ in $\mathcal{S}$.

For instance, suppose that $L = \Delta^1 \times \Delta^1$, and that the map $K \to L$ is the inclusion of its boundary (so that K is a simplicial square, and the map $K \to L$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$

presents the map $S^1 \to \mathrm{pt}_{\mathcal{S}}$ in $\mathcal{S}$). The minimal way to present this as a composition of pushouts of maps in I_{KQ} is as

$$K \rightarrowtail M \rightarrowtail N \rightarrowtail L,$$

where M is the 1-skeleton of L and the latter two maps are each obtained by attaching a 2-simplex. However, when we attempt to extend our given map $K \to W$ along $K \to M$ using Lemma 1.5.4, we may need to subdivide the 1-simplex that we're attaching, and so we only obtain an extension $M' \to W$ in $s\mathcal{S}$ for some factorization $K \to M' \xrightarrow{\approx} M$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. If *any* such subdivisions are required, then the two remaining holes to be filled in M' will now have at least four edges each, and so we are no closer to "filling the hole" than when we started.

On the other hand, in the case that the map $K \to L$ is $\varnothing_{s\mathcal{S}\mathrm{et}} \to \Delta^0$, then no subdivisions are required (indeed, subdivision preserves both $\varnothing_{s\mathcal{S}\mathrm{et}}$ and Δ^0, or alternatively we can see this from the fact that the map $W_0 \to |W|$ is a surjection). Thus, if we are given any $W \in s\mathcal{S}$ and any map $S^n \to |W|$ in $\mathcal{S}$, we can first present the composite $\mathrm{pt}_{\mathcal{S}} \to S^n \to |W|$ in $\mathcal{S}$ by some map $\Delta^0 \to W$ in $s\mathcal{S}$, and then taking $K = \Delta^0$ and forming the pushout

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & L \end{array}$$

in $s\mathcal{S}\mathrm{et}$, we are guaranteed a factorization $\Delta^0 \to L' \xrightarrow{\approx} L$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ and a map $L' \to W$ in $s\mathcal{S}$ presenting the chosen map $|L| \simeq S^n \to |W|$ in $\mathcal{S}$. Since so many arguments in $\mathcal{S}$ go by considering arbitrary maps into a space from a sphere (e.g. recall Lemma 1.5.1), the existence of such a minimal model $\Delta^n/\partial\Delta^n \in s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ for the object $S^n \in \mathcal{S}$ seems like a real stroke of luck.

In any case, we do not expect Lemma 1.5.4 to be particularly useful in the long run: it is effectively supplanted by the fundamental theorem of model ∞-categories (6.1.9). (In particular, see Corollary 1.6.2 (and Remark 1.6.3).)

1.6 Fibrancy, fibrations, and the Ex^∞ functor

In this section, we undertake a study of

- the subcategory $s\mathcal{S}^f_{\mathrm{KQ}} \subset s\mathcal{S}_{\mathrm{KQ}}$ of fibrant objects in §1.6.1,
- the subcategory $\mathbf{F}_{\mathrm{KQ}} \subset s\mathcal{S}_{\mathrm{KQ}}$ of fibrations in §1.6.2, and

- an Ex^∞ endofunctor on $s\mathcal{S}$ in §1.6.3.

As we will see, all of these behave quite analogously to their classical counterparts in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

1.6.1 Fibrancy

We begin by studying fibrant objects. First of all, Lemma 1.5.4 admits a much cleaner analog when $W \in s\mathcal{S}_{\mathrm{KQ}}$ is fibrant.

Lemma 1.6.1. *In Lemma 1.5.4, if $W \in s\mathcal{S}^f_{\mathrm{KQ}}$ then we may take $i = 0$. Hence, the map*

$$\hom_{s\mathcal{S}}(L, W) \to \lim \left(\begin{array}{ccc} & & \hom_{s\mathcal{S}}(K, W) \\ & & \downarrow \\ \hom_{\mathcal{S}}(|L|, |W|) & \longrightarrow & \hom_{\mathcal{S}}(|K|, |W|) \end{array} \right)$$

is a surjection.

Proof. We will argue using the diagram in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ shown in Figure 1.1, in which some of the objects and morphisms have yet to be constructed. First of all, recall

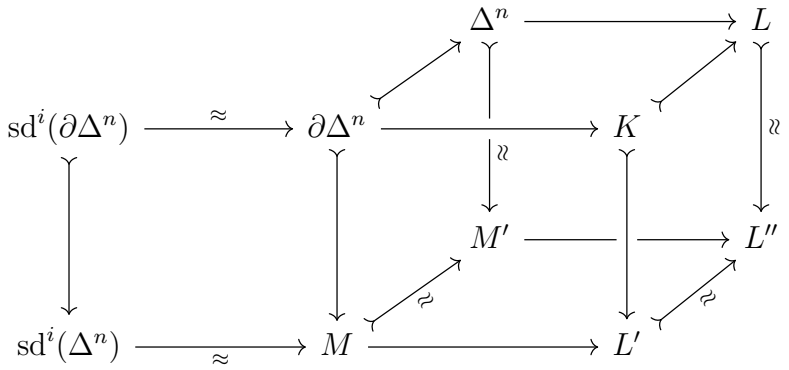
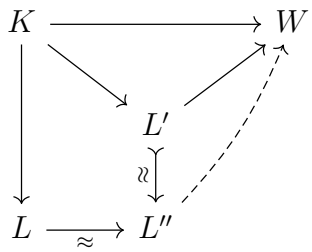


Figure 1.1: The diagram in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ used in the proof of Lemma 1.6.1.

that the top square of the cube is a pushout by the definition of L. Also, observe that we can also build L' via the iterated pushout in the front two squares, where we have $\mathrm{sd}^i(\Delta^n) \xrightarrow{\approx} M$ since $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is left proper. Next, choose any acyclic object $M' \in s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ admitting a cofibration from $M \coprod_{\partial\Delta^n} \Delta^n$, and use it to form the left face of the cube. (The maps from M and Δ^n to this pushout are cofibrations, which

is why the maps from M and Δ^n to M' are also cofibrations.) Then, define L'' by declaring that the bottom square of the cube is a pushout; by an easy diagram chase, the back square of the cube is therefore a pushout as well.

Now, since by assumption $W \to \mathrm{pt}_{s\mathcal{S}}$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$, we are guaranteed an extension



in $s\mathcal{S}$, and the composite map $L \to L'' \to W$ satisfies the same hypotheses as were required of the map $L' \to W$. Thus, we may take $i = 0$, as claimed.

Carrying out this same argument for all path components of the pullback implies that the indicated map is indeed a surjection. □

Corollary 1.6.2. *If $W \in s\mathcal{S}^f_{\mathrm{KQ}}$, then for any $K \rightarrowtail L$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, the map*

$$\hom_{s\mathcal{S}}(L, W) \to \lim \left(\begin{array}{ccc} & & \hom_{s\mathcal{S}}(K, W) \\ & & \downarrow \\ \hom_{\mathcal{S}}(|L|, |W|) & \longrightarrow & \hom_{\mathcal{S}}(|K|, |W|) \end{array} \right)$$

is a surjection.

Proof. First, we present the map $K \rightarrowtail L$ as a transfinite composition of pushouts of maps in I_{KQ}. Then, the result follows by transfinite induction, applying Lemma 1.6.1 at each successor ordinal and using the universal property of the colimit (which here is a colimit both in $s\mathcal{S}\mathrm{et}$ and in $s\mathcal{S}$) at each limit ordinal. □

Remark 1.6.3. In the special case that $K = \varnothing_{s\mathcal{S}\mathrm{et}}$, Corollary 1.6.2 reduces to the statement that for any $L \in s\mathcal{S}\mathrm{et} = s\mathcal{S}^c_{\mathrm{KQ}}$ and any $W \in s\mathcal{S}^f_{\mathrm{KQ}}$, the map $\hom_{s\mathcal{S}}(L, W) \to \hom_{\mathcal{S}}(|L|, |W|)$ is a surjection. This is a hint of the fundamental theorem of model ∞-categories (6.1.9) as applied to $s\mathcal{S}_{\mathrm{KQ}}$ (recall Remark 1.1.4).

Remark 1.6.4. Corollary 1.6.2 provides a basis for [BEBdBP, 30.10], which gives a complete characterization of the subcategory $\mathbf{W}^f_{\mathrm{KQ}} \subset s\mathcal{S}$ of weak equivalences between fibrant objects (and which is in turn the crucial ingredient of that paper).

1.6.2 Fibrations

We now turn from fibrant objects to fibrations:

- in §1.6.2.1 we lay out some general results on the interplay between fibrations and geometric realizations,
- in §1.6.2.2 we show that the left Quillen equivalence $\pi_0 : s\mathcal{S}_{\mathrm{KQ}} \to s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ preserves fibrations, and
- in §1.6.2.3 we give some comparisons with existing literature.

1.6.2.1 Fibrations and geometric realizations

The following result is crucial, and provides a basis for many of the convenient properties enjoyed by the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$. Its proof is relatively straightforward, though somewhat long (although not as long as it looks, since it contains so many diagrams).

Proposition 1.6.5. *Suppose the map $Y \to Z$ in $s\mathcal{S}$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$, and suppose we are given any point $\mathrm{pt}_{s\mathcal{S}} \xrightarrow{z} Z$. Let $F_z \in s\mathcal{S}$ be the fiber of $Y \to Z$ over z, and let $F_{|z|} \in \mathcal{S}$ be the fiber of $|Y| \to |Z|$ over $|z|$. Then the natural map $|F_z| \to F_{|z|}$ is an equivalence in $\mathcal{S}$.*

Proof. We use the criterion of Lemma 1.5.1. So, suppose that

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & |F_z| \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \longrightarrow & F_{|z|} \end{array}$$

is any commutative square in $\mathcal{S}$, for any $n \geq 0$. Since $F_z \to \mathrm{pt}_{s\mathcal{S}}$ also has $\mathrm{rlp}(J_{\mathrm{KQ}})$ as this property is closed under pullbacks, by Corollary 1.6.2 we may present the upper map in the above diagram as a map $\partial\Delta^n \to F_z$ in $s\mathcal{S}$.

From here, our argument will play back and forth between the diagrams shown in Figures 1.2 and 1.3. The former takes place in $s\mathcal{S}_{\mathrm{KQ}}$, while the latter takes place in $\mathcal{S}$; in both, many of the objects (and all of the dotted arrows) have yet to be constructed. For clarity, we proceed in steps.

(1) Given the composite map $\partial\Delta^n \to F_z \to Y$ in $s\mathcal{S}$ and its chosen extension

$$\begin{array}{ccccc} |\partial\Delta^n| & \longrightarrow & |F_z| & \longrightarrow & |Y| \\ \downarrow & & \downarrow & \nearrow & \\ |\Delta^n| & \longrightarrow & F_{|z|} & & \end{array}$$

in $\mathcal{S}$, by Lemma 1.5.4 there exists a factorization $\partial\Delta^n \to (\Delta^n)' \xrightarrow{\approx} \Delta^n$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ and a dotted arrow $(\Delta^n)' \to Y$ as in Figure 1.2 which models this extension in $\mathcal{S}$.

(2) For expository convenience, we consider the object $\Delta^0 \in s\mathcal{S}$ with its unique map $\Delta^0 \xrightarrow{\sim} \mathrm{pt}_{s\mathcal{S}}$ as selecting a composite map $|\Delta^0| \xrightarrow{\sim} |\mathrm{pt}_{s\mathcal{S}}| \xrightarrow{|z|} |Z|$.

(3) Choose any vertex of $(\Delta^n)'$, and use this to define $(\Delta^n)'' \in s\mathcal{S}\mathrm{et}$ by the pushout diagram

$$\begin{array}{ccc} \partial\Delta^1 & \xrightarrow{\approx} & (\Delta^n)' \sqcup \Delta^0 \\ \downarrow & & \downarrow \\ \Delta^1 & \xrightarrow[\approx]{} & (\Delta^n)'' \end{array}$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. Observe that both induced maps $\Delta^0 \to (\Delta^n)''$ and $(\Delta^n)' \to (\Delta^n)''$ are in $\mathbf{W}_{\mathrm{KQ}}$.

(4) Now, we have the solid commutative diagram in $\mathcal{S}$ of Figure 1.4, and hence we can obtain a dotted arrow $|(\Delta^n)''| \to |Z|$ therein making the entire diagram commute, as in Figure 1.3.

(5) Thus, we have a map $(\Delta^n)' \sqcup \Delta^0 \to Z$ in $s\mathcal{S}$ and a chosen extension

$$\begin{array}{ccc} |(\Delta^n)'| \sqcup |\Delta^0| & \longrightarrow & |Z| \\ \downarrow & \nearrow & \\ |(\Delta^n)''| & & \end{array}$$

in $\mathcal{S}$. (Note that the geometric realization functor $|-| : s\mathcal{S} \to \mathcal{S}$ commutes with colimits (being a left adjoint), and in particular with coproducts.) So by Lemma 1.5.4, there is a factorization $(\Delta^n)' \sqcup \Delta^0 \to (\Delta^n)''' \xrightarrow{\approx} (\Delta^n)''$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ and a dotted arrow $(\Delta^n)''' \to Z$ as in Figure 1.2 which models this extension in $\mathcal{S}$.

(6) It is easy to see that that in fact, we have $(\Delta^n)' \xrightarrow{\approx} (\Delta^n)'''$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ (for instance because $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is left proper, and so the defining map to $(\Delta^n)'''$ from some subdivision of Δ^1 is in $\mathbf{W}_{\mathrm{KQ}}$). Since the map $Y \to Z$ in $s\mathcal{S}$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$, it follows that there exists a lift $(\Delta^n)''' \to Y$ as in Figure 1.2.

(7) We now have the solid commutative diagram in $s\mathcal{S}$ of Figure 1.5, and so by the universal property of the pullback we can obtain a dotted arrow $\Delta^0 \to F_z$ making the entire diagram commute, as in Figure 1.2.

(8) Now, taking geometric realization gives the entire diagram in $\mathcal{S}$ of Figure 1.3 (including the dotted arrows). In particular, we obtain the desired lift

$$\begin{array}{ccc} |\partial\Delta^n| & \longrightarrow & |F_z| \\ \downarrow & \nearrow & \downarrow \\ |\Delta^n| & \longrightarrow & F_{|z|} \end{array}$$

in $\mathcal{S}$. (It is straightforward to see that this does indeed commute, as $F_{|z|}$ is defined as a pullback.) □

Remark 1.6.6. In either $s\mathcal{S}\mathrm{et}$ or $s\mathcal{S}$, one can always take the fiber of a map over a given point in its target. However, inasmuch as we are interested in simplicial sets and simplicial spaces as presenting spaces via geometric realization, we can view Proposition 1.6.5 as saying that in either case, this fiber is only "homotopically meaningful" – that is, it only computes the fiber in $\mathcal{S}$ – if the original map is a *fibration* in the corresponding Kan–Quillen model structure.

While Proposition 1.6.5 only addresses the question of when taking fibers commutes with geometric realization, we can use it to address the same question regarding more general pullbacks.

Corollary 1.6.7. *Suppose the map $Y \to Z$ in $s\mathcal{S}$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$. Then for any map $W \to Z$ in $s\mathcal{S}$, the natural map*

$$|W \times_Z Y| \to |W| \times_{|Z|} |Y|$$

is an equivalence in $\mathcal{S}$, i.e. the pullback of $Y \to Z$ along any map commutes with geometric realization.

Proof. It suffices to show that in the diagram

$$\begin{array}{ccc} |W \times_Z Y| & \longrightarrow & |Y| \\ \downarrow & & \downarrow \\ |W| & \longrightarrow & |Z| \end{array}$$

in $\mathcal{S}$, for every point of $|W|$ the upper map induces an equivalence on the corresponding fibers of the vertical maps.

To begin, note that by Lemma 1.5.4 (or since the map $W_0 \to |W|$ is a surjection), every point $\mathrm{pt}_{\mathcal{S}} \to |W|$ in $\mathcal{S}$ is represented by a point $\mathrm{pt}_{s\mathcal{S}} \simeq \Delta^0 \to W$ in $s\mathcal{S}$. For such a point $\mathrm{pt}_{s\mathcal{S}} \xrightarrow{w} W$, denote by $F_w \in s\mathcal{S}$ the fiber of the map $W \times_Z Y \to W$ over w.

Now, as $\mathrm{rlp}(J_{\mathrm{KQ}})$ is closed under pullbacks, then the map $W \times_Z Y \to W$ also has $\mathrm{rlp}(J_{\mathrm{KQ}})$. So in the diagram

$$\begin{array}{ccccc} F_w & \longrightarrow & W \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \mathrm{rlp}(J_{\mathrm{KQ}})} & & \downarrow{\scriptstyle \mathrm{rlp}(J_{\mathrm{KQ}})} \\ \mathrm{pt}_{s\mathcal{S}} & \xrightarrow[w]{} & W & \longrightarrow & Z \end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$, both the left square and the large rectangle are pullbacks, and by Proposition 1.6.5 these both remain pullbacks when we apply $|-| : s\mathcal{S} \to \mathcal{S}$. This proves the indicated sufficient condition. □

As stated in Remark 1.4.6, Corollary 1.6.7 already spells out one of the most useful consequences of the right properness of $s\mathcal{S}_{\mathrm{KQ}}$. However, for the sake of completeness, since we have not actually proved that this model structure is right proper, we do so now.

Corollary 1.6.8. $\mathbf{W}_{\mathrm{KQ}}$ *is preserved under pullback along maps that have* $\mathrm{rlp}(J_{\mathrm{KQ}})$.

Proof. Suppose that we have a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \mathrm{rlp}(J_{\mathrm{KQ}})} \\ W & \xrightarrow[\approx]{} & Z \end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$. By Corollary 1.6.7, the induced diagram

$$\begin{array}{ccc} |W \times_Z Y| & \longrightarrow & |Y| \\ \downarrow & & \downarrow \\ |W| & \xrightarrow{\sim} & |Z| \end{array}$$

in $\mathcal{S}$ is a pullback square. But this implies that the upper map is an equivalence in $\mathcal{S}$, i.e. that the map $W \times_Z Y \to Y$ is in $\mathbf{W}_{\mathrm{KQ}}$. □

1.6.2.2 Fibrations are preserved by π_0

We now proceed to give a necessary condition (Proposition 1.6.10) for a map to be a fibration in $s\mathcal{S}_{\mathrm{KQ}}$.

Lemma 1.6.9. *For any $Y \in s\mathcal{S}$ and any $K \xrightarrow{\approx} L$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, the canonical map $Y \to \pi_0(Y)$ in $s\mathcal{S}$ induces a π_0-isomorphism*

$$\mathrm{M}_L(Y) \to \lim \left(\begin{array}{ccc} & & \mathrm{M}_K(Y) \\ & & \downarrow \\ \mathrm{M}_L(\pi_0(Y)) & \longrightarrow & \mathrm{M}_K(\pi_0(Y)) \end{array} \right)$$

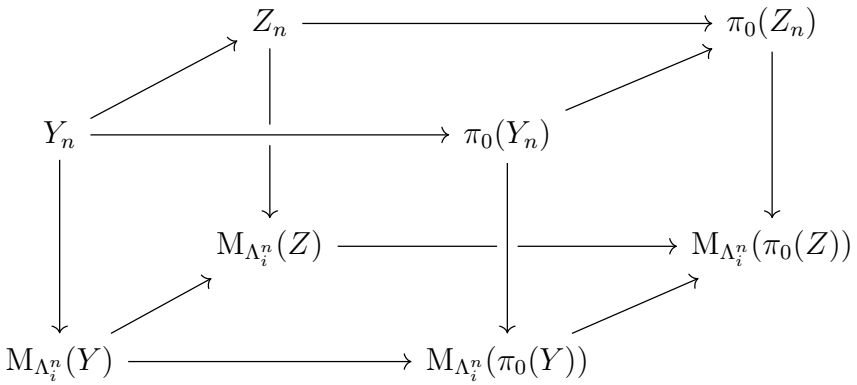
in $\mathcal{S}$. In particular, for any $Y \in s\mathcal{S}$, the canonical map $Y \to \pi_0(Y)$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$.

Proof. Letting J denote the collection of maps $K \to L$ in $s\mathcal{S}$ for which this induced map in $\mathcal{S}$ is a π_0-isomorphism, it suffices to prove that J contains the set $J_{\mathrm{KQ}} = \{\Lambda^n_i \to \Delta^n\}_{0 \leq i \leq n \geq 1}$. Note first that any map $\Delta^i \to \Delta^j$ is contained in J, since we have an equivalence $\mathrm{M}_{\Delta^n}(-) \simeq (-)_n$ in $\mathrm{Fun}(s\mathcal{S}, \mathcal{S})$ and pullbacks over discrete spaces commute with $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$. Note too that J is closed under pushouts and has the two-out-of-three property.

We now argue by induction: writing $J_{\leq n} = \{\Lambda^m_j \to \Delta^m\}_{0 \leq j \leq m \leq n} \subset J_{\mathrm{KQ}}$, we will show that $J_{\leq n} \subset J$ for all $n \geq 1$. We have already shown that $J_{\leq 1} \subset J$, since both maps $\Lambda^1_i \to \Delta^1$ are of the form $\Delta^0 \to \Delta^1$. So, suppose that $J_{\leq (n-1)} \subset J$, and let $\Lambda^n_i \to \Delta^n$ be any map in $J_{\leq n} \backslash J_{\leq (n-1)}$. Observe that the composite $\Delta^{\{i\}} \to \Lambda^n_i \to \Delta^n$ lies in J, and observe that the first map can be constructed as an iterated pushout of maps in $J_{\leq (n-1)}$ (note that pushouts along cofibrations in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ are also pushouts in $s\mathcal{S}$) and hence by assumption also lies in J since it is closed under pushouts. Since J also has the two-out-of-three property, it follows that the second map lies in J as well. This proves the claim. □

Proposition 1.6.10. *If a map $Y \to Z$ in $s\mathcal{S}$ has* $\mathrm{rlp}(J_{\mathrm{KQ}})$*, then so does* $\pi_0(Y) \to \pi_0(Z)$*.*

Proof. A map $\Lambda^n_i \to \Delta^n$ in J_{KQ} gives rise to a diagram



in $\mathcal{S}$. All four horizontal maps in this cube are π_0-isomorphisms by Lemma 1.6.9, and moreover all of their targets are discrete since the inclusion $\mathcal{S}\mathrm{et} \subset \mathcal{S}$ commutes with limits (being a right adjoint). Taking pullbacks of the cospans contained in the left and right faces, we obtain a commutative square

$$\begin{array}{ccc} Y_n & \longrightarrow & \pi_0(Y_n) \\ \downarrow & & \downarrow \\ \mathrm{M}_{\Lambda^n_i}(Y) \underset{\mathrm{M}_{\Lambda^n_i}(Z)}{\times} Z_n & \longrightarrow & \mathrm{M}_{\Lambda^n_i}(\pi_0(Y)) \underset{\mathrm{M}_{\Lambda^n_i}(\pi_0(Z))}{\times} \pi_0(Z_n) \end{array}$$

in which the upper map is a π_0-isomorphism and the lower map is a component of the canonical comparison map

$$\pi_0\left(\lim{}^{\mathcal{S}}(-)\right) \to \lim{}^{\mathcal{S}\mathrm{et}}\left(\pi_0^{\mathrm{lw}}(-)\right)$$

in $\mathrm{Fun}(\mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_2), \mathcal{S}), \mathcal{S}\mathrm{et})$. As this comparison map is a componentwise surjection, the surjectivity of the left map in the commutative square implies that of its right map. This proves the claim. □

Remark 1.6.11. The proof of Proposition 1.6.10 clearly illustrates the reason that its converse fails: the canonical comparison map

$$\pi_0\left(\lim{}^{\mathcal{S}}(-)\right) \to \lim{}^{\mathcal{S}\mathrm{et}}\left(\pi_0^{\mathrm{lw}}(-)\right)$$

in $\mathrm{Fun}(\mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_2), \mathcal{S}), \mathcal{S}\mathrm{et})$ is a componentwise surjection but not a componentwise isomorphism. On the other hand, if its component at the object

$$\left(\mathrm{M}_{\Lambda^n_i}(Y) \to \mathrm{M}_{\Lambda^n_i}(Z) \leftarrow Z_n\right) \in \mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_2), \mathcal{S})$$

happens to be an isomorphism (for instance, if $\mathrm{M}_{\Lambda^n_i}(Z) \in \mathcal{S}\mathrm{et} \subset \mathcal{S}$), then $\pi_0(Y) \to \pi_0(Z)$ having the right lifting property against the map $\Lambda^n_i \to \Delta^n$ implies that $Y \to Z$ has it as well. Assembling this observation over all maps in J_{KQ} then gives a partial converse to Proposition 1.6.10.

1.6.2.3 Comparisons with existing literature

Remark 1.6.12. In the unpublished note [Rez14], Rezk defines a *realization fibration* to be a map $Y \to Z$ in $s\mathcal{S}$ such that all pullbacks commute with geometric realization (see [Rez14, Definition 1.1]), and he completely characterizes them as being detected by iterated pullbacks along all possible composites $\Delta^{\{i\}} \to \Delta^n \to Z$ (see [Rez14, Proposition 5.10]). In this language, we can restate Corollary 1.6.7 as asserting that all maps in $\mathbf{F}_{\mathrm{KQ}} \subset s\mathcal{S}$ are realization fibrations. On the other hand, as geometric realization (being a sifted colimit) commutes with finite products in $\mathcal{S}$, every terminal map $Y \to \mathrm{pt}_{s\mathcal{S}}$ is a realization fibration, but it clearly need not be in $\mathbf{F}_{\mathrm{KQ}}$ in general.

Remark 1.6.13. There are two results, both of which appeared in the literature in 1978, which are strongly reminiscent of Corollary 1.6.7:

- one introduced by Bousfield–Friedlander in [BF78, Appendix B] (and recapitulated in [GJ99, §IV.4]) based on the notion of a simplicial space satisfying the *π_*-Kan condition*,
- the other introduced by Anderson in [And78] based on the notion of a simplicial groupoid being *fully fibrant*.

In fact, these two results are extremely similar to one another, and both rely on a common construction: given a space $Y \in \mathcal{S}$, a Grothendieck construction (see e.g. §3.1) applied to the resulting the functor

$$\Pi_1(Y) \xrightarrow{\prod_{n\geq 2} \pi_n(Y,-)} \mathcal{G}\mathrm{rp}$$

(from the fundamental (1-)groupoid of Y to the category of groups) yields a new groupoid, which we will denote by $\Pi_{\geq 1}(Y) \in \mathcal{G}\mathrm{pd}$. Using this, we can describe the conditions appearing in these results as follows.

- The π_*-*Kan condition* on a simplicial space $Y \in s\mathcal{S}$ demands of the simplicial groupoid $\Pi_{\geq 1}(Y_\bullet) \in s\mathcal{G}\mathrm{pd}$ that for every horn inclusion $\Lambda^n_i \xrightarrow{\approx} \Delta^n$ in J_{KQ}, the induced map

$$\Pi_{\geq 1}(Y_n) \cong \mathrm{M}_{\Delta^n}\left(\Pi_{\geq 1}(Y)\right) \to \mathrm{M}_{\Lambda^n_i}\left(\Pi_{\geq 1}(Y)\right)$$

  is full.

- The demand that the simplicial groupoid $\Pi_{\geq 1}(Y)_\bullet \in s\mathcal{G}\mathrm{pd}$ be *fully fibrant* amounts to the additional requirement that for every boundary inclusion $\partial\Delta^n \rightarrowtail \Delta^n$ in I_{KQ}, the induced map

$$\Pi_{\geq 1}(Y_n) \cong \mathrm{M}_{\Delta^n}\left(\Pi_{\geq 1}(Y)\right) \to \mathrm{M}_{\partial\Delta^n}\left(\Pi_{\geq 1}(Y)\right) = \mathrm{M}_n\left(\Pi_{\geq 1}(Y)\right)$$

  is full.[11]

Of course, a map in $\mathcal{G}\mathrm{pd}$ is full precisely if the induced maps on automorphism groups are all surjective, and so these conditions ultimately boil down to certain lifting criteria among the various homotopy groups $\{\pi_i(Y_j)\}_{j\geq 0, i\geq 1}$.

Then, [BF78, Theorem B.4] (resp. the main theorem of [And78]) asserts that if a map $Y \to Z$ in $s\mathcal{S}$ has that

- the induced map $\pi_0(Y) \to \pi_0(Z)$ lies in $\mathbf{F}_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}$ and
- both Y and Z satisfy the π_*-Kan condition (resp. both $\Pi_{\geq 1}(Y)$ and $\Pi_{\geq 1}(Z)$ are fully fibrant),

then the map $Y \to Z$ is a realization fibration (in the sense of Remark 1.6.12).

In light of Proposition 1.6.10 and Remark 1.6.11, it appears quite likely that these results are actually somehow secretly asking for the map $Y \to Z$ to be a fibration between fibrant objects, i.e. to lie in $\mathbf{F}^f_{\mathrm{KQ}} \subset s\mathcal{S}$.[12]

[11] This additional requirement can be rephrased as requiring that the simplicial groupoid be fibrant in $s(\mathcal{G}\mathrm{pd}_{\mathrm{can}})_{\mathrm{Reedy}}$, the Reedy model structure built on the canonical model structure on $\mathcal{G}\mathrm{pd}$ (which explains the presence of the word "fibrant" in the terminology "fully fibrant"). In turn, the canonical model structure, which in fact seems to have first appeared in [And78, §5], has that $\mathbf{W}_{\mathrm{can}} \subset \mathcal{G}\mathrm{pd}$ consists of the equivalences of groupoids, $\mathbf{C}_{\mathrm{can}} \subset \mathcal{G}\mathrm{pd}$ consists of those maps that are injective on objects, and $\mathbf{F}_{\mathrm{can}} \subset \mathcal{G}\mathrm{pd}$ consists of the isofibrations (i.e. those maps satisfying an "isomorphism lifting property").

[12] Anderson's notion of an *epifibration* (introduced in [And78, §6]) provides a model-categorical counterpart to our notion of a fibration in $s\mathcal{S}_{\mathrm{KQ}}$ (recall Remark 1.0.1), via which observation [And78, Theorem 6.2] appears to more-or-less imply Corollary 1.6.7. Unfortunately, there seem to be a number of issues with the proof given there. For instance, the two paragraphs following the statement imply that epifibrations present maps in $s\mathcal{S}$ which have not just $\mathrm{rlp}(J_{\mathrm{KQ}})$ but also have $\mathrm{rlp}(I_{\mathrm{KQ}})$. And then, the last sentence of the fourth paragraph of the proof of [And78, Lemma 6.5] has a counterexample given by the inclusion $\Delta^{\{i\}} \subset \Lambda^n_i$.

Both Bousfield–Friedlander and Anderson give classes of examples where their respective criteria hold:

- on the one hand, $Y \in s\mathcal{S}$ satisfies the π_*-Kan condition
  - if each $Y_n \in \mathcal{S}$ is connected, or
  - if each $Y_n \in \mathcal{S}$ is simple and for all $i \geq 1$ the map

$$[S^i, Y_\bullet]^{\mathrm{lw}}_{\mathcal{S}} \to [\mathrm{pt}_{\mathcal{S}}, Y_\bullet]^{\mathrm{lw}}_{\mathcal{S}} \cong \pi_0(Y)$$

    lies in $\mathbf{F}_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}$, so in particular
    - if it can be presented by a bisimplicial group,

  while

- on the other hand, $Y \in s\mathcal{S}$ has that $\Pi_{\geq 1}(Y)$ is fully fibrant
  - if each $Y_n \in \mathcal{S}$ is connected,
  - if it lies in $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$, or
  - if it can be presented by a simplicial topological group.

Remark 1.6.14. It is well known that for a simplicial space which is levelwise connected, taking loopspaces (with respect to any compatible choices of basepoints) commutes with geometric realization. (This follows from the results discussed in Remark 1.6.13, see e.g. [GJ99, Corollary IV.4.11].) In fact, *any* pullback in $s\mathcal{S}$ in which the common target in the cospan is levelwise connected commutes with geometric realization (see Lemma A.5.5.6.17). Of course, there are many interesting simplicial spaces which are not levelwise connected – for instance, any nontrivial simplicial set – and so this result is of somewhat limited use when manipulating simplicial spaces and their geometric realizations.

Remark 1.6.15. There are two papers which study certain classes of maps which are closely related to our notion of a fibration in $s\mathcal{S}_{\mathrm{KQ}}$.

- In [Sey80], Seymour studies a *continuous* lifting condition – that is, an enriched lifting condition with respect to the enrichment of $s\mathcal{T}\mathrm{op}$ over $\mathcal{T}\mathrm{op}$ – declaring that a map $x \xrightarrow{i} y$ has the "left lifting property" with respect to a map $z \xrightarrow{p} w$ if the induced map

$$\underline{\hom}_{s\mathcal{T}\mathrm{op}}(y, z) \to \underline{\hom}_{\mathrm{Fun}([1], s\mathcal{T}\mathrm{op})}(i, p)$$

  admits a section (instead of just being surjective). A "Kan fibration" in this sense – which for disambiguation we'll refer to as an "S-Kan fibration" – is

then defined to be a map in $s\mathcal{T}\mathrm{op}$ which satisfies this cotinuous right lifting property against the usual set of horn inclusions $J_{\mathrm{KQ}} = \{\Lambda^n_i \to \Delta^n\}_{0\leq i\leq n\geq 1}$ in $s\mathcal{S}\mathrm{et} \subset s\mathcal{T}\mathrm{op}$. Thus, aside from issues of homotopy coherence (which can presumably be handled using an appropriate model structure on $s\mathcal{T}\mathrm{op}$ (recall Remark 1.0.1)), it appears that these morphisms present a strict subset of those in the subcategory $\mathbf{F}_{\mathrm{KQ}} \subset s\mathcal{S}$ of fibrations.

The main result is then that S-Kan fibrations are stable under taking the internal hom into them from any other object; taking that source object to be $\Delta^1 \in s\mathcal{S}\mathrm{et} \subset s\mathcal{T}\mathrm{op}$, a "covering homotopy theorem" immediately follows (see [Sey80, Theorems 4.1 and 4.2]). Morally speaking, this is the case because the "acyclic cofibrations" in this setup are closed under taking the product with any identity map, a fact which is not true in $s\mathcal{S}_{\mathrm{KQ}}$ (but see Speculation 1.2.33 for an explanation of why this feature is in a certain sense undesirable).

- In [BS89], Brown–Szczarba study a continuous lifting condition which is similar to that of [Sey80] but is yet more restrictive: they additionally require lifting for all "sub-horns" (see [BS89, Definition 6.1]). Thus, it appears that their resulting "fibrations" – which for disambiguation we'll refer to as "BS-Kan fibrations" – present a strict subset of even those morphisms in $s\mathcal{S}$ which are presented by S-Kan fibrations.

  First of all, Brown–Szczarba prove an analogous result to Seymour's (see [BS89, Theorem 6.2]). Moreover, given a BS-Kan object $Y_\bullet \in s\mathcal{T}\mathrm{op}$ equipped with a basepoint $\mathrm{pt}_{s\mathcal{T}\mathrm{op}} \xrightarrow{y} Y_\bullet$, they define its "homotopy groups" to be those of the underlying pointed simplicial set, so that $\pi_n(Y, y)$ is a quotient of the subset

  $$\{y_n \in Y_n : \delta^n_0(y_n) = \delta^n_1(y_n) = \cdots = \delta^n_n(y_n) = y\} \subset Y_n,$$

  but is additionally topologized via the quotient topology. With respect to these homotopy groups, they obtain a "continuous long exact sequence associated to a fibration of Kan simplicial topological spaces" (in which the (strict) fiber is also a BS-Kan object) (see [BS89, Theorem 6.5 and Proposition 6.6]). They also develop notions of continuous (singular and de Rham) cohomology and of real homotopy type. Of course, all but the first of these accomplishments lie outside of the scope of what we seek to achieve here (and a comparison of the first with the present work is no different from that given above).

1.6.3 The Ex^∞ functor

We now return to the general theory. In the proof of Proposition 1.7.2 we will need to have a version of the Ex^∞ functor for simplicial spaces, so we take a moment to

develop that now. For simplicial sets, this was originally defined and explored in [Kan57, §3-4]; it is developed in more modern terminology in [GJ99, §III.4].

Definition 1.6.16. Recall that any $\Delta^n \in s\mathcal{S}\mathrm{et}$ admits a ***subdivision***, denoted $\mathrm{sd}(\Delta^n) \in s\mathcal{S}\mathrm{et}$; this is the nerve of its poset of nondegenerate simplices. Recall further that this admits a map $\mathrm{sd}(\Delta^n) \to \Delta^n$, called the ***last vertex*** map, induced by the map of posets given by taking a simplex to its last vertex. Recall still further that we can extend this definition to any $K \in s\mathcal{S}\mathrm{et}$ by defining

$$\mathrm{sd}(K) = \mathrm{colim}_{(\Delta^n \to K) \in \left(\mathbf{\Delta} \underset{s\mathcal{S}\mathrm{et}}{\times} s\mathcal{S}\mathrm{et}_{/K}\right)} \mathrm{sd}(\Delta^n),$$

and that we obtain an induced last vertex map $\mathrm{sd}(K) \to K$. We now extend this even further to any $Y \in s\mathcal{S}$ by defining

$$\mathrm{sd}(Y) = \mathrm{colim}_{(\Delta^n \to Y) \in \left(\mathbf{\Delta} \underset{s\mathcal{S}}{\times} s\mathcal{S}_{/Y}\right)} \mathrm{sd}(\Delta^n);$$

in the same way, this also admits a last vertex map $\mathrm{sd}(Y) \to Y$. Note that this does indeed extend the functor $\mathrm{sd} : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$, as $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ is a full subcategory (so that for any $K \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$, we have an equivalence

$$\mathbf{\Delta} \underset{s\mathcal{S}\mathrm{et}}{\times} s\mathcal{S}\mathrm{et}_{/K} \xrightarrow{\sim} \mathbf{\Delta} \underset{s\mathcal{S}}{\times} s\mathcal{S}_{/K}$$

of ∞-categories). This clearly defines a functor $\mathrm{sd} : s\mathcal{S} \to s\mathcal{S}$.

Definition 1.6.17. We define the ***extension*** of $Y \in s\mathcal{S}$ to be the object $\mathrm{Ex}(Y) \in s\mathcal{S}$ defined by $\mathrm{Ex}(Y)_n = \hom_{s\mathcal{S}}(\mathrm{sd}(\Delta^n), Y)$, with simplicial structure maps corepresented by the cosimplicial structure maps of $\mathrm{sd}(\Delta^\bullet) \in c(s\mathcal{S}\mathrm{et})$. This defines a functor $\mathrm{Ex} : s\mathcal{S} \to s\mathcal{S}$, which extends the usual functor $\mathrm{Ex} : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ (again since $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ is a full subcategory).

Notation 1.6.18. For any $i \geq 0$, we write $\mathrm{sd}^i = \mathrm{sd}^{\circ i}$ and $\mathrm{Ex}^i = \mathrm{Ex}^{\circ i}$ for the iterated composites of the indicated endofunctors on $s\mathcal{S}$ of Definition 1.6.16.

Lemma 1.6.19. *The functors* sd *and* Ex *define an adjunction* $\mathrm{sd} : s\mathcal{S} \rightleftarrows s\mathcal{S} : \mathrm{Ex}$, *and hence the functors* sd^i *and* Ex^i *define an adjunction* $\mathrm{sd}^i : s\mathcal{S} \rightleftarrows s\mathcal{S} : \mathrm{Ex}^i$ *for any* $i \geq 0$.

Proof. The first statement follows directly from the definitions and the fact that, as in any presheaf category, any $Y \in s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$ is recoverable as a colimit

$$Y \simeq \mathrm{colim}_{(\Delta^n \to Y) \in \left(\mathbf{\Delta} \underset{s\mathcal{S}}{\times} s\mathcal{S}_{/Y}\right)} \Delta^n$$

of representable presheaves. The second statement is obtained by composing the adjunction i times. □

Notation 1.6.20. By Lemma 1.6.19, for any $Y \in s\mathcal{S}$ the last vertex map $\mathrm{sd}(Y) \to Y$ is adjoint to a map $Y \to \mathrm{Ex}(Y)$. We write

$$\mathrm{Ex}^\infty(Y) = \mathrm{colim}(Y \to \mathrm{Ex}(Y) \to \mathrm{Ex}^2(Y) \to \cdots).$$

This defines an endofunctor $\mathrm{Ex}^\infty : s\mathcal{S} \to s\mathcal{S}$.

Remark 1.6.21. Using the classical theory of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, we can see that Ex^∞ cannot be a right adjoint. For instance, it does not commute with the countably infinite product of copies of the "simplicial infinite line", i.e. the nerve of the poset $(\mathbb{Z}, \leq)$. (This product is not acyclic, but the countably infinite product of any acyclic Kan complexes is again acyclic.)

We now give the result which we will need in the proof of Proposition 1.7.2.

Proposition 1.6.22. *For any $Y \in s\mathcal{S}$, $\mathrm{Ex}^\infty(Y) \in s\mathcal{S}^f_{\mathrm{KQ}}$.*

Proof. The proof of [GJ99, Lemma III.4.7], given as it is by a universal computation involving the map $\Lambda^n_i \to \Delta^n$, works equally well in our setting to show that for any $Y \in s\mathcal{S}$ and for any map $\Lambda^n_i \to \mathrm{Ex}(Y)$, there exists an extension

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Ex}(Y) \\ \downarrow & & \downarrow \\ \Delta^n & \dashrightarrow & \mathrm{Ex}^2(Y). \end{array}$$

By Corollary 1.5.3, it follows that $\mathrm{Ex}^\infty(Y) \to \mathrm{pt}_{s\mathcal{S}}$ is in $\mathrm{rlp}(J_{\mathrm{KQ}})$, i.e. that $\mathrm{Ex}^\infty(Y) \in s\mathcal{S}^f_{\mathrm{KQ}}$. □

Remark 1.6.23. Many of the usual results regarding the classical endofunctors $\mathrm{sd} : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ and $\mathrm{Ex} : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ extend to our setting.

For instance, the functor $\mathrm{Ex}^\infty : s\mathcal{S} \to s\mathcal{S}$ (with its canonical map from $\mathrm{id}_{s\mathcal{S}}$) is a fibrant replacement functor in $s\mathcal{S}_{\mathrm{KQ}}$. To see this, in light of Proposition 1.6.22 it suffices to show that the map $Y \to \mathrm{Ex}^\infty(Y)$ is in $\mathbf{W}_{\mathrm{KQ}}$. Since the subcategory $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ is closed under transfinite composition, it suffices to show that the map $Y \to \mathrm{Ex}(Y)$ is in $\mathbf{W}_{\mathrm{KQ}}$. For this, we first use the small object argument to produce

a map $Y' \to Y$ in $s\mathcal{S}$ that has $\mathrm{rlp}(I_{\mathrm{KQ}})$ with $Y' \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$. Then, we see that the map $\mathrm{Ex}(Y') \to \mathrm{Ex}(Y)$ also has $\mathrm{rlp}(I_{\mathrm{KQ}})$ since a commutative square

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathrm{Ex}(Y') \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Ex}(Y) \end{array}$$

is adjoint to a commutative square

$$\begin{array}{ccc} \mathrm{sd}(\partial\Delta^n) & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle \mathrm{rlp}(I_{\mathrm{KQ}})} \\ \mathrm{sd}(\Delta^n) & \longrightarrow & Y, \end{array}$$

and there is always a lift in the latter square (which is equivalent to a lift in the former square). It follows from Proposition 1.7.2 below that we have both $Y' \xrightarrow{\approx} Y$ and $\mathrm{Ex}(Y') \xrightarrow{\approx} \mathrm{Ex}(Y)$, and hence from the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\approx} & \mathrm{Ex}(Y') \\ \wr\downarrow & & \downarrow\wr \\ Y & \longrightarrow & \mathrm{Ex}(Y) \end{array}$$

we deduce that also $Y \xrightarrow{\approx} \mathrm{Ex}(Y)$ since $\mathbf{W}_{\mathrm{KQ}}$ satisfies the two-out-of-three property. On the other hand, our choice to use an unenriched lifting condition (and hence to be relatively restrictive about which maps are cofibrations) means that the map $Y \to \mathrm{Ex}(Y)$ is not generally in $\mathbf{C}_{\mathrm{KQ}}$.

From here, it is not hard to see that in fact we have a Quillen equivalence $\mathrm{sd} : s\mathcal{S}_{\mathrm{KQ}} \rightleftarrows s\mathcal{S}_{\mathrm{KQ}} : \mathrm{Ex}$. Indeed, it is straightforward to check that $\mathrm{sd} : s\mathcal{S} \to s\mathcal{S}$ preserves both I_{KQ}-cell and J_{KQ}-cell, so that this adjunction is a Quillen adjunction. Then, the condition for being a Quillen equivalence follows from the facts

- that $\mathrm{sd}(K) \xrightarrow{\approx} K$ for any $K \in s\mathcal{S}\mathrm{et} = s\mathcal{S}^c_{\mathrm{KQ}}$,
- that $Y \xrightarrow{\approx} \mathrm{Ex}(Y)$ for any $Y \in s\mathcal{S}$ (as shown above), and
- that $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ satisfies the two-out-of-three property.

One can similarly verify using standard arguments that Ex^∞ preserves $\mathbf{F}_{\mathrm{KQ}}$, finite limits, filtered colimits, and 0^{th} spaces.

1.7 The proof of the Kan–Quillen model structure

We now turn to the components of the proof of the main result of this chapter, Theorem 1.4.4. Recall that this appeals to the recognition theorem for cofibrantly generated model ∞-categories (1.3.11); we verify the various criteria in turn, as itemized in the proof of Theorem 1.4.4 above.

Proposition 1.7.1. $J_{\mathrm{KQ}}\text{-cof} \subset (I\text{-cof} \cap \mathbf{W})_{\mathrm{KQ}}$.

Proof. First, since $J_{\mathrm{KQ}} \subset I_{\mathrm{KQ}}$-cell, then J_{KQ}-inj $\supset$ (I_{KQ}-cell)-inj $= I_{\mathrm{KQ}}$-inj, so J_{KQ}-cof $\subset I_{\mathrm{KQ}}$-cof. So it remains to show that J_{KQ}-cof $\subset \mathbf{W}_{\mathrm{KQ}}$.

To show that J_{KQ}-cof $\subset \mathbf{W}_{\mathrm{KQ}}$, we claim that it suffices to show that J_{KQ}-cell $\subset \mathbf{W}_{\mathrm{KQ}}$; more precisely, we claim that any map in J_{KQ}-cof is a retract of a map in J_{KQ}-cell, and so the result follows from the fact that $\mathbf{W}_{\mathrm{KQ}}$ is closed under retracts. Indeed, by Corollary 1.5.3, we can apply the small object argument for J_{KQ} to any map in J_{KQ}-cof to factor it as a map in J_{KQ}-cell followed by a map in J_{KQ}-inj. Then, by the retract argument (in the form of [Hir03, Proposition 7.2.2(1)], whose proof for 1-categories carries over verbatim to ∞-categories), it follows that the original map in J_{KQ}-cof is a retract of the map in J_{KQ}-cell.

Finally, to see that J_{KQ}-cell $\subset \mathbf{W}_{\mathrm{KQ}}$, since a sequential colimit of equivalences is an equivalence, by transfinite induction it suffices to show that if we have a pushout square

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Z \end{array}$$

in $s\mathcal{S}$, then $|Y| \xrightarrow{\sim} |Z|$ in $\mathcal{S}$. But this follows from the fact that geometric realization (being a colimit) commutes with pushouts, so the induced square

$$\begin{array}{ccc} |\Lambda^n_i| & \longrightarrow & |Y| \\ \wr\downarrow & & \downarrow \\ |\Delta^n| & \longrightarrow & |Z| \end{array}$$

is a pushout in $\mathcal{S}$. □

Proposition 1.7.2. $I_{\mathrm{KQ}}\text{-inj} \subset (J\text{-inj} \cap \mathbf{W})_{\mathrm{KQ}}$.

Proof. First, since $J_{\mathrm{KQ}} \subset I_{\mathrm{KQ}}$-cell, then I_{KQ}-inj $= (I_{\mathrm{KQ}}\text{-cell})\text{-inj} \subset J_{\mathrm{KQ}}$-inj. So it remains to show that $I_{\mathrm{KQ}}\text{-inj} \subset \mathbf{W}_{\mathrm{KQ}}$.

So, suppose that the map $Y \to Z$ is in I_{KQ}-inj, i.e. that it has $\mathrm{rlp}(I_{\mathrm{KQ}})$. By Lemma 1.5.1, it suffices to show that any diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & |Y| \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \longrightarrow & |Z| \end{array}$$

in $\mathcal{S}$ admits a lift, for any $n \geq 0$. Using the observation in Remark 1.5.5, by Lemma 1.5.4 there exists a factorization $\varnothing_{s\mathcal{S}\mathrm{et}} \to K \xrightarrow{\approx} (\Delta^{n-1}/\partial\Delta^{n-1})$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ and a map $K \to Y$ in $s\mathcal{S}$ presenting the upper map in this diagram, where K has only finitely many nondegenerate simplices.

Now, the above diagram gives us a nullhomotopy of the composite $S^{n-1} \simeq |K| \to |Y| \to |Z|$ in $\mathcal{S}$, and we would like to extend the composite $K \to Y \to Z$ over a cofibration from K into a acyclic object of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ in a way which presents this nullhomotopy. To do this, we write $M = (K \times \Delta^1)/(K \times \Delta^{\{1\}})$ (with its natural inclusion $K \cong K \times \Delta^{\{0\}} \rightarrowtail M$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$), and then by Proposition 1.6.22 and Corollary 1.6.2 we conclude that there must exist an extension

$$\begin{array}{ccccc} K & \longrightarrow & Z & \longrightarrow & \mathrm{Ex}^{\infty}(Z) \\ \downarrow & & & \nearrow & \\ M & & & & \end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$ modeling the above nullhomotopy. However, since M also has only finitely many nondegenerate simplices, then by Lemma 1.5.2 there must exist a factorization

$$\begin{array}{ccccccc} K & \longrightarrow & L & \longrightarrow & \mathrm{Ex}^{i}(Z) & \longrightarrow & \mathrm{Ex}^{\infty}(Z) \\ \downarrow & & & \nearrow & & \nearrow & \\ M & & & & & & \end{array}$$

for some $i < \infty$. Via the adjunction $\mathrm{sd}^i : s\mathcal{S} \rightleftarrows s\mathcal{S} : \mathrm{Ex}^i$ of Lemma 1.6.19, the above extension yields the extension

$$\begin{array}{ccc} \mathrm{sd}^{i}(K) & \xrightarrow{\approx} & K \\ \downarrow & & \downarrow \\ \mathrm{sd}^{i}(M) & \dashrightarrow & Z \end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$, and plugging this back into the original diagram gives us the diagram

$$\begin{array}{ccccc} \mathrm{sd}^i(K) & \xrightarrow{\approx} & K & \longrightarrow & Y \\ \downarrow & & & & \downarrow \\ \mathrm{sd}^i(M) & & \longrightarrow & & Z \end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$. Now, since by assumption $Y \to Z$ has $\mathrm{rlp}(I_{\mathrm{KQ}})$, then there must exist a lift

$$\begin{array}{ccccc} \mathrm{sd}^i(K) & \xrightarrow{\approx} & K & \longrightarrow & Y \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{sd}^i(M) & & \longrightarrow & & Z \end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$, and upon extracting the outer rectangle and taking geometric realizations, this yields the desired lift

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & |Y| \\ \downarrow & \nearrow & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \longrightarrow & |Z| \end{array}$$

in $\mathcal{S}$. □

Remark 1.7.3. In the same spirit as Remark 1.6.13, the criterion of Proposition 1.7.2 is comparable to the lifting criterion coming from the generating cofibrations in the *Moerdijk model structure* on $ss\mathcal{S}\mathrm{et}$ (originally introduced in [Moe89, §1], but see also [GJ99, §IV.3.3]). However, to actually make a direct comparison requires a bit of care, and so we explain this in some detail.

First of all, the diagonal functor $\mathrm{diag} : \mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ induces an adjunction $\mathrm{diag}_! : s\mathcal{S}\mathrm{et} \rightleftarrows ss\mathcal{S}\mathrm{et} : \mathrm{diag}^*$, and the Moerdijk model structure on $ss\mathcal{S}\mathrm{et}$ is induced by the standard lifting theorem (on which Theorem 1.3.12 is based) applied to the model structure $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$; in fact, this yields a Quillen equivalence $\mathrm{diag}_! : s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Moer}} : \mathrm{diag}^*$. Denoting the external product by $-\widehat{\boxtimes}- : s\mathcal{S}\mathrm{et} \times s\mathcal{S}\mathrm{et} \to ss\mathcal{S}\mathrm{et}$, the generating cofibrations for this model structure are thus given by

$$I^{ss\mathcal{S}\mathrm{et}}_{\mathrm{Moer}} = \{\mathrm{diag}_!(\partial\Delta^n \to \Delta^n)\}_{n\geq 0} = \{\partial\Delta^n \widehat{\boxtimes} \partial\Delta^n \to \Delta^n \widehat{\boxtimes} \Delta^n\}_{n\geq 0},$$

and we have that $\mathrm{rlp}(I^{ss\mathcal{S}\mathrm{et}}_{\mathrm{Moer}}) \subset \mathbf{W}^{ss\mathcal{S}\mathrm{et}}_{\mathrm{Moer}}$.

Next, recall that we can present the ∞-category $s\mathcal{S}$ using the model category $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. Thus, any map $Y \to Z$ in $s\mathcal{S}$ can be presented as a map $\mathtt{Y} \to \mathtt{Z}$ in

$s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. If the latter map happens to have $\mathrm{rlp}(I^{ss\mathcal{S}\mathrm{et}}_{\mathrm{Moer}})$, then we will have that the induced map $\mathrm{diag}^*(\mathtt{Y}) \to \mathrm{diag}^*(\mathtt{Z})$ will be in $\mathbf{W}^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}}$; since in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ the diagonal always computes the homotopy colimit (see e.g. [GJ99, Exercise IV.1.6] or Example T.A.2.9.31), then this map in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ presents the map $|Y| \to |Z|$ in $\mathcal{S}$, which is therefore an equivalence.

Now, observe that the maps in $I^{ss\mathcal{S}\mathrm{et}}_{\mathrm{Moer}}$ are cofibrations (between cofibrant objects) when considered in the model category $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. Moreover, note that if

- the maps $A \xrightarrow{i} B$ and $Y \xrightarrow{p} Z$ in $s\mathcal{S}$ are respectively presented by the maps $\mathtt{A} \xrightarrow{\mathtt{i}} \mathtt{B}$ and $\mathtt{Y} \xrightarrow{\mathtt{p}} \mathtt{Z}$ in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$,
- $\mathtt{Z} \in s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})^f_{\mathrm{Reedy}}$, and
- $p \in \mathrm{rlp}(\{i\})$ in $s\mathcal{S}$,

then $\mathtt{p} \in \mathrm{rlp}(\{\mathtt{i}\})$ in $s(s\mathcal{S}\mathrm{et})$. (This follows from the fact that under these hypotheses, the model category $(s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}})_{\mathtt{A}//\mathtt{Z}}$ presents the ∞-category $s\mathcal{S}_{A//Z}$.) Together, these imply that if the map $Y \to Z$ in $s\mathcal{S}$ has the right lifting property against the set

$$I^{s\mathcal{S}}_{\mathrm{Moer}} = \{S^{n-1} \odot \partial\Delta^n \to \mathrm{pt}_{\mathcal{S}} \odot \Delta^n\}_{n\geq 0} = \{S^{n-1} \odot \partial\Delta^n \to \Delta^n\}_{n\geq 0},$$

then it can be presented by a map $\mathtt{Y} \to \mathtt{Z}$ in the model category $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ which has $\mathrm{rlp}(I^{ss\mathcal{S}\mathrm{et}}_{\mathrm{Moer}})$, and therefore $|Y| \xrightarrow{\sim} |Z|$ in $\mathcal{S}$. Hence, we obtain that $\mathrm{rlp}(I^{s\mathcal{S}}_{\mathrm{Moer}}) \subset \mathbf{W}_{\mathrm{KQ}}$. This, finally, allows us to make a direct comparison.

Of course, what we come to see now is that it appears much easier to have $\mathrm{rlp}(I^{s\mathcal{S}}_{\mathrm{KQ}})$ than to have $\mathrm{rlp}(I^{s\mathcal{S}}_{\mathrm{Moer}})$. The sets $I^{s\mathcal{S}}_{\mathrm{Moer}}$ and $I^{s\mathcal{S}}_{\mathrm{KQ}}$ of homotopy classes of maps in $s\mathcal{S}$ are illustrated in Figure 1.6, in which the various shapes and their positions are meant to vaguely indicate the different simplicial levels at which these spaces live as well as the simplicial structure maps between them:

- the maps in $I^{s\mathcal{S}}_{\mathrm{Moer}}$ are obtained by simultaneously coning off a $\partial\Delta^n$ worth of $(n-1)$-spheres and adding a new nondegenerate point in the n^{th} simplicial level, while
- the maps in $I^{s\mathcal{S}}_{\mathrm{KQ}}$ are simply maps of discrete simplicial spaces.

In particular, the maps in $I^{s\mathcal{S}}_{\mathrm{Moer}}$ have real homotopical content, and thus it appears that checking that a map has $\mathrm{rlp}(I^{s\mathcal{S}}_{\mathrm{Moer}})$ is indeed much more difficult than checking that it has $\mathrm{rlp}(I^{s\mathcal{S}}_{\mathrm{KQ}})$.

Remark 1.7.4. There is also the "$\overline{W}$ model structure" on $ss\mathcal{S}\mathrm{et}$ of [CR07], which admits a left Quillen equivalence to $ss\mathcal{S}\mathrm{et}_{\mathrm{Moer}}$ (see [CR07, Theorem 9]). However, this is of course also inherently 1-categorical, and hence any ∞-categorical lifting

criteria that come of it will likewise necessarily contain far more geometric content than the maps in I_{KQ} (compare with Remarks 1.7.3 and 1.7.5).

Remark 1.7.5. If one thinks of the 1-category of topological spaces or of simplicial sets in place of the ∞-category $\mathcal{S}$, then Proposition 1.7.2 may seem somewhat implausible. For instance, the functor disc : $s\mathcal{S}\mathrm{et} \to s\mathcal{S}$ of ∞-categories is modeled in $\mathfrak{Rel}\mathfrak{Cat}_{\mathrm{BK}}$ by an evident functor const : $s\mathcal{S}\mathrm{et}_{\mathrm{triv}} \to s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. On underlying 1-categories, this functor participates in an adjunction const : $s\mathcal{S}\mathrm{et} \rightleftarrows ss\mathcal{S}\mathrm{et} : ((-)_0)^{\mathrm{lw}}$ with the "levelwise 0-simplices" functor, which takes each constituent simplicial set to its set of 0-simplices. This right adjoint clearly doesn't know anything about the higher homotopical information in the bisimplicial set, and in particular cannot recover its geometric realization. Hence, one might deduce that asking for the right lifting property in $s\mathcal{S}$ against maps in the image of disc : $s\mathcal{S}\mathrm{et} \to s\mathcal{S}$ could not possibly tell us about the functor $|-| : s\mathcal{S} \to \mathcal{S}$. However, asking for the 0-simplices of a simplicial set isn't a homotopical operation in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, and so we cannot expect this maneuver to tell us anything ∞-categorical. Indeed, the above right adjoint is certainly not a relative functor, nor is it even a right Quillen functor (with respect to the indicated model structures), corresponding to the fact that the functor disc : $s\mathcal{S}\mathrm{et} \to s\mathcal{S}$ isn't a left adjoint.

Example 1.7.6. For a concrete nonexample of Proposition 1.7.2, we show that the map $\mathrm{const}(S^1) \to \mathrm{const}(\mathrm{pt}_{\mathcal{S}}) \simeq \mathrm{pt}_{s\mathcal{S}}$ in $s\mathcal{S}$ (which is not in $\mathbf{W}_{\mathrm{KQ}}$) does not have $\mathrm{rlp}(\{\partial\Delta^2 \to \Delta^2\})$. This illustrates the capability of "simplicial" spheres to detect "geometric" spheres in $s\mathcal{S}_{\mathrm{KQ}}$.

The quickest way to proceed is to use the adjunction $|-| = \mathrm{colim} : s\mathcal{S} \rightleftarrows \mathcal{S} : \mathrm{const}$. This gives a canonical commutative square

$$\begin{array}{ccc} \partial\Delta^2 & \longrightarrow & \mathrm{const}(S^1) \\ \downarrow & & \downarrow \\ \Delta^2 & \longrightarrow & \mathrm{const}(\mathrm{pt}_{\mathcal{S}}) \end{array}$$

in $s\mathcal{S}$ which corresponds to the evident commutative square

$$\begin{array}{ccc} |\partial\Delta^2| & \xrightarrow{\sim} & S^1 \\ \downarrow & & \downarrow \\ |\Delta^2| & \xrightarrow[\sim]{} & \mathrm{pt}_{\mathcal{S}} \end{array}$$

in $\mathcal{S}$. Moreover, a lift in either square yields a lift in the other, but a lift in the latter diagram would imply that its vertical maps are also equivalences, which is clearly false.

But we can also describe the above commutative square in $s\mathcal{S}$ more explicitly. Namely, we can define the upper map $\partial\Delta^2 \to \mathrm{const}(S^1)$ in $s\mathcal{S}$ by giving a *weak* natural transformation of simplicial topological spaces, as illustrated in Figure 1.7 (using the same schematics as were employed in Figure 1.6). Let us parametrize the circle as the group-theoretic quotient $\mathbb{R}/\mathbb{Z}$. Then, we begin at level 0 by sending $\Delta^{\{0\}}$ to 0, $\Delta^{\{1\}}$ to $1/3$, and $\Delta^{\{2\}}$ to $2/3$. Since $\partial\Delta^2$ is 1-skeletal, it remains to fill in the commutative diagram

$$\begin{array}{ccccc} L_1(\partial\Delta^2) & \longrightarrow & (\partial\Delta^2)_1 & \longrightarrow & M_1(\partial\Delta^2) \\ \downarrow & & \vdots & & \downarrow \\ L_1(\mathrm{const}(S^1)) & \longrightarrow & \mathrm{const}(S^1)_1 & \longrightarrow & M_1(\mathrm{const}(S^1)) \end{array}$$

in $\mathcal{S}$. To do this, we map the degenerate elements of $(\partial\Delta^2)_1$ so that the left square commutes on the nose. Then, we map the nondegenerate elements of $(\partial\Delta^2)_1$ to $\mathrm{const}(S^1)_1 = S^1$ by sending $\Delta^{\{01\}}$ to $1/6$, sending $\Delta^{\{12\}}$ to $1/2$, and sending $\Delta^{\{02\}}$ to $5/6$. To select a homotopy witnessing the homotopy commutativity of the right square, for $i \neq j$ we choose the evident paths of length $1/6$ from the image of each $\Delta^{\{ij\}}$ to the images of $\Delta^{\{i\}}$ and $\Delta^{\{j\}}$ (as indicated by the squiggly arrows in Figure 1.7). Again, it is clear that this map cannot be extended over Δ^2.[13]

Remark 1.7.7. In the proof of Proposition 1.7.2 above, one might be tempted to apply the small object argument for I_{KQ} to the map $K \to Z$ to obtain a factorization $K \to L \to Z$ with the map $K \to L$ in I_{KQ}-cell (so that $L \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$) and with the map $L \to Z$ in I_{KQ}-inj; then, we could proceed with the proof using standard techniques in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, using $L \in s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ as a replacement for $Z \in s\mathcal{S}_{\mathrm{KQ}}$. If this worked, it would allow us to sidestep the extension of the functor Ex^∞ from $s\mathcal{S}\mathrm{et}$ to $s\mathcal{S}$. However, such an argument would be circular: we can certainly obtain such a factorization $K \to L \to Z$, but to conclude that the map $L \to Z$ is in $\mathbf{W}_{\mathrm{KQ}}$ because it is in I_{KQ}-inj uses precisely the result that we are trying to prove.

Remark 1.7.8. Whereas Proposition 1.7.2 only allows us to detect acyclic fibrations in $s\mathcal{S}_{\mathrm{KQ}}$, the "weak equivalence criterion" of [BEBdBP, 30.10] gives a complete characterization of the subcategory $\mathbf{W}^f_{\mathrm{KQ}} \subset s\mathcal{S}$ of weak equivalences between fibrant objects (and thence also a complete (albeit rather abstract) characterization of the entire subcategory $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ of weak equivalences).

[13] This can also be realized as an actual natural transformation of simplicial topological spaces if we're willing to use a fatter model for the object $\partial\Delta^2 \in s\mathcal{S}$, despite the fact that the simplicial topological space which is constant at the circle isn't actually fibrant in the corresponding Reedy model structure.

In contrast with Remark 1.7.7, now that we have Proposition 1.7.2 in hand, we can use this technique of reducing to $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. We employ it in proving the following result, the last of this section.

Proposition 1.7.9. $(J\text{-inj} \cap \mathbf{W})_{\mathrm{KQ}} \subset I_{\mathrm{KQ}}\text{-inj}$.

Proof. Suppose that the map $Y \to Z$ in $s\mathcal{S}$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$ and geometrically realizes to an equivalence in $\mathcal{S}$. We must show that $Y \to Z$ also has $\mathrm{rlp}(I_{\mathrm{KQ}})$, i.e. that any commutative square

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Z \end{array}$$

in $s\mathcal{S}$ admits a lift, for any $n \geq 0$. We argue by constructing the diagram (and in particular the dotted arrow, which solves the above lifting problem) in $s\mathcal{S}_{\mathrm{KQ}}$ given in Figure 1.8, beginning with only the outermost square. For clarity, we proceed in steps.

(1) We use the small object argument for I_{KQ} to obtain the the factorization $\partial\Delta^n \rightarrowtail Y' \to Y$ in $s\mathcal{S}_{\mathrm{KQ}}$, where $Y' \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ and the latter map has $\mathrm{rlp}(I_{\mathrm{KQ}})$; by Proposition 1.7.2, this latter map is also in $\mathbf{W}_{\mathrm{KQ}}$.

(2) We use the small object argument for I_{KQ} to obtain the factorization $Y' \rightarrowtail Z' \to Z$ in $s\mathcal{S}_{\mathrm{KQ}}$ of the composite map $Y' \to Y \to Z$, where $Z' \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ and the latter map has $\mathrm{rlp}(I_{\mathrm{KQ}})$; again by Proposition 1.7.2, this latter map is also in $\mathbf{W}_{\mathrm{KQ}}$.

(3) Since the map $Z' \to Z$ has $\mathrm{rlp}(I_{\mathrm{KQ}})$, we are guaranteed a lift in the square

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & \nearrow & & & \downarrow{\scriptstyle \mathrm{rlp}(I_{\mathrm{KQ}})} \\ \Delta^n & & \longrightarrow & & Z. \end{array}$$

(4) We use the small object argument for J_{KQ} to obtain the factorization $Y' \overset{\approx}{\rightarrowtail} Y'' \twoheadrightarrow Z'$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} \subset s\mathcal{S}_{\mathrm{KQ}}$ of the map $Y' \to Z'$.

(5) Since the map $Y \to Z$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$, we are guaranteed a lift in the square

$$\begin{array}{ccccc} Y' & & \xrightarrow{\approx} & & Y \\ \wr\downarrow & \nearrow & & & \downarrow \mathrm{rlp}(J_{\mathrm{KQ}}) \\ Y'' & \dashrightarrow & Z' & \longrightarrow & Z. \end{array}$$

Since $\mathbf{W}_{\mathrm{KQ}}$ has the two-out-of-three property, then this lift is in $\mathbf{W}_{\mathrm{KQ}}$.

(6) Again since $\mathbf{W}_{\mathrm{KQ}}$ has the two-out-of-three property, the map $Y'' \to Z'$ must be in $\mathbf{W}_{\mathrm{KQ}}$.

(7) In $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ we have that $(\mathbf{W} \cap \mathbf{F})^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}} = \mathrm{rlp}(I^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}})$, so we must have a lift in the square

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & Y' & \longrightarrow & Y'' \\ \downarrow & & & \nearrow & \wr\downarrow \\ \Delta^n & & \longrightarrow & & Z', \end{array}$$

which gives us the dotted arrow in the diagram in Figure 1.8. □

1.8 The proof of Lemma 1.5.4

We now give the proof of Lemma 1.5.4, completing the proof of Theorem 1.4.4.

Proof of Lemma 1.5.4. We begin by observing that we have $L' \xrightarrow{\approx} L$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ (for any choice of $i \geq 0$) because this model category is left proper, so that these objects are both homotopy pushouts.

To prove the rest of the statement, we proceed in steps for clarity. To fix notation, suppose that the chosen point of the pullback selects a pair of maps

$$(\varphi, \varepsilon) \in \hom_{s\mathcal{S}}(K, W) \times \hom_{\mathcal{S}}(|L|, |W|).$$

We present the ∞-category $s\mathcal{S}$ via the model category $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$, and we denote by $\mathrm{const} : s\mathcal{S}\mathrm{et}_{\mathrm{triv}} \to s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ the evident right Quillen functor modeling the right adjoint $\mathrm{disc} : s\mathcal{S}\mathrm{et} \to s\mathcal{S}$ (though we will continue to suppress the latter).

(1) We choose an arbitrary fibrant representative $\mathtt{W} \in s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})^{f}_{\mathrm{Reedy}}$ for the object $W \in s\mathcal{S}$, and then we choose an arbitrary map $\mathrm{const}(K) \xrightarrow{\phi} \mathtt{W}$ in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ which presents the map $K \xrightarrow{\varphi} W$ in $s\mathcal{S}$.

(2) Recall that the diagonal functor $\mathrm{diag} : \mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ induces an adjunction

$$\mathrm{diag}_! : s\mathcal{S}\mathrm{et} \rightleftarrows ss\mathcal{S}\mathrm{et} : \mathrm{diag}^*.$$

Applying its right adjoint to ϕ yields a map

$$K \cong \mathrm{diag}^*(\mathrm{const}(K)) \xrightarrow{\mathrm{diag}^*(\phi)} \mathrm{diag}^*(\mathtt{W})$$

in $s\mathcal{S}\mathrm{et}$. Since this right adjoint $\mathrm{diag}^* : s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \to s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is a relative functor (see e.g. [GJ99, Proposition IV.1.9]) and considered in $\mathfrak{Rel}\mathfrak{Cat}_{\mathrm{BK}}$ models the functor $|-| : s\mathcal{S} \to \mathcal{S}$ (see e.g. [GJ99, Exercise IV.1.6] or Example T.A.2.9.31), then the map $\mathrm{diag}^*(\phi)$ in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ models the map $|K| \xrightarrow{|\varphi|} |W|$ in $\mathcal{S}$.

(3) By assumption, we have an extension

$$\begin{array}{ccc} |K| & \xrightarrow{|\varphi|} & |W| \\ \downarrow & \overset{\varepsilon}{\nearrow} & \\ |L| & & \end{array}$$

in $\mathcal{S}$. Since the ∞-category $\mathcal{S}_{|K|/}$ is presented by the model category $(s\mathcal{S}\mathrm{et}_{K/})_{\mathrm{KQ}}$, then this can be modeled as an extension

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & K & \xrightarrow{\mathrm{diag}^*(\phi)} & \mathrm{diag}^*(\mathtt{W}) \\ \downarrow & & \downarrow & & \downarrow \wr \\ \Delta^n & \longrightarrow & L & \overset{\epsilon}{\dashrightarrow} & \mathrm{Ex}^\infty(\mathrm{diag}^*(\mathtt{W})) \end{array}$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. To simplify our diagrams we will henceforth omit L, since it's defined as a pushout anyways.

(4) Recall that the map $\mathrm{diag}^*(\mathtt{W}) \xrightarrow{\approx} \mathrm{Ex}^\infty(\mathrm{diag}^*(\mathtt{W}))$ is defined as a transfinite composition

$$\mathrm{diag}^*(\mathtt{W}) \xrightarrow{\approx} \mathrm{Ex}(\mathrm{diag}^*(\mathtt{W})) \xrightarrow{\approx} \mathrm{Ex}^2(\mathrm{diag}^*(\mathtt{W})) \xrightarrow{\approx} \cdots$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. Since Δ^n is small as an object of $s\mathcal{S}\mathrm{et}_{\partial\Delta^n/}$, there must exist a factorization

$$\begin{array}{ccccc}
\partial\Delta^n & \longrightarrow & K & \xrightarrow{\mathrm{diag}^*(\phi)} & \mathrm{diag}^*(\mathtt{W}) \\
\downarrow & & & & \downarrow\wr \\
\Delta^n & & \dashrightarrow & & \mathrm{Ex}^i(\mathrm{diag}^*(\mathtt{W})) \\
& \searrow & & & \downarrow\wr \\
& & & & \mathrm{Ex}^\infty(\mathrm{diag}^*(\mathtt{W}))
\end{array}$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ for some $i < \infty$.

(5) Via the adjunction $\mathrm{sd}^i : s\mathcal{S}\mathrm{et} \rightleftarrows s\mathcal{S}\mathrm{et} : \mathrm{Ex}^i$, the extension in step (4) is equivalent to the extension

$$\begin{array}{ccccccc}
\mathrm{sd}^i(\partial\Delta^n) & \longrightarrow & \partial\Delta^n & \longrightarrow & K & \xrightarrow{\mathrm{diag}^*(\phi)} & \mathrm{diag}^*(\mathtt{W}) \\
\downarrow & & \downarrow & & & \nearrow & \\
\mathrm{sd}^i(\Delta^n) & \longrightarrow & \Delta^n, & & & &
\end{array}$$

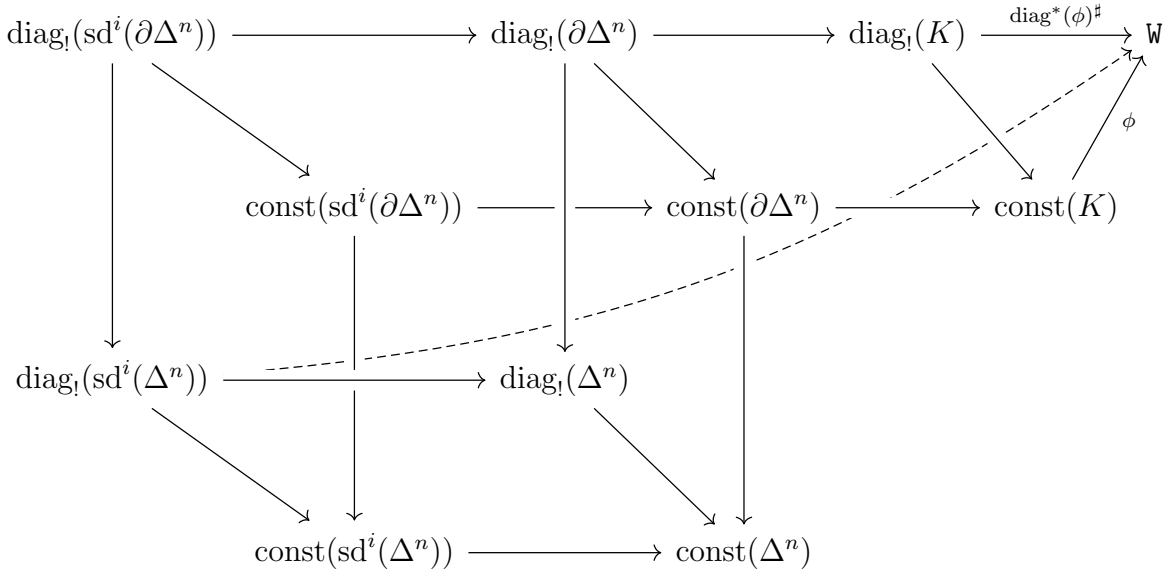
i.e. an extension

$$\begin{array}{ccc}
K & \xrightarrow{\mathrm{diag}^*(\phi)} & \mathrm{diag}^*(\mathtt{W}). \\
\downarrow & \nearrow & \\
L' & &
\end{array}$$

(6) Via the adjunction $\mathrm{diag}_! : s\mathcal{S}\mathrm{et} \rightleftarrows ss\mathcal{S}\mathrm{et} : \mathrm{diag}^*$, the extension in step (5) is equivalent to the extension

$$\begin{array}{ccccccc}
\mathrm{diag}_!(\mathrm{sd}^i(\partial\Delta^n)) & \longrightarrow & \mathrm{diag}_!(\partial\Delta^n) & \longrightarrow & \mathrm{diag}_!(K) & \xrightarrow{\mathrm{diag}^*(\phi)^\sharp} & \mathtt{W}. \\
\downarrow & & \downarrow & & & \nearrow & \\
\mathrm{diag}_!(\mathrm{sd}^i(\Delta^n)) & \longrightarrow & \mathrm{diag}_!(\Delta^n). & & & &
\end{array}$$

(7) The diagram in step (6) extends to a diagram

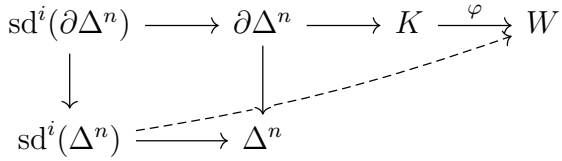


in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$, in which all unlabeled oblique arrows are components of the natural transformation

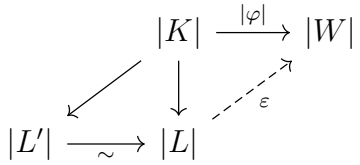
$$\mathrm{diag}_! \cong \mathrm{diag}_!\mathrm{diag}^*\mathrm{const} \to \mathrm{const}$$

in $\mathrm{Fun}(s\mathcal{S}\mathrm{et}, ss\mathcal{S}\mathrm{et})$ induced by the counit of the adjunction $\mathrm{diag}_! \dashv \mathrm{diag}^*$; indeed, the counit is initial among maps of the form $\mathrm{diag}^*(-)^\sharp$.

(8) The map $\mathrm{diag}_!(\mathrm{sd}^i(\Delta^n)) \to \mathrm{const}(\mathrm{sd}^i(\Delta^n))$ in the diagram of step (7) is a weak equivalence in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ by Lemma 1.8.1 below. Hence, upon applying the localization $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \to s\mathcal{S}$ to that diagram, we obtain the desired extension



in $s\mathcal{S}$. (Working back through the proof, it is clear that this does indeed model the extension



in $\mathcal{S}$.) $\square$

Lemma 1.8.1. *For any acyclic object $M \in s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, the component*

$$\mathrm{diag}_!(M) \cong \mathrm{diag}_!(\mathrm{diag}^*(\mathrm{const}(M))) \to \mathrm{const}(M)$$

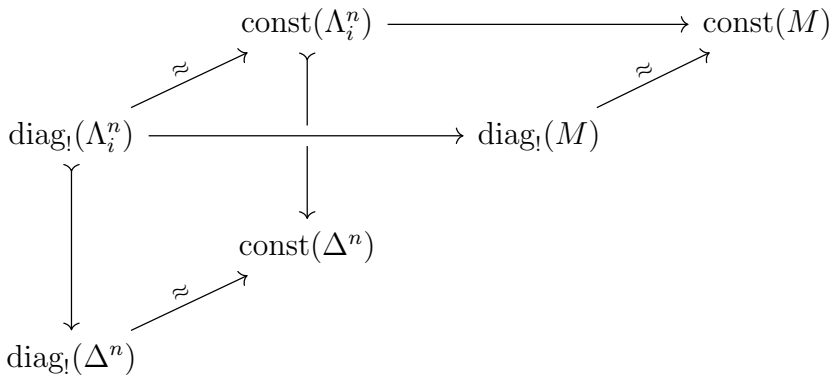
of the counit of the adjunction $\mathrm{diag}_! : s\mathcal{S}\mathrm{et} \rightleftarrows ss\mathcal{S}\mathrm{et} : \mathrm{diag}^*$ is a weak equivalence in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$.

Proof. We begin by choosing a presentation of M as a transfinite composition of pushouts of the generating acyclic cofibrations $J_{\mathrm{KQ}} = \{\Lambda^n_i \to \Delta^n\}_{0\leq i\leq n\geq 1}$. Note that $\mathrm{diag}_!$ is a left adjoint, so it commutes with pushouts; thus, this also gives us a presentation of $\mathrm{diag}_!(M)$ as a transfinite composition of pushouts of maps in $\mathrm{diag}_!(J_{\mathrm{KQ}}) = \{\mathrm{diag}_!(\Lambda^n_i) \to \mathrm{diag}_!(\Delta^n)\}_{0\leq i\leq n\geq 1}$.

We now argue by transfinite induction. Clearly the result holds if $M = \Delta^0$. To obtain the inductive step at any successor ordinal, we will show below that we have a commutative square

$$\begin{array}{ccc} \mathrm{diag}_!(\Lambda^n_i) & \xrightarrow{\approx} & \mathrm{const}(\Lambda^n_i) \\ \downarrow & & \downarrow \\ \mathrm{diag}_!(\Delta^n) & \xrightarrow[\approx]{} & \mathrm{const}(\Delta^n) \end{array}$$

in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. Then, since $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ is left proper (for instance because all its objects are cofibrant), the induced map between the pushouts of the front and back faces in the diagram



in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ will again be a weak equivalence. To obtain the inductive step at any limit ordinal, we observe that both colimits and weak equivalences in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ are defined levelwise, and that weak equivalences in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ are closed under transfinite composition (for instance by arguments in the style of steps (3)-(5) in the proof of Lemma 1.5.4 above).

So, it only remains to show that we have a commutative square in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ as claimed above. We verify the illustrated assertions in turn.

- Both vertical maps are monomorphisms and hence are cofibrations in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$.

- We have an isomorphism $\mathrm{diag}_!(\Delta^n) \cong \Delta^n \widehat{\boxtimes} \Delta^n$, under which identification the lower map is given by $\Delta^n \widehat{\boxtimes} \Delta^n \to \Delta^n \widehat{\boxtimes} \Delta^0$. In level j this is just the map $\coprod_{(\Delta^n)_j} \Delta^n \to \coprod_{(\Delta^n)_j} \Delta^0$, which is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. So the lower map is indeed a weak equivalence in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$.

- To see that the upper map is also a weak equivalence, we recall the explicit description of $\mathrm{diag}_!(\Lambda^n_i)$ (given both in the proof of [Moe89, Lemma 1.3] and in the text leading up to [GJ99, Lemma IV.3.10]), that

$$\mathrm{diag}_!(\Lambda^n_i)_{j,k} \cong \left\{ (\alpha, \beta) \in \hom_{\mathbf{\Delta}}([j],[n]) \times \hom_{\mathbf{\Delta}}([k],[n]) : \begin{array}{c} \text{there exists some} \\ l \in [n] \text{ with } l \neq i \\ \text{which is not in the} \\ \text{image of } \alpha \text{ or } \beta \end{array} \right\}.$$

  For any fixed $j \geq 0$, we must show that the map $\mathrm{diag}_!(\Lambda^n_i)_j \to \mathrm{const}(\Lambda^n_i)_j$ is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. The latter object is discrete (i.e. its only nondegenerate simplices are 0-simplices), and so this is equivalent to showing that the preimage of each such 0-simplex is acyclic in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. Such a 0-simplex is precisely the datum of a map $\alpha \in \hom_{\mathbf{\Delta}}([j],[n]) \cong \hom_{s\mathcal{S}\mathrm{et}}(\Delta^j, \Delta^n)$ whose image on 0-simplices does not cover $(\Delta^n)_0 \backslash \{\Delta^{\{i\}}\}$. Define the subset $T \subset (\Delta^n)_0$ to be the union of $\{\Delta^{\{i\}}\}$ and the image of α, and let $T^c \subset (\Delta^n)_0$ be its complement. Then, the preimage of the 0-simplex of $\mathrm{const}(\Lambda^n_i)_j$ corresponding to α is the subobject of $\Delta^n \in s\mathcal{S}\mathrm{et}$ consisting of those simplices whose 0-simplices do not contain all of $T^c \subset (\Delta^n)_0$, which is indeed acyclic in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ since T^c is nonempty. □

Remark 1.8.2. Lemma 1.8.1 fails drastically if we do not assume that $M \in s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is acyclic. In fact, in Remark 1.7.3, the stark difference between the sets of homotopy classes of maps $I^{s\mathcal{S}}_{\mathrm{Moer}}$ and $I^{s\mathcal{S}}_{\mathrm{KQ}}$ in $s\mathcal{S}$ is precisely due to the difference between (the weak equivalences classes of) the objects $\mathrm{diag}_!(\partial\Delta^n) \cong \partial\Delta^n \widehat{\boxtimes} \partial\Delta^n$ and $\mathrm{const}(\partial\Delta^n)$ in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$.

Remark 1.8.3. Using the arguments of Remark 1.7.3, one can use Lemma 1.8.1 to

give an alternative proof of Corollary 1.6.7. The key point is that we have a diagram

$$\begin{array}{ccc} \mathrm{diag}_!(\Lambda^n_i) & \xrightarrow{\approx} & \mathrm{const}(\Lambda^n_i) \\ \downarrow & & \downarrow \\ \mathrm{diag}_!(\Delta^n) & \xrightarrow[\approx]{} & \mathrm{const}(\Delta^n) \end{array}$$

in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. Hence, if the map $Y \to Z$ in $s\mathcal{S}$ has $\mathrm{rlp}(J_{\mathrm{KQ}})$, then it can be presented by a fibration $\mathtt{Y} \twoheadrightarrow \mathtt{Z}$ in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ that additionally has $\mathrm{rlp}(\mathrm{diag}_!(J_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}}))$, i.e. that is also in $\mathbf{F}^{ss\mathcal{S}\mathrm{et}}_{\mathrm{Moer}}$. Since both $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ and $ss\mathcal{S}\mathrm{et}_{\mathrm{Moer}}$ are right proper, it follows that all pullbacks of the map $\mathtt{Y} \to \mathtt{Z}$ in $ss\mathcal{S}\mathrm{et}$ simultaneously compute homotopy pullbacks in both model structures.

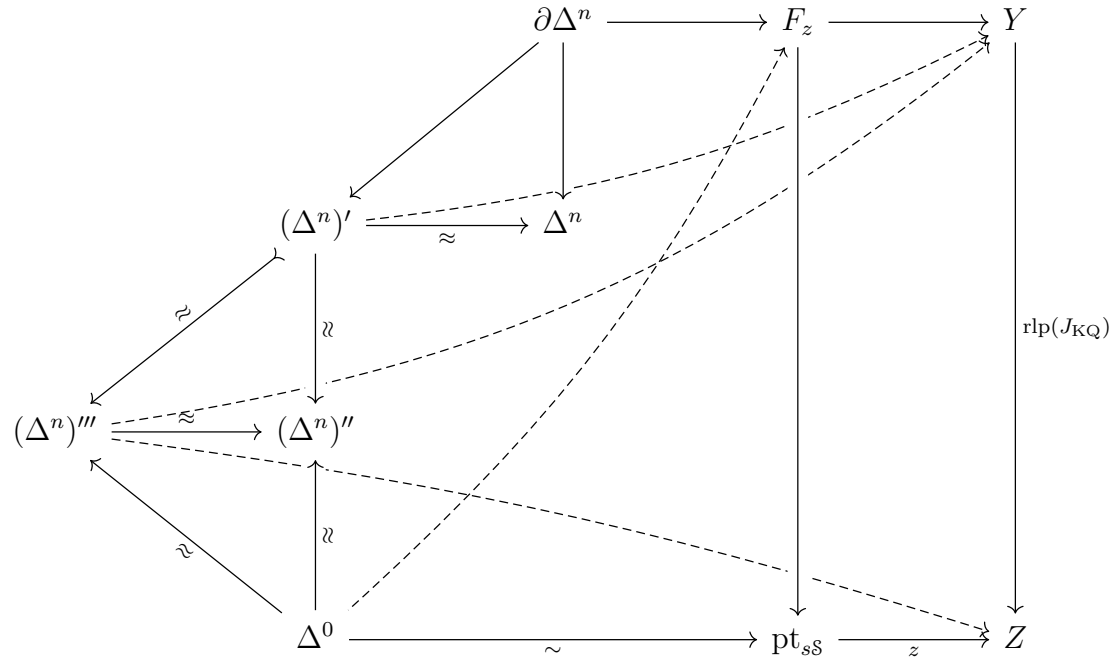


Figure 1.2: The diagram in $s\mathcal{S}_{\mathrm{KQ}}$ used in the proof of Proposition 1.6.5.

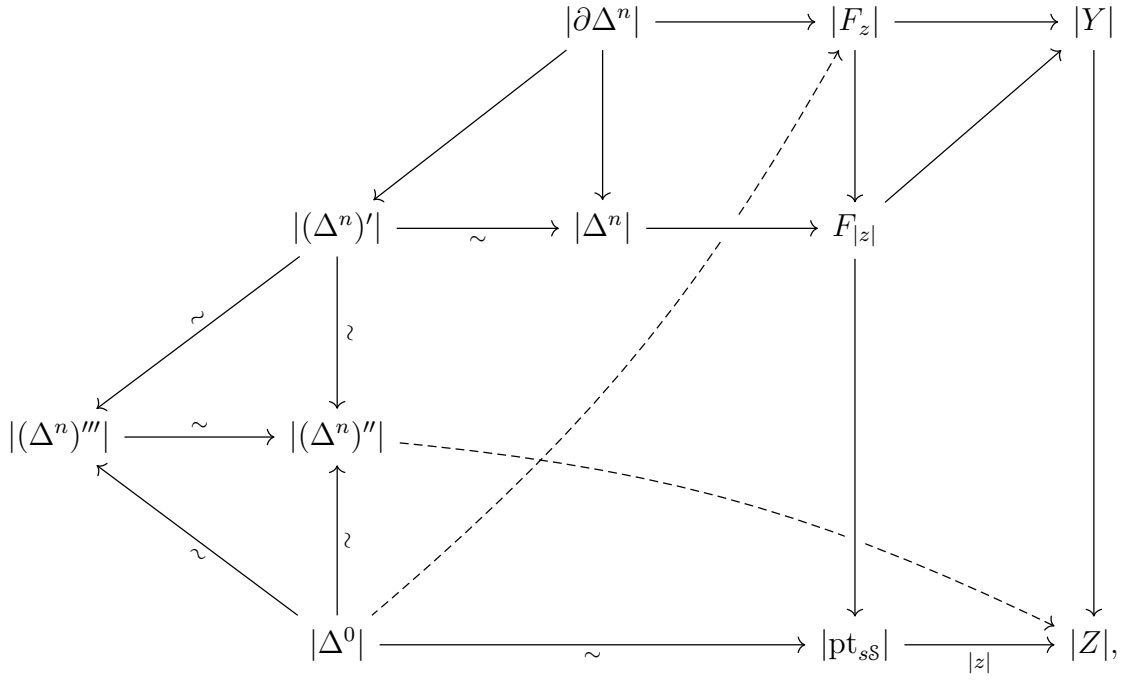


Figure 1.3: The diagram in $\mathcal{S}$ used in the proof of Proposition 1.6.5.

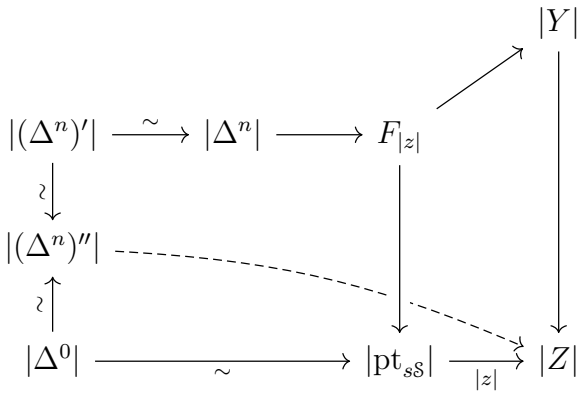


Figure 1.4: The subdiagram of the diagram in $\mathcal{S}$ of Figure 1.3 used in part (4) of the proof of Proposition 1.6.5.

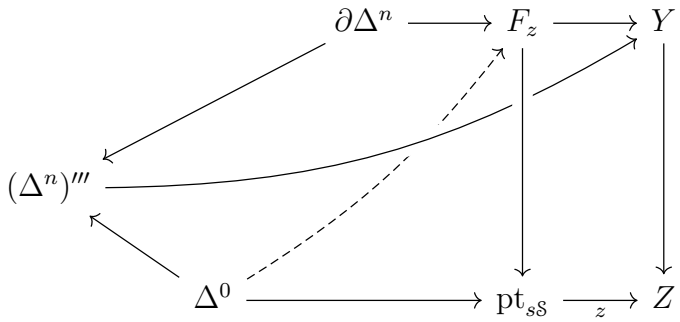


Figure 1.5: The subdiagram of the diagram in $s\mathcal{S}$ of Figure 1.2 used in part (7) of the proof of Proposition 1.6.5.

Figure 1.6: The maps in $I^{s\mathcal{S}}_{\mathrm{Moer}}$ and $I^{s\mathcal{S}}_{\mathrm{KQ}}$ at $n = 2$.

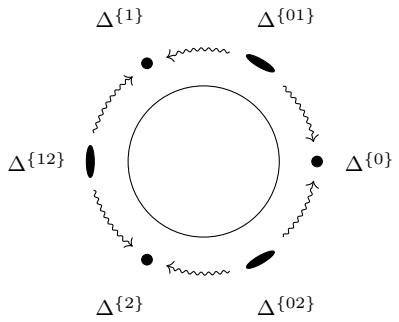


Figure 1.7: A weak natural transformation of simplicial topological spaces from $\partial\Delta^2$ to const(S^1).

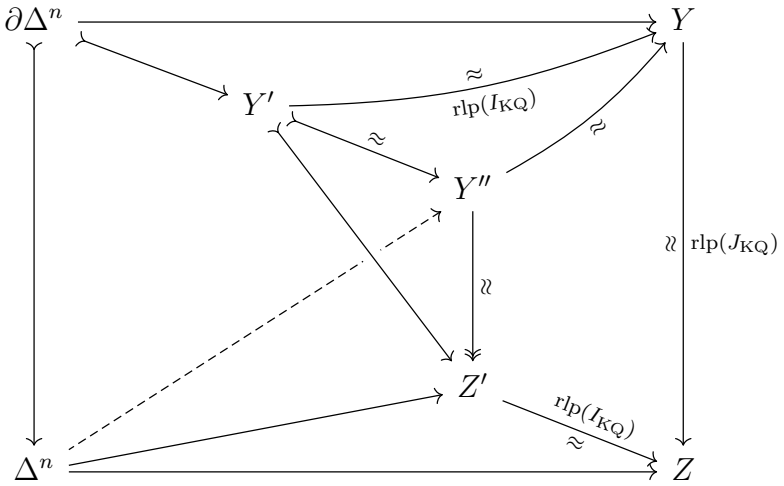


Figure 1.8: The diagram in $s\mathcal{S}_{\mathrm{KQ}}$ used in the proof of Proposition 1.7.9.

Chapter 2

The universality of the Rezk nerve

In this chapter, we functorially associate to each relative ∞-category $(\mathcal{R}, \mathbf{W})$ a simplicial space $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W})$, called its *Rezk nerve* (a straightforward generalization of Rezk's "classification diagram" construction for relative categories). We prove the following *local* and *global* universal properties of this construction: (i) that the complete Segal space generated by the Rezk nerve $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W})$ is precisely the one corresponding to the localization $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$; and (ii) that the Rezk nerve functor defines an equivalence $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty}[\![\mathbf{W}^{-1}_{\mathrm{BK}}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_{\infty}$ from a localization of the ∞-category of relative ∞-categories to the ∞-category of ∞-categories.

2.0 Introduction

2.0.1 The Rezk nerve

A *relative ∞-category* is a pair $(\mathcal{R}, \mathbf{W})$ of an ∞-category $\mathcal{R}$ and a subcategory $\mathbf{W} \subset \mathcal{R}$ containing all the equivalences, called the subcategory of *weak equivalences*. Freely inverting the weak equivalences, we obtain the *localization* of this relative ∞-category, namely the initial functor

$$\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$$

from $\mathcal{R}$ which sends all maps in $\mathbf{W}$ to equivalences. In general, it is extremely difficult to access the localization.[1] To ameliorate this state of affairs, in this chapter we

[1] For instance, even in the case that $\mathcal{R}$ is a one-object 1-category and we are only interested in its *1-categorical* localization, i.e. the composite $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!] \to \mathrm{ho}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \simeq \mathcal{R}[\mathbf{W}^{-1}]$ – that is, in the case that we are interested in freely inverting certain elements of a monoid –, obtaining a concrete description is nevertheless an intractable (in fact, computationally undecidable) task, closely related to the so-called "word problem" for generators and relations in abstract algebra.

provide a novel method of accessing this localization via Rezk's theory of ***complete Segal spaces***.

To describe this, let us first recall that the ∞-category $\mathcal{CSS}$ of complete Segal spaces participates in a diagram

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\rightleftarrows}} \mathcal{CSS} \underset{\mathrm{N}_\infty}{\overset{\mathrm{N}_\infty^{-1}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty.$$

That is, it sits as a reflective subcategory of the ∞-category $s\mathcal{S}$ of simplicial spaces, and it is equivalent to the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories. In particular, one can contemplate the complete Segal space (or equivalently, the ∞-category) *generated* by an arbitrary simplicial space Y, much as one can contemplate the 1-category generated by an arbitrary simplicial set: this is encoded by the unit

$$Y \xrightarrow{\eta} \mathrm{L}_{\mathcal{CSS}}(Y)$$

of the adjunction (where we omit the inclusion functor $\mathrm{U}_{\mathcal{CSS}}$ for brevity).

Now, given a relative ∞-category $(\mathcal{R}, \mathbf{W})$, its ***Rezk nerve*** is a certain simplicial space

$$\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}) \in s\mathcal{S}$$

which "wants to be" the complete Segal space

$$\mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \in \mathcal{CSS}$$

corresponding to its localization:

- it admits canonical maps

  $$\mathrm{N}_\infty(\mathcal{R}) \to \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}) \to \mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!]),$$

  and moreover

- its construction manifestly dictates that for any ∞-category $\mathcal{C}$, the restriction map

  $$\hom_{s\mathcal{S}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}), \mathrm{N}_\infty(\mathcal{C})) \to \hom_{s\mathcal{S}}(\mathrm{N}_\infty(\mathcal{R}), \mathrm{N}_\infty(\mathcal{C})) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C})$$

  factors through the subspace (i.e. collection of path components) of those functors $\mathcal{R} \to \mathcal{C}$ sending all maps in $\mathbf{W} \subset \mathcal{R}$ to equivalences in $\mathcal{C}$.

Unfortunately, life is not quite so simple: the Rezk nerve is not generally a complete Segal space (or even a Segal space).[2] Nevertheless, the second-best-possible thing is true.

[2] We provide sufficient conditions on $(\mathcal{R}, \mathbf{W})$ for its Rezk nerve $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W})$ to be a (complete) Segal space in Chapter 4.

Theorem (2.3.8)**.** *The above maps extend to a commutative diagram*

$$\begin{array}{ccccc} \mathrm{N}_\infty(\mathcal{R}) & \longrightarrow & \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W}) & \longrightarrow & \mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \\ {\scriptstyle\eta}\downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\eta} & & {\scriptstyle\wr}\downarrow{\scriptstyle\eta} \\ \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R})) & \longrightarrow & \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W})) & \xrightarrow[\sim]{} & \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!])). \end{array}$$

In other words, the complete Segal space generated by the Rezk nerve of $(\mathcal{R},\mathbf{W})$ is precisely the one corresponding to its localization.

This theorem provides a *local* universal property of the Rezk nerve: it asserts that the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS} \xrightarrow[\sim]{\mathrm{N}^{-1}_\infty} \mathcal{C}\mathrm{at}_\infty$$

takes each relative ∞-category $(\mathcal{R},\mathbf{W})$ to its localization $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$. However, it says nothing about the effect of this composite on *morphisms* of relative ∞-categories. To this end, we also prove the following.

Theorem (2.3.9 and 2.3.11)**.** *The above composite is canonically equivalent to the localization functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty.$$

In particular, denoting by $\mathbf{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *the subcategory of maps which it takes to equivalences, the above composite induces an equivalence*

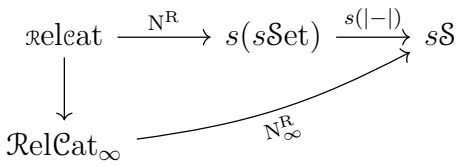
$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty[\![\mathbf{W}^{-1}_{\mathrm{BK}}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty.$$

In other words, the Rezk nerve functor does indeed *functorially* compute localizations of relative ∞-categories, and moreover the induced "homotopy theory" on the ∞-category $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ of relative ∞-categories – that is, the relative ∞-category structure $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty,\mathbf{W}_{\mathrm{BK}})$ that results therefrom – gives a presentation of the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories. We therefore deem this result as capturing the *global* universal property of the Rezk nerve.

Remark 2.0.1. The Rezk nerve functor is a close cousin of Rezk's "classification diagram" functor of [Rez01, 3.3]; to emphasize the similarity, we denote the latter functor by

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xrightarrow{\mathrm{N}^{\mathrm{R}}} ss\mathcal{S}\mathrm{et}$$

and refer to it as the *1-categorical Rezk nerve*. In fact, as we explain in Remark 2.3.2, this is essentially just the restriction of the ∞-categorical Rezk nerve functor, in the sense that there is a canonical commutative diagram



in $\mathcal{C}\mathrm{at}_\infty$. In Remark 2.3.10, we use this observation to show that our global universal property of the ∞-categorical Rezk nerve can be seen as a generalization of work of Barwick–Kan.

2.0.2 Outline

We now provide a more detailed outline of the contents of this chapter.

- In §2.1, we undertake a study of relative ∞-categories and their localizations.
- In §2.2, we briefly review the theory of complete Segal spaces.
- In §2.3, we introduce the Rezk nerve and state its local and global universal properties. We give a proof of the global universal property which relies on the local one, but we defer the proof of the local one to §2.4.
- In §2.4, we prove the local universal property of the Rezk nerve. Though much of the proof is purely formal, at its heart it ultimately relies on some rather delicate model-categorical arguments.

2.0.3 Acknowledgments

We heartily thank Zhen Lin Low, Eric Peterson, Chris Schommer-Pries, and Mike Shulman for many (sometimes extremely extended) discussions regarding the material in this chapter, particularly the proof of Lemma 2.4.3, and we are grateful to Adeel Khan Yusufzai for providing helpful comments on a preliminary draft. It is also our pleasure to thank Katherine de Kleer for writing a Python script verifying the identities for the simplicial homotopies defined therein.[3] We would also like to thank an anonymous referee for a very careful report, as well as for pointing out both

[3]This script is readily available upon request.

a subtle error and a means of fixing it. Lastly, we thank the NSF graduate research fellowship program (grant DGE-1106400) for its financial support during the time that this work was carried out.

2.1 Relative ∞-categories and their localizations

Given an ∞-category and some chosen subset of its morphisms, we are interested in freely inverting those morphisms. In order to codify these initial data, we introduce the following.

Definition 2.1.1. A ***relative ∞-category*** is a pair $(\mathcal{R}, \mathbf{W})$ of an ∞-category $\mathcal{R}$ and a subcategory $\mathbf{W} \subset \mathcal{R}$, called the subcategory of ***weak equivalences***, such that $\mathbf{W}$ contains all the equivalences (and in particular, all the objects) in $\mathcal{R}$. These form the evident ∞-category $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.[4] Weak equivalences will be denoted by the symbol $\xrightarrow{\approx}$. Though we will of course write $\mathcal{R}$ for the ∞-category obtained by forgetting $\mathbf{W}$, to ease notation we will also sometimes simply write $\mathcal{R}$ for the pair $(\mathcal{R}, \mathbf{W})$. We write $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ for the full subcategory on those relative ∞-categories $(\mathcal{R}, \mathbf{W})$ such that $\mathcal{R} \in \mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty$.

Remark 2.1.2. As we are working invariantly, our Definition 2.1.1 is not quite a generalization of the 1-category $\mathfrak{rel}\mathfrak{cat}$ as given e.g. in [BK12b, 3.1] or [LMG15, Definition 3.1], an object of which is a *strict* category $\mathcal{R} \in \mathfrak{cat}$ (i.e. a simplicial set satisfying the Segal condition (see subitem A(4)(c))) equipped with a wide subcategory $\mathbf{W} \subset \mathcal{R}$ (i.e. one containing all the objects). For emphasis, we will therefore sometimes refer to objects of $\mathfrak{rel}\mathfrak{cat}$ as *strict relative categories*.

In addition to being the only meaningful variant in the invariant world, Definition 2.1.1 allows for a clean and aesthetically appealing definition of localization, namely as a left adjoint (see Definition 2.1.8). In any case, as we are ultimately only interested in relative ∞-categories because we are interested in their localizations, this requirement is no real loss.

Despite these differences, there is an evident functor

$$\mathfrak{rel}\mathfrak{cat} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at},$$

to which we will refer on occasion.

Notation 2.1.3. In order to disambiguate our notation associated to various relative ∞-categories, we introduce the following conventions.

[4] To be precise, one can view $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \simeq \mathrm{Fun}^{\mathrm{surj\ mono}}([1], \mathcal{C}\mathrm{at}_\infty) \subset \mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$ as the full subcategory on those functors selecting the inclusion of a surjective monomorphism.

- When multiple relative ∞-categories are under discussion, we will sometimes decorate them for clarity. For instance, we may write $(\mathcal{R}_1, \mathbf{W}_1)$ and $(\mathcal{R}_2, \mathbf{W}_2)$ to denote two arbitrary relative ∞-categories, or we may instead write $(\mathcal{I}, \mathbf{W}_{\mathcal{I}})$ and $(\mathcal{J}, \mathbf{W}_{\mathcal{J}})$.

- Moreover, we will eventually study certain "named" relative ∞-categories; for example, there is a *Barwick–Kan relative structure* on $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ itself (see Definition 2.1.16). We will always subscript the subcategory of weak equivalences of such a relative ∞-category with (an abbreviation of) its name; for example, we will write $\mathbf{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. We may also merely similarly subscript the ambient ∞-category to denote the relative ∞-category; for example, we will write $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} = (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathbf{W}_{\mathrm{BK}})$.

- Finally, there will occasionally be two different ∞-categories with relative structures of the same name. In such cases, if disambiguation is necessary we will additionally superscript the subcategory of weak equivalences with the name of the ambient ∞-category. For instance, we would write $\mathbf{W}_{\mathrm{BK}}^{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ to distinguish it from the subcategory $\mathbf{W}_{\mathrm{BK}}^{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$.

We have the following fundamental source of examples of relative ∞-categories.

Example 2.1.4. If $\mathcal{R} \to \mathcal{C}$ is any functor of ∞-categories, we can define a relative ∞-category $(\mathcal{R}, \mathbf{W})$ by declaring $\mathbf{W} \subset \mathcal{R}$ to be the subcategory on those maps that are sent to equivalences in $\mathcal{C}$. Note that $\mathbf{W} \subset \mathcal{R}$ will automatically have the two-out-of-three property.

Definition 2.1.5. In the situation of Example 2.1.4, we will say that the functor $\mathcal{R} \to \mathcal{C}$ ***creates*** the subcategory $\mathbf{W} \subset \mathcal{R}$.

We will make heavy use of the following construction.

Notation 2.1.6. Given any $(\mathcal{R}_1, \mathbf{W}_1), (\mathcal{R}_2, \mathbf{W}_2) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, we define

$$\left(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\right) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$$

by setting

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}} \subset \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)$$

to be the full subcategory on those functors which send $\mathbf{W}_1 \subset \mathcal{R}_1$ into $\mathbf{W}_2 \subset \mathcal{R}_2$, and setting

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}$$

to be the (generally non-full) subcategory on the natural weak equivalences.[5] It is not hard to see that this defines an internal hom bifunctor for $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \times)$.

It will be useful to have the following terminology.

Definition 2.1.7. If $\mathcal{C}$ is any ∞-category, we call $(\mathcal{C}, \mathcal{C}^{\simeq})$ the associated ***minimal relative ∞-category*** and we call $(\mathcal{C}, \mathcal{C})$ the associated ***maximal relative ∞-category***. These define fully faithful inclusions

$$\mathcal{C}\mathrm{at}_\infty \underset{\perp}{\overset{\min}{\underset{\max}{\leftarrow}}} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$$

which are respectively left and right adjoint to the forgetful functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{U}_{\mathcal{R}\mathrm{el}}} \mathcal{C}\mathrm{at}_\infty$ sending $(\mathcal{R}, \mathbf{W})$ to $\mathcal{R}$. For $[n] \in \mathbf{\Delta} \subset \mathcal{C}\mathrm{at}_\infty$, we will use the abbreviation $[n]_{\mathbf{W}} = \max([n])$, since these relative categories will appear quite often; correspondingly, we will also make the implicit identification $[n] = \min([n])$.

We now come to our central object of interest.

Definition 2.1.8. The functor $\min : \mathcal{C}\mathrm{at}_\infty \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ also admits a left adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty,$$

which we refer to as the ***localization*** functor on relative ∞-categories. For a relative ∞-category $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, we will often write $\mathcal{R}[\![\mathbf{W}^{-1}]\!] = \mathscr{L}(\mathcal{R}, \mathbf{W})$; we only write $\mathscr{L}$ since the notation $(-)[\![(-)^{-1}]\!]$ is a bit unwieldy. Explicitly, its value on $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ can be obtained as the pushout

$$\mathcal{R}[\![\mathbf{W}^{-1}]\!] \simeq \mathrm{colim} \left(\begin{array}{ccc} \mathbf{W} & \longrightarrow & \mathcal{R} \\ \downarrow & & \\ \mathbf{W}^{\mathrm{gpd}} & & \end{array} \right)$$

in $\mathcal{C}\mathrm{at}_\infty$ (and the functor itself can be obtained by applying this construction in families).

[5] If we consider $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \subset \mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$, then $\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}$ is simply the ∞-category of natural transformations.

Remark 2.1.9. Using model categories, one can of course compute the pushout in $\mathcal{C}\mathrm{at}_\infty$ of Definition 2.1.8 by working in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ (which is left proper), for instance after presenting the map $\mathbf{W} \to \mathbf{W}^{\mathrm{gpd}}$ using the derived unit of the Quillen adjunction $\mathrm{id} : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} : \mathrm{id}$, i.e. after taking a fibrant replacement via a cofibration in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ of a quasicategory presenting $\mathbf{W}$. However, note that this derived unit can be quite difficult to describe in practice, and moreover the resulting pushout will generally still be very far from being a quasicategory. Equally inexplicitly, one can also obtain a quasicategory presenting $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ by computing a fibrant replacement in the marked model structure of Proposition T.3.1.3.7 (i.e. in the specialization of the model structure given there to the case where the base is the terminal object $\mathrm{pt}_{s\mathcal{S}\mathrm{et}}$).

Remark 2.1.10. We will also use the term "localization" to refer to the canonical map $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$ satisfying the universal property that for any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, the restriction

$$\hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!], \mathcal{C}) \to \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C})$$

defines an equivalence onto the subspace

$$\hom_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty}((\mathcal{R}, \mathbf{W}), \min(\mathcal{C})) \subset \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C})$$

of those functors which take $\mathbf{W}$ into $\mathcal{C}^{\simeq}$.[6] Thus, by definition the map $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ is an epimorphism in $\mathcal{C}\mathrm{at}_\infty$.

Example 2.1.11. The localization of a minimal relative ∞-category $\min(\mathcal{C}) = (\mathcal{C}, \mathcal{C}^{\simeq})$ is simply the identity functor $\mathcal{C} \xrightarrow{\sim} \mathcal{C}$.

Example 2.1.12. The localization of a maximal relative ∞-category $\max(\mathcal{C}) = (\mathcal{C}, \mathcal{C})$ is the groupoid completion functor $\mathcal{C} \to \mathcal{C}^{\mathrm{gpd}}$ (i.e. the component at $\mathcal{C}$ of the unit of the adjunction $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{S} : \mathrm{U}_{\mathcal{S}}$).

Example 2.1.13. Given a left localization adjunction $\mathrm{L} : \mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C} : \mathrm{U}$ (i.e. an adjunction with fully faithful right adjoint), if we define $\mathbf{W} \subset \mathcal{C}$ to be created by $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$, then the localization of $(\mathcal{C}, \mathbf{W})$ is precisely $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$: that is, the functor $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$ induces an equivalence $\mathcal{C}[\![\mathbf{W}^{-1}]\!] \xrightarrow{\sim} \mathrm{L}\mathcal{C}$, which is in fact inverse to the composite $\mathrm{L}\mathcal{C} \xrightarrow{\mathrm{U}} \mathcal{C} \to \mathcal{C}[\![\mathbf{W}^{-1}]\!]$. This follows from Proposition T.5.2.7.12, or alternatively from Lemma 2.1.24 (see Remark 2.1.25). Of course, a dual statement holds for right localization adjunctions.

[6] This map can be obtained either by applying $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ to the counit $\min(\mathcal{R}) \to (\mathcal{R}, \mathbf{W})$ of the adjunction $\min \dashv \mathrm{U}_{\mathcal{R}\mathrm{el}}$, or by applying $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{U}_{\mathcal{R}\mathrm{el}}} \mathcal{C}\mathrm{at}_\infty$ to the unit $(\mathcal{R}, \mathbf{W}) \to \min(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$ of the adjunction $\mathscr{L} \dashv \min$.

For an arbitrary relative ∞-category $(\mathcal{R}, \mathbf{W})$, note that the localization map $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ might *not* create the subcategory $\mathbf{W} \subset \mathcal{R}$: there might be strictly more maps in $\mathcal{R}$ which are sent to equivalences in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$. This leads us to the following notion.

Definition 2.1.14. A relative ∞-category $(\mathcal{R}, \mathbf{W})$ is called ***saturated*** if the localization map $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ creates the subcategory $\mathbf{W} \subset \mathcal{R}$.

Remark 2.1.15. If a relative ∞-category $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ has its subcategory of weak equivalences $\mathbf{W} \subset \mathcal{R}$ created by *any* functor $\mathcal{R} \to \mathcal{C}$, then $(\mathcal{R}, \mathbf{W})$ will automatically be saturated. To see this, consider the induced factorization

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathcal{C} \\ & \searrow \quad \nearrow & \\ & \mathcal{R}[\![\mathbf{W}^{-1}]\!] & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$. This implies that any morphism in $\mathcal{R}$ which is sent to an equivalence in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ must also be sent to an equivalence in $\mathcal{C}$ (by the functoriality of inverse morphisms), so that by definition it lies in the subcategory $\mathbf{W} \subset \mathcal{R}$.

Now, we will be using relative ∞-categories as "presentations of ∞-categories", namely of their localizations. However, a map of relative ∞-categories may induce an equivalence on localizations without itself being an equivalence in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. This leads us to the following notion.

Definition 2.1.16. We define the subcategory $\mathbf{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ of ***Barwick–Kan weak equivalences*** to be created by the localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$. We denote the resulting relative ∞-category by $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} = (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathbf{W}_{\mathrm{BK}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.

The following result then justifies our usage of relative ∞-categories as "presentations of ∞-categories".

Proposition 2.1.17. *The functors in the left localization adjunction* $\mathscr{L} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{C}\mathrm{at}_\infty : \min$ *induce inverse equivalences*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty[\![\mathbf{W}_{\mathrm{BK}}^{-1}]\!] \simeq \mathcal{C}\mathrm{at}_\infty$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proof. This is a special case of Example 2.1.13. □

We have the following strengthening of Remark 2.1.10.

Proposition 2.1.18. *For any* $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *and any* $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$*, the restriction*

$$\mathrm{Fun}(\mathcal{R}[\![\mathbf{W}^{-1}]\!], \mathcal{C}) \to \mathrm{Fun}(\mathcal{R}, \mathcal{C})$$

along the localization functor $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ *defines an equivalence onto the full subcategory of* $\mathrm{Fun}(\mathcal{R}, \mathcal{C})$ *spanned by those functors which take* $\mathbf{W}$ *into* $\mathcal{C}^{\simeq}$*.*

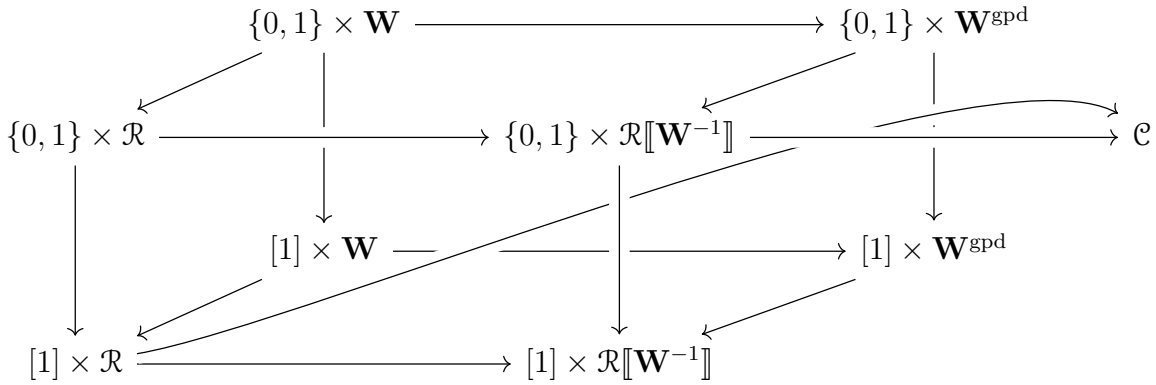
Proof. We begin by observing that this functor is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$: this is because we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{R}[\![\mathbf{W}^{-1}]\!], \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathbf{W}^{\mathrm{gpd}}, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{R}, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathbf{W}, \mathcal{C}) \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ in which the right arrow is clearly a monomorphism, and monomorphisms are closed under pullback. So in particular, this functor is the inclusion of a subcategory. Then, to see that it is full, suppose we are given two functors $\mathcal{R}[\![\mathbf{W}^{-1}]\!] \rightrightarrows \mathcal{C}$, considered as objects of $\mathrm{Fun}(\mathcal{R}[\![\mathbf{W}^{-1}]\!], \mathcal{C})$. A natural transformation between their images in $\mathrm{Fun}(\mathcal{R}, \mathcal{C})$ is given by a functor $[1] \times \mathcal{R} \to \mathcal{C}$ which restricts to the the two composites $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!] \rightrightarrows \mathcal{C}$ on the two objects $0, 1 \in [1]$. Since we already know that $\mathrm{Fun}(\mathcal{R}[\![\mathbf{W}^{-1}]\!], \mathcal{C}) \subset \mathrm{Fun}(\mathcal{R}, \mathcal{C})$ is the inclusion of a subcategory, it suffices to obtain an extension

$$\begin{array}{ccc} [1] \times \mathcal{R} & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ [1] \times \mathcal{R}[\![\mathbf{W}^{-1}]\!] & & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$. For this, consider the diagram



in $\mathcal{C}\mathrm{at}_\infty$ containing and extending the above data. The bottom square is a pushout since the functor $[1] \times - : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is a left adjoint, and the back square is a pushout by Lemma 2.1.20. Together, these observations guarantee the desired extension. □

Remark 2.1.19. Proposition 2.1.18 implies that Definition 2.1.8 agrees with Definition A.1.3.4.1.

We now make an easy observation regarding the localization functor, which is necessary for the argument of Proposition 2.1.18 but will also be useful in its own right.

Lemma 2.1.20. *The localization functor* $\mathscr{L} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ *commutes with finite products.*

For the proof of Lemma 2.1.20, it will be convenient to have the following notion.

Definition 2.1.21. Let $(\mathcal{C}, \otimes)$ be a closed symmetric monoidal ∞-category with internal hom bifunctor

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} \mathcal{C}.$$

A collection of objects I of $\mathcal{C}$ is called an ***exponential ideal*** if we have $\underline{\hom}_{\mathcal{C}}(Y, Z) \in I$ for any $Y \in \mathcal{C}$ and any $Z \in I$. We will use this same terminology to refer to a full subcategory $\mathcal{D} \subset \mathcal{C}$ whose objects form an exponential ideal.

The following straightforward result explains why we are interested in exponential ideals.

Lemma 2.1.22. *Suppose that* $(\mathcal{C}, \otimes)$ *is a closed symmetric monoidal* ∞*-category, and let* $\mathrm{L} : \mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C} : \mathrm{U}$ *be a left localization with unit map* $\mathrm{id}_{\mathcal{C}} \xrightarrow{\eta} \mathrm{L}$ *in* $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ *(where we implicitly consider* $\mathrm{L}\mathcal{C} \subset \mathcal{C}$*). Then, the full subcategory* $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ *is an exponential ideal if and only if the natural map* $\mathrm{L}(\eta \otimes \eta)$ *is an equivalence in* $\mathrm{Fun}(\mathcal{C} \times \mathcal{C}, \mathcal{C})$ *(i.e. we have*

$$\mathrm{L}(Y \otimes Z) \xrightarrow{\sim} \mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z))$$

in $\mathrm{L}\mathcal{C}$ *for all* $Y, Z \in \mathcal{C}$*). In particular, if* $\mathrm{L}\mathcal{C}$ *is closed under the monoidal structure, then* $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ *is an exponential ideal if and only if*

$$\mathrm{L}(Y \otimes Z) \simeq \mathrm{L}(Y) \otimes \mathrm{L}(Z)$$

in $\mathrm{L}\mathcal{C}$ *for all* $Y, Z \in \mathcal{C}$.

Proof. Suppose that $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ is an exponential ideal. Then, for any $Y, Z \in \mathcal{C}$ and any test object $W \in \mathrm{L}\mathcal{C}$, we have the string of natural equivalences

$$\begin{aligned}\hom_{\mathcal{C}}(\mathrm{L}(Y \otimes Z), W) &\simeq \hom_{\mathcal{C}}(Y \otimes Z, W) \simeq \hom_{\mathcal{C}}(Y, \underline{\hom}_{\mathcal{C}}(Z, W)) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y), \underline{\hom}_{\mathcal{C}}(Z, W)) \\ &\simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes Z, W) \simeq \hom_{\mathcal{C}}(Z \otimes \mathrm{L}(Y), W) \simeq \hom_{\mathcal{C}}(Z, \underline{\hom}_{\mathcal{C}}(\mathrm{L}(Y), W)) \\ &\simeq \hom_{\mathcal{C}}(\mathrm{L}(Z), \underline{\hom}_{\mathcal{C}}(\mathrm{L}(Y), W)) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Z) \otimes \mathrm{L}(Y), W) \\ &\simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes \mathrm{L}(Z), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z)), W).\end{aligned}$$

Hence, we have an equivalence $\mathrm{L}(Y \otimes Z) \simeq \mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z))$ by the Yoneda lemma applied to the ∞-category $\mathrm{L}\mathcal{C}$ (and it is straightforward to check that this equivalence is indeed induced by the specified map). So $\mathrm{L}(\eta \otimes \eta)$ is an equivalence in $\mathrm{Fun}(\mathcal{C} \times \mathcal{C}, \mathcal{C})$, as desired.

On the other hand, suppose that $\mathrm{L}(Y \otimes Z) \xrightarrow{\sim} \mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z))$ for all $Y, Z \in \mathcal{C}$. Then, we have the string of natural equivalences

$$\begin{aligned}\hom_{\mathcal{C}}(Y, \underline{\hom}_{\mathcal{C}}(Z, W)) &\simeq \hom_{\mathcal{C}}(Y \otimes Z, W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y \otimes Z), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z)), W) \\ &\simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes \mathrm{L}(Z), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y)) \otimes \mathrm{L}(Z), W) \\ &\simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(\mathrm{L}(Y)) \otimes \mathrm{L}(Z)), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y) \otimes Z), W) \\ &\simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes Z, W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y), \underline{\hom}_{\mathcal{C}}(Z, W)).\end{aligned}$$

Hence, for any map $Y \to Y'$ in $\mathcal{C}$ which localizes to an equivalence $\mathrm{L}(Y) \xrightarrow{\sim} \mathrm{L}(Y')$ in $\mathrm{L}\mathcal{C} \subset \mathcal{C}$, we obtain an equivalence $\hom_{\mathcal{C}}(Y, \underline{\hom}_{\mathcal{C}}(Z, W)) \xleftarrow{\sim} \hom_{\mathcal{C}}(Y', \underline{\hom}_{\mathcal{C}}(Z, W))$. It follows that the object $\underline{\hom}_{\mathcal{C}}(Z, W) \in \mathcal{C}$ is local with respect to the left localization, i.e. that in fact $\underline{\hom}_{\mathcal{C}}(Z, W) \in \mathrm{L}\mathcal{C} \subset \mathcal{C}$. So $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ is an exponential ideal. □

With Lemma 2.1.22 in hand, we now proceed to prove Lemma 2.1.20.

Proof of Lemma 2.1.20. The right adjoint $\min : \mathcal{C}\mathrm{at}_\infty \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ induces an equivalence onto the full subcategory of minimal relative ∞-categories. It is easy to see that this is an exponential ideal in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \times)$, and so the result follows from Lemma 2.1.22. □

The following useful construction relies on Lemma 2.1.20.

Remark 2.1.23. Let $(\mathcal{R}_1, \mathbf{W}_1), (\mathcal{R}_2, \mathbf{W}_2) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Then the identity map

$$\left(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\right) \to \left(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\right)$$

is adjoint to an evaluation map

$$(\mathcal{R}_1, \mathbf{W}_1) \times \left(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\right) \to (\mathcal{R}_2, \mathbf{W}_2).$$

By Lemma 2.1.20, applying the localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ yields a map

$$\mathcal{R}_1[\![\mathbf{W}_1^{-1}]\!] \times \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}[\![\big(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\big)^{-1}]\!] \to \mathcal{R}_2[\![\mathbf{W}_2^{-1}]\!],$$

which is itself adjoint to a canonical map

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}[\![\big(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\big)^{-1}]\!] \to \mathrm{Fun}(\mathcal{R}_1[\![\mathbf{W}_1^{-1}]\!], \mathcal{R}_2[\![\mathbf{W}_2^{-1}]\!]).$$

In particular, precomposing with the localization map for the internal hom-object yields a canonical map

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}} \to \mathrm{Fun}(\mathcal{R}_1[\![\mathbf{W}_1^{-1}]\!], \mathcal{R}_2[\![\mathbf{W}_2^{-1}]\!]).$$

Lemma 2.1.20 also allows us to prove the following result, which will be useful later and which gives a sense of the interplay between relative ∞-categories and their localizations.

Lemma 2.1.24. *Given any $(\mathcal{R}_1\mathbf{W}_1), (\mathcal{R}_2, \mathbf{W}_2) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ and any pair of maps $\mathcal{R}_1 \rightrightarrows \mathcal{R}_2$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, a natural weak equivalence between them induces an equivalence between their induced functors $\mathcal{R}_1[\![\mathbf{W}_1^{-1}]\!] \rightrightarrows \mathcal{R}_2[\![\mathbf{W}_2^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$.*

Proof. A natural weak equivalence corresponds to a map $[1]_{\mathbf{W}} \times \mathcal{R}_1 \to \mathcal{R}_2$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. By Lemma 2.1.20 (and Example 2.1.12), this gives rise to a map $[1]^{\mathrm{gpd}} \times \mathcal{R}_1[\![\mathbf{W}_1^{-1}]\!] \to \mathcal{R}_2[\![\mathbf{W}_2^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$, which precisely selects the desired equivalence. □

Remark 2.1.25. Lemma 2.1.24 allows for a simple proof of Proposition T.5.2.7.12, that a left localization is in particular a free localization. Indeed, given a left localization adjunction $\mathrm{L} : \mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C} : \mathrm{U}$, write $\mathbf{W} \subset \mathcal{C}$ for the subcategory created by the functor $\mathrm{L} : \mathcal{C} \to \mathrm{L}\mathcal{C}$. Then, this adjunction gives rise to a pair of maps $(\mathcal{C}, \mathbf{W}) \xrightarrow{\mathrm{L}} \min(\mathrm{L}\mathcal{C})$ and $\min(\mathrm{L}\mathcal{C}) \xrightarrow{\mathrm{U}} (\mathcal{C}, \mathbf{W})$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Moreover, the composite

$$\min(\mathrm{L}\mathcal{C}) \xrightarrow{\mathrm{U}} (\mathcal{C}, \mathbf{W}) \xrightarrow{\mathrm{L}} \min(\mathrm{L}\mathcal{C})$$

is an equivalence, while the composite

$$(\mathcal{C}, \mathbf{W}) \xrightarrow{\mathrm{L}} \min(\mathrm{L}\mathcal{C}) \xrightarrow{\mathrm{U}} (\mathcal{C}, \mathbf{W})$$

is connected to $\mathrm{id}_{(\mathcal{C},\mathbf{W})}$ by the unit of the adjunction, which is a componentwise weak equivalence (since for any $Y \in \mathcal{C}$, applying $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$ to the map $Y \to \mathrm{L}(Y)$ gives an equivalence $\mathrm{L}(Y) \xrightarrow{\sim} \mathrm{L}(\mathrm{L}(Y))$). Hence, it follows that these functors induce inverse equivalences $\mathcal{C}[\![\mathbf{W}^{-1}]\!] \simeq \mathrm{L}\mathcal{C}$. (From here, one can obtain the actual statement of Proposition T.5.2.7.12 by appealing to Proposition 2.1.18.)

Lemma 2.1.24 also has the following special case which will be useful to us.

Lemma 2.1.26. *Given any $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ and any pair of maps $\mathcal{C} \rightrightarrows \mathcal{D}$, a natural transformation between them induces an equivalence between the induced maps $\mathcal{C}^{\mathrm{gpd}} \rightrightarrows \mathcal{D}^{\mathrm{gpd}}$ in $\mathcal{S}$.*

Proof. In light of Example 2.1.12, this follows from applying Lemma 2.1.24 in the special case that $(\mathcal{R}_1, \mathbf{W}_1) = \max(\mathcal{C})$ and $(\mathcal{R}_2, \mathbf{W}_2) = \max(\mathcal{D})$. □

Remark 2.1.27. Lemma 2.1.26 can also be seen as following from applying Lemma 2.1.22 to the left localization $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{S} : \mathrm{U}_{\mathcal{S}}$. Namely, since the full subcategory $\mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$ is an exponential ideal for $(\mathcal{C}\mathrm{at}_\infty, \times)$, then the left adjoint $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ commutes with finite products, and hence a natural transformation $[1] \times \mathcal{C} \to \mathcal{D}$ gives rise to a map $([1] \times \mathcal{C})^{\mathrm{gpd}} \simeq [1]^{\mathrm{gpd}} \times \mathcal{C}^{\mathrm{gpd}} \to \mathcal{D}^{\mathrm{gpd}}$ which selects the desired equivalence in $\hom_{\mathcal{S}}(\mathcal{C}^{\mathrm{gpd}}, \mathcal{D}^{\mathrm{gpd}})$.

In turn, Lemma 2.1.26 has the following useful further special case.

Corollary 2.1.28. *An adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces inverse equivalences $F^{\mathrm{gpd}} : \mathcal{C}^{\mathrm{gpd}} \xrightarrow{\sim} \mathcal{D}^{\mathrm{gpd}}$ and $\mathcal{C}^{\mathrm{gpd}} \xleftarrow{\sim} \mathcal{D}^{\mathrm{gpd}} : G^{\mathrm{gpd}}$ in $\mathcal{S}$.*

Proof. The adjunction $F \dashv G$ has unit and counit natural transformations $\mathrm{id}_{\mathcal{C}} \to G \circ F$ and $F \circ G \to \mathrm{id}_{\mathcal{D}}$, and so the claim follows from Lemma 2.1.26. □

We note the following interaction between taking localizations and taking homotopy categories.

Remark 2.1.29. Observe that the composite left adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{(\mathrm{ho}(-),\mathrm{ho}(-))} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xrightarrow{(-)[(-)^{-1}]} \mathcal{C}\mathrm{at}$$

coincides with the composite left adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)[\![(-)^{-1}]\!]} \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{ho}} \mathcal{C}\mathrm{at},$$

since they share a right adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \hookleftarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xleftarrow{\min} \mathcal{C}\mathrm{at}.$$

Hence, for any $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ we have a natural equivalence

$$\mathrm{ho}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \xrightarrow{\sim} \mathrm{ho}(\mathcal{R})[\mathrm{ho}(\mathbf{W})^{-1}]$$

in $\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty$.

We end this section with the following observation (which partly echoes Example 1.2.11).

Remark 2.1.30. Suppose $(\mathcal{R}, \mathbf{W})$ is a relative ∞-category. Then $(\mathrm{ho}(\mathcal{R}), \mathrm{ho}(\mathbf{W}))$ is a relative category (so is in particular a relative ∞-category). However, its localization $\mathrm{ho}(\mathcal{R})[\![\mathrm{ho}(\mathbf{W})^{-1}]\!]$ need not recover $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$. This is for the same reason as such facts always are, namely that we lose coherence data when we pass from $\mathcal{R}$ to $\mathrm{ho}(\mathcal{R})$. (Commutative diagrams in $\mathrm{ho}(\mathcal{R})$ need not come from commutative diagrams in $\mathcal{R}$, and when they do they might do so in multiple, inequivalent ways.) An explicit counterexample is provided by the minimal relative ∞-category $(\mathcal{R}, \mathbf{W}) = (\mathcal{R}, \mathcal{R}^{\simeq})$: then

$$\mathrm{ho}(\mathbf{W}) \simeq \mathrm{ho}(\mathcal{R}^{\simeq}) \simeq \mathrm{ho}(\mathcal{R})^{\simeq} \subset \mathrm{ho}(\mathcal{R})$$

since the equivalences in $\mathcal{R}$ are created by $\mathcal{R} \to \mathrm{ho}(\mathcal{R})$, and hence $\mathrm{ho}(\mathcal{R})[\![\mathrm{ho}(\mathbf{W})^{-1}]\!] \simeq \mathrm{ho}(\mathcal{R})$ (while of course $\mathcal{R}[\![\mathbf{W}^{-1}]\!] \simeq \mathcal{R}$). One might therefore refer to the ∞-category $\mathrm{ho}(\mathcal{R})[\![\mathrm{ho}(\mathbf{W})^{-1}]\!]$ as an "exotic enrichment" of the homotopy category $\mathrm{ho}(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$.

2.2 Complete Segal spaces

We now give an extremely brief review of the theory of complete Segal spaces. This section exists more-or-less solely to fix notation; we refer the reader seeking a more thorough discussion either to the original paper [Rez01] (which uses model categories) or to [Lur09c, §1] (which uses ∞-categories).

Let us write $\mathbf{\Delta} \xrightarrow{[\bullet]} \mathbb{c}\mathrm{at}$ for the standard cosimplicial (strict) category. Then, recall that the *nerve* of $\mathcal{C} \in \mathbb{c}\mathrm{at}$ is by definition the simplicial set $\mathrm{N}(\mathcal{C})_\bullet = \hom^{\mathrm{lw}}_{\mathbb{c}\mathrm{at}}([\bullet], \mathcal{C})$. This defines a fully faithful embedding $\mathrm{N} : \mathbb{c}\mathrm{at} \to s\mathcal{S}\mathrm{et}$, with image those simplicial sets which admit *unique* lifts for the inner horn inclusions $\{\Lambda^n_i \to \Delta^n\}_{0<i<n\geq 0}$. In fact, this functor is a right adjoint.

The situation with ∞-categories is completely analogous.

Definition 2.2.1. The (∞-categorical) ***nerve*** of an ∞-category $\mathcal{C}$ is the simplicial space

$$\mathrm{N}_\infty(\mathcal{C})_\bullet = \hom^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}_\infty}([\bullet], \mathcal{C}),$$

i.e. the composite

$$\mathbf{\Delta}^{op} \xrightarrow{[\bullet]^{op}} (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\hom_{\mathcal{C}\mathrm{at}_\infty}(-,\mathcal{C})} \mathcal{S}.$$

This defines a fully faithful embedding $\mathrm{N}_\infty : \mathcal{C}\mathrm{at}_\infty \hookrightarrow s\mathcal{S}$, with image the full subcategory $\mathcal{CSS} \subset s\mathcal{S}$ of ***complete Segal spaces***, i.e. those simplicial spaces satisfying

the *Segal condition* and the *completeness condition*. This inclusion fits into a left localization adjunction $\mathrm{L}_{\mathcal{CSS}} : s\mathcal{S} \rightleftarrows \mathcal{CSS} : \mathrm{U}_{\mathcal{CSS}}$. Hence, we obtain an equivalence

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow[\sim]{\mathrm{N}_\infty} \mathcal{CSS},$$

whose inverse

$$\mathcal{CSS} \xrightarrow[\sim]{\mathrm{N}_\infty^{-1}} \mathcal{C}\mathrm{at}_\infty$$

takes an object $Y_\bullet \in \mathcal{CSS}$ to the coend

$$\int^{[n]\in\mathbf{\Delta}} Y_n \times [n]$$

in $\mathcal{C}\mathrm{at}_\infty$. (These claims respectively follow from [Proposition A.A.7.10], [JT07, Theorem 4.12], [Rez01, Theorem 7.2], and [JT07, Theorem 4.12] again.) This equivalence identifies the subcategory $\mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$ with the subcategory of *constant* simplicial spaces (which are automatically complete Segal spaces).

Remark 2.2.2. Complete Segal spaces provide an extremely efficient way of computing the hom-spaces in an ∞-category: if $x, y \in \mathcal{C}$, then there is a natural equivalence

$$\hom_{\mathcal{C}}(x,y) \simeq \lim \left(\begin{array}{ccc} & & \mathrm{N}_\infty(\mathcal{C})_1 \\ & & \downarrow{\scriptstyle (s,t)} \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[(x,y)]{} & \mathrm{N}_\infty(\mathcal{C})_0 \times \mathrm{N}_\infty(\mathcal{C})_0 \end{array} \right)$$

in $\mathcal{S}$, where we use the notations $s = \delta_1$ and $t = \delta_0$ to emphasize the roles that these two face maps play in this theory. (Note that $\mathrm{N}_\infty(\mathcal{C})_0 = \hom_{\mathcal{C}\mathrm{at}_\infty}([0], \mathcal{C}) \simeq \mathcal{C}^{\simeq}$ is simply the maximal subgroupoid of $\mathcal{C}$, while $\mathrm{N}_\infty(\mathcal{C})_1 = \hom_{\mathcal{C}\mathrm{at}_\infty}([1], \mathcal{C}) \simeq \mathrm{Fun}([1], \mathcal{C})^{\simeq}$ is the space morphisms in $\mathcal{C}$.)

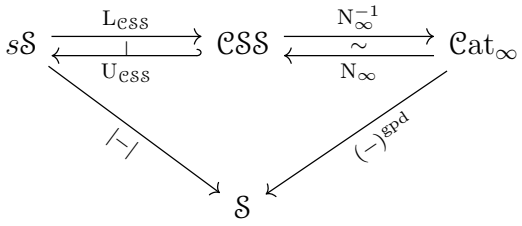
Remark 2.2.3. There is a canonical involution $\mathbf{\Delta} \xrightarrow{\sim} \mathbf{\Delta}$ in $\mathcal{C}\mathrm{at}$, which is the identity on objects but acts on morphisms by "reversing the coordinates": a map $[m] \xrightarrow{\varphi} [n]$ is taken to the map

$$[m] \xrightarrow{i \mapsto (n - \varphi(m-i))} [n].$$

Taking opposites, this induces an involution $\mathbf{\Delta}^{op} \xrightarrow{\sim} \mathbf{\Delta}^{op}$, which in turn induces an involution of $s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$ by precomposition. Unwinding the definitions, we see that this involution $s\mathcal{S} \xrightarrow{\sim} s\mathcal{S}$ restricts to an involution $\mathcal{CSS} \xrightarrow{\sim} \mathcal{CSS}$ which corresponds to the involution $(-)^{op} : \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$.

For future use, we record the following observation.

Proposition 2.2.4. *The diagram*



commutes: that is,

- *geometric realization of complete Segal spaces models groupoid completion of ∞-categories, and*
- *for any $Y \in s\mathcal{S}$, the localization map $Y \to \mathrm{L}_{\mathcal{CSS}}(Y)$ becomes an equivalence upon geometric realization.*

Proof. For the first claim, note that the functor $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ is a left localization, and the composite

$$\mathcal{S} \xhookrightarrow{\mathrm{U}_{\mathcal{S}}} \mathcal{C}\mathrm{at}_\infty \xrightarrow[\sim]{\mathrm{N}_\infty} \mathcal{CSS} \xhookrightarrow{\mathrm{U}_{\mathcal{CSS}}} s\mathcal{S}$$

agrees with the functor $\mathrm{const} : \mathcal{S} \to s\mathcal{S}$. Hence, the equivalence

$$|-| \circ \mathrm{U}_{\mathcal{CSS}} \circ \mathrm{N}_\infty \simeq (-)^{\mathrm{gpd}}$$

in $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$ follows from the uniqueness of left adjoints.

For the second claim, note that the reflective inclusion $\mathrm{const} : \mathcal{S} \hookrightarrow s\mathcal{S}$ factors through the reflective inclusion $\mathrm{U}_{\mathcal{CSS}} : \mathcal{CSS} \hookrightarrow s\mathcal{S}$. Hence, the factorization $\mathcal{S} \hookrightarrow \mathcal{CSS}$ is also a reflective inclusion. The equivalence

$$|-| \simeq |-| \circ \mathrm{U}_{\mathcal{CSS}} \circ \mathrm{L}_{\mathcal{CSS}}$$

in $\mathrm{Fun}(s\mathcal{S}, \mathcal{S})$ now also follows from the uniqueness of left adjoints. □

Remark 2.2.5. We may interpret Proposition 2.2.4 as saying that, while a simplicial space $Y \in s\mathcal{S}$ can be thought of as generating an ∞-category (namely the one corresponding to $\mathrm{L}_{\mathcal{CSS}}(Y) \in \mathcal{CSS}$), we can already directly extract its groupoid completion from Y itself. This is analogous to the fact that an arbitrary simplicial set can be thought of as generating a quasicategory via fibrant replacement in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, and the replacement map lies in $\mathbf{W}_{\mathrm{Joyal}} \subset \mathbf{W}_{\mathrm{KQ}}$ (i.e. it induces an equivalence on geometric realizations).

Remark 2.2.6. Given a strict category $\mathcal{C} \in \mathbb{c}\mathrm{at}$, the maps $\hom_{\mathbb{c}\mathrm{at}}([n], \mathcal{C}) \to \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C})$ from hom-sets to hom-spaces collect into a map

$$\mathrm{N}(\mathcal{C}) \to \mathrm{N}_\infty(\mathcal{C})$$

in $s\mathcal{S}$; in turn, these maps assemble into a natural transformation $\mathrm{N} \to \mathrm{N}_\infty$ in $\mathrm{Fun}(\mathbb{c}\mathrm{at}, s\mathcal{S})$. This map will be an equivalence in $s\mathcal{S}$ if and only if $\mathcal{C}$ is *gaunt*: while the nerve $\mathrm{N}(\mathcal{C}) \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ is always a Segal space, it only satisfies the completeness condition when every isomorphism in $\mathcal{C}$ is actually an identity map.[7] However, by [Rez01, Remark 7.8], the above map induces an equivalence

$$\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}(\mathcal{C})) \xrightarrow{\sim} \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{C})) \simeq \mathrm{N}_\infty(\mathcal{C})$$

in $\mathcal{CSS} \subset s\mathcal{S}$. In particular, it therefore follows from Proposition 2.2.4 that it also induces an equivalence

$$|\mathrm{N}(\mathcal{C})| \xrightarrow{\sim} |\mathrm{N}_\infty(\mathcal{C})|$$

in $\mathcal{S}$.

2.3 The Rezk nerve

Recall that the *localization* of a relative ∞-category $(\mathcal{R}, \mathbf{W})$ is the initial ∞-category $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ equipped with a functor from $\mathcal{R}$ which sends the subcategory $\mathbf{W} \subset \mathcal{R}$ of weak equivalences to equivalences. Meanwhile, given an arbitrary ∞-category $\mathcal{C}$, observe that the n^{th} space of its nerve can be considered as

$$\mathrm{N}_\infty(\mathcal{C})_n = \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C}) \simeq \mathrm{Fun}([n], \mathcal{C})^{\simeq} \subset \mathrm{Fun}([n], \mathcal{C}),$$

the subcategory of $\mathrm{Fun}([n], \mathcal{C})$ whose morphisms are the natural *equivalences*. Putting these two facts together, one is led to suspect that the n^{th} space of the nerve $\mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!])_\bullet$ should somehow contain the subcategory

$$\mathrm{Fun}([n], \mathcal{R})^{\mathbf{W}} \subset \mathrm{Fun}([n], \mathcal{R})$$

of $\mathrm{Fun}([n], \mathcal{R})$ whose morphisms are the natural *weak equivalences*. Of course, this will not generally form a space, but will instead be an ∞-category. On the other hand, there is a universal choice for a space admitting a map from this ∞-category, namely its groupoid completion. We are thus naturally led to make the following construction, a direct generalization of the "classification diagram" construction for relative categories defined in [Rez01, 3.3].

[7] Note that the Segal condition in $s\mathcal{S}\mathrm{et}$ can be equivalently checked in $s\mathcal{S}$ since the inclusion $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ is a right adjoint.

Definition 2.3.1. Given a relative ∞-category $(\mathcal{R}, \mathbf{W})$, its ($\infty$-categorical) ***Rezk pre-nerve*** is the simplicial ∞-category

$$\mathrm{preN}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W})_\bullet = \mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\mathbf{W}},$$

i.e. the composite

$$\mathbf{\Delta}^{op} \xrightarrow{[\bullet]^{op}} (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\min^{op}} (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\mathrm{Fun}(-,\mathcal{R})^{\mathbf{W}}} \mathcal{C}\mathrm{at}_\infty.$$

This defines a functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_{\infty}} s\mathcal{C}\mathrm{at}_\infty.$$

Then, the (∞-categorical) ***Rezk nerve*** functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_{\infty}} s\mathcal{S}$$

is given by the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_{\infty}} s\mathcal{C}\mathrm{at}_\infty \xrightarrow{s(-)^{\mathrm{gpd}}} s\mathcal{S}.$$

Remark 2.3.2. Recall that Rezk's "classification diagram" construction of [Rez01, 3.3], which we will denote by

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xrightarrow{\mathrm{N}^{\mathrm{R}}} s(s\mathcal{S}\mathrm{et})$$

and refer to as the *1-categorical Rezk nerve* functor, is given by the formula

$$\mathrm{N}^{\mathrm{R}}(\mathcal{R}, \mathbf{W})_\bullet = \mathrm{N}\left(\mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\mathbf{W}}\right).$$

Of course, we would like to think of this as a simplicial space using the model category $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. Indeed, combining Proposition 2.2.4 and Remark 2.2.6, we obtain a canonical commutative diagram

$$\begin{array}{ccccc} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} & \xrightarrow{\mathrm{N}^{\mathrm{R}}} & s(s\mathcal{S}\mathrm{et}) & \xrightarrow{s(|-|)} & s\mathcal{S} \\ \downarrow & & & \nearrow_{\mathrm{N}^{\mathrm{R}}_{\infty}} & \\ \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty & & & & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$; in fact, this even refines to a canonical commutative diagram

$$\begin{array}{ccccccc} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} & \xrightarrow{\mathrm{N}^{\mathrm{R}}} & s(s\mathcal{S}\mathrm{et}) & \longrightarrow & s\mathcal{C}\mathrm{at}_\infty & \xrightarrow{s(-)^{\mathrm{gpd}}} & s\mathcal{S} \\ \downarrow & & & \nearrow_{\mathrm{preN}^{\mathrm{R}}_{\infty}} & & \nearrow_{\mathrm{N}^{\mathrm{R}}_{\infty}} & \\ \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty & & & & & & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ (in which the functor $s(s\mathcal{S}\mathrm{et}) \to s\mathcal{C}\mathrm{at}_\infty$ is obtained by applying $s(-) = \mathrm{Fun}(\mathbf{\Delta}^{op}, -)$ to the localization $s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{\mathrm{Joyal}}^{-1}]\!] \simeq \mathcal{C}\mathrm{at}_\infty$). Thus, at least as far as homotopical content is concerned, the ∞-categorical Rezk nerve functor strictly generalizes its 1-categorical counterpart.

Remark 2.3.3. In turn, the 1-categorical Rezk nerve functor of Remark 2.3.2 suggests a similar model-dependent definition of a Rezk nerve functor for "marked quasicategories" (once again landing in $ss\mathcal{S}\mathrm{et}$). In fact, as the first step in the proof of Lemma 2.4.3, we will show that this construction is a model-categorical presentation

- of the ∞-categorical Rezk nerve when considered in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$, and in fact
- of the ∞-categorical Rezk pre-nerve when considered in $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$.

Remark 2.3.4. We have the following slight reformulation of Definition 2.3.1: in view of Proposition 2.2.4, the Rezk nerve functor can also be described as a composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} s\mathcal{C}\mathrm{at}_\infty \simeq s\mathcal{C}\mathcal{S}\mathcal{S} \xleftarrow{s(\mathrm{U}_{\mathcal{C}ss})} s(s\mathcal{S}) \xrightarrow{s(|-|)} s\mathcal{S}.$$

Note that the composite functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to s(s\mathcal{S})$ is a right adjoint, whose left adjoint is the left Kan extension

$$\begin{array}{ccc} \mathbf{\Delta} \times \mathbf{\Delta} & \xrightarrow{\mathrm{m}\times\mathrm{m}=(([m],[n])\mapsto[m]\times[n]\mathbf{w})} & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \\ {\scriptstyle \lrcorner}\downarrow & \nearrow_{\lrcorner!(\mathrm{m}\times\mathrm{m})} & \\ s(s\mathcal{S}) & & \end{array}$$

along the Yoneda embedding, where we write $\mathrm{m} \times \mathrm{m}$ for the upper "min $\times$ max" functor for brevity. On the other hand, the functor $s(|-|) : s(s\mathcal{S}) \to s\mathcal{S}$ is a left adjoint. Hence, as the Rezk nerve functor is the composite of a right adjoint followed by a left adjoint, understanding its behavior in general is a rather difficult task. (In fact, it follows that $\mathrm{preN}^{\mathrm{R}}_\infty : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to s\mathcal{C}\mathrm{at}_\infty$ is also a right adjoint, while $s(-)^{\mathrm{gpd}} : s\mathcal{C}\mathrm{at}_\infty \to s\mathcal{S}$ is of course also a left adjoint.)

We have the following identifications of the Rezk nerves of minimal and maximal relative ∞-categories: in both of these extremal cases, the Rezk nerve does indeed compute the localization.

Proposition 2.3.5. *The Rezk nerve functor acts on the full subcategories of* $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *spanned by the minimal and maximal relative* ∞*-categories (both of which can be*

indentified with $\mathcal{C}\mathrm{at}_\infty$*) according to the canonical commutative diagram*

$$\begin{array}{ccccc} \mathcal{C}\mathrm{at}_\infty & \xhookrightarrow{\min} & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty & \xhookleftarrow{\max} & \mathcal{C}\mathrm{at}_\infty \\ {\scriptstyle \mathrm{N}_\infty}\downarrow\wr & & \downarrow{\scriptstyle \mathrm{N}^{\mathrm{R}}_\infty} & & \downarrow{\scriptstyle (-)^{\mathrm{gpd}}} \\ \mathcal{CSS} & \xhookrightarrow[\mathrm{U}_{\mathcal{CSS}}]{} & s\mathcal{S} & \xhookleftarrow[\mathrm{const}]{} & \mathcal{S} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proof. To see that the left square commutes, given any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ we compute that

$$\mathrm{preN}^{\mathrm{R}}_\infty(\min(\mathcal{C}))_n = \mathrm{Fun}([n], \min(\mathcal{C}))^{\mathbf{W}} \simeq \mathrm{Fun}([n], \mathcal{C})^{\simeq} \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C}) = \mathrm{N}_\infty(\mathcal{C})_n$$

(in a way compatible with the evident simplicial structure maps on both sides), i.e. we even have a canonical equivalence

$$\mathrm{preN}^{\mathrm{R}}_\infty(\min(\mathcal{C}))_\bullet \simeq \mathrm{N}_\infty(\mathcal{C})_\bullet$$

in $s\mathcal{C}\mathrm{at}_\infty$. As $s(-)^{\mathrm{gpd}} : s\mathcal{C}\mathrm{at}_\infty \rightleftarrows s\mathcal{S} : s(\mathrm{U}_{\mathcal{S}})$ is a left localization adjunction, it follows that we also have a canonical equivalence

$$\mathrm{N}^{\mathrm{R}}_\infty(\min(\mathcal{C}))_\bullet \simeq \mathrm{N}_\infty(\mathcal{C})_\bullet$$

in $s\mathcal{S}$.

To see that the right square commutes, given any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ we first compute that

$$\mathrm{preN}^{\mathrm{R}}_\infty(\max(\mathcal{C}))_n = \mathrm{Fun}([n], \max(\mathcal{C}))^{\mathbf{W}} \simeq \mathrm{Fun}([n], \mathcal{C}).$$

Moreover, note that every face-then-degeneracy composite

$$\mathrm{Fun}([n], \mathcal{C}) \xrightarrow{\delta_i} \mathrm{Fun}([n-1], \mathcal{C}) \xrightarrow{\sigma_j} \mathrm{Fun}([n], \mathcal{C})$$

admits a natural transformation either to or from $\mathrm{id}_{\mathrm{Fun}([n],\mathcal{C})}$ (depending on i and j).[8] By Lemma 2.1.26, it follows that all the structure maps of $\mathrm{N}^{\mathrm{R}}_\infty(\max(\mathcal{C})) \in s\mathcal{S}$ are equivalences, and hence (since $\mathbf{\Delta}^{op}$ is sifted so in particular $(\mathbf{\Delta}^{op})^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$) it follows that this simplicial space is constant. The commutativity of the right square now follows from the computation

$$\mathrm{N}^{\mathrm{R}}_\infty(\max(\mathcal{C}))_0 = \left(\mathrm{Fun}([0], \max(\mathcal{C}))^{\mathbf{W}}\right)^{\mathrm{gpd}} \simeq \mathcal{C}^{\mathrm{gpd}},$$

which gives rise to a canonical equivalence $\mathrm{N}^{\mathrm{R}}_\infty(\max(\mathcal{C}))_\bullet \simeq \mathrm{const}(\mathcal{C}^{\mathrm{gpd}}) \simeq \mathrm{N}_\infty(\mathcal{C}^{\mathrm{gpd}})_\bullet$ in $s\mathcal{S}$. □

[8] We refer the reader to Lemma 4.3.5 for a more general statement (whose proof of course does not rely on the present discussion in any way).

Now, recall that *any* relative ∞-category $(\mathcal{R}, \mathbf{W})$ admits a natural map $\min(\mathcal{R}) = (\mathcal{R}, \mathcal{R}^{\simeq}) \to (\mathcal{R}, \mathbf{W})$ (namely the unit of the adjunction $\min \dashv \mathrm{U}_{\mathcal{R}\mathrm{el}}$). Hence, by Proposition 2.3.5 we obtain a natural map

$$\mathrm{N}_\infty(\mathcal{R}) \to \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W})$$

in $s\mathcal{S}$.[9] This immediately suggests the following two questions.

Question 2.3.6. When does this map in $s\mathcal{S}$ (or equivalently, its target) actually lie in the full subcategory $\mathcal{CSS} \subset s\mathcal{S}$?

Question 2.3.7. In light of the composite adjunction

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\rightleftarrows}} \mathcal{CSS} \underset{\mathrm{N}_\infty}{\overset{\mathrm{N}_\infty^{-1}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty,$$

what is the ∞-categorical significance of this map?

We give a partial answer to Question 2.3.6 in Chapter 4 (see the *calculus theorem* (4.5.1)). Meanwhile, the essence of the present chapter consists in the following complete answer to Question 2.3.7, the ***local universal property of the Rezk nerve***.

Theorem 2.3.8. *For any* $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *and any* $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$*, we have a commutative square*

$$\begin{array}{ccc} \hom_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty}((\mathcal{R}, \mathbf{W}), \min(\mathcal{C})) & \hookrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C}) \\ \wr\downarrow & & \downarrow\wr \\ \hom_{s\mathcal{S}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}), \mathrm{N}_\infty(\mathcal{C})) & \longrightarrow & \hom_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}), \mathrm{N}_\infty(\mathcal{C})). \end{array}$$

In other words, the natural map

$$\mathrm{N}_\infty(\mathcal{R}) \simeq \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R})) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}))$$

in $\mathcal{CSS}$ *corresponds to the localization map* $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ *in* $\mathcal{C}\mathrm{at}_\infty$.

We will give a proof of Theorem 2.3.8 in §2.4.

Using Theorem 2.3.8 as input, we now prove the first half of the *global* universal property of the Rezk nerve.

[9] This can also be obtained from the levelwise inclusion $\hom^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}_\infty}([\bullet], \mathcal{R}) \simeq \left(\mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\mathbf{W}}\right)^{\simeq} \hookrightarrow \mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\mathbf{W}}$ of maximal subgroupoids.

Proposition 2.3.9. *The composite functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS} \xrightarrow[\sim]{\mathrm{N}^{-1}_\infty} \mathcal{C}\mathrm{at}_\infty$$

is canonically equivalent in $\mathrm{Fun}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathcal{C}\mathrm{at}_\infty)$ *to the localization functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty.$$

Proof. Consider the commutative triangle

$$\begin{array}{ccc} & \min \circ \mathrm{U}_{\mathcal{R}\mathrm{el}} & \\ \swarrow & & \searrow \\ \mathrm{id}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} & \longrightarrow & \min \circ \mathrm{N}^{-1}_\infty \circ \mathrm{L}_{\mathcal{CSS}} \circ \mathrm{N}^{\mathrm{R}}_\infty \end{array}$$

in $\mathrm{Fun}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)$. Postcomposing with the functor $\mathscr{L}$ yields a commutative triangle

$$\begin{array}{ccc} & \mathrm{U}_{\mathcal{R}\mathrm{el}} & \\ \swarrow & & \searrow \\ \mathscr{L} & \longrightarrow & \mathrm{N}^{-1}_\infty \circ \mathrm{L}_{\mathcal{CSS}} \circ \mathrm{N}^{\mathrm{R}}_\infty \end{array}$$

in $\mathrm{Fun}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathcal{C}\mathrm{at}_\infty)$. By Theorem 2.3.8, the horizontal morphism in this commutative triangle is an equivalence. □

Remark 2.3.10. Proposition 2.3.9 can be seen as a generalization of work of Barwick–Kan. To see this, consider the composite pair of Quillen adjunctions

$$s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} \rightleftarrows \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}},$$

where

- the first is the left Bousfield localization which defines the Rezk model structure (see [Rez01, Theorem 7.2]) and presents the adjunction $\mathrm{L}_{\mathcal{CSS}} : s\mathcal{S} \rightleftarrows \mathcal{CSS} : \mathrm{U}_{\mathcal{CSS}}$, and
- the second is the Quillen equivalence which defines the Barwick–Kan model structure (see [BK12b, Theorem 6.1]).

As the latter is constructed using the lifting theorem for cofibrantly generated model categories, its right adjoint preserves all weak equivalences by definition. Moreover, Barwick–Kan provide a natural weak equivalence in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ (and hence also

in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$) from the Rezk nerve functor to the right adjoint of their Quillen equivalence (see [BK12b, Lemma 5.4]).

Now, consider the commutative triangle

$$\begin{array}{ccc} s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} & \xleftarrow{\mathrm{N}^{\mathrm{R}}} & \mathfrak{Re}\mathrm{lcat}_{\mathrm{triv}} \\ & {\scriptstyle \mathrm{id}_{ss\mathcal{S}\mathrm{et}}} \searrow \quad \swarrow & \\ & ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} & \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ (in which we take $\mathfrak{Re}\mathrm{lcat}$ with the *trivial* model structure since we are interested in relative categories themselves here). Applying the localization functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty} \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_{\infty},$$

this yields a commutative triangle

$$\begin{array}{ccc} s\mathcal{S} & \xleftarrow{s(|-|)\circ \mathrm{N}^{\mathrm{R}}} & \mathfrak{Re}\mathrm{lcat} \\ & {\scriptstyle \mathrm{L}_{\mathcal{CSS}}} \searrow \quad \swarrow {\scriptstyle \mathrm{N}_{\infty}\circ\mathscr{L}} & \\ & \mathcal{CSS} & \end{array}$$

in $\mathcal{C}\mathrm{at}_{\infty}$, in which

- the upper map coincides with the composite

  $$\mathfrak{Re}\mathrm{lcat} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty} \xrightarrow{\mathrm{N}^{\mathrm{R}}_{\infty}} s\mathcal{S}$$

  by Remark 2.3.2, and

- the map $\mathfrak{Re}\mathrm{lcat} \to \mathcal{CSS}$ can be identified as indicated since by what we have just seen it is equivalent to the projection

  $$\mathfrak{Re}\mathrm{lcat} \to \mathfrak{Re}\mathrm{lcat}[\mathbf{W}_{\mathrm{BK}}^{-1}] \simeq \mathcal{C}\mathrm{at}_{\infty}$$

  to the underlying ∞-category (which is indeed given by localization).

It follows that we obtain a commutative diagram

$$\begin{array}{ccccc} \mathfrak{Re}\mathrm{lcat} & \longrightarrow & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty} & \xrightarrow{\mathrm{N}^{\mathrm{R}}_{\infty}} & s\mathcal{S} \\ {\scriptstyle \mathscr{L}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{L}_{\mathcal{CSS}}} \\ \mathcal{C}\mathrm{at}_{\infty} & & \xrightarrow[\mathrm{N}_{\infty}]{\sim} & & \mathcal{CSS} \end{array}$$

in $\mathcal{C}\mathrm{at}_{\infty}$, which is precisely the restriction of the assertion of Proposition 2.3.9 to the category $\mathfrak{Re}\mathrm{lcat}$, as claimed.

We now prove the second half of the global universal property of the Rezk nerve.

Proposition 2.3.11. *The composite functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{C}\mathcal{S}\mathcal{S}}} \mathcal{C}\mathcal{S}\mathcal{S} \simeq \mathcal{C}\mathrm{at}_\infty$$

induces an equivalence

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty[\mathbf{W}_{\mathrm{BK}}^{-1}] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty.$$

In the proof of Proposition 2.3.11, it will be convenient to have the following terminology.

Definition 2.3.12. We define the subcategory $\mathbf{W}_{\mathrm{Rezk}} \subset ss\mathcal{S}$ of ***Rezk weak equivalences*** to be created by the composite

$$s(s\mathcal{S}) \xrightarrow{s(|-|)} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{C}\mathcal{S}\mathcal{S}}} \mathcal{C}\mathcal{S}\mathcal{S} \simeq \mathcal{C}\mathrm{at}_\infty.$$

(This name is meant to be suggestive of Rezk's "complete Segal space" model structure on the category $ss\mathcal{S}\mathrm{et}$ of bisimplicial *sets*.) We denote the resulting relative ∞-category by $ss\mathcal{S}_{\mathrm{Rezk}} = (ss\mathcal{S}, \mathbf{W}_{\mathrm{Rezk}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Since left localizations are in particular free localizations (recall Example 2.1.13), this composite left adjoint induces an equivalence

$$ss\mathcal{S}[\mathbf{W}_{\mathrm{Rezk}}^{-1}] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proof of Proposition 2.3.11. Recalling Remark 2.3.4, we have a composite adjunction

$$ss\mathcal{S} \underset{\mathrm{preN}^{\mathrm{R}}_\infty}{\overset{\mathfrak{Z}_!(\mathrm{m}\times\mathrm{m})}{\rightleftarrows}} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \underset{\mathrm{min}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty.$$

Moreover, it follows from Proposition 2.3.5 that the right adjoint of this composite adjunction is precisely that of the composite adjunction

$$s(s\mathcal{S}) \underset{s(\mathrm{const})}{\overset{s(|-|)}{\rightleftarrows}} s\mathcal{S} \underset{\mathrm{U}_{\mathcal{C}\mathcal{S}\mathcal{S}}}{\overset{\mathrm{L}_{\mathcal{C}\mathcal{S}\mathcal{S}}}{\rightleftarrows}} \mathcal{C}\mathcal{S}\mathcal{S} \underset{\mathrm{N}_\infty}{\overset{\mathrm{N}_\infty^{-1}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty$$

whose left adjoint defines $\mathbf{W}_{\mathrm{Rezk}} \subset ss\mathcal{S}$, and hence in particular it follows that the right adjoint of our original composite adjunction defines a weak equivalence

$$ss\mathcal{S}_{\mathrm{Rezk}} \xleftarrow[\approx]{\mathrm{min} \circ \mathrm{preN}^{\mathrm{R}}_\infty} \mathrm{min}(\mathcal{C}\mathrm{at}_\infty)$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$.

Next, we claim that the right adjoint $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} ss\mathcal{S}$ is a relative functor. To see this, first note that given any $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, we obtain a counit map

$$(\mathcal{R}, \mathbf{W}) \xrightarrow{\approx} \min(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$ from the adjunction $\mathscr{L} \dashv \min$. Theorem 2.3.8 and Proposition 2.3.5 then together imply that applying the functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} ss\mathcal{S}$ to this map yields a weak equivalence

$$\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}) \xrightarrow{\approx} \mathrm{preN}^{\mathrm{R}}_\infty(\min(\mathcal{R}[\![\mathbf{W}^{-1}]\!])) \simeq \mathrm{const}^{\mathrm{lw}}(\mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!]))$$

in $ss\mathcal{S}_{\mathrm{Rezk}}$. Hence, any weak equivalence $(\mathcal{R}_1, \mathbf{W}_1) \xrightarrow{\approx} (\mathcal{R}_2, \mathbf{W}_2)$ in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}_1, \mathbf{W}_1) & \longrightarrow & \mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}_2, \mathbf{W}_2) \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{const}^{\mathrm{lw}}(\mathrm{N}_\infty(\mathcal{R}_1[\![\mathbf{W}_1^{-1}]\!])) & \xrightarrow{\sim} & \mathrm{const}^{\mathrm{lw}}(\mathrm{N}_\infty(\mathcal{R}_2[\![\mathbf{W}_2^{-1}]\!])) \end{array}$$

in $ss\mathcal{S}_{\mathrm{Rezk}}$, and then the top arrow in this square is also in $\mathbf{W}_{\mathrm{Rezk}} \subset ss\mathcal{S}$ since it has the two-out-of-three property. So this does indeed define a relative functor

$$(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} ss\mathcal{S}_{\mathrm{Rezk}}.$$

From here, it follows that the right adjoints of our original composite adjunction form a commutative diagram

$$\begin{array}{ccc} ss\mathcal{S}_{\mathrm{Rezk}} & \xleftarrow[\approx]{\mathrm{preN}^{\mathrm{R}}_\infty \circ \min} & \min(\mathcal{C}\mathrm{at}_\infty) \\ & {\scriptstyle \mathrm{preN}^{\mathrm{R}}_\infty} \nwarrow \quad \swarrow {\scriptstyle \approx \atop \min} & \\ & (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} & \end{array}$$

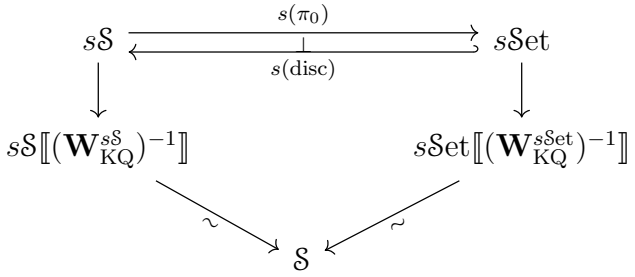
in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$, and so the entire diagram lies in $\mathbf{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ since it has the two-out-of-three property. Hence, we obtain a commutative diagram

$$\begin{array}{ccccc} & & \overset{\approx}{\frown} & & \\ ss\mathcal{S}_{\mathrm{Rezk}} & \xrightarrow{s(|-|)} & s\mathcal{S} & \xrightarrow{\mathrm{L}\mathcal{CSS}} & \mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty \\ {\scriptstyle \mathrm{preN}^{\mathrm{R}}_\infty}\uparrow\wr & \nearrow {\scriptstyle \mathrm{N}^{\mathrm{R}}_\infty} & & & \\ (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} & & & & \end{array}$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$, which proves the claim. □

Remark 2.3.13. It does not appear possible to give a completely hands-off proof of Proposition 2.3.11, i.e. one not relying on Theorem 2.3.8 (or perhaps even one that would prove Theorem 2.3.8 as a formal consequence). More specifically, adjunctions of underlying ∞-categories do not necessarily play well with relative ∞-category structures, even if one of the adjoints is a relative functor: one must have some control over the behavior of *both* adjoints.

For instance, the geometric realization functor $s\mathcal{S} \xrightarrow{|-|} \mathcal{S}$ and its restriction to the subcategory $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ create subcategories of weak equivalences which define the *Kan–Quillen relative ∞-category* structures $(s\mathcal{S}, \mathbf{W}_{\mathrm{KQ}}^{s\mathcal{S}}), (s\mathcal{S}\mathrm{et}, \mathbf{W}_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ (which underlie their respective Kan–Quillen model structures (see §1.4)). Moreover, these relative ∞-categories give rise to a diagram



in which the right adjoint commutes with the respective localization functors: in other words, it induces a weak equivalence

$$(s\mathcal{S}_{\mathrm{KQ}}, \mathbf{W}_{\mathrm{KQ}}^{s\mathcal{S}}) \xleftarrow{\approx} (s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}, \mathbf{W}_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}})$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$. Nevertheless, the left adjoint is clearly very far from also defining a weak equivalence in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$.

Remark 2.3.14. Taken together, Propositions 2.3.9 and 2.3.11 imply that in fact the adjunction

$$ss\mathcal{S} \underset{\mathrm{preN}_\infty^{\mathrm{R}}}{\overset{\mathfrak{K}!(\mathrm{m}\times\mathrm{m})}{\rightleftarrows}} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$$

has

- that both adjoints are relative functors (with respect to their respective Rezk and Barwick–Kan relative structures), and
- that the unit and counit are both natural weak equivalences.

This can be seen as follows.

First of all, recall that in the proof of Proposition 2.3.11, we already saw that the right adjoint is a relative functor. On the other hand, the left adjoint is a relative

functor because the composite left adjoint

$$ss\mathcal{S} \xrightarrow{\text{ɟ}_!(\mathrm{m}\times\mathrm{m})} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$$

agrees with the left adjoint

$$ss\mathcal{S} \xrightarrow{s(|-|)} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{C}SS}} \mathcal{C}S\mathcal{S} \xrightarrow[\sim]{\mathrm{N}_\infty^{-1}} \mathcal{C}\mathrm{at}_\infty$$

(since we have seen in the proof of Proposition 2.3.11 that they share a right adjoint), and so in fact the subcategory $\mathbf{W}_{\mathrm{Rezk}} \subset ss\mathcal{S}$ is created by pulling back the subcategory $\mathbf{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.

Next, we can see that the counit map

$$\text{ɟ}_!(\mathrm{m}\times\mathrm{m})(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W})) \to (\mathcal{R},\mathbf{W})$$

is a weak equivalence in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$ as follows. Applying the functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$, we obtain a map

$$\mathscr{L}(\text{ɟ}_!(\mathrm{m}\times\mathrm{m})(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W}))) \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$$

in $\mathcal{C}\mathrm{at}_\infty$. Then, again appealing to the fact that these composite left adjoints $ss\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$ agree, we can reidentify the source as

$$\mathscr{L}(\text{ɟ}_!(\mathrm{m}\times\mathrm{m})(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W})) \simeq \mathrm{N}_\infty^{-1}(\mathrm{L}_{\mathcal{C}SS}(s(|-|)(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W})))) \simeq \mathrm{N}_\infty^{-1}(\mathrm{L}_{\mathcal{C}SS}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W}))).$$

So, we can reidentify this map as

$$\mathrm{N}_\infty^{-1}(\mathrm{L}_{\mathcal{C}SS}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W}))) \to \mathcal{R}[\![\mathbf{W}^{-1}]\!],$$

which is an equivalence by Theorem 2.3.8. So the counit map is indeed a weak equivalence in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$, i.e. the counit is a natural weak equivalence.

Finally, we can see that the unit map

$$\mathrm{preN}^{\mathrm{R}}_\infty(\text{ɟ}_!(\mathrm{m}\times\mathrm{m})(Y)) \to Y$$

is a weak equivalence in $ss\mathcal{S}_{\mathrm{Rezk}}$ as follows. Applying the composite left adjoint

$$ss\mathcal{S} \xrightarrow{\mathrm{N}_\infty^{-1}\circ \mathrm{L}_{\mathcal{C}SS}\circ s(|-|)} \mathcal{C}\mathrm{at}_\infty$$

and appealing to Proposition 2.3.9, we obtain a map

$$\mathscr{L}(\text{ɟ}_!(\mathrm{m}\times\mathrm{m})(Y)) \to \mathrm{N}_\infty^{-1}(\mathrm{L}_{\mathcal{C}SS}(s(|-|)(Y)))$$

in $\mathcal{C}\mathrm{at}_\infty$, and the same equivalence of composite left adjoints $ss\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$ implies that this is an equivalence. So the unit map is indeed a weak equivalence in $ss\mathcal{S}_{\mathrm{Rezk}}$, i.e. the unit is a natural weak equivalence.

2.4 The proof of Theorem 2.3.8

Let $(\mathcal{R}, \mathbf{W})$ be an arbitrary relative ∞-category. In this section, we show that as a simplicial space, its Rezk nerve $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W})$ enjoys the desired universal property for mapping *into* complete Segal spaces: for any $\mathcal{C} \in \mathcal{C}\mathrm{at}_{\infty}$, we have a commutative diagram

$$\begin{array}{ccc} \hom_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty}}((\mathcal{R}, \mathbf{W}), \min(\mathcal{C})) & \hookrightarrow & \hom_{\mathcal{C}\mathrm{at}_{\infty}}(\mathcal{R}, \mathcal{C}) \\ \downarrow\wr & & \downarrow\wr \\ \hom_{s\mathcal{S}}(\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W}), \mathrm{N}_{\infty}(\mathcal{C})) & \longrightarrow & \hom_{\mathcal{C}s\mathcal{S}}(\mathrm{N}_{\infty}(\mathcal{R}), \mathrm{N}_{\infty}(\mathcal{C})) \end{array}$$

in $\mathcal{S}$, as asserted in Theorem 2.3.8.

Most of the proof is reasonably straightforward, and we can give it immediately. But there will be one technical result (Lemma 2.4.3) that is necessary for the proof which will occupy us for the remainder of the section.

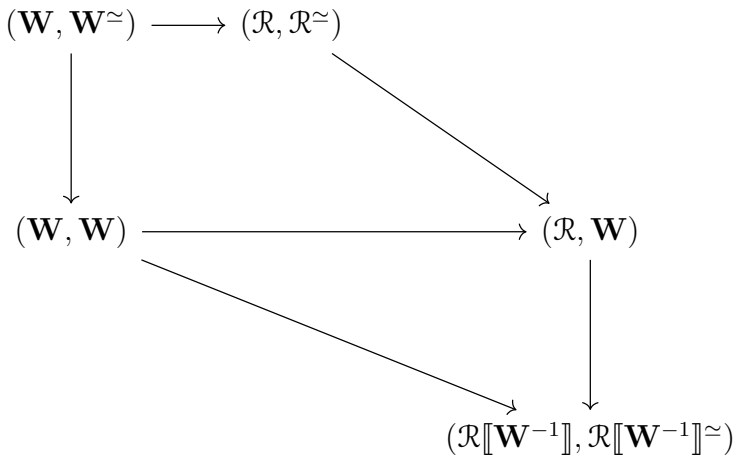
Proof of Theorem 2.3.8. By definition, the localization $\mathcal{R}[\![\mathbf{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_{\infty}$ is given as the pushout

$$\begin{array}{ccc} \mathbf{W} & \longrightarrow & \mathcal{R} \\ \downarrow & & \downarrow \\ \mathbf{W}^{\mathrm{gpd}} & \longrightarrow & \mathcal{R}[\![\mathbf{W}^{-1}]\!] \end{array}$$

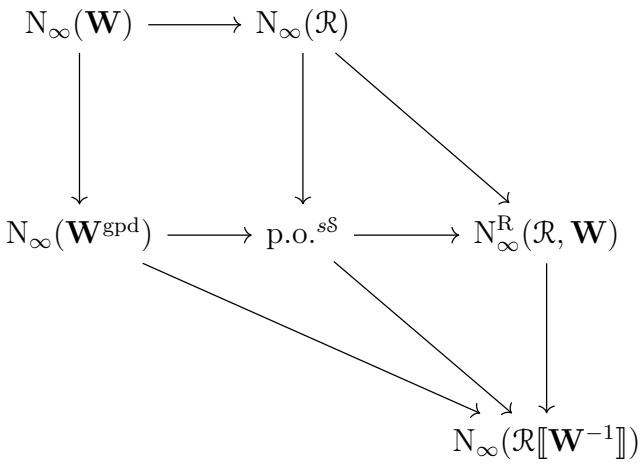
in $\mathcal{C}\mathrm{at}_{\infty}$; under the equivalence $\mathrm{N}_{\infty} : \mathcal{C}\mathrm{at}_{\infty} \xrightarrow{\sim} \mathcal{C}\mathcal{S}\mathcal{S}$, this corresponds to a pushout diagram

$$\begin{array}{ccc} \mathrm{N}_{\infty}(\mathbf{W}) & \longrightarrow & \mathrm{N}_{\infty}(\mathcal{R}) \\ \downarrow & & \downarrow \\ \mathrm{N}_{\infty}(\mathbf{W}^{\mathrm{gpd}}) & \longrightarrow & \mathrm{N}_{\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \end{array}$$

in $\mathcal{CSS} \subset s\mathcal{S}$. On the other hand, there is an evident commutative diagram



in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Applying the functor $\mathrm{N}^{\mathrm{R}}_\infty : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to s\mathcal{S}$ and taking the pushout of the upper left span, in light of Proposition 2.3.5 we obtain a commutative diagram



in $s\mathcal{S}$,

- where $\mathrm{p.o.}^{s\mathcal{S}}$ denotes the pushout in $s\mathcal{S}$ of the upper left span, and
- which contains as a subdiagram the above pushout square in $\mathcal{CSS} \subset s\mathcal{S}$ (namely the upper left span along with the object $\mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$).

Our goal is to prove that the induced map

$$\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W})) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!])) \simeq \mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$$

is an equivalence in $\mathcal{CSS} \subset s\mathcal{S}$.

For notational convenience, let us simply write

$$(s\mathcal{S})^{op} \xrightarrow{\text{ɬ}_{(s\mathcal{S})^{op}}} \mathrm{Fun}(s\mathcal{S},\mathcal{S}) \xrightarrow{-\circ \mathrm{U}_{\mathcal{CSS}}} \mathrm{Fun}(\mathcal{CSS},\mathcal{S})$$
$$\text{ɬ}_{\mathcal{CSS}^{op}}$$

for the restricted contravariant Yoneda functor, so that for any $Y \in s\mathcal{S}$ we have

$$\text{ɬ}_{\mathcal{CSS}^{op}}(Y) = \hom_{s\mathcal{S}}(Y, \mathrm{U}_{\mathcal{CSS}}(-)) \simeq \hom_{\mathcal{CSS}}(\mathrm{L}_{\mathcal{CSS}}(Y), -)$$

in $\mathrm{Fun}(\mathcal{CSS},\mathcal{S})$. Then, by Yoneda's lemma, our aforestated goal is equivalent to proving that the map

$$\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R},\mathbf{W}) \to \mathrm{N}_{\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$$

in $s\mathcal{S}$ induces an equivalence

$$\text{ɬ}_{\mathcal{CSS}^{op}}(\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R},\mathbf{W})) \leftarrow \text{ɬ}_{\mathcal{CSS}^{op}}(\mathrm{N}_{\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]))$$

in $\mathrm{Fun}(\mathcal{CSS},\mathcal{S})$. Moreover, as the functor $s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS}$ commutes with pushouts (being a left adjoint), it follows that the map

$$\mathrm{p.o.}^{s\mathcal{S}} \to \mathrm{N}_{\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$$

in $s\mathcal{S}$ induces an equivalence

$$\mathrm{L}_{\mathcal{CSS}}\left(\mathrm{p.o.}^{s\mathcal{S}}\right) \xrightarrow{\sim} \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_{\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!])) \simeq \mathrm{N}_{\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$$

in $\mathcal{CSS} \subset s\mathcal{S}$, and so the above diagram in $s\mathcal{S}$ gives rise to a retraction diagram

$$\begin{array}{ccc} \text{ɬ}_{\mathcal{CSS}^{op}}(\mathrm{p.o.}^{s\mathcal{S}}) & \longleftarrow & \text{ɬ}_{\mathcal{CSS}^{op}}(\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R},\mathbf{W})) \\ & \nwarrow^{\sim} & \uparrow \\ & & \text{ɬ}_{\mathcal{CSS}^{op}}(\mathrm{N}_{\infty}(\mathcal{R}[\![\mathbf{W}^{-1}]\!])) \end{array}$$

in $\mathrm{Fun}(\mathcal{CSS},\mathcal{S})$ into which this map which we must show to be an equivalence fits, and which it therefore suffices to show is in fact a diagram of equivalences.

Now, observe that $\mathcal{CSS}$ is complete and hence in particular is cotensored over $\mathcal{S}$, and observe moreover that the functor

$$(s\mathcal{S})^{op} \xrightarrow{\text{ɬ}_{\mathcal{CSS}^{op}}} \mathrm{Fun}(\mathcal{CSS},\mathcal{S})$$

factors through the contravariant Yoneda embedding and hence takes values in functors which commute with cotensors. So by Lemma 2.4.1, it suffices to show that after postcomposition with $\mathcal{S} \xrightarrow{\pi_0} \mathcal{S}\mathrm{et}$, the above retraction diagram in $\mathrm{Fun}(\mathcal{C}\mathcal{S}\mathcal{S}, \mathcal{S})$ becomes a diagram of natural isomorphisms in $\mathrm{Fun}(\mathcal{C}\mathcal{S}\mathcal{S}, \mathcal{S}\mathrm{et})$. Hence, it suffices to show that the induced map

$$\big(\pi_0 \circ \text{ɟ}_{\mathcal{C}\mathcal{S}\mathcal{S}^{op}}(\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W}))\big) \to \big(\pi_0 \circ \text{ɟ}_{\mathcal{C}\mathcal{S}\mathcal{S}^{op}}(\mathrm{p.o.}^{s\mathcal{S}})\big)$$

is a natural monomorphism in $\mathrm{Fun}(\mathcal{C}\mathcal{S}\mathcal{S}, \mathcal{S}\mathrm{et})$. This follows from the stronger statement that the composite

$$\big(\pi_0 \circ \text{ɟ}_{\mathcal{C}\mathcal{S}\mathcal{S}^{op}}(\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W}))\big) \to \big(\pi_0 \circ \text{ɟ}_{\mathcal{C}\mathcal{S}\mathcal{S}^{op}}(\mathrm{p.o.}^{s\mathcal{S}})\big) \to \big(\pi_0 \circ \text{ɟ}_{\mathcal{C}\mathcal{S}\mathcal{S}^{op}}(\mathrm{N}_{\infty}(\mathcal{R}))\big)$$

is a natural monomorphism in $\mathrm{Fun}(\mathcal{C}\mathcal{S}\mathcal{S}, \mathcal{S}\mathrm{et})$, which in turn follows from Lemma 2.4.3. □

We needed the following easy result in the proof of Theorem 2.3.8.

Lemma 2.4.1. *Let $\mathcal{C}$ be an ∞-category admitting a cotensoring*

$$\mathcal{S}^{op} \times \mathcal{C} \xrightarrow{-\pitchfork-} \mathcal{C},$$

and suppose we are given two space-valued functors $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ that commute with cotensors. Then, a natural transformation $F \to G$ is a natural equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ if and only if its postcomposition $\pi_0 F \to \pi_0 G$ with $\mathcal{S} \xrightarrow{\pi_0} \mathcal{S}\mathrm{et}$ is a natural isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$.

Proof. The "only if" direction is clear. So, suppose we are given a natural transformation $F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ such that the induced natural transformation $\pi_0 F \to \pi_0 G$ is a natural equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$. Since equivalences in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ are determined componentwise, it suffices to show that for any $Y \in \mathcal{C}$, the map $F(Y) \to G(Y)$ is an equivalence in $\mathcal{S}$. In turn, since equivalences in $\mathcal{S}$ are created in $\mathrm{ho}(\mathcal{S})$, by Yoneda's lemma it suffices to show that for any $Z \in \mathcal{S}$, the induced map $[Z, F(Y)]_{\mathcal{S}} \to [Z, G(Y)]_{\mathcal{S}}$ is an isomorphism in $\mathcal{S}\mathrm{et}$. But since $\mathcal{C}$ admits cotensors, then we can reidentify this map via the canonical commutative square

$$\begin{array}{ccc} \pi_0(F(Z \pitchfork Y)) & \xrightarrow{\cong} & \pi_0(G(Z \pitchfork Y)) \\ \wr\downarrow & & \downarrow\wr \\ [Z, F(Y)]_{\mathcal{S}} & \longrightarrow & [Z, G(Y)]_{\mathcal{S}} \end{array}$$

in $\mathcal{S}\mathrm{et}$, in which the top arrow is an isomorphism by the assumption that $\pi_0 F \to \pi_0 G$ is a natural isomorphism and the vertical arrows are isomorphisms by the assumption that F and G commute with cotensors. This proves the claim. □

Before moving on to Lemma 2.4.3, it will be convenient to have the following bit of terminology.

Definition 2.4.2. A morphism in a model category $\mathcal{M}$ is called a ***homotopy epimorphism*** if it presents an epimorphism in the underlying ∞-category $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$.

We now proceed to the technical heart of the proof of Theorem 2.3.8. We warn the reader that our proof of the following result is (perhaps unexpectedly, and certainly unsatisfyingly) complicated.

Lemma 2.4.3. *The map* $\mathrm{N}_\infty(\mathcal{R}) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}))$ *is an epimorphism in* $\mathcal{CSS}$.

Proof. Our proof will proceed using model categories – primarily $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ and $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, but also a number of others auxiliarily –, and will also use the language of marked simplicial sets (see e.g. §T.3.1).

We begin by recalling the two Quillen equivalences between $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ and $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ given in [JT07].

(1) Let us write $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{\mathrm{pr}_2} \mathbf{\Delta}^{op}$ for the second projection map and $\mathbf{\Delta}^{op} \xrightarrow{i_2} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ for the functor $\mathrm{const}([0]^\circ) \times \mathrm{id}_{\mathbf{\Delta}^{op}}$. Pullbacks along these two functors induce the Quillen equivalence

$$\mathrm{pr}_2^* : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} : i_2^*$$

of [JT07, Theorem 4.11].

(2) Let us write $(\Delta^i)^{gpd} \in s\mathcal{S}\mathrm{et}$ for the nerve of the strict (i.e. objects-preserving) groupoid completion of $[i] \in \mathbb{C}\mathrm{at}$, and let us write $t_! : ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ for the left Kan extension

$$\begin{array}{ccc} \mathbf{\Delta} \times \mathbf{\Delta} & \xrightarrow{([n],[i]) \mapsto \Delta^n \times (\Delta^i)^{gpd}} & s\mathcal{S}\mathrm{et} \\ \downarrow & \nearrow & \\ ss\mathcal{S}\mathrm{et} & & \end{array}$$

along the (1-categorical) Yoneda embedding. This has a right adjoint $t^! : s\mathcal{S}\mathrm{et} \to ss\mathcal{S}\mathrm{et}$ given by

$$t^!(Y) = \{\{\hom_{s\mathcal{S}\mathrm{et}}(\Delta^n \times (\Delta^i)^{gpd}, Y)\}_{i \geq 0}\}_{n \geq 0},$$

and together these fit into the Quillen equivalence

$$t_! : ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} : t^!$$

of [JT07, Theorem 4.12].

Now, suppose that $\mathtt{R} \in s\mathcal{S}\mathrm{et}^{f}_{\mathrm{Joyal}}$ is a quasicategory presenting $\mathcal{R} \in \mathcal{C}\mathrm{at}_\infty$, and let $(\mathtt{R}, \mathtt{W}) \in s\mathcal{S}\mathrm{et}^{+}$ be the marked simplicial set obtained by marking precisely those edges of $\mathtt{R}$ which present maps in $\mathbf{W} \subset \mathcal{R}$. For any $n \geq 0$, the ∞-category $\mathrm{Fun}([n], \mathcal{R})$ is presented by the object

$$\underline{\hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, \mathtt{R}) = \{\hom_{s\mathcal{S}\mathrm{et}}(\Delta^n \times \Delta^i, \mathtt{R})\}_{i \geq 0} \in s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}},$$

and hence its subcategory

$$\mathrm{Fun}([n], \mathcal{R})^{\mathbf{W}} \subset \mathrm{Fun}([n], \mathcal{R})$$

is presented by the object

$$\{\hom_{s\mathcal{S}\mathrm{et}^+}((\Delta^n)^\flat \times (\Delta^i)^\sharp, (\mathtt{R}, \mathtt{W}))\}_{i \geq 0} \in s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}.$$

These constructions are contravariantly functorial in $[n] \in \mathbf{\Delta}$, and hence we obtain that the Rezk pre-nerve

$$\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}) = \mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\mathbf{W}} \in s\mathcal{C}\mathrm{at}_\infty$$

is presented by the object

$$\{\{\hom_{s\mathcal{S}\mathrm{et}^+}((\Delta^n)^\flat \times (\Delta^i)^\sharp, (\mathtt{R}, \mathtt{W}))\}_{i \geq 0}\}_{n \geq 0} \in s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}.$$

From here, we observe that the Quillen adjunction

$$\mathrm{id}_{ss\mathcal{S}\mathrm{et}} : s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \rightleftarrows s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} : \mathrm{id}_{ss\mathcal{S}\mathrm{et}}$$

presents the left localization adjunction $s((-)^{\mathrm{gpd}}) : s\mathcal{C}\mathrm{at}_\infty \rightleftarrows s\mathcal{S} : s(\mathrm{U}_{\mathcal{S}})$; as all objects of $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ are cofibrant, it follows that when considered as an object of $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$, this same bisimplicial set presents $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}) \in s\mathcal{S}$. Moreover, in light of the left Bousfield localization

$$\mathrm{id}_{ss\mathcal{S}\mathrm{et}} : s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} : \mathrm{id}_{ss\mathcal{S}\mathrm{et}}$$

presenting the left localization adjunction $\mathrm{L}_{\mathcal{CSS}} : s\mathcal{S} \rightleftarrows \mathcal{CSS} : \mathrm{U}_{\mathcal{CSS}}$, when considered as an object of $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$, this same bisimplicial set presents the Rezk nerve

$$\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}) = \left(\mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\mathbf{W}}\right)^{\mathrm{gpd}} \in \mathcal{CSS}.$$

We will denote this bisimplicial set by $\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}) \in ss\mathcal{S}\mathrm{et}$.[10] In particular, note that we have a natural isomorphism $\mathrm{N}^{\mathrm{R}}(\mathtt{R}^\natural) \cong t^!(\mathtt{R})$ in $ss\mathcal{S}\mathrm{et}$, and hence we see that the right Quillen equivalence

$$t^! : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \to ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$$

[10] When $(\mathtt{R}, \mathtt{W}) \in s\mathcal{S}\mathrm{et}^+$ is the "marked nerve" of a relative 1-category, this recovers the 1-categorical Rezk nerve of Remark 2.3.2 (as an object of $ss\mathcal{S}\mathrm{et}$), and so there is no ambiguity in the notation.

presents the equivalence $\mathrm{N}_\infty : \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{CSS}$ of ∞-categories.

Now, the natural map

$$\mathtt{R}^\natural \to (\mathtt{R}, \mathtt{W})$$

in $s\mathcal{S}\mathrm{et}^+$ induces a map

$$\mathrm{N}^{\mathrm{R}}(\mathtt{R}^\natural) \to \mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})$$

in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$, which by what we have seen presents the map

$$\mathrm{N}_\infty(\mathcal{R}) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W}))$$

in $\mathcal{CSS}$. So, to prove that this latter map is an epimorphism in $\mathcal{CSS}$, it suffices to prove that the former map is a homotopy epimorphism in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$. However, note that there is a natural isomorphism $t_!(\mathrm{pr}_2^*(\mathtt{R})) \xrightarrow{\cong} \mathtt{R}$ in $s\mathcal{S}\mathrm{et}$, which is in particular a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$; via the Quillen equivalence of item (2), this corresponds to a weak equivalence $\mathrm{pr}_2^*(\mathtt{R}) \xrightarrow{\simeq} t^!(\mathtt{R})$ in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$. So, it also suffices to show that the composite map

$$\mathrm{pr}_2^*(\mathtt{R}) \xrightarrow{\simeq} t^!(\mathtt{R}) \cong \mathrm{N}^{\mathrm{R}}(\mathtt{R}^\natural) \to \mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})$$

is a homotopy epimorphism in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$.

For this, let us also recall the "usual" geometric realization functor $ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ (a homotopy colimit functor with respect to $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$): this is the left Kan extension

$$\begin{array}{ccc} \mathbf{\Delta} \times \mathbf{\Delta} & \xrightarrow{([n],[i]) \mapsto \Delta^n \times \Delta^i} & s\mathcal{S}\mathrm{et} \\ \downarrow & \nearrow & \\ ss\mathcal{S}\mathrm{et} & & \end{array}$$

along the (1-categorical) Yoneda embedding, but by [GJ99, Exercise IV.1.6] this is (naturally isomorphic to) the functor $\mathrm{diag}^* : ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$, where $\mathbf{\Delta}^{op} \xrightarrow{\mathrm{diag}} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ denotes the diagonal functor. Now, the evident morphisms $\Delta^n \times \Delta^i \to \Delta^n \times (\Delta^i)^{gpd}$ in $s\mathcal{S}\mathrm{et}$ induce a natural transformation $\mathrm{diag}^* \to t_!$ in $\mathrm{Fun}(ss\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et})$. Moreover, it is not hard to see that upon precomposition with $s\mathcal{S}\mathrm{et} \xrightarrow{\mathrm{pr}_2^*} ss\mathcal{S}\mathrm{et}$, this induces the identity natural transformation from $\mathrm{id}_{s\mathcal{S}\mathrm{et}}$ to itself in $\mathrm{Fun}(s\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et})$ (up to isomorphism). Applying these observations to the above composite map in $ss\mathcal{S}\mathrm{et}$, we obtain a commutative square

$$\begin{array}{ccc} \mathrm{diag}^*(\mathrm{pr}_2^*(\mathtt{R})) & \xrightarrow{\alpha} & \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \\ \wr \downarrow & & \downarrow \beta \\ t_!(\mathrm{pr}_2^*(\mathtt{R})) & \xrightarrow[\gamma]{} & t_!(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \end{array}$$

in $s\mathcal{S}\mathrm{et}$, where both objects on the left are (compatibly) isomorphic to $\mathtt{R}$ itself. Since $t_! : ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is a left Quillen equivalence and all objects of $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ are cofibrant, it suffices to show that the map γ is a homotopy epimorphism in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. For this, it suffices to prove that when considered in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, the map α is a weak equivalence and the map β is a homotopy epimorphism. This, finally, is what we will show.

We begin with the second assertion, that the map

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})) \xrightarrow{\beta} t_!(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))$$

is a homotopy epimorphism in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. In fact, we will show that the natural transformation $\mathrm{diag}^* \to t_!$ in $\mathrm{Fun}(ss\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})$ is a componentwise homotopy epimorphism. Just for the duration of this sub-proof, let us "reverse" our simplicial coordinates, so that the one we have been denoting by "i" will be the *outer* coordinate while the one we have been denoting by "n" will be the *inner* coordinate. Now, observe that we can rewrite these two functors as

$$\mathrm{diag}^* \cong \int^{[i]\in\mathbf{\Delta}} (-)_i \times \Delta^i : s(s\mathcal{S}\mathrm{et}) \to s\mathcal{S}\mathrm{et}$$

and

$$t_! \cong \int^{[i]\in\mathbf{\Delta}} (-)_i \times (\Delta^i)^{gpd} : s(s\mathcal{S}\mathrm{et}) \to s\mathcal{S}\mathrm{et},$$

under which identifications our natural transformation $\mathrm{diag}^* \to t_!$ is induced by the evident map $\Delta^\bullet \to (\Delta^\bullet)^{gpd}$ in $c(s\mathcal{S}\mathrm{et})$. Moreover, by Proposition T.A.2.9.26, we obtain a left Quillen bifunctor

$$\int^{[i]\in\mathbf{\Delta}} (-)_i \times (-)^i : s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \times c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$$

(since $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is cartesian, i.e. the product bifunctor is left Quillen).[11] As every object of $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ is cofibrant, for any object

$$Y_\bullet \in s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$$

the above left Quillen bifunctor induces a left Quillen functor

$$\int^{[i]\in\mathbf{\Delta}} Y_i \times (-)^i : c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}.$$

[11] Note that since we have flipped our simplicial coordinates, this model structure $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ is *different* from the model structure $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ that appeared earlier (with respect to the fixed copy of the underlying category $ss\mathcal{S}\mathrm{et}$ in which we have been working).

Moreover, the cofibrant objects of $c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ are exactly those of $c(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ (since the cofibrations in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ are exactly those of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$), and so in particular the objects $\Delta^\bullet, (\Delta^\bullet)^{gpd} \in c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ are cofibrant by [Hir03, Corollary 15.9.10].

Now, epimorphisms (being determined by a colimit condition) are preserved by left adjoint functors of ∞-categories. Moreover, by [MG16, Theorem 2.1], a left Quillen functor between model categories induces a left adjoint functor between ∞-categories, which is presented (in $\mathfrak{Rel}\mathcal{C}\mathrm{at}_{\mathrm{BK}}$) by the restriction of the left Quillen functor to the subcategory of cofibrant objects. So, it suffices to show that the map $\Delta^\bullet \to (\Delta^\bullet)^{gpd}$ is a homotopy epimorphism in $c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$.

For this, observe that the model category $c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ presents the ∞-category $c\mathcal{C}\mathrm{at}_\infty$. Since epimorphisms in $c\mathcal{C}\mathrm{at}_\infty = \mathrm{Fun}(\mathbf{\Delta}, \mathcal{C}\mathrm{at}_\infty)$ are determined componentwise, it suffices to show that each $\Delta^i \to (\Delta^i)^{gpd}$ is a homotopy epimorphism in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. But this is clear: this map in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ presents the terminal map

$$[i] \to [i]^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$$

in $\mathcal{C}\mathrm{at}_\infty$, which on an arbitrary ∞-category $\mathcal{C}$ corepresents the inclusion

$$\mathcal{C}^{\simeq} \hookrightarrow \hom_{\mathcal{C}\mathrm{at}_\infty}([i], \mathcal{C})$$

of the subspace of length-i sequences of composable *equivalences* (inside of the space of arbitrary length-i sequences of composable morphisms). Thus, the natural transformation $\mathrm{diag}^* \to t_!$ in $\mathrm{Fun}(ss\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})$ is indeed a componentwise homotopy epimorphism, and so in particular we obtain that the map β (which is its component at the object $\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}) \in ss\mathcal{S}\mathrm{et}$) is a homotopy epimorphism, as claimed.

So, it only remains to show that the map

$$\mathtt{R} \cong \mathrm{diag}^*(\mathrm{pr}_2^*(\mathtt{R})) \xrightarrow{\alpha} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. Unwinding the definitions, we see that via the evident cosimplicial object

$$\mathbf{\Delta} \xrightarrow{(\Delta^\bullet)^\flat \times (\Delta^\bullet)^\sharp} s\mathcal{S}\mathrm{et}^+,$$

we obtain a canonical isomorphism

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}^+}((\Delta^\bullet)^\flat \times (\Delta^\bullet)^\sharp, (\mathtt{R}, \mathtt{W})).$$

Moreover, via the canonical isomorphisms

$$\mathtt{R} \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}}(\Delta^\bullet, \mathtt{R}) \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}^+}((\Delta^\bullet)^\flat, \mathtt{R}^\flat) \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}^+}((\Delta^\bullet)^\flat, (\mathtt{R}, \mathtt{W})),$$

this map α is corepresented by the collection of first projection maps

$$(\Delta^n)^\flat \times (\Delta^n)^\sharp \to (\Delta^n)^\flat,$$

which assemble to a natural transformation in $\mathrm{Fun}(\mathbf{\Delta}, s\mathcal{S}\mathrm{et}^+)$. On the other hand, the collection of diagonal maps

$$(\Delta^n)^\flat \to (\Delta^n)^\flat \times (\Delta^n)^\sharp$$

(or more precisely, the unique maps in $s\mathcal{S}\mathrm{et}^+$ which recover the diagonal maps in $s\mathcal{S}\mathrm{et}$ under the forgetful functor $s\mathcal{S}\mathrm{et}^+ \to s\mathcal{S}\mathrm{et}$) also assemble into a natural transformation in $\mathrm{Fun}(\mathbf{\Delta}, s\mathcal{S}\mathrm{et}^+)$, which likewise corepresents a map

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\rho} \mathtt{R}$$

in $s\mathcal{S}\mathrm{et}$. Clearly, the composite

$$\mathtt{R} \xrightarrow{\alpha} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\rho} \mathtt{R}$$

is the identity map, since this is true of the composite

$$(\Delta^n)^\flat \to (\Delta^n)^\flat \times (\Delta^n)^\sharp \to (\Delta^n)^\flat$$

of the diagonal map followed by the first projection. On the other hand, we will show that the composite

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\rho} \mathtt{R} \xrightarrow{\alpha} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$$

is connected to $\mathrm{id}_{\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))}$ by the zigzag of simplicial homotopies illustrated in Figure 2.1, whose components (i.e. whose values on the vertices of (the source copies of) $\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$) are all degenerate edges of (the target copy of) $\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$. Postcomposing with an arbitrary fibrant replacement

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\approx} \mathbb{R}(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))) \twoheadrightarrow \mathrm{pt}_{s\mathcal{S}\mathrm{et}}$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, we obtain a composite

$$\Lambda^2_2 \to \underline{\hom}_{s\mathcal{S}\mathrm{et}}(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})), \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))) \to \underline{\hom}_{s\mathcal{S}\mathrm{et}}(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})), \mathbb{R}(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))))$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ which, by [Joyb, Chapter 5, Theorem C] (and [Joyb, Proposition 4.8] (and the fact that $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is cartesian)), presents a zigzag of natural equivalences in $\mathcal{C}\mathrm{at}_\infty$ between the functors presented by the maps $\mathrm{id}_{\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))}$ and $\alpha\rho$ in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$.

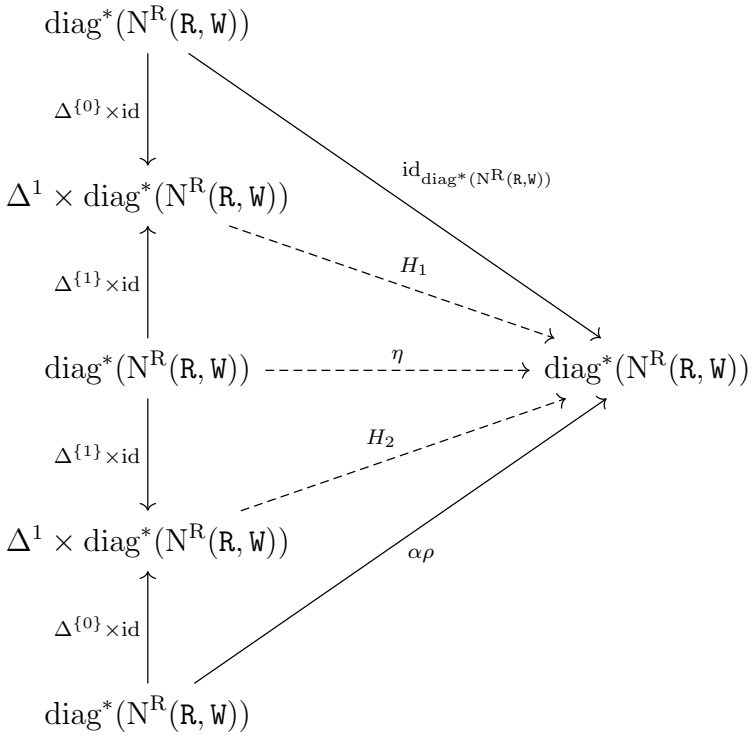


Figure 2.1: The zigzag of simplicial homotopies in $s\mathcal{S}\mathrm{et}$ in the proof of Lemma 2.4.3.

In turn, this zigzag (along with the natural equivalence in $\mathcal{C}\mathrm{at}_\infty$ presented by the identification $\rho\alpha = \mathrm{id}_{\mathtt{R}}$) witnesses the fact that the maps α and ρ in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ present inverse equivalences in $\mathcal{C}\mathrm{at}_\infty$, from which we conclude that in particular the map α is indeed a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$.

Now, all three of η, H_1, and H_2 will be corepresented by maps between the various objects $(\Delta^n)^\flat \times (\Delta^n)^\sharp \in s\mathcal{S}\mathrm{et}^+$; in turn, all of these maps will be obtained by applying the evident "marked nerve" functor $\mathrm{N}^+ : \mathfrak{Relcat} \to s\mathcal{S}\mathrm{et}^+$ to maps between the various objects $[n] \times [n]_{\mathbf{W}} \in \mathfrak{Relcat}$.

We begin by defining the map $\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\eta} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$: this is corepresented by the marked nerves of the maps

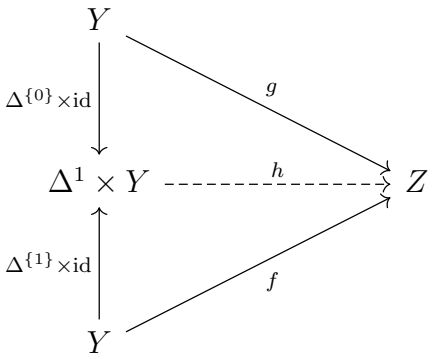
$$[n] \times [n]_{\mathbf{W}} \xrightarrow{\eta^n} [n] \times [n]_{\mathbf{W}}$$

in $\mathfrak{Relcat}$ given by

$$\eta^n(i,j) = \begin{cases} (i,i), & i \geq j \\ (i,j), & i < j. \end{cases}$$

It is easy to verify that this does indeed define a map in $\mathfrak{relcat}$, and moreover that assembling these maps for all $n \geq 0$ yields an endomorphism of the object $[\bullet]\times[\bullet]_{\mathbf{W}} \in c\mathfrak{Rel}\mathfrak{cat}$.

In order to define the simplicial homotopies H_1 and H_2, we first recall a combinatorial reformation of the definition of a simplicial homotopy (see e.g. [May92, Definitions 5.1]): for any $Y, Z \in s\mathcal{S}\mathrm{et}$ and any $f, g \in \hom_{s\mathcal{S}\mathrm{et}}(Y, Z)$, a simplicial homotopy



is equivalently given by a family of maps

$$\{h_{i,n} \in \hom_{\mathcal{S}\mathrm{et}}(Y_n, Z_{n+1})\}_{0\leq i\leq n\geq 0}$$

which satisfy the identities

$$\delta_0 h_{0,n} = f_n,$$

$$\delta_{n+1} h_{n,n} = g_n,$$

$$\delta_i h_{j,n} = \begin{cases} h_{j-1,n-1}\delta_i, & i < j \\ \delta_i h_{i-1,n}, & i = j \neq 0 \\ h_{j,n-1}\delta_{i-1}, & i > j+1, \end{cases}$$

and

$$\sigma_i h_{j,n} = \begin{cases} h_{j+1,n+1}\sigma_i, & i \leq j \\ h_{j,n+1}\sigma_{i-1}, & i > j. \end{cases}$$

So, for $\varepsilon \in \{1, 2\}$, we will define the simplicial homotopies

$$\Delta^1 \times \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{H_\varepsilon} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$$

to be corepresented by the marked nerves of families of maps

$$\{H_\varepsilon^{i,n} \in \hom_{\mathfrak{relcat}}([n+1]\times[n+1]_{\mathbf{W}}, [n]\times[n]_{\mathbf{W}})\}_{0\leq i\leq n\geq 0}$$

satisfying the opposites of the identities given above (with the first two "boundary condition" identities being dictated by their respective sources and targets). Namely, we define

$$H_1^{i,n}(j,k) = \begin{cases} (j,k), & 0 \leq j,k \leq i \\ (j-1,j-1), & j > i \text{ and } j \geq k \\ (j,k-1), & k > i \geq j \\ (j-1,k-1), & k > j > i \end{cases}$$

and

$$H_2^{i,n}(j,k) = \begin{cases} (j,j), & j \leq i \\ (j-1,j-1), & j > i \text{ and } j \geq k \\ (j-1,k-1), & k > j > i. \end{cases}$$

It is a straightforward (but lengthy) process to verify

- that these satisfy the opposites of the identities given above,
- that they restrict along their boundaries to the various maps

  $$\mathrm{id}_{\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))}, \eta, \alpha\rho \in \hom_{s\mathcal{S}\mathrm{et}}(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})), \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})))$$

  as indicated in Figure 2.1, and
- that their values on vertices are all degenerate edges,

as claimed. This completes the proof. □

Chapter 3

On the Grothendieck construction for ∞-categories

In this chapter we provide, among other things: (i) a Bousfield–Kan formula for colimits in ∞-categories (generalizing the 1-categorical formula for a colimit as a coequalizer of maps between coproducts); (ii) ∞-categorical generalizations of Barwick–Kan's Theorem B_n and Dwyer–Kan–Smith's Theorem C_n (regarding homotopy pullbacks in the Thomason model structure, which themselves vastly generalize Quillen's Theorem B); and (iii) an articulation of the simultaneous and interwoven functoriality of colimits (or dually, of limits) for natural transformations and for pullback along maps of diagram ∞-categories.

3.0 Introduction

3.0.1 Outline

As the title and abstract suggest, this is essentially an omnibus chapter in which we collect a number of useful results in ∞-category theory, all having to do in some way or another with the ***Grothendieck construction***. This is instantiated in quasicategories by Lurie's *unstraightening construction*, although we work model-independently.

- In §3.1, we fix some notation and terminology surrounding the Grothendieck construction. We also give some examples, and we highlight some of its important features – notably its *naturality* (as proved by Gepner–Haugseng–Nikolaus).

- In §3.2, we define *lax* and *oplax* natural transformations of functors $\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ via the Grothendieck construction. Using these, we then construct a ***global colimit functor*** for a cocomplete ∞-category $\mathcal{C}$: this is a functor

$$\mathcal{L}\mathrm{ax}(\mathcal{C}) \to \mathcal{C}$$

from the *lax overcategory* of $\mathcal{C}$

  - which sends an object

$$(\mathcal{D} \xrightarrow{F} \mathcal{C}) \in \mathcal{L}\mathrm{ax}(\mathcal{C})$$

    to its colimit

$$\mathrm{colim}_{\mathcal{D}}(F) \in \mathcal{C},$$

    and
  - which encodes the simultaneous and interwoven functoriality of colimits in $\mathcal{C}$
    * for *natural transformations* – that is, for maps in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ – and
    * for *pullback along maps of diagram ∞-categories* – that is, for maps in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$.

  This immediately dualizes to give an analogous ***global limit functor***

$$\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{D})^{op} \xrightarrow{\lim} \mathcal{D}$$

  for a complete ∞-category $\mathcal{D}$ (now running from the *opposite* of its *oplax* overcategory).

- In §3.3, we explore the relationship between the Grothendieck construction (and its "two-sided" generalization) and colimits in the ∞-category of spaces. For instance, we record an ∞-categorical version of Thomason's *homotopy colimit theorem*. Some of these results are suggestive of the $(\infty, 2)$-categorical functoriality of the Grothendieck construction (e.g. we say the word "modification").

- In §3.4, we prove ∞-categorical versions of Barwick–Kan's ***Theorem B_n*** and Dwyer–Kan–Smith's ***Theorem C_n***, thus further extending the following sequence of increasingly general results in 1-category theory.

– Given a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ satisfying a certain property B, Quillen's *Theorem B* gives a simple description of the fibers

$$\lim \left(\begin{array}{ccc} & & \mathcal{D}^{\mathrm{gpd}} \\ & & \downarrow{\scriptstyle F^{\mathrm{gpd}}} \\ \{c\}^{\mathrm{gpd}} & \hookrightarrow & \mathcal{C}^{\mathrm{gpd}} \end{array} \right)$$

of the induced map on (∞-)groupoid completions.

– Given a functor satisfying a certain property B_n (which recovers property B when $n = 1$ but becomes weaker as n grows),

  * Dwyer–Kan–Smith's *Theorem* B_n gives a description of the fibers of the induced map on groupoid completions (which recovers Quillen's description when $n = 1$ but in trade becomes more complicated as n grows), while
  * their *Theorem* C_n asserts that if $\mathcal{C}$ satisfies a certain property C_n, then *any* functor $\mathcal{D} \to \mathcal{C}$ satisfies property B_n.

– Given a cospan $\mathcal{D} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{E}$ such that F satisfies property B_n (but without any conditions on G), Barwick–Kan's *enhanced* Theorem B_n gives a description of the pullback

$$\lim \left(\begin{array}{ccc} & & \mathcal{D}^{\mathrm{gpd}} \\ & & \downarrow{\scriptstyle F^{\mathrm{gpd}}} \\ \mathcal{E}^{\mathrm{gpd}} & \xrightarrow[G^{\mathrm{gpd}}]{} & \mathcal{C}^{\mathrm{gpd}} \end{array} \right)$$

of the induced cospan on groupoid completions (which likewise becomes more complicated as n grows).

- In §3.5, we prove a ***Bousfield–Kan formula*** for colimits in ∞-categories. This generalizes the 1-categorical formula for a colimit as a coequalizer of maps between coproducts. We also illustrate its application with concrete examples.

- In §3.6, we construct a ***Thomason model structure*** on the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories. Aside from its intrinsic interest, this model ∞-category $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ provides a convenient language for the various results which appear

throughout this chapter: it gives a presentation of the ∞-category $\mathcal{S}$ of spaces, and moreover its localization functor

$$\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty[\mathbf{W}_{\mathrm{Th}}^{-1}] \simeq \mathcal{S}$$

can be canonically identified with the groupoid completion functor. In particular, the subcategory $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$ of Thomason weak equivalences consists of precisely those maps which become equivalences upon groupoid completion.

This model structure is analogous to the classical Thomason model structure on the category $\mathcal{C}\mathrm{at}$ of categories; however, it is (in a sense) better behaved, and moreover it completely accounts for a certain quirk that prevents the latter from being lifted directly along the nerve functor. On the other hand, this model structure bears some rather surprising features of its own.

3.0.2 Acknowledgments

We heartily thank David Ayala, David Gepner, Saul Glasman, Zhen Lin Low, and Charles Rezk for their helpful input on the material presented in this chapter. We are also grateful to the NSF graduate research fellowship program (grant DGE-1106400) for its financial support during the time that this work was carried out, and to the Pacific Science Institute for the hospitality and kite-surfing that it afforded during the time that this chapter was being written.

3.1 The Grothendieck construction

In this section, we recall some basic notions involving the Grothendieck construction and the various sorts of fibrations that it involves: we discuss co/cartesian (and left/right) fibrations in §3.1.1, and we discuss the Grothendieck construction itself in §3.1.2. For background, we refer the reader to our companion paper [MG], which contains

- model-independent definitions of co/cartesian morphisms and co/cartesian fibrations,
- proofs that these model-independent definitions are suitably compatible with their quasicategorical counterparts, and
- an extended informal discussion the Grothendieck construction.

3.1.1 Fibrations

We begin by fixing the following notation (without really giving any definitions).

Notation 3.1.1. Let $\mathcal{C}$ be an ∞-category.

- We denote by $\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})$ the ∞-category of *left fibrations* over $\mathcal{C}$; by the dual of Corollary T.2.2.3.12 (see Remark T.2.1.4.12), this is the underlying ∞-category of the covariant model structure of Proposition T.2.1.4.7, and is well-defined by Remark T.2.1.4.11.

- We denote by $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$ the ∞-category of *cocartesian fibrations* over $\mathcal{C}$; by the dual of Proposition T.3.1.4.1, this is the underlying ∞-category of the cocartesian model structure which is dual to that of Proposition T.3.1.3.7 (see Remark T.3.1.3.9), and is well-defined by Proposition T.3.3.1.1.

By Theorem T.3.1.5.1, these two ∞-categories sit in a sequence of adjunctions

$$(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}} \underset{\mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\overset{\mathrm{L}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\rightleftarrows}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \underset{\mathrm{U}_{\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\overset{\mathrm{L}_{\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\rightleftarrows}} \mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C}) \underset{\mathrm{U}_{\mathcal{L}(\mathcal{C})}}{\overset{\mathrm{L}_{\mathcal{L}(\mathcal{C})}}{\rightleftarrows}} \mathcal{S}_{/\mathcal{C}^{\mathrm{gpd}}}$$

in $\mathcal{C}\mathrm{at}_\infty$.[1]

Of course, there are dual notions of *right fibrations* and of *cartesian fibrations*, defined to be those functors that respectively become left fibrations or cocartesian fibrations under the involution $(-)^{op} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$. These then assemble into an analogous string of adjunctions

$$(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}} \underset{\mathrm{U}_{\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\overset{\mathrm{L}_{\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\rightleftarrows}} \mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \underset{\mathrm{U}_{\mathcal{R}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\overset{\mathrm{L}_{\mathcal{R}\mathcal{F}\mathrm{ib}(\mathcal{C})}}{\rightleftarrows}} \mathcal{R}\mathcal{F}\mathrm{ib}(\mathcal{C}) \underset{\mathrm{U}_{\mathcal{R}(\mathcal{C})}}{\overset{\mathrm{L}_{\mathcal{R}(\mathcal{C})}}{\rightleftarrows}} \mathcal{S}_{/\mathcal{C}^{\mathrm{gpd}}}$$

in $\mathcal{C}\mathrm{at}_\infty$. By taking opposites, it will often suffice to leave observations about these latter notions implicit.

We now assemble a number of useful observations.

Remark 3.1.2. The adjunctions $\mathrm{L}_{\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})} \dashv \mathrm{U}_{\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})}$ and $\mathrm{L}_{\mathcal{L}(\mathcal{C})} \dashv \mathrm{U}_{\mathcal{L}(\mathcal{C})}$ are both left localization adjunctions.[2] However, the adjunction $\mathrm{L}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})} \dashv \mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}$ is not: given

[1] The identification of $\mathcal{S}_{/\mathcal{C}^{\mathrm{gpd}}}$ as the underlying ∞-category of the corresponding model category follows from the fact that $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is right proper, and hence *any* weak equivalence induces a Quillen adjunction on overcategories. In particular, the overcategory of a quasicategory already has the correct underlying ∞-category, even without replacing the quasicategory by a Kan complex.

[2] To see this, note that the first is presented by the composite of a left Bousfield localization followed by a Quillen equivalence, while the second is presented by a left Bousfield localization.

two objects $(\mathcal{D} \xrightarrow{p} \mathcal{C}), (\mathcal{E} \xrightarrow{q} \mathcal{C}) \in \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$, a morphism

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{E} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{C} & \end{array}$$

in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$ must take p-cocartesian morphisms in $\mathcal{D}$ to q-cocartesian morphisms in $\mathcal{E}$ in order to define a morphism in $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$. (However, the right adjoint $\mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}$ is nevertheless the inclusion of a (non-full) subcategory, as indicated.) In §3.2, we will see that its failure to be a left localization gives rise to the notion of a *lax natural transformation* between objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$.

Remark 3.1.3. Given any functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$, the resulting horizontal composite in the diagram

$$\begin{array}{ccccc} \mathcal{D} \times_{\mathcal{C}} \mathrm{Fun}([1], \mathcal{C}) & \longrightarrow & \mathrm{Fun}([1], \mathcal{C}) & \xrightarrow{t} & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle s} & & \\ \mathcal{D} & \xrightarrow[F]{} & \mathcal{C} & & \end{array}$$

is a cocartesian fibration by (the dual of) Corollary T.2.4.7.12. In fact, by (the dual of) [GHN, Theorem 4.5], this is precisely the *free cocartesian fibration* on F: this construction gives the left adjoint $\mathrm{L}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}$.

Remark 3.1.4. In the sequence

$$(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}} \xrightarrow{\mathrm{L}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \xrightarrow{\mathrm{L}_{\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})}} \mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C}) \xrightarrow{\mathrm{L}_{\mathcal{L}(\mathcal{C})}} \mathcal{S}_{\mathcal{C}^{\mathrm{gpd}}}$$

of left adjoints, all of the (possibly composite) left adjoints with target $\mathcal{S}_{/\mathcal{C}^{\mathrm{gpd}}}$ are given by taking an object $\mathcal{D} \to \mathcal{C}$ to the object $\mathcal{D}^{\mathrm{gpd}} \to \mathcal{C}^{\mathrm{gpd}}$; this follows directly from the definitions of the overlying Quillen adjunctions of Theorem T.3.1.5.1.

Remark 3.1.5. When in fact $\mathcal{C} \in \mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$, the adjunctions $\mathrm{L}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})} \dashv \mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}$ and $\mathrm{L}_{\mathcal{L}(\mathcal{C})} \dashv \mathrm{U}_{\mathcal{L}(\mathcal{C})}$ become adjoint equivalences by Theorem T.3.1.5.1.

3.1.2 The Grothendieck construction

We now give the main definition of this section.

Definition 3.1.6. We define the ***Grothendieck construction*** to be either equivalence of ∞-categories in the diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) & \xrightarrow[\sim]{\mathrm{Gr}} & \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \\ \uparrow & & \uparrow \\ \mathrm{Fun}(\mathcal{C}, \mathcal{S}) & \xrightarrow[\mathrm{Gr}]{\sim} & \mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C}); \end{array}$$

here, the upper (resp. lower) equivalence underlies the right adjoint of the Quillen equivalence which is dual to that of Theorem T.3.2.0.1 (resp. that of Theorem T.2.2.1.2) in the special case that the functor of $s\mathcal{S}$et-enriched categories is the identity. The fact that the diagram commutes follows from the construction (see Definition T.3.2.1.2). Of course, there are also Grothendieck constructions for cartesian fibrations and right fibrations; we will denote these by

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{C}\mathrm{at}_\infty) & \xrightarrow[\sim]{\mathrm{Gr}^-} & \mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \\ \uparrow & & \uparrow \\ \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) & \xrightarrow[\mathrm{Gr}^-]{\sim} & \mathcal{R}\mathcal{F}\mathrm{ib}(\mathcal{C}). \end{array}$$

When we need to distinguish between these two types of Grothendieck constructions, we will refer to the former sort as ***covariant*** and to the latter sort as ***contravariant***.

For any $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$, when we need to refer to it we will write

$$\mathrm{Gr}(F) \xrightarrow{\mathrm{pr}_{\mathrm{Gr}(F)}} \mathcal{C}$$

for the cocartesian fibration that it classifies. Similarly, given any $\mathcal{C}^{op} \xrightarrow{G} \mathcal{C}\mathrm{at}_\infty$, when we need to refer to it we will write

$$\mathrm{Gr}^-(G) \xrightarrow{\mathrm{pr}_{\mathrm{Gr}^-(G)}} \mathcal{C}$$

for the cartesian fibration that it classifies.

The following characterization of the Grothendieck construction may at first appear rather abstract. However, it gives excellent geometric intuition, as we illustrate in Example 3.1.8 (the first nontrivial case).

Remark 3.1.7. The covariant Grothendieck construction can be characterized as a *lax colimit*: by [GHN, Theorem 7.4], for any functor $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ we have an equivalence

$$\mathrm{Gr}(F) \simeq \int^{c \in \mathcal{C}} \mathcal{C}_{c/} \times F(c)$$

of its covariant Grothendieck construction with its colimit weighted by $\mathcal{C}^{op} \xrightarrow{\mathcal{C}_{-/}} \mathcal{C}\mathrm{at}_\infty$. Thus, whereas the ordinary colimit of the diagram F can be viewed as "gluing together" all of the ∞-categories $F(c)$ – that is, taking all of the values $F(c) \in \mathcal{C}\mathrm{at}_\infty$ and, for every map $c \xrightarrow{\varphi} c'$ in $\mathcal{C}$ adding in an *equivalence*

$$y \xrightarrow{\sim} (F(\varphi))(y)$$

for every $y \in F(c)$ –, in a *lax* colimit, we now only add in a *noninvertible* morphism

$$y \to (F(\varphi))(y)$$

corresponding to such data. Dually, the contravariant Grothendieck construction can be characterized as an *oplax colimit*: by [GHN, Corollary 7.6], for any functor $\mathcal{C}^{op} \xrightarrow{G} \mathcal{C}\mathrm{at}_\infty$ we have an equivalence

$$\mathrm{Gr}^-(G) \simeq \int^{c^\circ \in \mathcal{C}^{op}} \mathcal{C}_{/c} \times G(c^\circ)$$

of its contravariant Grothendieck construction with its colimit weighted by $\mathcal{C} \xrightarrow{\mathcal{C}_{/-}} \mathcal{C}\mathrm{at}_\infty$.[3]

Example 3.1.8. Suppose that $[1] \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ selects a functor $\mathcal{C}_0 \xrightarrow{f} \mathcal{C}_1$. Then, its covariant Grothendieck construction can be identified as

$$\mathrm{Gr}(F) \simeq \mathrm{colim} \left(\begin{array}{ccc} & \mathcal{C}_0 & \xrightarrow{\quad f \quad} \mathcal{C}_1 \\ {\scriptstyle \mathrm{id}_{\mathcal{C}_0} \times \{1\}} \big\downarrow & & \\ & \mathcal{C}_0 \times [1] & \end{array} \right),$$

a "directed mapping cylinder" for f. Dually, if $[1]^{op} \xrightarrow{G} \mathcal{C}\mathrm{at}_\infty$ selects a functor $\mathcal{D}_0 \xleftarrow{g} \mathcal{D}_1$, then its contravariant Grothendieck construction can be identified as

$$\mathrm{Gr}^-(G) \simeq \mathrm{colim} \left(\begin{array}{ccc} & \mathcal{D}_1 & \xrightarrow{\quad g \quad} \mathcal{D}_0 \\ {\scriptstyle \mathrm{id}_{\mathcal{D}_1} \times \{0\}} \big\downarrow & & \\ & \mathcal{D}_1 \times [1] & \end{array} \right),$$

a "reversed directed mapping cylinder" for g.

[3] One can also identify the ∞-*categories of sections* of Grothendieck constructions with op/lax *limits* (see e.g. [GHN, Proposition 7.1]), but this is less essential for geometric intuition.

We now list a few more basic examples of the Grothendieck construction.

Example 3.1.9. The equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$$

necessarily preserves terminal objects. In the source, the terminal object is $\mathrm{const}(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})$, while in the target, the terminal object is the identity functor $\mathrm{id}_{\mathcal{C}}$ (which is a cocartesian fibration). Similarly, the identity functor $\mathrm{id}_{\mathcal{C}}$ is the terminal object of $\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$.

Example 3.1.10. Given any two functors $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$, consider the pullback diagram

$$\begin{array}{ccc} \mathrm{Gr}(F) \times_{\mathcal{C}} \mathrm{Gr}(G) & \longrightarrow & \mathrm{Gr}(F) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\mathrm{Gr}(F)}} \\ \mathrm{Gr}(G) & \xrightarrow[\mathrm{pr}_{\mathrm{Gr}(G)}]{} & \mathcal{C} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$. Note first that the composite functor

$$\mathrm{Gr}(F) \times_{\mathcal{C}} \mathrm{Gr}(G) \to \mathcal{C}$$

is again a cocartesian fibration by the dual of (parts (2) and (3) of) Proposition T.2.4.2.3. Moreover, this is the product of the objects $\mathrm{pr}_{\mathrm{Gr}(F)}$ and $\mathrm{pr}_{\mathrm{Gr}(G)}$ in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$, and since the inclusion $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \subset (\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$ is a right adjoint, this must also be their product in $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$. Thus, this cocartesian fibration must be classified by the composite functor

$$\mathcal{C} \xrightarrow{(F,G)} \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty \xrightarrow{-\times-} \mathcal{C}\mathrm{at}_\infty,$$

i.e. the product $F \times G \in \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$.

Example 3.1.11. In the special case that $\mathcal{C} = \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$, the Grothendieck construction yields an equivalence

$$\mathcal{C}\mathrm{at}_\infty \simeq \mathrm{Fun}(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}).$$

By [Toë05, Théorèm 6.3], this must be inverse to the composite

$$\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}) \xrightarrow{\sim} (\mathcal{C}\mathrm{at}_\infty)_{/\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}} \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$$

of two forgetful equivalences.

Example 3.1.12. Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, and let $c \in \mathcal{C}$. Then, the forgetful functor $\mathcal{C}_{c/} \to \mathcal{C}$ from the undercategory is a left fibration by Corollary T.2.1.2.2; in fact, it follows from Proposition T.4.4.4.5 that we can identify this as

$$\mathcal{C}_{c/} \simeq \mathrm{Gr}\left(\mathcal{C} \xrightarrow{\hom_{\mathcal{C}}(c,-)} \mathcal{S}\right).$$

Dually, we have that $(\mathcal{C}_{/c} \to \mathcal{C}) \in \mathcal{RF}\mathrm{ib}(\mathcal{C})$, and we have the identification

$$\mathcal{C}_{/c} \simeq \mathrm{Gr}^-\left(\mathcal{C}^{op} \xrightarrow{\hom_{\mathcal{C}}(-,c)} \mathcal{S}\right).$$

Remark 3.1.13. An important property of the Grothendieck construction is its *naturality*: it assembles to a functor

$$(\mathcal{C}\mathrm{at}_\infty)^{op} \to \mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$$

which sends an ∞-category $\mathcal{C}$ to the object

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$$

of $\mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$ by (the dual of) [GHN, Corollary A.31]. By its construction (see the proof of [GHN, Proposition A.30]), over $0 \in [1]$ this functor is given by precomposition of functors to $\mathcal{C}\mathrm{at}_\infty$, while over $1 \in [1]$ this functor is given by pullback of cocartesian fibrations. So for instance, a map $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathcal{C}\mathrm{at}_\infty$ determines a map

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{D}, \mathcal{C}\mathrm{at}_\infty) & \xrightarrow[\sim]{\mathrm{Gr}} & \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{D}) \\ {\scriptstyle -\circ F}\downarrow & & \downarrow{\scriptstyle F^*} \\ \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) & \xrightarrow[\mathrm{Gr}]{\sim} & \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \end{array}$$

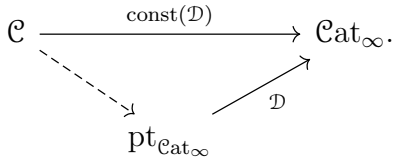
in $\mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$ (where we read this square in $\mathcal{C}\mathrm{at}_\infty$ vertically in order to consider it as a morphism between the two horizontal arrows). Of course, by duality the contravariant Grothendieck construction enjoys analogous naturality.

Since this observation will arise so frequently for us, we codify it.

Definition 3.1.14. We refer to the phenomenon described in Remark 3.1.13 as the ***naturality*** of the Grothendieck construction.

This has the following easy consequence.

Example 3.1.15. Consider the tautological factorization



By Example 3.1.11, we have a canonical equivalence

$$\mathrm{Gr}\left(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{\mathcal{D}} \mathcal{C}\mathrm{at}_\infty\right) \simeq \mathcal{D}.$$

Hence, the naturality of the Grothendieck construction implies that the pullback square

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ provides a canonical equivalence

$$\mathrm{Gr}\left(\mathcal{C} \xrightarrow{\mathrm{const}(\mathcal{D})} \mathcal{C}\mathrm{at}_\infty\right) \simeq \mathcal{C} \times \mathcal{D},$$

with the cocartesian fibration down to $\mathcal{C}$ identifying as $\mathrm{pr}_{\mathrm{Gr}(\mathrm{const}(\mathcal{D}))} \simeq \mathrm{pr}_{\mathcal{C}}$, the projection onto the first factor. (From here, we can recover Example 3.1.9 as the special case where $\mathcal{D} = \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$.) Similarly, we have that

$$\mathrm{Gr}^-\left(\mathcal{C}^{op} \xrightarrow{\mathrm{const}(\mathcal{D})} \mathcal{C}\mathrm{at}_\infty\right) \simeq \mathcal{C} \times \mathcal{D},$$

with $\mathrm{pr}_{\mathrm{Gr}^-(\mathrm{const}(\mathcal{D}))} \simeq \mathrm{pr}_{\mathcal{C}}$: in other words, the projection $\mathcal{C} \times \mathcal{D} \xrightarrow{\mathrm{pr}_{\mathcal{C}}} \mathcal{C}$ is simultaneously a cocartesian fibration and a cartesian fibration, in either case classified by the constant functor at the object $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$.

Definition 3.1.16. Fix some $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$. By the naturality of the Grothendieck construction (and Example 3.1.11), for any $x \in \mathcal{C}$ there is a canonical pullback square

$$\begin{array}{ccc} F(x) & \longrightarrow & \mathrm{Gr}(F) \\ \downarrow & & \downarrow \\ \{x\} & \hookrightarrow & \mathcal{C} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$. We refer to $F(x)$ as the ***fiber*** of the cocartesian fibration over the object $x \in \mathcal{C}$, and we refer to the above commutative square as a ***fiber inclusion***.[4]

Remark 3.1.17. A fiber inclusion will not generally be the inclusion of a full subcategory: it will not contain those morphisms covering nontrivial endomorphisms. In fact, in the extreme case that we take $\mathcal{C} = Y \in \mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$ to be an ∞-groupoid, the functor corepresented by an object $y \in Y$ will have

$$\mathrm{Gr}\left(Y \xrightarrow{\hom_Y(y,-)} \mathcal{S}\right) \simeq Y_{y/} \simeq \mathrm{pt}_{\mathcal{S}}$$

(recall Example 3.1.12); then, the fiber inclusion over the object $y \in Y$ itself will be given by the pullback square

$$\begin{array}{ccc} \Omega Y & \longrightarrow & \mathrm{pt}_{\mathcal{S}} \\ \downarrow & & \downarrow \\ \{y\} & \hookrightarrow & Y \end{array}$$

in $\mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$, the "inclusion" of the based loopspace of Y at y into the terminal space.

We have the following concrete identification of the left localization which takes cocartesian fibrations to left fibrations.

Proposition 3.1.18. *Under the equivalences given by the Grothendieck construction, the left localization*

$$\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \xrightarrow{\mathrm{L}_{\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})}} \mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C})$$

corresponds to the functor $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ *given by postcomposition with*

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{\mathrm{gpd}}} \mathcal{S}.$$

Proof. This follows from the uniqueness of left adjoints and the commutativity of the diagram in Definition 3.1.6: the adjunction $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ comes from applying the functor $\mathrm{Fun}(\mathcal{C}, -) : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ to the adjunction $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{S}$. □

[4] At the level of quasicategories, this can be computed by the inclusion of the fiber over a vertex corresponding to $x \in \mathcal{C}$, which is a homotopy fiber in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ by the Reedy trick (and the implications of Remark T.2.0.0.5).

3.2 Op/lax natural transformations and the global co/limit functor

In ordinary category theory, functors with the same source and target can be related by natural transformations between them. When the target (and possibly the source) is a 2-category, this notion can be relaxed in two different ways, yielding notions of *lax* and *oplax* natural transformations.

We will be interested in this phenomenon in the ∞-categorical context. However, we will concern ourselves exclusively with the special case in which the source is only an ∞-category and the target is $\mathcal{C}\mathrm{at}_\infty$, considered as an $(\infty, 2)$-category via the closure of its symmetric monoidal structure $(\mathcal{C}\mathrm{at}_\infty, \times)$. In this case, the Grothendieck construction allows us to easily and concisely define such transformations without reference to an ambient theory of $(\infty, 2)$-categories: heuristically speaking, it "reduces category level by one".[5]

In §3.2.1 we define op/lax natural transformations via the Grothendieck construction, and then in §3.2.2 we apply this framework to study the ($(\infty, 2)$-categorical) functoriality of colimits in an arbitrary but fixed ∞-category.

3.2.1 Op/lax natural transformations

We begin with the following definition.

Definition 3.2.1. Suppose we are given a pair of functors $F, G : \mathcal{C} \rightrightarrows \mathcal{C}\mathrm{at}_\infty$. A ***lax natural transformation*** from F to G, denoted $F \rightsquigarrow G$ for short, is a map

$$\begin{array}{ccc} \mathrm{Gr}(F) & \longrightarrow & \mathrm{Gr}(G) \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$. Meanwhile, an ***oplax natural transformation*** from F to G, denoted $F \overset{op}{\rightsquigarrow} G$ for short, is a map

$$\begin{array}{ccc} \mathrm{Gr}^-(F) & \longrightarrow & \mathrm{Gr}^-(G) \\ & \searrow \quad \swarrow & \\ & \mathcal{C}^{op} & \end{array}$$

[5] The notion we introduce here is manifestly equivalent to the more general notion of lax natural transformations between functors among $(\infty, 2)$-categories studied in [GR17, Chapter 10, §3].

in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}^{op}}$.

To provide some intuition, we illustrate Definition 3.2.1 in the simplest nontrivial case.

Example 3.2.2. Recall from Example 3.1.8 that we can think of the equivalence

$$\mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1])$$

as a sort of "directed mapping cylinder" construction. Hence, if $[1] \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ selects a functor $\mathcal{C}_0 \xrightarrow{f} \mathcal{C}_1$ and $[1] \xrightarrow{G} \mathcal{C}\mathrm{at}_\infty$ selects a functor $\mathcal{D}_0 \xrightarrow{g} \mathcal{D}_1$, then a lax natural transformation $\alpha : F \rightsquigarrow G$ is equivalent to a square

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{f} & \mathcal{C}_1 \\ {\scriptstyle \alpha_0}\downarrow & \nearrow & \downarrow{\scriptstyle \alpha_1} \\ \mathcal{D}_0 & \xrightarrow[g]{} & \mathcal{D}_1 \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$, i.e. the data of

- a functor $\mathcal{C}_0 \xrightarrow{\alpha_0} \mathcal{D}_0$,
- a functor $\mathcal{C}_1 \xrightarrow{\alpha_1} \mathcal{D}_1$, and
- a natural transformation $g \circ \alpha_0 \to \alpha_1 \circ f$ in $\mathrm{Fun}(\mathcal{C}_0, \mathcal{D}_1)$.

A similar analysis shows that an oplax natural transformation $\alpha : F \overset{op}{\rightsquigarrow} G$ simply reverses the direction of the natural transformation (so that it runs down and to the left, instead of up and to the right).

We will be interested in the special case of Example 3.2.2 in which $G = \mathrm{const}(\mathcal{C})$ (so that $g = \mathrm{id}_{\mathcal{C}}$) for some chosen ∞-category $\mathcal{C}$. For a fixed such choice, these can be organized in the following way.

Definition 3.2.3. For any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, we define the ***lax overcategory*** of $\mathcal{C}$ to be

$$\mathcal{L}\mathrm{ax}(\mathcal{C}) = \mathrm{Gr}^- \left((\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\mathrm{Fun}(-,\mathcal{C})} \mathcal{C}\mathrm{at}_\infty \right).$$

Thus, the objects of $\mathcal{L}\mathrm{ax}(\mathcal{C})$ are functors with target $\mathcal{C}$, and a morphism from $\mathcal{D} \xrightarrow{F} \mathcal{C}$ to $\mathcal{E} \xrightarrow{G} \mathcal{C}$ in $\mathcal{L}\mathrm{ax}(\mathcal{C})$ is given by a triangle

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\quad H \quad} & \mathcal{E} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathcal{C} & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$. We write

$$\mathcal{L}\mathrm{ax}(\mathcal{C}) \xrightarrow{s_{\mathcal{L}\mathrm{ax}(\mathcal{C})}} \mathcal{C}\mathrm{at}_\infty$$

for the canonical projection map, and refer to it as the ***source projection***; this is by definition a cartesian fibration, with fiber over $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ given by $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Similarly, we define the ***oplax overcategory*** of $\mathcal{C}$ to be

$$\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C}) = \mathrm{Gr}^- \left((\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\mathrm{Fun}((-)^{op}, \mathcal{C}^{op})} \mathcal{C}\mathrm{at}_\infty \right).$$

Thus, the objects of $\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})$ can be canonically identified (via the involution $(-)^{op} : \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$) with functors with target $\mathcal{C}$, and a morphism from (the object identified with) $\mathcal{D} \xrightarrow{F} \mathcal{C}$ to (the object identified with) $\mathcal{E} \xrightarrow{G} \mathcal{C}$ in $\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})$ can be canonically identified with a triangle

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\quad H \quad} & \mathcal{E} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathcal{C} & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$. We write

$$\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C}) \xrightarrow{s_{\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})}} \mathcal{C}\mathrm{at}_\infty$$

for the canonical projection map, and also refer to it as the ***source projection***; this is again by definition a cartesian fibration, with fiber over $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ given by $\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{C}^{op}) \simeq \mathrm{Fun}(\mathcal{D}, \mathcal{C})^{op}$.

Remark 3.2.4. A map $\mathcal{C}_1 \to \mathcal{C}_2$ induces a natural transformation $\mathrm{Fun}(-, \mathcal{C}_1) \to \mathrm{Fun}(-, \mathcal{C}_2)$ in $\mathrm{Fun}((\mathcal{C}\mathrm{at}_\infty)^{op}, \mathcal{C}\mathrm{at}_\infty)$, which in turn gives rise to a commutative triangle

$$\begin{array}{ccc} \mathcal{L}\mathrm{ax}(\mathcal{C}_1) & \longrightarrow & \mathcal{L}\mathrm{ax}(\mathcal{C}_2) \\ & {\scriptstyle s_{\mathcal{L}\mathrm{ax}(\mathcal{C}_1)}}\searrow \quad \swarrow{\scriptstyle s_{\mathcal{L}\mathrm{ax}(\mathcal{C}_2)}} & \\ & \mathcal{C}\mathrm{at}_\infty & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$; altogether, we obtain a functor

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathcal{L}\mathrm{ax}(-)} \mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}\mathrm{at}_\infty).$$

Similarly, we obtain a functor

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{op}\mathcal{L}\mathrm{ax}(-)} \mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}\mathrm{at}_\infty).$$

Remark 3.2.5. Expanding out the definition, we also can write

$$\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C}) = \mathrm{Gr}^- \left((\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{((-)^{op})^{op}} (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\mathrm{Fun}(-,\mathcal{C}^{op})} \mathcal{C}\mathrm{at}_\infty \right)$$

(where the first functor is obtained by applying the involution $\mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{op}} \mathcal{C}\mathrm{at}_\infty$ to the morphism $\mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{op}} \mathcal{C}\mathrm{at}_\infty$). By the naturality of the Grothendieck construction, it follows that we have a pullback square

$$\begin{array}{ccc} \mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{L}\mathrm{ax}(\mathcal{C}^{op}) \\ {\scriptstyle s_{\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})}}\downarrow & & \downarrow{\scriptstyle s_{\mathcal{L}\mathrm{ax}(\mathcal{C}^{op})}} \\ \mathcal{C}\mathrm{at}_\infty & \xrightarrow[(-)^{op}]{\sim} & \mathcal{C}\mathrm{at}_\infty \end{array}$$

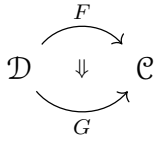
in $\mathcal{C}\mathrm{at}_\infty$.

Remark 3.2.6. As an alternative to the construction of Definition 3.2.3, both $\mathcal{L}\mathrm{ax}(\mathcal{C})$ and $\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})$ are simultaneously encoded in the "(op)lax square" of $\mathcal{C}\mathrm{at}_\infty$ as constructed in [JFS, §5]. While this alternative construction is both clean and aesthetically pleasing, we have chosen our own exposition in pursuit of the meta-goal of this chapter (namely, to connect as many different concepts as possible to the Grothendieck construction).

3.2.2 The global co/limit functor

Suppose we are given an arbitrary cocomplete ∞-category $\mathcal{C}$. Then, the operation of taking colimits in $\mathcal{C}$ should be functorial in two different senses.

- On the one hand, colimits are functorial for natural transformations. For instance, a natural transformation



induces a canonical map $\mathrm{colim}_{\mathcal{D}}(F) \to \mathrm{colim}_{\mathcal{D}}(G)$ in $\mathcal{C}$: a colimiting cocone $\mathcal{D}^{\triangleright} \to \mathcal{C}$ extending G composes to give an arbitrary cocone $\mathcal{D}^{\triangleright} \to \mathcal{C}$ extending F, and then the desired map arises from the definition of a colimit as an initial cocone extending the given diagram. Thus, taking colimits should give rise to a functor

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \xrightarrow{\mathrm{colim}} \mathcal{C}.$$

- On the other hand, colimits are also functorial for commutative diagrams of ∞-categories over $\mathcal{C}$. For instance, a commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{E} \\ & {\scriptstyle F} \searrow \quad \swarrow {\scriptstyle G} & \\ & \mathcal{C} & \end{array}$$

in $\mathcal{C}\mathrm{at}_{\infty}$ induces a canonical map $\mathrm{colim}_{\mathcal{D}}(F) \to \mathrm{colim}_{\mathcal{E}}(G)$ in $\mathcal{C}$: a colimiting cocone $\mathcal{E}^{\triangleright} \to \mathcal{C}$ extending G composes to give an arbitrary cocone $\mathcal{D}^{\triangleright} \to \mathcal{E}^{\triangleright} \to \mathcal{C}$ extending F, and then the desired map once again arises from the definition of a colimit as an initial cocone extending the given diagram. Thus, taking colimits should also give rise to a functor

$$(\mathcal{C}\mathrm{at}_{\infty})_{/\mathcal{C}} \xrightarrow{\mathrm{colim}} \mathcal{C}.$$

In fact, we have already seen a construction in Definition 3.2.3 which unifies these two situations: via the cartesian fibration $\mathcal{L}\mathrm{ax}(\mathcal{C}) \xrightarrow{s_{\mathcal{L}\mathrm{ax}(\mathcal{C})}} \mathcal{C}\mathrm{at}_{\infty}$, an arbitrary morphism

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{H} & \mathcal{E} \\ & {\scriptstyle F} \searrow \quad \swarrow {\scriptstyle G} & \\ & \mathcal{C} & \end{array}$$

in $\mathcal{L}\mathrm{ax}(\mathcal{C})$ gives rise to a unique diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{H} & \mathcal{E} \\ & {\scriptstyle F} \searrow \quad \nearrow {\scriptstyle G \circ H} & \\ & \mathcal{C} & \end{array}$$

in which the inner (commutative) triangle determines a cartesian arrow and the indicated natural transformation is a fiber morphism (lying over the object $\mathcal{D} \in$

$\mathcal{C}\mathrm{at}_\infty$). Thus, one might expect that it is possible to unify the above two senses in which colimits are functorial by means of a single functor

$$\mathcal{L}\mathrm{ax}(\mathcal{C}) \xrightarrow{\mathrm{colim}} \mathcal{C}.$$

The purpose of this subsection is to construct precisely such a *global colimit functor*. In fact, we will achieve this (as Proposition 3.2.12) for an arbitrary ∞-category $\mathcal{C}$ (i.e. when $\mathcal{C}$ is not necessarily cocomplete), although in this more general setting we will of course need to restrict to the full subcategory of $\mathcal{L}\mathrm{ax}(\mathcal{C})$ spanned by those diagrams in $\mathcal{C}$ which admit a colimit (see Notation 3.2.11).

Remark 3.2.7. A similar analysis suggests that when $\mathcal{C}$ is complete, there should exist a corresponding *global limit functor* running

$$\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})^{op} \xrightarrow{\lim} \mathcal{C}.$$

Indeed, this can be obtained from Proposition 3.2.12 simply by taking opposites: if $\mathcal{C}$ is complete then $\mathcal{C}^{op}$ is cocomplete, and combining the resulting global colimit functor

$$\mathcal{L}\mathrm{ax}(\mathcal{C}^{op}) \xrightarrow{\mathrm{colim}} \mathcal{C}^{op}$$

with the equivalence of Remark 3.2.5 and taking opposites yields a composite

$$\mathrm{op}\mathcal{L}\mathrm{ax}(\mathcal{C})^{op} \xrightarrow{\sim} \mathcal{L}\mathrm{ax}(\mathcal{C}^{op})^{op} \to \mathcal{C}$$

which one can easily verify is the desired global limit functor. (And of course, this equally well generalizes to the case that $\mathcal{C}$ only admits certain limits.) We will therefore henceforth focus our attention only on the global *colimit* functor.

The first step in constructing the global colimit functor is to make the following reidentification of the nerve of the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories.

Lemma 3.2.8. *There is a canonical identification of* $\mathrm{N}_\infty(\mathcal{C}\mathrm{at}_\infty)_\bullet \in \mathcal{CSS} \subset s\mathcal{S}$ *with the composite*

$$\mathbf{\Delta}^{op} \hookrightarrow (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(-)^{\simeq}} \mathcal{S}.$$

Proof. For any $[n] \in \mathbf{\Delta}$, the Grothendieck construction provides an equivalence

$$\mathrm{Fun}([n], \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([n]);$$

passing to maximal subgroupoids, we obtain an equivalence

$$\mathrm{N}_\infty(\mathcal{C}\mathrm{at}_\infty)_n = \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([n])^{\simeq}.$$

The claim then follows from the naturality of the Grothendieck construction. □

Next, we would like to correspondingly identify the nerve of the lax overcategory of $\mathcal{C}$ (along with that of its source projection). For this, observe (recalling Example 3.2.2) that the datum of a morphism in $\mathcal{L}\mathrm{ax}(\mathcal{C})$ is specified by the pair of

- its image $[1] \xrightarrow{H} \mathcal{C}\mathrm{at}_\infty$ under the source projection, which by Lemma 3.2.8 is equivalent to specifying a point
$$\Big(\mathrm{Gr}(H) \xrightarrow{\mathrm{pr}_{\mathrm{Gr}(H)}} [1]\Big) \in \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1])^{\simeq} \in \mathcal{S},$$
along with
- a map $\mathrm{Gr}(H) \to \mathcal{C}$ from (the underlying ∞-category of) the "directed mapping cylinder" $\mathrm{Gr}(H)$ into our fixed target ∞-category $\mathcal{C}$.

Moreover, a similar observation holds when we replace the object $[1] \in \mathbf{\Delta}$ by an arbitrary object $[n] \in \mathbf{\Delta}$. Altogether, this identifies the canonical maps
$$\mathrm{N}_\infty(\mathcal{L}\mathrm{ax}(\mathcal{C}))_n \to \mathrm{N}_\infty(\mathcal{C}\mathrm{at}_\infty)_n$$
(obtained by applying the functor
$$\mathrm{N}_\infty(-)_n \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}([n], -) \in \mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$$
to the source projection map $\mathcal{L}\mathrm{ax}(\mathcal{C}) \xrightarrow{s_{\mathcal{L}\mathrm{ax}(\mathcal{C})}} \mathcal{C}\mathrm{at}_\infty$) as being the cocartesian fibration associated to a certain map
$$\mathrm{N}_\infty(\mathcal{C}\mathrm{at}_\infty)_n \to \mathcal{S}.$$
This motivates our desired identification (Lemma 3.2.10), but in order to state it precisely we first introduce the following notation.

Notation 3.2.9. To ease notation, for any ∞-category $\mathcal{C}$ we denote by $\mathrm{U}_{\mathcal{C}}$ any sub-composite of the composite
$$\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})^{\simeq} \hookrightarrow \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \xrightarrow{\mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}} (\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}} \to \mathcal{C}\mathrm{at}_\infty$$
of the inclusion of the maximal subgroupoid with the two evident forgetful functors. Moreover, we denote by $\mathrm{U}^\dagger_{\mathcal{C}}$ any sub-composite beginning at $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})^{\simeq}$ of the composite
$$\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})^{\simeq} \xrightarrow{\sim} (\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})^{\simeq})^{op} \hookrightarrow (\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}))^{op} \xrightarrow{\mathrm{U}^{op}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})}} \big((\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}\big)^{op} \to (\mathcal{C}\mathrm{at}_\infty)^{op}$$
of the canonical equivalence followed by the opposite of the above composite. We also use this same notation when restricting to the subcategory $\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{C}) \subset \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$ or to its maximal subgroupoid.

Lemma 3.2.10. *Fix any* $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$.

(1) For any $n \geq 0$, *there is a canonical equivalence*

$$\mathrm{N}_\infty(\mathcal{L}\mathrm{ax}(\mathcal{C}))_n \simeq \mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([n])^\simeq}\left(\mathrm{Gr}\left(\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([n])^\simeq \xrightarrow{\mathrm{U}^\dagger_{[n]}(-)} (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\hom_{\mathcal{C}\mathrm{at}_\infty}(-,\mathcal{C})} \mathcal{S}\right)\right)$$

in $\mathcal{S}$.

(2) The equivalences of part (1) assemble into a canonical identification of $\mathrm{N}_\infty(\mathcal{L}\mathrm{ax}(\mathcal{C}))_\bullet \in \mathcal{C}\mathcal{S}\mathcal{S} \subset s\mathcal{S}$ *with the composite*

$$\mathbf{\Delta}^{op} \hookrightarrow (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(-)^\simeq}\left(\mathrm{Gr}\left(\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(-)^\simeq \xrightarrow{\mathrm{U}^\dagger_{(-)}(=)} (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\hom_{\mathcal{C}\mathrm{at}_\infty}(=,\mathcal{C})} \mathcal{S}\right)\right)} \mathcal{S}.$$

Proof. Part (1) follows directly from Definition 3.2.3, and part (2) follows from the naturality of the Grothendieck construction. □

Notation 3.2.11. We denote by $\mathcal{L}\mathrm{ax}(\mathcal{C})^{\mathrm{colim}} \subset \mathcal{L}\mathrm{ax}(\mathcal{C})$ the full subcategory on those functors $\mathcal{D} \xrightarrow{F} \mathcal{C}$ which admit a colimit in $\mathcal{C}$.

We can now give the main result of this section, whose output we refer to as the ***global colimit functor*** for $\mathcal{C}$.

Proposition 3.2.12. *For any* $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, *there is a functor*

$$\mathcal{L}\mathrm{ax}(\mathcal{C})^{\mathrm{colim}} \xrightarrow{\mathrm{colim}} \mathcal{C}$$

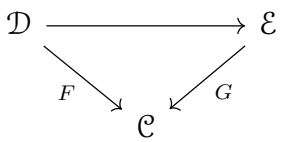
which takes an object $(\mathcal{D} \xrightarrow{F} \mathcal{C}) \in \mathcal{L}\mathrm{ax}(\mathcal{C})^{\mathrm{colim}}$ *to* $\mathrm{colim}_{\mathcal{D}}(F) \in \mathcal{C}$. *Moreover, this functor*

- *restricts to the usual colimit functor on each fiber*

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C})^{\mathrm{colim}} = \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \cap \mathcal{L}\mathrm{ax}(\mathcal{C})^{\mathrm{colim}},$$

and

- *takes a commutative diagram*



(considered as a cartesian morphism in $\mathcal{L}\mathrm{ax}(\mathcal{C})^{\mathrm{colim}}$*) to the canonical induced map*

$$\mathrm{colim}_{\mathcal{D}} F \to \mathrm{colim}_{\mathcal{E}} G$$

in $\mathcal{C}$*.*

Proof. In order to prove the claim, we make the following construction. Define $\mathcal{L}\mathrm{ax}(\mathcal{C})' \in \mathcal{C}\mathrm{at}_\infty$ to be the unique ∞-category such that $\mathrm{N}_\infty(\mathcal{L}\mathrm{ax}(\mathcal{C})')_\bullet \in \mathcal{CSS} \subset s\mathcal{S}$ is given by

$$\mathbf{\Delta}^{op} \hookrightarrow (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\mathrm{U}_{\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(-)^{\simeq}}\left(\mathrm{Gr}\left(\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(-)^{\simeq} \xrightarrow{\mathrm{U}^{\dagger}_{(-)}\left(=\underset{(-)}{\diamond}(-)\right)} (\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\hom_{\mathcal{C}\mathrm{at}_\infty}(=,\mathcal{C})} \mathcal{S}\right)\right)} \mathcal{S},$$

where the subscript decorating the second functor indicates that we are restricting to the subspace corresponding to those pairs of a functor $[n] \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ and a map

$$\mathrm{U}^{\dagger}_{[n]}\left(\mathrm{Gr}(F) \underset{[n]}{\diamond} [n]\right) \to \mathcal{C}$$

such that the induced composite diagram

$$\begin{array}{ccc} \mathrm{U}^{\dagger}_{[n]}(\mathrm{Gr}(F)) & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathrm{U}^{\dagger}_{[n]}\left(\mathrm{Gr}(F) \underset{[n]}{\diamond} [n]\right) & & \end{array}$$

defines a left Kan extension (along the inclusion of a full subcategory). The canonical inclusions

$$\mathrm{Gr}(F) \hookrightarrow \mathrm{Gr}(F) \underset{[n]}{\diamond} [n] \hookleftarrow [n]$$

induce maps

$$\mathrm{N}_\infty(\mathcal{L}\mathrm{ax}(\mathcal{C}))_\bullet \leftarrow \mathrm{N}_\infty(\mathcal{L}\mathrm{ax}(\mathcal{C})')_\bullet \to \mathrm{N}_\infty(\mathcal{C})_\bullet \times \mathrm{N}_\infty(\mathcal{C}\mathrm{at}_\infty)_\bullet$$

in $\mathcal{CSS} \subset s\mathcal{S}$, where the identification of $\mathrm{N}_\infty(\mathcal{L}\mathrm{ax}(\mathcal{C}))_\bullet$ comes from Lemma 3.2.10(2) and the identification of $\mathrm{N}_\infty(\mathcal{C})_\bullet \times \mathrm{N}_\infty(\mathcal{C}\mathrm{at}_\infty)_\bullet$ follows from Lemma 3.2.8 and Example 3.1.15. It then follows from Proposition T.4.3.2.15 that we have an induced

factorization

$$\begin{array}{ccc} \mathcal{L}\mathrm{ax}(\mathcal{C}) & \longleftarrow & \mathcal{L}\mathrm{ax}(\mathcal{C})' \\ \uparrow & \swarrow_{\sim} & \\ \mathcal{L}\mathrm{ax}(\mathcal{C})^{\mathrm{colim}} & & \end{array}$$

via an equivalence in $\mathcal{C}\mathrm{at}_\infty$. Moreover, combining this diagram with the composite $\mathcal{L}\mathrm{ax}(\mathcal{C})' \to \mathcal{C} \times \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}$ with the projection map gives us a unique functor

$$\begin{array}{ccccc} \mathcal{L}\mathrm{ax}(\mathcal{C}) & \longleftarrow & \mathcal{L}\mathrm{ax}(\mathcal{C})' & \longrightarrow & \mathcal{C} \\ \uparrow & \swarrow_{\sim} & & \nearrow & \\ \mathcal{L}\mathrm{ax}(\mathcal{C})^{\mathrm{colim}} & & & & \end{array}$$

making the diagram commute. By Proposition T.4.3.3.10, this is precisely the desired functor; moreover, it is clear from the construction that it encodes the asserted functorialities. □

Remark 3.2.13. Clearly, the global colimit functor (Proposition 3.2.12) is itself functorial in the following sense: if $\mathcal{C}_1 \to \mathcal{C}_2$ is a functor which commutes with all colimits existing in $\mathcal{C}_1$, then we obtain a commutative square

$$\begin{array}{ccc} \mathcal{L}\mathrm{ax}(\mathcal{C}_1)^{\mathrm{colim}} & \xrightarrow{\mathrm{colim}^{\mathcal{C}_1}} & \mathcal{C}_1 \\ \downarrow & & \downarrow \\ \mathcal{L}\mathrm{ax}(\mathcal{C}_2)^{\mathrm{colim}} & \xrightarrow[\mathrm{colim}^{\mathcal{C}_2}]{} & \mathcal{C}_2 \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$.

Remark 3.2.14. One could also construct the global colimit functor (i.e. prove Proposition 3.2.12) in the following way. First of all, the Grothendieck construction of the source projection

$$\mathcal{L}\mathrm{ax}(\mathcal{C}) \xrightarrow{s_{\mathcal{L}\mathrm{ax}(\mathcal{C})}} \mathcal{C}\mathrm{at}_\infty$$

produces a "tautological bundle" over $\mathcal{L}\mathrm{ax}(\mathcal{C})$, a cocartesian fibration whose fiber over an object $(\mathcal{D} \xrightarrow{F} \mathcal{C}) \in \mathcal{L}\mathrm{ax}(\mathcal{C})$ is $\mathcal{D}$ itself. This can moreover be shown to admit a tautological map to $\mathcal{C}$ which, restricted to such a fiber, is precisely the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$. The global colimit functor can then be produced by appealing to Proposition T.4.2.2.7. However, this method is in fact quite a bit more involved than the route we have taken here.

3.3 The Grothendieck construction and colimits of spaces

In this section, we study the relationship between the Grothendieck construction and colimits in the ∞-category $\mathcal{S}$ of spaces: in §3.3.1 we give some basic results, in §3.3.2 we extend these to the "two-sided" Grothendieck construction, and in §3.3.3 we collect some results which are suggestive of the $(\infty, 2)$-categorical functoriality of the Grothendieck construction. We state these results in terms of the *Thomason model structure* on $\mathcal{C}\mathrm{at}_\infty$ of Definition 3.6.2: a weak equivalence in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ is a functor that induces an equivalence on groupoid completions.

3.3.1 The Grothendieck construction and colimits

We begin with the following fundamental fact, on which all further results in this direction are based. Its 1-categorical version, Thomason's *homotopy colimit theorem*, first appeared as [Tho79, Theorem 1.2].

Proposition 3.3.1. *For any $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$, the Grothendieck construction computes its homotopy colimit when considered as diagram in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$, i.e.*

$$\mathrm{Gr}(F)^{\mathrm{gpd}} \simeq \mathrm{colim}\left(\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{\mathrm{gpd}}} \mathcal{S}\right).$$

Proof. This follows from Corollary T.3.3.4.3 and the fact that groupoid completion (being a left adjoint) commutes with colimits. □

Corollary 3.3.2. *Suppose that we are given a pair of functors $F, G : \mathcal{C} \rightrightarrows (\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ and a natural weak equivalence $F \xrightarrow{\simeq} G$. Then the induced map $\mathrm{Gr}(F) \to \mathrm{Gr}(G)$ is a weak equivalence in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$.*

Proof. Combining the equivalence

$$(-)^{\mathrm{gpd}} \circ F \xrightarrow{\sim} (-)^{\mathrm{gpd}} \circ G$$

in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ with two applications of the equivalence of Proposition 3.3.1, we obtain a composite equivalence

$$\mathrm{Gr}(F_1)^{\mathrm{gpd}} \simeq \mathrm{colim}\left(\mathcal{C} \xrightarrow{F_1} \mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{\mathrm{gpd}}} \mathcal{S}\right) \xrightarrow{\sim} \mathrm{colim}\left(\mathcal{C} \xrightarrow{F_2} \mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{\mathrm{gpd}}} \mathcal{S}\right) \simeq \mathrm{Gr}(F_2)^{\mathrm{gpd}}$$

in $\mathcal{S}$. □

3.3.2 The two-sided Grothendieck construction and colimits

We will also be interested in the following variant of the Grothendieck construction and its interaction with colimits in $\mathcal{S}$.

Definition 3.3.3. Given any (ordered) pair of functors $\mathcal{C}^{op} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ and $\mathcal{C} \xrightarrow{G} \mathcal{C}\mathrm{at}_\infty$, we define the ***two-sided Grothendieck construction*** of F and G to be the pullback

$$\mathrm{Gr}(F, \mathcal{C}, G) = \lim \left(\begin{array}{ccc} & & \mathrm{Gr}^-(F) \\ & & \downarrow \\ \mathrm{Gr}(G) & \longrightarrow & \mathcal{C} \end{array} \right)$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proposition 3.3.4. *Suppose that we are given*

- *a pair of functors $F, F' : \mathcal{C}^{op} \rightrightarrows (\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ and a natural weak equivalence $F \xrightarrow{\approx} F'$, and*
- *a pair of functors $G, G' : \mathcal{C} \rightrightarrows (\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ and a natural weak equivalence $G \xrightarrow{\approx} G'$.*

Then, the induced map

$$\mathrm{Gr}(F, \mathcal{C}, G) \to \mathrm{Gr}(F', \mathcal{C}, G')$$

is a weak equivalence in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$.

Proof. The naturality of the Grothendieck construction induces an evident naturality of the two-sided Grothendieck construction; in light of this, by duality and since $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$ is closed under composition, it suffices to prove the claim in the special case that $G = G'$ and that the natural weak equivalence $G \xrightarrow{\approx} G'$ is simply id_G. Then, by the naturality of the Grothendieck construction we have an equivalence

$$\mathrm{Gr}(F, \mathcal{C}, G) = \lim \left(\begin{array}{ccc} & & \mathrm{Gr}^-(F) \\ & & \downarrow \\ \mathrm{Gr}(G) & \longrightarrow & \mathcal{C} \end{array} \right) \simeq \mathrm{Gr}^-(F \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)})$$

in $\mathcal{C}\mathcal{F}\mathrm{ib}(\mathrm{Gr}(G))$, and similarly we have an equivalence

$$\mathrm{Gr}(F', \mathcal{C}, G) \simeq \mathrm{Gr}^-(F' \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)})$$

in $\mathcal{C}\mathcal{F}\mathrm{ib}(\mathrm{Gr}(G))$. Moreover, by assumption, the map

$$F \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)} \to F' \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)}$$

in $\mathrm{Fun}(\mathrm{Gr}(G)^{op}, \mathcal{C}\mathrm{at}_\infty)$ has that for every $y \in \mathrm{Gr}(G)^{op}$, the induced map

$$(F \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)})(y)^{\mathrm{gpd}} \to (F' \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)})(y)^{\mathrm{gpd}}$$

is an equivalence in $\mathcal{S}$: in other words, the map

$$(-)^{\mathrm{gpd}} \circ F \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)} \to (-)^{\mathrm{gpd}} \circ F' \circ \mathrm{pr}^{op}_{\mathrm{Gr}(G)}$$

is an equivalence in $\mathrm{Fun}(\mathrm{Gr}(G)^{op}, \mathcal{S})$. The claim now follows from (the dual of) Corollary 3.3.2. □

3.3.3 Higher-categorical functoriality of the Grothendieck construction and colimits

We now assemble a few results which are suggestive of the $(\infty, 2)$-categorical functoriality of the Grothendieck construction. However, we do not pursue such functoriality in any systematic way.

Proposition 3.3.5. *A diagram*

$$\mathcal{C} \underset{F}{\overset{E}{\rightrightarrows}} \mathcal{D} \xrightarrow{G} \mathcal{C}\mathrm{at}_\infty \qquad (\alpha \Downarrow)$$

induces a commutative triangle

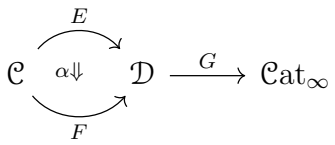
$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{C}}((-)^{\mathrm{gpd}} \circ G \circ E) & & \\ \downarrow & \searrow & \\ & & \mathrm{colim}_{\mathcal{D}}((-)^{\mathrm{gpd}} \circ G) \\ \mathrm{colim}_{\mathcal{C}}((-)^{\mathrm{gpd}} \circ G \circ F) & \nearrow & \end{array}$$

in $\mathcal{S}$.

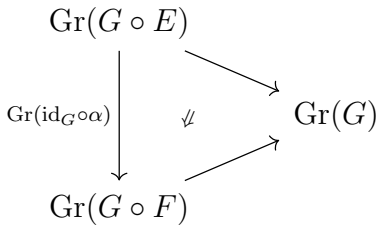
Proof. This follows by combining Proposition 3.3.1, Lemma 3.3.6, and Lemma 2.1.26. □

The following result is an ingredient of the proof of Proposition 3.3.5.

Lemma 3.3.6. *A diagram*

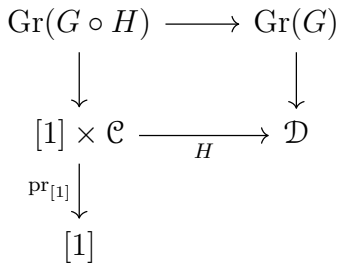


induces a morphism



in $\mathcal{L}\mathrm{ax}(\mathrm{Gr}(G))$.

Proof. We begin by encoding α as a functor $[1] \times \mathcal{C} \xrightarrow{H} \mathcal{D}$. By the naturality of the Grothendieck construction, this gives rise to a diagram



in which both left vertical arrows are cocartesian fibrations (the upper by pullback, the lower the Grothendieck construction of the functor $[1] \xrightarrow{\mathrm{const}(\mathcal{C})} \mathcal{C}\mathrm{at}_\infty$). Hence, the composite left vertical arrow is a cocartesian fibration as well, namely the one classified by the map $[1] \to \mathcal{C}\mathrm{at}_\infty$ selecting the functor

$$\mathrm{Gr}(G \circ E) \xrightarrow{\mathrm{Gr}(\mathrm{id}_G \circ \alpha)} \mathrm{Gr}(G \circ F).$$

Now, in order to extract this functor as a map *between* cocartesian fibrations (as opposed to being contained *within* a cocartesian fibration), we obtain a morphism in

$\mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$ with this map as its target by precomposing with the map $[1] \times [1] \to [1]$ given by

$$(i,j) \mapsto \begin{cases} 0, & (i,j) \neq (1,1) \\ 1, & (i,j) = (1,1); \end{cases}$$

by adjunction, this yields a functor $[1] \to \mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$ which, considered as a map in $\mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$, has source $[1] \xrightarrow{\mathrm{const}(\mathrm{Gr}(G \circ E))} \mathcal{C}\mathrm{at}_\infty$ and has target selecting this same functor $\mathrm{Gr}(G \circ E) \xrightarrow{\mathrm{Gr}(\mathrm{id}_G \circ \alpha)} \mathrm{Gr}(G \circ F)$. From here, the equivalence $\mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1])$ gives rise to the diagram of Figure 3.1, in which for clarity we include both fiber inclusions of each of these objects of $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1])$ as well as the induced maps between them. Moreover, there is a canonical natural transformation in $\mathrm{Fun}(\mathrm{Gr}(G \circ$

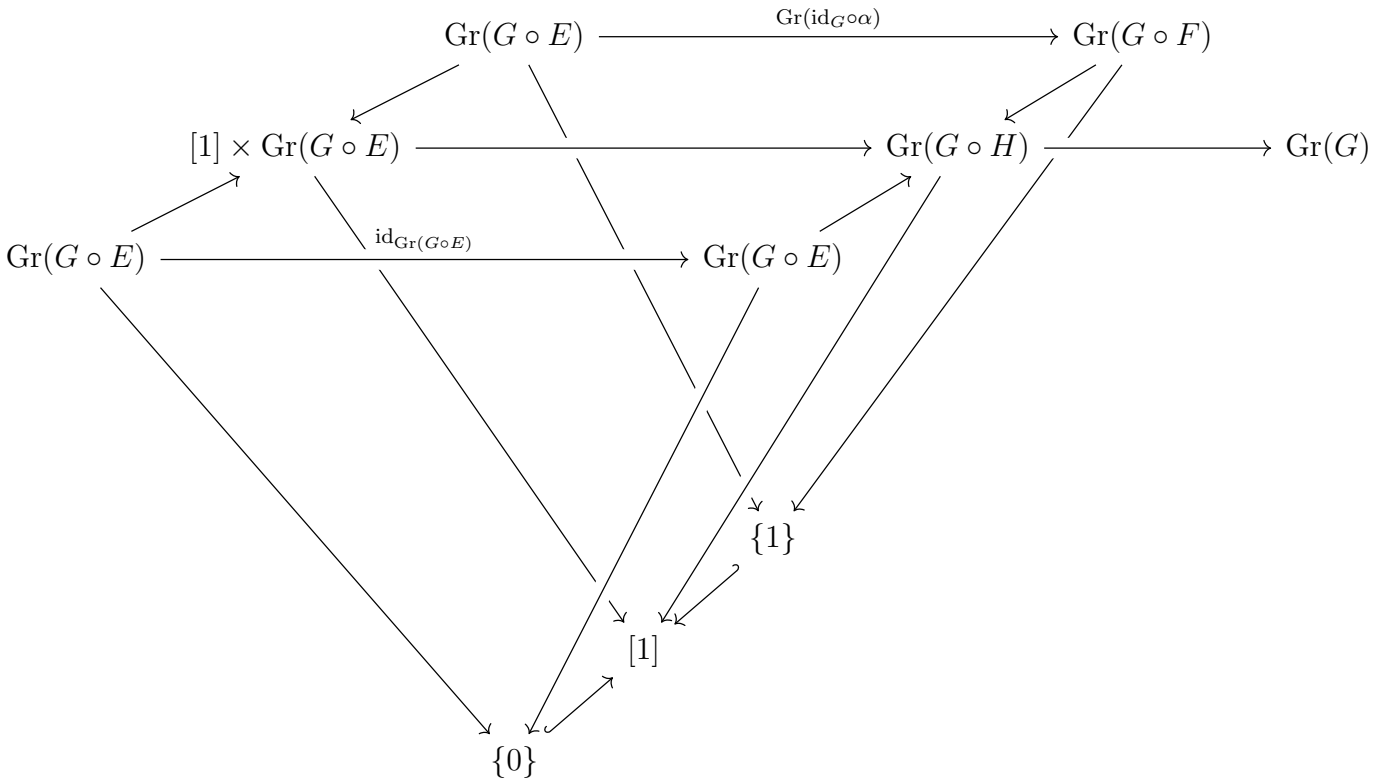


Figure 3.1: The diagram in $\mathcal{C}\mathrm{at}_\infty$ used in the proof of Lemma 3.3.6.

$E), [1] \times \mathrm{Gr}(G \circ E))$ from the inclusion of the fiber over $0 \in [1]$ to the inclusion of the

fiber over $1 \in [1]$ (selected by the identity map $\mathrm{id}_{[1] \times \mathrm{Gr}(G \circ E)}$). Taking the horizontal composite

$$\mathrm{Gr}(G \circ E) \overset{\{0\} \times \mathrm{id}_{\mathrm{Gr}(G \circ E)}}{\underset{\{1\} \times \mathrm{id}_{\mathrm{Gr}(G \circ E)}}{\Downarrow}} [1] \times \mathrm{Gr}(G \circ E) \longrightarrow \mathrm{Gr}(G \circ H) \longrightarrow \mathrm{Gr}(G)$$

of this natural transformation with the horizontal composite in the diagram of Figure 3.1 then yields the desired morphism in $\mathcal{L}\mathrm{ax}(\mathrm{Gr}(G))$. □

In manipulating colimits, we will also make use of the following notion (which is actually only a special case of a more general $(\infty, 2)$-categorical phenomenon).

Definition 3.3.7. Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, let $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$, and let $\alpha, \beta \in \hom_{\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)}(F, G)$. A ***modification*** from α to β is a map

$$\mathrm{const}([1]) \times F \xrightarrow{\mu} G$$

in $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$ whose restriction along $\mathrm{const}(\{0\}) \to \mathrm{const}([1])$ recovers α and whose restriction along $\mathrm{const}(\{1\}) \to \mathrm{const}([1])$ recovers β.

The following result describes the effect of the Grothendieck construction on modifications.

Proposition 3.3.8. *Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, let $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$, and let $\alpha, \beta \in \hom_{\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)}(F, G)$. A modification μ from α to β induces a natural transformation*

$$\mathrm{Gr}(F) \overset{\mathrm{Gr}(\alpha)}{\underset{\mathrm{Gr}(\beta)}{\mathrm{Gr}(\mu)\Downarrow}} \mathrm{Gr}(G)$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proof. Applying the Grothendieck construction to the modification μ, we obtain a map

$$\mathrm{Gr}(\mathrm{const}([1]) \times F) \xrightarrow{\mathrm{Gr}(\mu)} \mathrm{Gr}(G)$$

in $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$. But this source can be identified as

$$\mathrm{Gr}(\mathrm{const}([1]) \times F) \simeq \mathrm{Gr}(\mathrm{const}([1])) \times_{\mathcal{C}} \mathrm{Gr}(F) \simeq ([1] \times \mathcal{C}) \times_{\mathcal{C}} \mathrm{Gr}(F) \simeq [1] \times \mathrm{Gr}(F),$$

where the first equivalence follows from the fact that $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$ is an equivalence (so commutes with products) and the fact that the forgetful functor $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \to (\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$ is a right adjoint and hence commutes with products. So, this becomes a map

$$[1] \times \mathrm{Gr}(F) \xrightarrow{\mathrm{Gr}(\mu)} \mathrm{Gr}(G),$$

as desired. □

3.4 Homotopy pullbacks in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$, finality, and Theorems A, B_n, and C_n

Via the Thomason model structure on $\mathcal{C}\mathrm{at}_\infty$ of §3.6, we can consider ∞-categories as "presentations of spaces"; the corresponding localization functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty[\mathbf{W}_{\mathrm{Th}}^{-1}] \simeq \mathcal{S}$ is that of groupoid completion. Being a left adjoint, this functor commutes with colimits, but in general its interplay with limits is much more complicated. In this section, we describe certain sufficient conditions under which it commutes with a given pullback.

In the 1-categorical case, there is a long history of results of this variety, going back to Quillen's celebrated [Qui73, Theorem B]. The current state of the art seems to be Barwick–Kan's pair of results [BK, Theorems B_n (5.6) and C_n (5.8)] (the former generalizing Dwyer–Kan–Smith's [DKS89, Theorem B_n (6.2)], the latter identical to their [DKS89, Theorem C_n (6.4)]), as described in §3.0.1.

The main goal of this section is to give ∞-categorical generalizations of these results (which we prove in an essentially analogous fashion); these appear in §3.4.3. In §3.4.1 we work towards this goal with a pair of foundational results surrounding homotopy pullbacks in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ (whose 1-categorical analogs constitute the main input to the proof of [Qui73, Theorem B]), and in §3.4.2 we take a moment to briefly restate Joyal's quasicategorical analog (namely Theorem T.4.1.3.1) of Quillen's [Qui73, Theorem A] in invariant language (i.e. stated in $\mathcal{C}\mathrm{at}_\infty$ instead of in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$).

3.4.1 Homotopy pullbacks in $(\mathcal{C}\mathrm{at}_\infty)_{\mathbf{Th}}$: a first pass

In and of themselves, pullbacks among ∞-categories are relatively understandable; for instance, limits commute with the right adjoint of the composite adjunction

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\rightleftarrows}} \mathcal{CSS} \underset{\mathrm{N}_\infty}{\overset{\mathrm{N}_\infty^{-1}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty.$$

On the other hand, it is a subtle question to determine when such a pullback commutes with groupoid completion (or, working in complete Segal spaces, with geometric realization (recall Proposition 2.2.4)). In this subsection, we address this question in the special case that one of the maps in the pullback is a special sort of cocartesian fibration.

We begin with the relevant definition, an analog of [Qui73, 4.4].

Definition 3.4.1. We say that a functor $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ has ***property Q*** if it factors through $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$.

The importance of Definition 3.4.1 stems from the following result, which we call ***Lemma Q***, an analog of [BK, Lemma 4.5] (there called "Quillen's lemma").

Lemma 3.4.2. *If $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ factors through $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$, then for any $x \in \mathcal{C}$, the fiber inclusion*

$$\begin{array}{ccc} F(x) & \longrightarrow & \mathrm{Gr}(F) \\ \downarrow & & \downarrow \\ \{x\} & \hookrightarrow & \mathcal{C} \end{array}$$

is a homotopy pullback square in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$, i.e. it gives rise to a pullback square

$$\begin{array}{ccc} F(x)^{\mathrm{gpd}} & \longrightarrow & \mathrm{Gr}(F)^{\mathrm{gpd}} \\ \downarrow & & \downarrow \\ \{x\}^{\mathrm{gpd}} & \hookrightarrow & \mathcal{C}^{\mathrm{gpd}} \end{array}$$

in $\mathcal{S}$.

Proof. Let $\mathtt{C} \in s\mathcal{S}\mathrm{et}^f_{\mathrm{Joyal}}$ be a quasicategory presenting $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, let $\mathtt{D} \to \mathtt{C}$ be a JL-cocartesian fibration presenting $(\mathrm{Gr}(F) \to \mathcal{C}) \in \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$, let $\mathtt{D}' \to \mathtt{C}$ be a JL-left fibration presenting $(\mathrm{L}_{\mathcal{LF}\mathrm{ib}(\mathcal{C})}(\mathrm{Gr}(F)) \to \mathcal{C}) \in \mathcal{LF}\mathrm{ib}(\mathcal{C})$, and let $\mathtt{x} \in \mathtt{C}_0$ be a vertex

corresponding to $x \in \mathcal{C}$ (see [MG, Definition 3.2]). Let us define $\mathtt{F}, \mathtt{F}' \in s\mathcal{S}\mathrm{et}$ via the pullback squares

$$\begin{array}{ccc} \mathtt{F} & \longrightarrow & \mathtt{D} \\ \downarrow & & \downarrow \\ \mathrm{pt}_{s\mathcal{S}\mathrm{et}} & \xrightarrow[\mathtt{x}]{} & \mathtt{C} \end{array}$$

and

$$\begin{array}{ccc} \mathtt{F}' & \longrightarrow & \mathtt{D}' \\ \downarrow & & \downarrow \\ \mathrm{pt}_{s\mathcal{S}\mathrm{et}} & \xrightarrow[\mathtt{x}]{} & \mathtt{C} \end{array}$$

in $s\mathcal{S}\mathrm{et}$. Considered in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, these present fiber inclusions, the first of which is

$$\begin{array}{ccc} F(x) & \longrightarrow & \mathrm{Gr}(F) \\ \downarrow & & \downarrow \\ \{x\} & \hookrightarrow & \mathcal{C} \end{array}$$

and the second of which, by Proposition 3.1.18, we can identify as

$$\begin{array}{ccc} F(x)^{\mathrm{gpd}} & \longrightarrow & \mathrm{L}_{\mathcal{LF}\mathrm{ib}(\mathcal{C})}(\mathrm{Gr}(F)) \\ \downarrow & & \downarrow \\ \{x\} & \hookrightarrow & \mathcal{C}. \end{array}$$

Moreover, by Proposition T.2.1.3.1, the assertion that

$$\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{\mathrm{gpd}}} \mathcal{S}$$

factors through $\mathcal{S}^{\simeq} \subset \mathcal{S}$ is equivalent to the assertion that the map $\mathtt{D}' \to \mathtt{C}$ is in fact in $\mathbf{F}_{\mathrm{KQ}}$. Since $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is right proper (see [Hir03, Theorem 13.1.13]), it follows that the pullback square

$$\begin{array}{ccc} \mathtt{F}' & \longrightarrow & \mathtt{D}' \\ \downarrow & & \downarrow\!\!\downarrow \\ \mathrm{pt}_{s\mathcal{S}\mathrm{et}} & \xrightarrow[\mathtt{x}]{} & \mathtt{C} \end{array}$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ is also a homotopy pullback square, which implies that

$$\begin{array}{ccc} F(x)^{\mathrm{gpd}} & \longrightarrow & \mathrm{L}_{\mathcal{F}\mathrm{ib}}(\mathrm{Gr}(F))^{\mathrm{gpd}} \\ \downarrow & & \downarrow \\ \{x\}^{\mathrm{gpd}} & \hookrightarrow & \mathcal{C}^{\mathrm{gpd}} \end{array}$$

is a pullback square in $\mathcal{S}$. By Remark 3.1.4 we obtain an equivalence

$$\mathrm{Gr}(F)^{\mathrm{gpd}} \xrightarrow{\sim} (\mathrm{L}_{\mathcal{LF}\mathrm{ib}(\mathcal{C})}(\mathrm{Gr}(F)))^{\mathrm{gpd}}$$

in $\mathcal{S}_{/\mathcal{C}^{\mathrm{gpd}}}$, which completes the proof. □

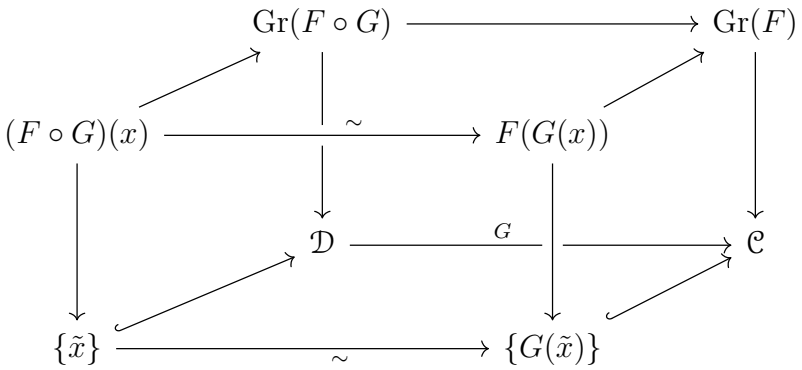
We also have the following parametrized version of Lemma Q (3.4.2).

Corollary 3.4.3. *If $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ has property Q, then for any $\mathcal{D} \xrightarrow{G} \mathcal{C}$, the resulting pullback $\mathcal{D} \times_{\mathcal{C}} \mathrm{Gr}(F)$ is a homotopy pullback in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$.*

Proof. By the naturality of the Grothendieck construction, we have a commutative square

$$\begin{array}{ccc} \mathrm{Gr}(F \circ G) & \longrightarrow & \mathrm{Gr}(F) \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow[G]{} & \mathcal{C} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$, where both of the vertical maps are cocartesian fibrations. We would like to show that this becomes a pullback square upon application of $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$. For this, note that any map $\mathrm{pt}_{\mathcal{S}} \xrightarrow{x} \mathcal{D}^{\mathrm{gpd}}$ in $\mathcal{S}$ comes from a map $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{\tilde{x}} \mathcal{D}$ in $\mathcal{C}\mathrm{at}_\infty$. Then, we have the diagram



in $\mathcal{C}\mathrm{at}_\infty$, whose back face is the above commutative square and in which the upper oblique arrows are the fiber inclusions over $\tilde{x} \in \mathcal{D}$ and $G(\tilde{x}) \in \mathcal{C}$. By Lemma Q (3.4.2), we obtain that applying $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ to this commutative diagram yields a commutative diagram in $\mathcal{S}$ in which the oblique squares are pullbacks. Hence, the square

$$\begin{array}{ccc} \mathrm{Gr}(F \circ G)^{\mathrm{gpd}} & \longrightarrow & \mathrm{Gr}(F)^{\mathrm{gpd}} \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{gpd}} & \xrightarrow[G^{\mathrm{gpd}}]{} & \mathcal{C}^{\mathrm{gpd}} \end{array}$$

is a pullback in $\mathcal{S}$, since for every point of $\mathcal{D}^{\mathrm{gpd}}$ the induced map on corresponding fibers is an equivalence. □

3.4.2 Finality and Theorem A

In this subsection, we briefly recall a few definitions and results from §T.4.1, restating them in invariant language.

Definition 3.4.4. A functor $\mathcal{I} \xrightarrow{F} \mathcal{J}$ is called ***final*** if, for any functor $\mathcal{J} \xrightarrow{G} \mathcal{C}$ such that $\mathrm{colim}_{\mathcal{J}} G$ exists, the colimit $\mathrm{colim}_{\mathcal{I}}(G \circ F)$ also exists and the natural map

$$\mathrm{colim}_{\mathcal{I}}(G \circ F) \to \mathrm{colim}_{\mathcal{J}} G$$

(in the sense of the global colimit functor (Proposition 3.2.12)) is an equivalence in $\mathcal{C}$.

Dually, a functor $\mathcal{I} \xrightarrow{F} \mathcal{J}$ is called ***initial*** if, for any functor $\mathcal{J} \xrightarrow{G} \mathcal{C}$ such that $\lim_{\mathcal{J}} G$ exists, the limit $\lim_{\mathcal{I}}(G \circ F)$ also exists and the natural map

$$\lim_{\mathcal{J}}(G) \to \lim_{\mathcal{I}}(G \circ F)$$

(in the sense of the global limit functor (the dual of Proposition 3.2.12)) is an equivalence in $\mathcal{C}$.

Remark 3.4.5. A functor $\mathcal{I} \to \mathcal{J}$ is initial if and only if its opposite $\mathcal{I}^{op} \to \mathcal{J}^{op}$ is final.

Remark 3.4.6. The notion of finality given in Definition 3.4.4 is also sometimes called "cofinality" or "right cofinality", while that initiality is also sometimes called "cocofinality" or "left cofinality". We have chosen our terminology because it seems most natural: the simplest example of a final functor is the inclusion $\{\mathrm{pt}_{\mathcal{I}}\} \hookrightarrow \mathcal{I}$ of a final object, while the simplest example of an initial functor is the inclusion $\{\varnothing_{\mathcal{I}}\} \hookrightarrow \mathcal{I}$ of an initial object.

Remark 3.4.7. By Proposition T.4.1.1.8 (see also Corollary T.4.1.1.10), the notion of finality given in Definition 3.4.4 is an invariant version of the quasicategorical notion of "cofinality" (given e.g. as Definition T.4.1.1.1).

Proposition 3.4.8. *Every final functor lies in* $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_{\infty}$.

Proof. This follows from Proposition T.4.1.1.3(3). □

Proposition 3.4.9. *If* $\mathcal{I}_1 \xrightarrow{F_1} \mathcal{J}_1$ *and* $\mathcal{I}_2 \xrightarrow{F_2} \mathcal{J}_2$ *are both final, then so is* $\mathcal{I}_1 \times \mathcal{I}_2 \xrightarrow{F_1 \times F_2} \mathcal{J}_1 \times \mathcal{J}_2$.

Proof. By Corollary T.4.1.1.13, both of the functors in the composite

$$\mathcal{I}_1 \times \mathcal{I}_2 \xrightarrow{F_1 \times \mathrm{id}_{\mathcal{I}_2}} \mathcal{J}_1 \times \mathcal{I}_2 \xrightarrow{\mathrm{id}_{\mathcal{J}_1} \times F_2} \mathcal{J}_1 \times \mathcal{J}_2$$

are final, and hence by Proposition T.4.1.1.3(2) their composite $F_1 \times F_2$ is also final.[6] □

The next result is the main point of this subsection, Joyal's ∞-categorical analog of [Qui73, Theorem A]; we refer to it simply as ***Theorem A***.

Theorem 3.4.10. *A functor $\mathcal{I} \xrightarrow{F} \mathcal{J}$ is final iff for every object $j \in \mathcal{J}$,*

$$(\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/})^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}.$$

Proof. This follows from Theorem T.4.1.3.1; by the Reedy trick (and the implications of Remark T.2.0.0.5 and Corollary T.2.1.2.2), the pullback given there is a homotopy pullback in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. □

Corollary 3.4.11. *If $\mathcal{J} \in \mathcal{C}\mathrm{at}_\infty$ has a terminal object, then $\mathcal{J}^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$.*

Proof. We apply Theorem 3.4.10 to deduce that the functor $\{\mathrm{pt}_{\mathcal{J}}\} \hookrightarrow \mathcal{J}$ is final: for any $j \in \mathcal{J}$, we have that $\{\mathrm{pt}_{\mathcal{J}}\} \times_{\mathcal{J}} \mathcal{J}_{j/} \simeq \hom_{\mathcal{J}}(j, \mathrm{pt}_{\mathcal{J}}) \simeq \mathrm{pt}_{\mathcal{S}}$. Hence, the claim follows from Proposition 3.4.8. □

Remark 3.4.12. One could also prove Corollary 3.4.11 by observing that a functor $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \to \mathcal{J}$ is a right adjoint if (and only if) it selects a terminal object, and then appealing to Corollary 2.1.28.

3.4.3 Theorems B_n and C_n: a second pass at homotopy pullbacks in $(\mathcal{C}\mathrm{at}_\infty)_{\mathbf{Th}}$

In this final subsection, we provide ∞-categorical generalizations of Barwick–Kan's pair of results [BK, Theorems B_n (5.6) and C_n (5.8)].

Remark 3.4.13. The results of this subsection will be used in §6.9. In fact, there we will actually employ their dual formulations. Our choice of variances in this subsection are so that our exposition adheres as closely as possible to that of [BK, §5].

[6] This is of course closely related to Fubini's theorem for colimits, but to deduce it from that result would require slightly trickier manipulations depending on which of the various colimits actually exist.

We begin with the following.

Notation 3.4.14. For $n \geq 1$, we define the strict category $\underline{\mathbf{z}}_n \in \mathbb{c}\mathrm{at}$ by the pattern

$$\begin{aligned}
\underline{\mathbf{z}}_1 &= (s \to t)\,, \\
\underline{\mathbf{z}}_2 &= (s \leftarrow \bullet \to t)\,, \\
\underline{\mathbf{z}}_3 &= (s \to \bullet \leftarrow \bullet \to t)\,, \\
\underline{\mathbf{z}}_4 &= (s \leftarrow \bullet \to \bullet \leftarrow \bullet \to t)\,,
\end{aligned}$$

etc. (where we have named only the leftmost and rightmost objects of these categories).

We now give the following omnibus definition.

Definition 3.4.15. For any $n \geq 1$ and any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, we define $(\mathcal{C} \downarrow_n \mathcal{C}) = \mathrm{Fun}(\underline{\mathbf{z}}_n, \mathcal{C})$. Evaluation at the objects $s, t \in \underline{\mathbf{z}}_n$ induces maps

$$\mathcal{C} \xleftarrow{s} (\mathcal{C} \downarrow_n \mathcal{C}) \xrightarrow{t} \mathcal{C}.$$

More generally, for any functors $\mathcal{D} \xrightarrow{F} \mathcal{C}$ and $\mathcal{E} \xrightarrow{G} \mathcal{C}$ and any objects $d \in \mathcal{D}$ and $e \in \mathcal{E}$, we define new ∞-categories and maps between them via the induced diagram

$$\begin{array}{ccccccc}
(F(d) \downarrow_n G(e)) & \longrightarrow & (F(\mathcal{D}) \downarrow_n G(e)) & \longrightarrow & (\mathcal{C} \downarrow_n G(e)) & \longrightarrow & \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle e} \\
(F(d) \downarrow_n G(\mathcal{E})) & \longrightarrow & (F(\mathcal{D}) \downarrow_n G(\mathcal{E})) & \longrightarrow & (\mathcal{C} \downarrow_n G(\mathcal{E})) & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle G} \\
(F(d) \downarrow_n \mathcal{C}) & \longrightarrow & (F(\mathcal{D}) \downarrow_n \mathcal{C}) & \longrightarrow & (\mathcal{C} \downarrow_n \mathcal{C}) & \xrightarrow[t]{} & \mathcal{C} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle s} & & \\
\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[d]{} & \mathcal{D} & \xrightarrow[F]{} & \mathcal{C} & &
\end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ in which all squares are pullbacks. Thus, we may consider $(F(\mathcal{D}) \downarrow_n G(\mathcal{E}))$ as simultaneously generalizing all three constructions $(F(\mathcal{D}) \downarrow_n \mathcal{C})$, $(\mathcal{C} \downarrow_n G(\mathcal{E}))$ and $(\mathcal{C} \downarrow_n \mathcal{C})$, with the convention that if either F or G is simply $\mathrm{id}_{\mathcal{C}}$ then we omit it from the notation; we refer to any of these constructions (but especially to $(F(\mathcal{D}) \downarrow_n G(\mathcal{E}))$) as a ***potential homotopy pullback ∞-category***. Similarly, the construction $(F(d) \downarrow_n G(\mathcal{E}))$ generalizes the construction $(F(d) \downarrow_n \mathcal{C})$ (in the case that $G = \mathrm{id}_{\mathcal{C}}$), while the construction $(F(\mathcal{D}) \downarrow_n G(e))$ generalizes the construction

$(\mathcal{C} \downarrow_n G(e))$ (in the case that $F = \mathrm{id}_{\mathcal{C}}$). Additionally, we denote all vertical maps landing at $\mathcal{D}$ (and in particular, all vertical maps in the third column landing at $\mathcal{C}$) by s and refer to these as ***source*** maps, and we denote all horizontal maps landing at $\mathcal{E}$ (and in particular, all horizontal maps in the third row landing at $\mathcal{C}$) by t and refer to these as ***target*** maps.

Remark 3.4.16. To make sense of the terminology, one should think of the construction $(F(\mathcal{D}) \downarrow_n G(\mathcal{E}))$ of Definition 3.4.15 as a sort of "directed" analog of the standard explicit construction of a *homotopy pullback* of topological spaces (as the space of pairs of points in the two sources of the cospan equipped with a path between their images in the common target). The question, of course, is whether this actually computes the homotopy pullback in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ (which explains the word "potential" in the name); a sufficient condition for this to be the case is precisely the content of Theorem B_n (3.4.23). Continuing along these lines, one might think of $(F(\mathcal{D}) \downarrow_n \mathcal{C})$ and $(\mathcal{C} \downarrow_n G(\mathcal{E}))$ as "directed" analogs of the *mapping path space* construction, i.e. the standard explicit factorization of an arbitrary map of topological spaces as a weak equivalence followed by a fibration. The reader may find these analogies helpful to keep in mind while reading the rest of this subsection.

In order to simultaneously deal with the case when n is even and when n is odd, we also introduce the following.

Notation 3.4.17. Inspired by the half-open intervals $[0, 1)$ and $(0, 1]$, we will enclose an expression by $[-)$ when we mean for it to be read only when n is even, while when we will enclose an expression by $(-]$ when we mean for it to be read only when n is odd. So for instance, $\mathcal{C}^{[op)}$ denotes $\mathcal{C}^{op}$ when n is even, and simply denotes $\mathcal{C}$ when n is odd.

Lemma 3.4.18. *Let $n \geq 1$, let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ and $\mathcal{E} \xrightarrow{G} \mathcal{C}$ be any functors.*

(1) We have

$$\Big((F(\mathcal{D}) \downarrow_n G(\mathcal{E})) \xrightarrow{s} \mathcal{D}\Big) \in [\mathrm{co})\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{D})$$

and

$$\Big((F(\mathcal{D}) \downarrow_n G(\mathcal{E})) \xrightarrow{t} \mathcal{E}\Big) \in \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{E}).$$

(2) For any objects $d \in \mathcal{D}$ and $e \in \mathcal{E}$, we have

$$\Big((F(\mathcal{D}) \downarrow_n G(e)) \xrightarrow{s} \mathcal{D}\Big) \in [\mathrm{co})\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{D})$$

and

$$\Big((F(d) \downarrow_n G(\mathcal{E})) \xrightarrow{t} \mathcal{E}\Big) \in \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{E}).$$

Proof. In both parts, we will only prove the second of the two claims; the first claims follow from nearly identical arguments. Moreover, since cocartesian fibrations are stable under pullback, it suffices to prove these statements in the case that $G = \mathrm{id}_{\mathcal{C}}$. For this, let us denote by $t' \in \underline{\mathbf{z}}_n$ the penultimate object (reading from left to right), and let us denote by $\underline{\mathbf{z}}'_n \subset \underline{\mathbf{z}}_n$ the full subcategory on all the objects besides $t \in \underline{\mathbf{z}}_n$, so that we can identify $\underline{\mathbf{z}}_n$ as a pushout

$$\underline{\mathbf{z}}_n \simeq \underline{\mathbf{z}}'_n \coprod_{t', \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}, 0} [1]$$

in $\mathcal{C}\mathrm{at}_\infty$. We therefore obtain a diagram

$$\begin{array}{ccccccccc}
(F(d) \downarrow_n \mathcal{C}) & \longrightarrow & (F(\mathcal{D}) \downarrow_n \mathcal{C}) & \longrightarrow & (\mathcal{C} \downarrow_n \mathcal{C}) & \longrightarrow & \mathrm{Fun}([1], \mathcal{C}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{C} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} & & \\
\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \underset{F(d),\mathcal{C},\mathrm{ev}_s}{\times} \mathrm{Fun}(\underline{\mathbf{z}}'_n, \mathcal{C}) & \longrightarrow & \mathcal{D} \underset{F,\mathcal{C},\mathrm{ev}_s}{\times} \mathrm{Fun}(\underline{\mathbf{z}}'_n, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\underline{\mathbf{z}}'_n, \mathcal{C}) & \xrightarrow[\mathrm{ev}_{t'}]{} & \mathcal{C} & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_s} & & & & \\
\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[d]{} & \mathcal{D} & \xrightarrow[F]{} & \mathcal{C} & & & &
\end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ in which all squares are pullbacks and the upper row contains as composites the two functors $(F(\mathcal{D}) \downarrow_n \mathcal{C}) \xrightarrow{t} \mathcal{C}$ and $(F(d) \downarrow_n \mathcal{C}) \xrightarrow{t} \mathcal{C}$ which we would like to show are cocartesian fibrations. Both claims now follow by applying the dual of Corollary T.2.4.7.12 to the corresponding composites

$$\mathcal{D} \underset{F,\mathcal{C},\mathrm{ev}_s}{\times} \mathrm{Fun}(\underline{\mathbf{z}}'_n, \mathcal{C}) \to \mathcal{C}$$

and

$$\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \underset{F(d),\mathcal{C},\mathrm{ev}_s}{\times} \mathrm{Fun}(\underline{\mathbf{z}}'_n, \mathcal{C}) \to \mathcal{C}$$

in the middle row. □

Notation 3.4.19. We denote the functors classifying the co/cartesian fibrations of Lemma 3.4.18(1) by

$$\mathcal{D}^{(op]} \xrightarrow{(F(-)\downarrow_n G(\mathcal{E}))} \mathcal{C}\mathrm{at}_\infty$$

and

$$\mathcal{E} \xrightarrow{(F(\mathcal{D})\downarrow_n G(-))} \mathcal{C}\mathrm{at}_\infty,$$

and we denote the functors classifying the co/cartesian fibrations of Lemma 3.4.18(2) by

$$\mathcal{D}^{(op]} \xrightarrow{(F(-)\downarrow_n G(e))} \mathcal{C}\mathrm{at}_\infty$$

and

$$\mathcal{E} \xrightarrow{(F(d)\downarrow_n G(-))} \mathcal{C}\mathrm{at}_\infty$$

(again omitting the functor F or G if it is just $\mathrm{id}_{\mathcal{C}}$). Note that this is indeed consistent with the notation given in Definition 3.4.15, and identifies all of the various pullbacks over $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$ in the diagram given there as fiber inclusions. (In particular, the ∞-category $(F(d) \downarrow_n G(e))$ includes as a fiber of two different co/cartesian fibrations.)

Remark 3.4.20. We may consider Lemma 3.4.18(2) as equipping the first two functors defined in Notation 3.4.19 with lifts

$$\begin{array}{ccc} & & \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{E}) \\ & \nearrow & \downarrow \\ \mathcal{D}^{(op]} & \xrightarrow[(F(-)\downarrow_n G(\mathcal{E}))]{} & \mathcal{C}\mathrm{at}_\infty \end{array}$$

and

$$\begin{array}{ccc} & & [\mathrm{co})\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{D}) \\ & \nearrow & \downarrow \\ \mathcal{E} & \xrightarrow[(F(\mathcal{D})\downarrow_n G(-))]{} & \mathcal{C}\mathrm{at}_\infty \end{array}$$

(through the evident forgetful functors). (Preservation of co/cartesian morphisms follows from Corollary T.2.4.7.12.)

We then have the following elementary but important observations.

Proposition 3.4.21. *Let $n \geq 1$, and choose any functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ and $\mathcal{E} \xrightarrow{G} \mathcal{C}$.*

(1) The unique functor $\underline{\mathbf{z}}_n \to \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$ induces a common section

$$(\mathcal{C} \downarrow_n \mathcal{C}) \overset{q}{\underset{t}{\overset{\overset{s}{\longrightarrow}}{\longrightarrow}}} \mathcal{C}$$

to the source and target maps.

(2) The common section of part (1) induces sections

$$(F(\mathcal{D}) \downarrow_n \mathcal{C}) \underset{s}{\overset{q}{\rightleftarrows}} \mathcal{D}$$

and

$$(\mathcal{C} \downarrow_n G(\mathcal{E})) \underset{t}{\overset{q}{\rightleftarrows}} \mathcal{E}.$$

(3) The section diagrams of part (2) (and in particular, those of part (1)) define homotopy equivalences in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$.

Proof. Part (1) follows from the fact that both composites $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \rightrightarrows \underline{\mathbf{z}}_n \to \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$ are canonically equivalent to $\mathrm{id}_{\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}}$. Then, part (2) follows from part (1) and the definitions of $(F(\mathcal{D}) \downarrow_n \mathcal{C})$ and $(\mathcal{C} \downarrow_n G(\mathcal{E}))$ as pullbacks. For part (3), observe first that either composite $\underline{\mathbf{z}}_n \to \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \rightrightarrows \underline{\mathbf{z}}_n$ is connected by a zigzag of natural transformations to $\mathrm{id}_{\underline{\mathbf{z}}_n}$; moreover, working with $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{s} \underline{\mathbf{z}}_n$ this zigzag can be taken in $\mathcal{C}\mathrm{at}_*$ where we point our categories using their source objects (i.e. such that all the constituent natural transformations of the zigzag have the map id_s as their component at s), and similarly for working with $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{t} \underline{\mathbf{z}}_n$. By applying Lemma 4.3.5 (where we take $\mathcal{C}$ to be equipped with the maximal relative structure), we see that either composite $(\mathcal{C} \downarrow_n \mathcal{C}) \rightrightarrows \mathcal{C} \to (\mathcal{C} \downarrow_n \mathcal{C})$ is in turn connected to $\mathrm{id}_{(\mathcal{C}\downarrow_n\mathcal{C})}$ by a zigzag of natural transformations, such that all of the constituent natural transformations commute with the chosen projection $(\mathcal{C} \downarrow_n \mathcal{C}) \to \mathcal{C}$ (either the source or target map). Since the functor $\mathcal{C}\mathrm{at}_\infty \xrightarrow{-\times[1]} \mathcal{C}\mathrm{at}_\infty$ commutes with pullbacks (being a limit), by the functoriality of pullbacks these induce zigzags of natural transformations between the composite

$$(F(\mathcal{D}) \downarrow_n \mathcal{C}) \xrightarrow{s} \mathcal{D} \to (F(\mathcal{D}) \downarrow_n \mathcal{C})$$

and $\mathrm{id}_{(F(\mathcal{D})\downarrow_n\mathcal{C})}$ and between the composite

$$(\mathcal{C} \downarrow_n G(\mathcal{E})) \xrightarrow{t} \mathcal{E} \to (\mathcal{C} \downarrow_n G(\mathcal{E}))$$

and $\mathrm{id}_{(\mathcal{C}\downarrow_n G(\mathcal{E}))}$. Thus, the claim follows from Lemma 2.1.26. □

We now define the key concept of this subsection.

Definition 3.4.22. We say that a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has ***property $\boldsymbol{B}_n$*** if the functor

$$\mathcal{C} \xrightarrow{(F(\mathcal{D})\downarrow_n -)} \mathcal{C}\mathrm{at}_\infty$$

has property Q.

We now give the main result of this subsection, which we refer to as ***Theorem B_n*** (for homotopy pullbacks (in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$)).

Theorem 3.4.23. *If the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has property B_n, then for any functor $\mathcal{E} \xrightarrow{G} \mathcal{C}$, the (not generally commutative) square*

$$\begin{array}{ccc} (F(\mathcal{D}) \downarrow_n G(\mathcal{E})) & \xrightarrow{s} & \mathcal{D} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{E} & \xrightarrow[G]{} & \mathcal{C} \end{array}$$

is a homotopy pullback square in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$, i.e. it induces a (commutative) pullback square

$$\begin{array}{ccc} (F(\mathcal{D}) \downarrow_n G(\mathcal{E}))^{\mathrm{gpd}} & \xrightarrow{s^{\mathrm{gpd}}} & \mathcal{D}^{\mathrm{gpd}} \\ {\scriptstyle t^{\mathrm{gpd}}}\downarrow & & \downarrow{\scriptstyle F^{\mathrm{gpd}}} \\ \mathcal{E}^{\mathrm{gpd}} & \xrightarrow[G^{\mathrm{gpd}}]{} & \mathcal{C}^{\mathrm{gpd}} \end{array}$$

in $\mathcal{S}$.

Proof. To say that $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has property B_n is to say that the functor

$$\mathcal{C} \xrightarrow{(F(\mathcal{D})\downarrow_n -)} \mathcal{C}\mathrm{at}_\infty$$

has property Q. By the naturality of the Grothendieck construction, the functor

$$\mathcal{E} \xrightarrow{(F(\mathcal{D})\downarrow_n G(-))} \mathcal{C}\mathrm{at}_\infty$$

is precisely the composite

$$\mathcal{E} \xrightarrow{G} \mathcal{C} \xrightarrow{(F(\mathcal{D})\downarrow_n -)} \mathcal{C}\mathrm{at}_\infty,$$

and therefore also has property Q. Hence, by Lemma Q (3.4.2), for any objects $c \in \mathcal{C}$ and $e \in \mathcal{E}$ the fiber inclusions

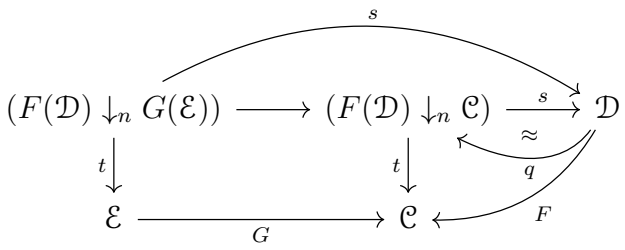
$$\begin{array}{ccc} (F(\mathcal{D}) \downarrow_n c) & \longrightarrow & (F(\mathcal{D}) \downarrow_n \mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle t} \\ \{c\} & \hookrightarrow & \mathcal{C} \end{array}$$

and

$$\begin{array}{ccc} (F(\mathcal{D}) \downarrow_n G(e)) & \longrightarrow & (F(\mathcal{D}) \downarrow_n G(\mathcal{E})) \\ \downarrow & & \downarrow{\scriptstyle t} \\ \{e\} & \longrightarrow & \mathcal{E} \end{array}$$

are both homotopy pullback squares in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$.

Now, observe that we have a diagram



in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$, in which

- the map labeled q comes from Proposition 3.4.21(2),
- every bounded connected region is commutative except for the one containing the symbol $\approx$, which is homotopy commutative (and is hence bounded by weak equivalences in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$) by Proposition 3.4.21(3), and
- the square is by definition a pullback square in $\mathcal{C}\mathrm{at}_\infty$.

Our goal, then, reduces to showing that the commutative square in this diagram is also a homotopy pullback square in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$. For this, it suffices to verify that in the induced commutative square

$$\begin{array}{ccc} (F(\mathcal{D}) \downarrow_n G(\mathcal{E}))^{\mathrm{gpd}} & \longrightarrow & (F(\mathcal{D}) \downarrow_n \mathcal{C})^{\mathrm{gpd}} \\ {\scriptstyle t^{\mathrm{gpd}}}\downarrow & & \downarrow{\scriptstyle t^{\mathrm{gpd}}} \\ \mathcal{E}^{\mathrm{gpd}} & \xrightarrow[G^{\mathrm{gpd}}]{} & \mathcal{C}^{\mathrm{gpd}} \end{array}$$

in $\mathcal{S}$, we obtain an equivalence on fibers over every point of $\mathcal{E}^{\mathrm{gpd}}$.

Now, observe first that any point $\mathrm{pt}_{\mathcal{S}} \to \mathcal{E}^{\mathrm{gpd}}$ is represented by a map $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{e} \mathcal{E}$ in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$. Moreover, we have just seen that in the resulting commutative diagram

$$\begin{array}{ccccc} (F(\mathcal{D}) \downarrow_n G(e)) & \longrightarrow & (F(\mathcal{D}) \downarrow_n G(\mathcal{E})) & \longrightarrow & (F(\mathcal{D}) \downarrow_n \mathcal{C}) \\ \downarrow & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ \{e\} & \hookrightarrow & \mathcal{E} & \xrightarrow[G]{} & \mathcal{C} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$, both the left square and the outer rectangle are fiber inclusions which are moreover homotopy pullback squares in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ (where we take $c = G(e) \in \mathcal{C}$). So we do indeed obtain an equivalence on fibers over every point of $\mathcal{E}^{\mathrm{gpd}}$ in the above commutative square in $\mathcal{S}$, which completes the proof. □

Remark 3.4.24. As [DKS89, Theorem B_n (6.2)] specializes to [Qui73, Theorem B] in the case that $n = 1$, our Theorem B_n (3.4.23) also generalizes [Barb, Theorem B for ∞-categories (5.3)].

The following definition allows us to formulate a useful sufficient condition for a functor to have property B_n.

Definition 3.4.25. We say that $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ has ***property C_n*** if every functor $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \to \mathcal{C}$ has property B_n.

The sufficient condition is then provided by the following main supporting result of this section, which we refer to as ***Theorem C_n***.

Theorem 3.4.26. *If $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ has property C_n, then any functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has property B_n.*

Proof. We must show that for any map $c_1 \xrightarrow{\varphi} c_2$ in $\mathcal{C}$, the resulting functor

$$(F(\mathcal{D}) \downarrow_n c_1) \to (F(\mathcal{D}) \downarrow_n c_2)$$

is in $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$. Recall from Remark 3.4.20 that this functor can be considered as the image of the map φ under a functor

$$\mathcal{C} \xrightarrow{(F(\mathcal{D})\downarrow_n -)} [\mathrm{co})\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{D}).$$

Via the Grothendieck construction

$$\mathrm{Fun}(\mathcal{D}^{(op]}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}} [\mathrm{co})\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{D})$$

this is classified by a natural transformation

$$(F(-) \downarrow_n c_1) \xrightarrow{\mathrm{Gr}^{-1}((F(\mathcal{D})\downarrow_n \varphi))} (F(-) \downarrow_n c_2)$$

in $\mathrm{Fun}(\mathcal{D}^{(op]}, \mathcal{C}\mathrm{at}_\infty)$, and by Corollary 3.3.2 it suffices to show that the components of this natural transformation lie in $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$. This is just the assertion that for any object $d \in \mathcal{D}$, the induced map

$$(F(d) \downarrow_n c_1) \to (F(d) \downarrow_n c_2)$$

is in $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$, which follows by applying the definition of property C_n to the functor $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{F(d)} \mathcal{C}$. □

3.5 The Bousfield–Kan colimit formula

In 1-category theory, there's an extremely useful formula which expresses an arbitrary colimit as a coequalizer of maps between coproducts (at least when the ambient category is cocomplete). Namely, if we are given $\mathcal{C}, \mathcal{D} \in \mathbb{C}\mathrm{at}$, $\mathcal{C}$ admits coproducts and coequalizers, and $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is any functor, then we have an isomorphism

$$\mathrm{colim}_{\mathcal{D}} F \cong \mathrm{coeq}\left(\coprod_{(d_1 \to d_2) \in \mathrm{N}(\mathcal{D})_1} F(d_1) \rightrightarrows \coprod_{d \in \mathrm{N}(\mathcal{D})_0} F(d) \right)$$

in $\mathcal{C}$, where on the summand corresponding to $(d_1 \xrightarrow{\varphi} d_2) \in \mathrm{N}(\mathcal{D})_1$, one map is given by

$$F(d_1) \xrightarrow{\mathrm{id}_{F(d_1)}} F(d_1)$$

(landing on the summand corresponding to $d_1 \in \mathrm{N}(\mathcal{D})_0$) and the other map is given by

$$F(d_1) \xrightarrow{F(\varphi)} F(d_2)$$

(landing on the summand corresponding to $d_2 \in \mathrm{N}(\mathcal{D})_0$). (In fact, it follows from this formula that $\mathcal{C}$ is cocomplete if (and only if) it admits coproducts and coequalizers.)

In this section, we generalize this colimit formula to ∞-categories. Two things must be changed. First of all, this coequalizer will be replaced by a geometric realization.[7] Moreover, the coproducts over the *sets* of objects and morphisms of $\mathcal{D}$ will be replaced by the colimits over the *spaces* of objects, morphisms, pairs of two composable morphisms, etc., of the diagram ∞-category – in other words, over the constituents of its ∞-categorical nerve. To be more precise, the *simplicial replacement* of a diagram $\mathcal{D} \xrightarrow{F} \mathcal{C}$ in $\mathcal{C}$ will be a simplicial object $\mathrm{srep}(F)_\bullet \in s\mathcal{C}$ which in level n is the colimit of the composite

$$\mathrm{N}_\infty(\mathcal{D})_n \xrightarrow{\{0\}} \mathrm{N}_\infty(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{F} \mathcal{C}$$

(which of course only need exist when $\mathcal{C}$ has a sufficient supply of colimits). Then, the geometric realization of the simplicial replacement will compute the colimit $\mathrm{colim}_{\mathcal{D}} F$ of the original diagram in $\mathcal{C}$.

[7] Recall that $\mathbf{\Delta}_{\leq 1}^{op}$ is the walking reflexive pair, so that colimits over it are precisely reflexive coequalizers. In fact, the above pair of parallel arrows in $\mathcal{C}$ is indeed a reflexive pair: a common section is given by taking the summand $F(d)$ over the element $d \in \mathrm{N}(\mathcal{D})_0$ to summand $F(d)$ over the element $\mathrm{id}_d \in \mathrm{N}(\mathcal{D})_1$ via the identity map $\mathrm{id}_{F(d)}$ in $\mathcal{C}$.

This section is organized as follows. In §3.5.1, we carefully construct the simplicial replacement and prove the *Bousfield–Kan colimit formula* (Theorem 3.5.8). In §3.5.2, we provide some examples to illustrate the usage of this formula. Finally, in §3.5.3, we provide a functoriality result.

Remark 3.5.1. Of course, one can dualize this entire section to obtain analogous constructions and results concerning limits in a complete ∞-category.

3.5.1 The Bousfield–Kan colimit formula

In this subsection, we construct the simplicial replacement and prove the main result of this section. We begin with the following observation.

Remark 3.5.2. The structure maps of the diagrams in $\mathcal{C}$ assembling to the simplicial replacement are not strictly compatible: we will only have a lax natural transformation $\mathrm{N}_\infty(\mathcal{D})_\bullet \rightsquigarrow \mathrm{const}(\mathcal{D})$ in $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C}\mathrm{at}_\infty)$, i.e. a map

$$\begin{array}{ccc} \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) & \dashrightarrow & \mathcal{D} \times \mathbf{\Delta}^{op} \\ & \searrow \quad \swarrow & \\ & \mathbf{\Delta}^{op} & \end{array}$$

making the triangle commute, which by precomposition will induce the passage to the simplicial replacement. For instance, a point $\varphi \in \mathrm{N}_\infty(\mathcal{D})_1$ corresponds to a morphism $x \xrightarrow{\varphi} y$ in $\mathcal{D}$, and the above map should send this to the point $x \in \mathcal{D} \times \{[1]^\circ\}$. On the other hand, the two simplicial structure maps take the point $\varphi \in \mathrm{N}_\infty(\mathcal{D})_1$ to the points $x, y \in \mathrm{N}_\infty(\mathcal{D})_0$, and the map $\mathrm{N}_\infty(\mathcal{D})_0 \to \mathcal{D} \times \{[0]^\circ\}$ is simply the inclusion of the maximal subgroupoid. So in the diagram

$$\begin{array}{ccc} \mathrm{N}_\infty(\mathcal{D})_1 & \xrightarrow{\{0\}} & \mathcal{D} \\ {\scriptstyle \delta_0}\downarrow & & \wr\downarrow{\scriptstyle \delta_0} \\ \mathrm{N}_\infty(\mathcal{D})_0 & \xrightarrow[\{0\}]{} & \mathcal{D}, \end{array}$$

going across and then down gives $\varphi \mapsto x \mapsto x$, while going down and then across gives $\varphi \mapsto y \mapsto y$. Hence, the diagram does not strictly commute. However, it will commute *up to a natural transformation* (running down and to the left), whose component at the point $\varphi \in \mathrm{N}_\infty(\mathcal{D})_1$ is simply the map φ itself.

Remark 3.5.3. In light of Remark 3.5.2, one sees that the "decomposition of colimits" results of §T.4.2.3 do not suffice for our purposes here: they only apply to *strict* diagrams of diagram ∞-categories lying over our diagram ∞-category $\mathcal{D}$.

We now construct the lax natural transformation of Remark 3.5.2. In fact, we will construct it in a way which is functorial in $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$.[8]

Construction 3.5.4. We construct the map $\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \to \mathcal{D} \times \mathbf{\Delta}^{op}$ in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathbf{\Delta}^{op}}$ as follows. By the universal property of products, it suffices to construct only the map $\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \to \mathcal{D}$ in $\mathcal{C}\mathrm{at}_\infty$. This we construct on nerves, i.e. as a map $\mathrm{N}_\infty(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet))_\bullet \to \mathrm{N}_\infty(\mathcal{D})_\bullet$ in $\mathcal{CSS} \subset s\mathcal{S}$. For this, we will unwind the definitions of these two constructions as functors of $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$.

Of course, the target is corepresented in level n by $[n] \in \mathbf{\Delta} \subset \mathcal{C}\mathrm{at}_\infty$.

On the other hand, every string of composable morphisms in $\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet)$ is uniquely determined by its source and its image in $\mathbf{\Delta}^{op}$ (since the functor $\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \to \mathbf{\Delta}^{op}$ is a left fibration and hence all maps are cocartesian), and so we obtain an equivalence

$$\mathrm{N}_\infty(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet))_n \simeq \coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \mathrm{N}_\infty(\mathcal{D})_{\alpha(0)},$$

where we consider $\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n \cong \hom_{\mathcal{C}\mathrm{at}}([n], \mathbf{\Delta}^{op})$. Hence, the composite functor

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}_\infty(-)_\bullet} s\mathcal{S} \simeq \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}) \xrightarrow{\mathrm{Gr}} \mathcal{LF}\mathrm{ib}(\mathbf{\Delta}^{op}) \xrightarrow{\mathrm{U}_{\mathcal{L}(\mathbf{\Delta}^{op})}} (\mathcal{C}\mathrm{at}_\infty)_{/\mathbf{\Delta}^{op}} \to \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}_\infty(-)_\bullet} s\mathcal{S},$$

considered by adjunction as a simplicial object in $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$, is given in each level by a coproduct of corepresentable objects. In level n, it is given by

$$\coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \text{ɬ}(\alpha(0)^\circ),$$

where we write $\text{ɬ} = \text{ɬ}_{(\mathcal{C}\mathrm{at}_\infty)^{op}}$ for the contravariant Yoneda functor

$$(\mathcal{C}\mathrm{at}_\infty)^{op} \xrightarrow{\hom_{\mathcal{C}\mathrm{at}_\infty}(-,=)} \mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$$

for brevity. To describe its simplicial structure maps, given a map $[n] \xrightarrow{\alpha} \mathbf{\Delta}^{op}$, for any $0 \leq i \leq j \leq n$ let us denote the corresponding map in $\mathbf{\Delta}^{op}$ selected by α by

$$\alpha(i) \xrightarrow{\alpha^\circ_{i,j}} \alpha(j)$$

(i.e. the image under α of the unique element of $\hom_{[n]}(i,j)$). Then, associated to a map $[n]^\circ \xrightarrow{\varphi^\circ} [m]^\circ$ in $\mathbf{\Delta}^{op}$, the structure map

$$\left(\coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \text{ɬ}(\alpha(0)^\circ) \right) \to \left(\coprod_{\beta \in \mathrm{N}(\mathbf{\Delta}^{op})_m} \text{ɬ}(\beta(0)^\circ) \right)$$

[8] This is essentially a homotopy-invariant analog of the "first vertex projection" from the opposite of the category of simplices of a quasicategory. The main difficulty lies in keeping careful track of all coherence data.

of this simplicial object in $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$ is given by taking the summand $\text{ɬ}(\alpha(0)^\circ)$ indexed by $\alpha \in \hom_{\mathbb{c}\mathrm{at}}([n], \mathbf{\Delta}^{op})$ to the summand $\text{ɬ}(\beta(0)^\circ)$ indexed by $\beta = \alpha \circ \varphi \in \hom_{\mathbb{c}\mathrm{at}}([m], \mathbf{\Delta}^{op})$ via the map

$$\text{ɬ}(\alpha(0)^\circ) \xrightarrow{\text{ɬ}(\alpha^\circ_{0,\varphi(0)})} \text{ɬ}(\alpha(\varphi(0))^\circ)$$

in $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$. Since the objects of $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$ corepresented by gaunt categories and their finite coproducts generate a full subcategory which is just a 1-category, no higher coherence issues arise. (We will implicitly appeal to this fact for our further manipulations as well.)

We can now describe our desired map $\mathrm{N}_\infty(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet))_\bullet \to \mathrm{N}_\infty(\mathcal{D})_\bullet$ in $\mathcal{CSS} \subset s\mathcal{S}$. In level n, it is obtained from the map

$$\left(\coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \text{ɬ}(\alpha(0)^\circ) \right) \to \text{ɬ}([n]^\circ)$$

which on the summand $\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n$ is the map

$$\text{ɬ}\left(\alpha(0)^\circ \xrightarrow{((i \mapsto \alpha_{0,i}(0))_{0 \le i \le n})^\circ} [n]^\circ \right)$$

in $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$. That is, in level n, on the summand α it is corepresented by the map $[n] \to \alpha(0)^\circ$ in $\mathbf{\Delta} \subset \mathcal{C}\mathrm{at}_\infty$ given by $i \mapsto \alpha_{0,i}(0)$, i.e. the map taking $i \in [n]$ to the image in $\alpha(0)^\circ$ of the object $0 \in \alpha(i)^\circ$ under the composite

$$\alpha(0)^\circ \xleftarrow{\alpha_{0,1}} \cdots \xleftarrow{\alpha_{i-1,i}} \alpha(i)^\circ$$

in $\mathbf{\Delta}$.

We have now associated to the object $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ a map $\mathrm{N}_\infty(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet)) \to \mathrm{N}_\infty(\mathcal{D})$ in $\mathcal{CSS}$. In fact, this map clearly commutes with the induced projections to $\mathrm{N}_\infty(\mathbf{\Delta}^{op})$, and moreover, this association is clearly functorial in $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ since it is entirely corepresented. Hence, we obtain a functor

$$\mathcal{C}\mathrm{at}_\infty \to \mathrm{Fun}([1], \mathcal{CSS}_{/\mathrm{N}_\infty(\mathbf{\Delta}^{op})}),$$

which immediately (and equivalently) produces our desired functor

$$\mathcal{C}\mathrm{at}_\infty \to \mathrm{Fun}([1], (\mathcal{C}\mathrm{at}_\infty)_{/\mathbf{\Delta}^{op}})$$

which takes the object $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ to a commutative triangle as depicted in Remark 3.5.2 (considered as an object of $\mathrm{Fun}([1], (\mathcal{C}\mathrm{at}_\infty)_{/\mathbf{\Delta}^{op}})$).

Example 3.5.5. To illustrate the combinatorics of the corepresenting map in Construction 3.5.4, we return to the situation described in Remark 3.5.2. Namely, let us restrict our attention to the case that $n = 1$, $\alpha(0) = [1]^\circ$, and $\alpha(1) = [0]^\circ$. Then, there are two possibilities for the map $\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_1 = \hom_{\mathcal{C}\mathrm{at}}([1], \mathbf{\Delta}^{op})$: it selects either $[1]^\circ \xrightarrow{\delta_0} [0]^\circ$ or $[1]^\circ \xrightarrow{\delta_1} [0]^\circ$. These are respectively opposite to the map $[0] \to [1]$ in $\mathbf{\Delta}$ given by $0 \mapsto 1$ or $0 \mapsto 0$. Hence, the corresponding corepresenting map $[1] \to \alpha(0)^\circ = [1]$ in $\mathbf{\Delta}$ is respectively either $\mathrm{id}_{[1]}$ or $\mathrm{const}(0)$.

We now define our main object of interest, the simplicial replacement.

Definition 3.5.6. Suppose that $\mathcal{C}$ is cocomplete, and let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be a functor. We construct a composite

$$\begin{array}{ccccc} \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) & \longrightarrow & \mathcal{D} \times \mathbf{\Delta}^{op} & \xrightarrow{F \times \mathrm{id}_{\mathbf{\Delta}^{op}}} & \mathcal{C} \times \mathbf{\Delta}^{op} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbf{\Delta}^{op} & & \end{array}$$

in which the first horizontal map is given by Construction 3.5.4. By Proposition T.4.2.2.7, there is a unique lift in the diagram

$$\begin{array}{ccc} \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) & \longrightarrow & \mathcal{C} \times \mathbf{\Delta}^{op} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} & \longrightarrow & \mathbf{\Delta}^{op} \end{array}$$

such that the composite

$$\mathbf{\Delta}^{op} \to \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \to \mathcal{C} \times \mathbf{\Delta}^{op}$$

takes each $[n]^\circ \in \mathbf{\Delta}^{op}$ to a colimit of the composite

$$\mathrm{N}_\infty(\mathcal{D})_n \xrightarrow{\{0\}} \mathrm{N}_\infty(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

We define the object $\mathrm{srep}(F)_\bullet \in s\mathcal{C}$ to be the composite

$$\mathbf{\Delta}^{op} \to \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \to \mathcal{C} \times \mathbf{\Delta}^{op} \to \mathcal{C},$$

and refer to it as the ***simplicial replacement*** of the functor F.

Lemma 3.5.7. *Suppose that* $\mathcal{C}_1 \xrightarrow{\chi} \mathcal{C}_2$ *is a cocontinuous functor between cocomplete* ∞*-categories, and suppose that* $\mathcal{D} \xrightarrow{F} \mathcal{C}_1$ *is any functor. Then the composite*

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{srep}(F)_\bullet} \mathcal{C}_1 \xrightarrow{\chi} \mathcal{C}_2$$

is canonically equivalent to the object $\mathrm{srep}(\chi \circ F)_\bullet \in s(\mathcal{C}_2)$.

Proof. This follows directly from Definition 3.5.6. □

We can now give the main result of this section, the ***Bousfield–Kan colimit formula***.

Theorem 3.5.8. *Let* $\mathcal{C}$ *be cocomplete, and let* $\mathcal{D} \xrightarrow{F} \mathcal{C}$ *be a functor. Then there is a canonical equivalence*

$$\mathrm{colim}_{\mathcal{D}} F \simeq |\mathrm{srep}(F)_\bullet|$$

in $\mathcal{C}$.

Proof. Observe that the colimit of a $\mathcal{D}$-shaped diagram is functorial in cocomplete ∞-categories under $\mathcal{D}$ and cocontinuous functors between them. Hence, in light of Lemma 3.5.7, it suffices to prove the statement in the universal (i.e. initial) case, namely taking the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ to be the canonical map from $\mathcal{D}$ into its free cocompletion, i.e. the Yoneda embedding $\mathcal{D} \xrightarrow{\text{ɣ}} \mathcal{P}(\mathcal{D}) = \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$.

First of all, we claim that there is a canonical equivalence $\mathrm{colim}_{\mathcal{D}}(\text{ɣ}) \simeq \mathrm{const}(\mathrm{pt}_{\mathcal{S}})$ in $\mathcal{P}(\mathcal{D})$. To see this, observe that for any $d \in \mathcal{D}$, we have a string of equivalences

$$(\mathrm{colim}_{\mathcal{D}} \text{ɣ})(d) \simeq \mathrm{colim}_{\mathcal{D}}(\mathrm{ev}_d \circ \text{ɣ}) \simeq \mathrm{colim}_{\mathcal{D}}(\mathrm{hom}_{\mathcal{D}}(d, -))$$

in $\mathcal{S}$, where the first equivalence is because colimits in $\mathcal{P}(\mathcal{D})$ are computed pointwise and the second is simply because $\mathrm{ev}_d \circ \text{ɣ} \simeq \mathrm{hom}_{\mathcal{D}}(d, -)$ in $\mathrm{Fun}(\mathcal{D}, \mathcal{S})$. Appealing to Proposition 3.3.1 and the canonical equivalence

$$\mathrm{Gr}(\mathrm{hom}_{\mathcal{D}}(d, -)) \simeq \mathcal{D}_{d/}$$

in $\mathcal{L}\mathcal{F}\mathrm{ib}(\mathcal{D})$ of Example 3.1.12, we obtain a string of equivalences

$$\mathrm{colim}_{\mathcal{D}}(\mathrm{hom}_{\mathcal{D}}(d, -)) \simeq \mathrm{Gr}(\mathrm{hom}_{\mathcal{D}}(d, -))^{\mathrm{gpd}} \simeq (\mathcal{D}_{d/})^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}},$$

where the last equivalence follows from the dual of Corollary 3.4.11. Hence, $(\mathrm{colim}_{\mathcal{D}} \text{ɣ})(d) \simeq \mathrm{pt}_{\mathcal{S}}$ for every $d \in \mathcal{D}$, and so it follows that the terminal map $\mathrm{colim}_{\mathcal{D}}(\text{ɣ}) \to \mathrm{const}(\mathrm{pt}_{\mathcal{S}})$ in $\mathcal{P}(\mathcal{D})$ is indeed an equivalence.[9]

[9] This is also proved as Lemma T.5.3.3.2.

On the other hand, recall that $\mathrm{srep}(\text{ɟ})_n \in \mathcal{P}(\mathcal{D})$ is the colimit of the composite

$$\mathrm{N}_\infty(\mathcal{D})_n \xrightarrow{\{0\}} \mathrm{N}_\infty(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{\text{ɟ}} \mathcal{P}(\mathcal{D}).$$

For any $d \in \mathcal{D}$, the composite functor

$$\mathrm{N}_\infty(\mathcal{D})_n \xrightarrow{\{0\}} \mathrm{N}_\infty(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{\text{ɟ}} \mathcal{P}(\mathcal{D}) \xrightarrow{\mathrm{ev}_d} \mathcal{S}$$

classifies the left fibration which is the upper composite map

$$\begin{array}{ccccc} \{d\} \underset{\mathrm{N}_\infty(\mathcal{D})_0,\{0\}}{\times} \mathrm{N}_\infty(\mathcal{D})_{n+1} & \longrightarrow & \mathrm{N}_\infty(\mathcal{D})_{n+1} & \xrightarrow{\delta_0} & \mathrm{N}_\infty(\mathcal{D})_n \\ \downarrow & & \downarrow{\scriptstyle \{0\}} & & \\ \{d\} & \hookrightarrow & \mathrm{N}_\infty(\mathcal{D})_0. & & \end{array}$$

Again appealing to Proposition 3.3.1 (and the fact that colimits in $\mathcal{P}(\mathcal{D})$ are computed pointwise), it follows that

$$\begin{aligned} \mathrm{srep}(\text{ɟ})_n(d) &\simeq \left(\mathrm{colim}\left(\mathrm{N}_\infty(\mathcal{D})_n \xrightarrow{\{0\}} \mathrm{N}_\infty(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{\text{ɟ}} \mathcal{P}(\mathcal{D})\right)\right)(d) \\ &\simeq \mathrm{colim}\left(\mathrm{N}_\infty(\mathcal{D})_n \xrightarrow{\{0\}} \mathrm{N}_\infty(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{\text{ɟ}} \mathcal{P}(\mathcal{D}) \xrightarrow{\mathrm{ev}_d} \mathcal{S}\right) \\ &\simeq \left(\{d\} \underset{\mathrm{N}_\infty(\mathcal{D})_0,\{0\}}{\times} \mathrm{N}_\infty(\mathcal{D})_{n+1}\right)^{\mathrm{gpd}} \\ &\simeq \{d\} \underset{\mathrm{N}_\infty(\mathcal{D})_0,\{0\}}{\times} \mathrm{N}_\infty(\mathcal{D})_{n+1} \end{aligned}$$

(where the last equivalence follows from the fact that the inclusion $\mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$ is a right adjoint and hence commutes with pullbacks (and the fact that $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ is idempotent)). Unwinding the definitions of the simplicial structure maps of $\mathrm{srep}(\text{ɟ})_\bullet$, we obtain that in fact

$$\mathrm{srep}(\text{ɟ})_\bullet(d) \simeq \left(\{d\} \underset{\mathrm{N}_\infty(\mathcal{D})_0,\{0\}}{\times} \mathrm{N}_\infty(\mathcal{D})_{\bullet+1}\right) \simeq \mathrm{N}_\infty(\mathcal{D}_{d/})_\bullet.$$

Hence, it follows that

$$|\mathrm{srep}(\text{ɟ})_\bullet|(d) \simeq |\mathrm{srep}(\text{ɟ})_\bullet(d)| \simeq |\mathrm{N}_\infty(\mathcal{D}_{d/})_\bullet| \simeq (\mathcal{D}_{d/})^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}},$$

where again the last equivalence follows from the dual of Corollary 3.4.11, and the second-to-last equivalence follows from Proposition 2.2.4. Therefore, the terminal

map $|\mathrm{srep}(\text{ﻉ})_\bullet| \to \mathrm{const}(\mathrm{pt}_{\mathcal{S}})$ in $\mathcal{P}(\mathcal{D})$ is also an equivalence. It follows that we have a canonical equivalence

$$\mathrm{colim}_{\mathcal{D}} \text{ﻉ} \simeq \mathrm{const}(\mathrm{pt}_{\mathcal{S}}) \simeq |\mathrm{srep}(\text{ﻉ})_\bullet|,$$

which proves the claim. □

Remark 3.5.9. Our Bousfield–Kan colimit formula (Theorem 3.5.8) is certainly inspired by the classical Bousfield–Kan formula (see the original source [BK72, Chapter XII, §5], or e.g. [Hir03, §18.1] for a more modern treatment), but it is actually rather different: the latter computes a *homotopy* colimit in a simplicial model category. Moreover, even in the case that both $\mathcal{C}$ and $\mathcal{D}$ are only 1-categories, it does not generally agree with the formula for a colimit as a coequalizer of coproducts: this additionally requires that $\mathcal{D}$ be *gaunt*. (Note that that formula is actually evil: it is not invariant under replacing $\mathcal{D}$ by an equivalent category, referring as it does to its actual sets $\mathrm{N}(\mathcal{D})_0$ and $\mathrm{N}(\mathcal{D})_1$ of objects and of morphisms.) On the other hand, when $\mathcal{D}$ is a (1- or ∞-)groupoid, then our simplicial replacement (Definition 3.5.6) is trivial: in that case we have a canonical equivalence $\mathrm{N}_\infty(\mathcal{D})_\bullet \simeq \mathrm{const}(\mathcal{D})$ in $s\mathcal{C}$, whence it follows that the simplicial replacement $\mathrm{srep}(F)_\bullet \in s\mathcal{C}$ of a diagram $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is already constant at the object $\mathrm{colim}_{\mathcal{D}} F \in \mathcal{C}$.

3.5.2 Examples of the Bousfield–Kan colimit formula

The Bousfield–Kan colimit formula (Theorem 3.5.8) is most interesting (and novel) when the diagram ∞-category $\mathcal{D}$ is not merely a 1-category. For instance, applying it to a pushout diagram and canceling out the redundancies in the resulting geometric realization yields nothing but the original pushout diagram. We therefore give two inherently ∞-categorical examples to illustrate its application.

Example 3.5.10. Choose any space $Y \in \mathcal{S}$, and let us take $\mathcal{D}$ to be its "categorical suspension": this is an ∞-category with two objects d_1 and d_2, which is determined by the prescriptions that

- $\hom_{\mathcal{D}}(d_1, d_1) \simeq \hom_{\mathcal{D}}(d_2, d_2) \simeq \mathrm{pt}_{\mathcal{S}}$,
- $\hom_{\mathcal{D}}(d_2, d_1) \simeq \varnothing_{\mathcal{S}}$, and
- $\hom_{\mathcal{D}}(d_1, d_2) \simeq Y$.

Then, a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ selects the data of

- a pair of objects $c_1, c_2 \in \mathcal{C}$, and
- a map $Y \to \hom_{\mathcal{S}}(c_1, c_2)$ in $\mathcal{S}$.

Canceling out redundancies (and assuming $\mathcal{C}$ is cocomplete), the Bousfield–Kan colimit formula (Theorem 3.5.8) then gives equivalences

$$\mathrm{colim}_{\mathcal{D}}(F) \simeq |\mathrm{srep}(F)_\bullet| \simeq \mathrm{colim} \begin{pmatrix} c_1 \odot Y & \longrightarrow & c_1 \\ \downarrow & & \\ c_2 & & \end{pmatrix}$$

in $\mathcal{C}$, where in the pushout

- the horizontal map is given by the tensoring $c_1 \odot (Y \to \mathrm{pt}_{\mathcal{S}})$, and
- the vertical map is the adjunct of the chosen map $Y \to \hom_{\mathcal{C}}(c_1, c_2)$.

Example 3.5.11. Choose any space $Y \in \mathcal{S}$, and let us take $\mathcal{D} = Y^{\triangleleft} \in \mathcal{C}\mathrm{at}_\infty$ to be the left cone on it (considered as an ∞-groupoid). Assume for simplicity that $\mathcal{C}$ is bicomplete. Then, a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is determined by

- its restriction $Y \xrightarrow{F'} \mathcal{C}$, and
- a map $c \to \lim_Y(F')$ from some object $c \in \mathcal{C}$ (selected by the cone point) to the limit of this restriction.[10]

Canceling out redundancies, the Bousfield–Kan colimit formula (Theorem 3.5.8) then gives equivalences

$$\mathrm{colim}_{\mathcal{D}}(F) \simeq |\mathrm{srep}(F)_\bullet| \simeq \mathrm{colim} \begin{pmatrix} c \odot Y & \longrightarrow & c \\ \downarrow & & \\ \mathrm{colim}_Y(F') & & \end{pmatrix}$$

in $\mathcal{C}$, where in the pushout

- the horizontal map is given by $c \odot (Y \to \mathrm{pt}_{\mathcal{S}})$, and
- the vertical map is induced by the natural transformation $\mathrm{const}_Y(c) \to F'$ in $\mathrm{Fun}(Y, \mathcal{C})$ which is adjunct to the chosen map $c \to \lim_Y(F')$.

On an object $c' \in \mathcal{C}$, this pushout corepresents the data of

[10] If $Y \xrightarrow{F'} \mathcal{C}$ is constant, then $\lim_Y(F')$ reduces to a cotensor. However, there are interesting examples where this is not the case: for instance, if Y is a 1-type, this limit computes the *fixed points* of the corresponding group action. Of course, a dual observation applies to $\mathrm{colim}_Y(F')$, which appears here as well. (Another interesting example of a colimit over an ∞-groupoid is the *Thom spectrum* construction.)

- a morphism $\mathrm{colim}_Y(F') \to c'$ in $\mathcal{C}$ (or equivalently, a morphism $F' \to \mathrm{const}_Y(c')$ in $\mathrm{Fun}(Y, \mathcal{C})$, along with
- a trivialization

$$\begin{array}{ccc} Y & \longrightarrow & \hom_{\mathcal{C}}(c, c') \\ \downarrow & \nearrow & \\ \mathrm{pt}_{\mathcal{S}} & & \end{array}$$

of the adjunct to the induced composite

$$c \odot Y \to \mathrm{colim}_Y(F') \to c'.$$

3.5.3 Functoriality of the Bousfield–Kan colimit formula

Of course, it is perfectly reasonable to expect that the Bousfield–Kan colimit formula enjoys good functoriality properties, along the lines of those explored in § 3.2.2. However, for expedience, rather than pursue a full treatment, in this subsection we exhibit only the mere shadow of such functoriality that we will actually need.

We begin by identifying the simplicial replacement as a left Kan extension.

Lemma 3.5.12. *Suppose that $\mathcal{C}$ is cocomplete, let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be any diagram. Then, in the commutative diagram*

$$\begin{array}{ccccc} & & \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) & \longrightarrow & \mathcal{C} \times \mathbf{\Delta}^{op} \longrightarrow \mathcal{C} \\ & & \downarrow & \nearrow & \\ \mathbf{\Delta}^{op} & \longrightarrow & \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} & & \end{array}$$

containing the simplicial replacement $\mathbf{\Delta}^{op} \xrightarrow{\mathrm{srep}(F)_\bullet} \mathcal{C}$ as a composite, the vertical functor is a full inclusion and the dotted arrow is a left Kan extension along it.

Proof. To see that the vertical functor is a full inclusion, we simply unwind the definition of its target to obtain

$$\begin{aligned} &\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \\ &= \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \coprod_{\left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\times} \mathbf{\Delta}^{op} \times \{0\}\right)} \left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\times} \mathbf{\Delta}^{op} \times [1]\right) \coprod_{\left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\times} \mathbf{\Delta}^{op} \times \{1\}\right)} \mathbf{\Delta}^{op} \end{aligned}$$

$$\simeq (\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \times [1]) \coprod_{(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet)\times\{1\})} \mathbf{\Delta}^{op}.$$

That is, this target is precisely the cocartesian fibration over $[1]$ classified by the functor $[1] \to \mathcal{C}\mathrm{at}_\infty$ which selects the projection $\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \to \mathbf{\Delta}^{op}$, and our vertical functor is the fiber inclusion over the object $0 \in [1]$; in particular, it is full (as the object $0 \in [1]$ admits no nontrivial automorphisms).

Now, to check that the dotted arrow is a left Kan extension along this full inclusion, by Remark T.4.3.2.3 it suffices to show that for any object

$$[n]^\circ \in \mathbf{\Delta}^{op} \subset \left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}\right),$$

the corresponding fiber inclusion $\mathrm{N}_\infty(\mathcal{D})_n \hookrightarrow \mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet)$ induces a functor

$$\mathrm{N}_\infty(\mathcal{D})_n \to \left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}\right)}{\times} \left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}\right)_{/[n]^\circ} \right)$$

which is final. This is straightforwardly verified using Theorem A (3.4.10): all of the comma ∞-categories whose groupoid completions must be shown to be contractible are easily seen to possess initial objects, and hence the equivalent condition follows from the opposite of Corollary 3.4.11. □

We can now describe our desired shadow of functoriality.

Proposition 3.5.13. *Let $\mathcal{C}$ be cocomplete, and suppose that*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{H} & \mathcal{E} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathcal{C} & \end{array}$$

is a commutative diagram in $\mathcal{C}\mathrm{at}_\infty$.

(1) There is a canonical induced map

$$\mathrm{srep}(F)_\bullet \to \mathrm{srep}(G)_\bullet$$

in $s\mathcal{C}$, which is functorial in the variable $\mathcal{C}$ for cocontinuous functors between cocomplete ∞-categories.

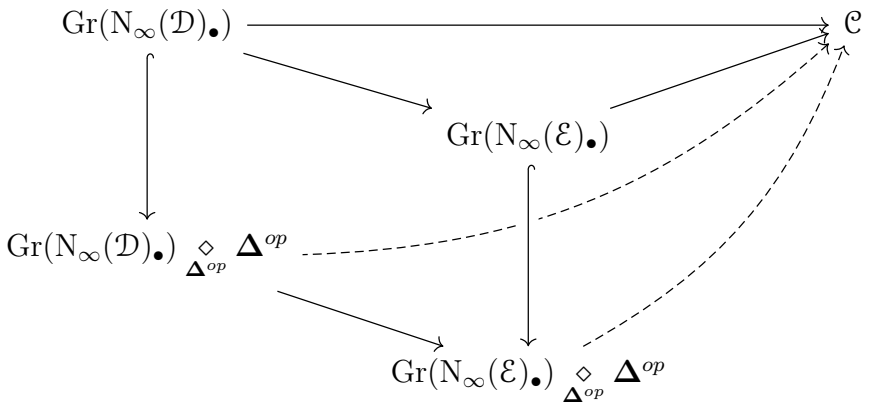
(2) We have a commutative square

$$\begin{array}{ccc} \operatorname{colim}_{\mathcal{D}} F & \longrightarrow & \operatorname{colim}_{\mathcal{E}} G \\ \wr\downarrow & & \downarrow\wr \\ |\mathrm{srep}(F)_\bullet| & \longrightarrow & |\mathrm{srep}(G)_\bullet| \end{array}$$

in $\mathcal{C}$, in which

- *the upper map is the induced map on colimits of the global colimit functor (Proposition 3.2.12),*
- *the lower map is the geometric realization of the canonical map of part (1), and*
- *the vertical equivalences are those of Theorem 3.5.8.*

Proof. For part (1), by Lemma 3.5.12 and the functoriality provided by Construction 3.5.4, we have a commutative diagram



in which the vertical arrows are full inclusions and the dotted arrows are left Kan extensions therealong. As the composite

$$\left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}\right) \to \left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{E})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}\right) \to \mathcal{C}$$

also extends the map $\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \to \mathcal{C}$ along the given full inclusion, it therefore admits a canonical natural transformation from the dotted map

$$\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \to \mathcal{C}.$$

Restricting to the full subcategory

$$\mathbf{\Delta}^{op} \subset \left(\mathrm{Gr}(\mathrm{N}_\infty(\mathcal{D})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \right),$$

we obtain a natural transformation

$$\begin{array}{ccc} & \mathrm{srep}(F)_\bullet & \\ \curvearrowright & & \curvearrowright \\ \mathbf{\Delta}^{op} & \Downarrow & \mathcal{C} \\ \curvearrowright & & \curvearrowright \\ & \mathrm{srep}(G)_\bullet & \end{array}$$

which is precisely our desired map in $s\mathcal{C}$. Moreover, the asserted functoriality follows easily from this argument (recall Remark 3.2.13).

For part (2), note that there is a unique cocontinuous functor $\mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{E})$ making the diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{H} & \mathcal{E} \\ {\scriptstyle \text{ɬ}_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \text{ɬ}_{\mathcal{E}}} \\ \mathcal{P}(\mathcal{D}) & \dashrightarrow & \mathcal{P}(\mathcal{E}) \end{array}$$

commute.[11] From here, by Lemma 3.5.7 and the functoriality asserted in part (1), it suffices to verify the claim in the case that the functor $\mathcal{E} \xrightarrow{G} \mathcal{C}$ is the Yoneda embedding $\mathcal{E} \xrightarrow{\text{ɬ}_{\mathcal{E}}} \mathcal{P}(\mathcal{E})$, so that the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is the composite $\mathcal{D} \xrightarrow{\text{ɬ}_{\mathcal{E}} \circ H} \mathcal{P}(\mathcal{E})$. Hence, it remains to show that we have a commutative square

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{D}}(\text{ɬ}_{\mathcal{E}} \circ H) & \longrightarrow & \mathrm{colim}_{\mathcal{E}} \text{ɬ}_{\mathcal{E}} \\ \wr\downarrow & & \downarrow\wr \\ |\mathrm{srep}(\text{ɬ}_{\mathcal{E}} \circ H)_\bullet| & \longrightarrow & |\mathrm{srep}(\text{ɬ}_{\mathcal{E}})_\bullet| \end{array}$$

in $\mathcal{P}(\mathcal{E})$ satisfying the described criteria. But as we have seen in the proof of Theorem 3.5.8, if we consider these two vertical maps as objects of $\mathrm{Fun}([1], \mathcal{P}(\mathcal{E}))$, then the one on the right determines a terminal object. As the datum of this commutative square is equivalent to that of the corresponding morphism in $\mathrm{Fun}([1], \mathcal{P}(\mathcal{E}))$ (reading the square from left to right), it follows that in fact the square must commute in a canonical and unique way. □

[11] In fact, by Lemma T.5.1.5.5(1), this functor must be precisely the left Kan extension $(\text{ɬ}_{\mathcal{D}})_!(\text{ɬ}_{\mathcal{E}} \circ H)$.

3.6 The Thomason model structure on the ∞-category of ∞-categories

In this final section, we equip the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories with a *Thomason model structure* analogous to the classical Thomason model structure on $\mathcal{C}\mathrm{at}$ (though see Remark 3.6.6) and observe some of its basic features. (We refer the reader to §1.1 for the definition of a model structure on an ∞-category, and to §1.4 for the definition of the Kan–Quillen model structure on the ∞-category $s\mathcal{S}$ of simplicial spaces.) This model structure provides a convenient language for a number of the results in the main body of the chapter.

We begin by constructing it.

Theorem 3.6.1. *The Kan–Quillen model structure on $s\mathcal{S}$ lifts along the composite adjunction*

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\rightleftarrows}} \mathcal{CSS} \underset{\mathrm{N}_\infty}{\overset{\mathrm{N}_\infty^{-1}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty.$$

Moreover, this Quillen adjunction is a Quillen equivalence.

Proof. For notational convenience, we prove the statement for the adjunction $\mathrm{L}_{\mathcal{CSS}} \dashv \mathrm{U}_{\mathcal{CSS}}$ (which is of course equivalent).

For the first claim, we verify the hypotheses of the lifting theorem for cofibrantly generated model ∞-categories (1.3.12) in turn.

First of all, for any $Y \in s\mathcal{S}$ and $Z \in \mathcal{CSS}$ we have that

$$\hom_{\mathcal{CSS}}(\mathrm{L}_{\mathcal{CSS}}(Y), Z) \simeq \hom_{s\mathcal{S}}(Y, \mathrm{U}_{\mathcal{CSS}}(Z)),$$

and so the fact that the sets $\mathrm{L}_{\mathcal{CSS}}(I_{\mathrm{KQ}})$ and $\mathrm{L}_{\mathcal{CSS}}(J_{\mathrm{KQ}})$ of homotopy classes of maps in $\mathcal{CSS}$ permit the small object argument follows from Lemma 1.5.2.

Next, we show that the right adjoint $s\mathcal{S} \xleftarrow{\mathrm{U}_{\mathcal{CSS}}} \mathcal{CSS}$ takes relative $\mathrm{L}_{\mathcal{CSS}}(J_{\mathrm{KQ}})$-cell complexes into $\mathbf{W}_{\mathrm{KQ}}$. For this, let us begin by supposing that

$$\begin{array}{ccc} \mathrm{L}_{\mathcal{CSS}}(\Lambda^n_i) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{L}_{\mathcal{CSS}}(\Delta^n) & \longrightarrow & Z \end{array}$$

is a pushout square in $\mathcal{CSS}$. Since $s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS}$ commutes with pushouts (being a left adjoint), we have that

$$Z \simeq \operatorname{colim}^{\mathcal{CSS}} \begin{pmatrix} \mathrm{L}_{\mathcal{CSS}}(\Lambda^n_i) & \longrightarrow & \mathrm{L}_{\mathcal{CSS}}(\mathrm{U}_{\mathcal{CSS}}(Y)) \\ \downarrow & & \\ \mathrm{L}_{\mathcal{CSS}}(\Delta^n) & & \end{pmatrix} \simeq \mathrm{L}_{\mathcal{CSS}} \left(\operatorname{colim}^{s\mathcal{S}} \begin{pmatrix} \Lambda^n_i & \longrightarrow & \mathrm{U}_{\mathcal{CSS}}(Y) \\ \downarrow & & \\ \Delta^n & & \end{pmatrix} \right).$$

Using Proposition 2.2.4 and the fact that colimits commute with colimits, we then obtain the string of equivalences

$$\begin{aligned} |\mathrm{U}_{\mathcal{CSS}}(Z)| &\simeq \left| \mathrm{U}_{\mathcal{CSS}} \left(\mathrm{L}_{\mathcal{CSS}} \left(\operatorname{colim}^{s\mathcal{S}} \begin{pmatrix} \Lambda^n_i & \longrightarrow & \mathrm{U}_{\mathcal{CSS}}(Y) \\ \downarrow & & \\ \Delta^n & & \end{pmatrix} \right) \right) \right| \\ &\simeq \left| \operatorname{colim}^{s\mathcal{S}} \begin{pmatrix} \Lambda^n_i & \longrightarrow & \mathrm{U}_{\mathcal{CSS}}(Y) \\ \downarrow & & \\ \Delta^n & & \end{pmatrix} \right| \\ &\simeq \operatorname{colim}^{\mathcal{S}} \begin{pmatrix} |\Lambda^n_i| & \longrightarrow & |\mathrm{U}_{\mathcal{CSS}}(Y)| \\ \wr\downarrow & & \\ |\Delta^n| & & \end{pmatrix} \\ &\simeq |\mathrm{U}_{\mathcal{CSS}}(Y)|. \end{aligned}$$

Hence, the map

$$\mathrm{U}_{\mathcal{CSS}}(Y) \to \mathrm{U}_{\mathcal{CSS}}(Z)$$

lies in $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$. Now, to prove the claim for more general transfinite compositions, it then suffices to observe that the composite

$$\mathcal{CSS} \xhookrightarrow{\mathrm{U}_{\mathcal{CSS}}} s\mathcal{S} \xrightarrow{|-|} \mathcal{S}$$

is a left adjoint and hence commutes with colimits. Thus, $\mathrm{U}_{\mathcal{CSS}}$ does indeed take relative $\mathrm{L}_{\mathcal{CSS}}(J_{\mathrm{KQ}})$-cell complexes into $\mathbf{W}_{\mathrm{KQ}}$.

So, the Kan–Quillen model structure $s\mathcal{S}_{\mathrm{KQ}}$ does indeed lift along the adjunction as claimed, and it remains to check that the resulting Quillen adjunction is a Quillen

equivalence. For this, suppose we are given *any* objects $Y \in s\mathcal{S}$ and $Z \in \mathcal{CSS}$. Then, a map

$$Y \to \mathrm{U}_{\mathcal{CSS}}(Z)$$

in $s\mathcal{S}$ corresponds via the adjunction to the map

$$\mathrm{L}_{\mathcal{CSS}}(Y) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{U}_{\mathcal{CSS}}(Z)) \simeq Z$$

in $\mathcal{CSS}$. Since the lifting theorem (1.3.12) produces a model structure for which the right adjoint creates the weak equivalences, the claim follows from Proposition 2.2.4. □

Definition 3.6.2. We refer to the model structure on $\mathcal{C}\mathrm{at}_\infty$ defined by Theorem 3.6.1 as the ***Thomason model structure***, denoted $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$.

Remark 3.6.3. Proposition 2.2.4 implies that the subcategory $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$ is created by the groupoid completion functor $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$. As this functor is a left localization, it therefore induces an equivalence $\mathcal{C}\mathrm{at}_\infty[\mathbf{W}_{\mathrm{Th}}^{-1}] \xrightarrow{\sim} \mathcal{S}$. Moreover, it is not hard to see directly that the adjoint functors $\mathrm{N}_\infty^{-1} \circ \mathrm{L}_{\mathcal{CSS}} \dashv \mathrm{U}_{\mathcal{CSS}} \circ \mathrm{N}_\infty$ induce inverse equivalences $s\mathcal{S}[\mathbf{W}_{\mathrm{KQ}}^{-1}] \simeq \mathcal{C}\mathrm{at}_\infty[\mathbf{W}_{\mathrm{Th}}^{-1}]$.

Remark 3.6.4. Let us explore what it means for an object $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ to be fibrant in the Thomason model structure. By definition, this means that $\mathrm{N}_\infty(\mathcal{C}) \in \mathcal{CSS} \subset s\mathcal{S}$ has the extension property for the set $J_{\mathrm{KQ}} = \{\Lambda^n_i \to \Delta^n\}_{0 \leq i \leq n \geq 1}$ of horn inclusions in $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$.

In fact, the Segal condition on a simplicial space implies that it admits *unique* fillers for the set $\{\Lambda^n_i \to \Delta^n\}_{0<i<n\geq 2}$ of *inner* horn inclusions. To see this, observe that the inclusion

$$\left(\Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}\right) \to \Lambda^n_i$$

of subobjects of $\Delta^n \in s\mathcal{S}\mathrm{et}$ can be constructed as a (finite) composition of pushouts of inner horn inclusions $\Lambda^k_j \to \Delta^k$ for $k < n$, and the Segal condition is precisely the assertion that a given simplicial space admits unique extensions for the inclusion of the above source into Δ^n (the "n^{th} spine inclusion"). From here, the assertion follows by induction.

On the other hand, it is already clear that if a (complete) Segal space has the extension property for the outer horn inclusion $\Lambda^2_0 \to \Delta^2$, then it must be the nerve of an ∞-category whose morphisms are all invertible. Hence, the fibrant objects in the model structure on $\mathcal{CSS}$ of Theorem 3.6.1 are precisely the *constant* complete Segal spaces (i.e. those that are constant as simplicial spaces). It follows that we

can identify the subcategory of fibrant objects in the Thomason model structure as $(\mathcal{C}\mathrm{at}_\infty)^f_{\mathrm{Th}} = \mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$, the subcategory of ∞-groupoids (i.e. spaces).

In fact, we have the following strengthening of Remark 3.6.4.[12,13]

Proposition 3.6.5. *The Thomason model structure on $\mathcal{C}\mathrm{at}_\infty$ is a left localization model structure (in the sense of Example 1.2.12) with respect to the left localization adjunction*

$$\mathcal{C}\mathrm{at}_\infty \underset{\mathrm{U}_{\mathcal{S}}}{\overset{(-)^{\mathrm{gpd}}}{\rightleftarrows}} \mathcal{S}.$$

Proof. Observe that a model structure on an ∞-category is clearly determined by its subcategories of weak equivalences and of cofibrations (just as a model structure on a 1-category). So, it only remains to check that all maps in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ are cofibrations. For this, it suffices to present any map in $\mathcal{CSS}$ as the image under $s\mathcal{S} \xrightarrow{\mathrm{L}\mathcal{CSS}} \mathcal{CSS}$ of a cofibration in $s\mathcal{S}_{\mathrm{KQ}}$. Note that the inclusion $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ induces an inclusion $\mathbf{C}^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}} \subset \mathbf{C}^{s\mathcal{S}}_{\mathrm{KQ}}$; in fact, we will present any map in $\mathcal{CSS}$ as the image of a cofibration in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

To accomplish this, we begin by observing that the composite functor

$$s\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S} \xrightarrow{\mathrm{L}\mathcal{CSS}} \mathcal{CSS}$$

consists of a right adjoint followed by a left adjoint. In fact, these are both presented by Quillen functors: the first functor is presented by the right Quillen functor in the evident Quillen adjunction

$$\pi_0^{\mathrm{lw}} : s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{triv}} : \mathrm{const}^{\mathrm{lw}}$$

(where in the left adjoint we slightly abuse notation by using the symbol π_0 to refer to the composite $s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}[\![\mathbf{W}^{-1}_{\mathrm{KQ}}]\!] \simeq \mathcal{S} \xrightarrow{\pi_0} \mathcal{S}\mathrm{et}$), while the second functor is presented by the left Quillen functor in the left Bousfield localization

$$\mathrm{id}_{ss\mathcal{S}\mathrm{et}} : s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} : \mathrm{id}_{ss\mathcal{S}\mathrm{et}}$$

(see (the proof of) [Rez01, Theorem 7.2]). Moreover, all objects of $s\mathcal{S}\mathrm{et}_{\mathrm{triv}}$ are fibrant while all objects of $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ are cofibrant, so this composite of Quillen functors

[12] Proposition 3.6.5 immediately implies the conclusion of Remark 3.6.4 (that $(\mathcal{C}\mathrm{at}_\infty)^f_{\mathrm{Th}} = \mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$), but we will want to build on the explicit geometric arguments given there in Remark 3.6.8.

[13] Proposition 3.6.5 does *not* follow from the discussion of Example 1.2.12. For instance, this left localization is certainly not left exact; the entire point of §3.4 is to obtain sufficient conditions under which it commutes with pullbacks. See also Remark 3.6.9.

does not need to be corrected at either stage in order to compute the value of the corresponding adjoint functor of ∞-categories. On the other hand, this composite functor

$$s\mathcal{S}\mathrm{et}_{\mathrm{triv}} \xrightarrow{\mathrm{const}^{\mathrm{lw}}} s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \xrightarrow{\mathrm{id}_{ss\mathcal{S}\mathrm{et}}} ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$$

of model categories is precisely the functor underlying the left Quillen equivalence $\mathrm{const}^{\mathrm{lw}} : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \to ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ of [JT07, Theorem 4.11]. It therefore follows that the above composite functor of ∞-categories induces an equivalence

$$s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{Joyal}}^{-1}] \xrightarrow{\sim} \mathcal{CSS}$$

from the localization of $s\mathcal{S}\mathrm{et}$ at the subcategory $\mathbf{W}_{\mathrm{Joyal}} \subset s\mathcal{S}\mathrm{et}$ to the ∞-category of complete Segal spaces.[14]

We can now easily achieve our goal. Since $\mathbf{C}_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}} = \mathbf{C}_{\mathrm{Joyal}}^{s\mathcal{S}\mathrm{et}}$, it follows that we can simply choose a cofibration in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ presenting our given map in $\mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty$: considering this map of simplicial sets as a map of discrete simplicial spaces, its image under the functor $s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS}$ is precisely the chosen map. □

Remark 3.6.6. We observe that the Thomason model structure on $\mathcal{C}\mathrm{at}_\infty$ does *not* extend the original Thomason model structure on $\mathbb{C}\mathrm{at}$: the model category $\mathbb{C}\mathrm{at}_{\mathrm{Th}}$ is not a model subcategory of the model ∞-category $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ (in the sense of Definition 1.4.11, and ignoring the fact that $\mathbb{C}\mathrm{at}$ is not, strictly speaking, a subcategory of $\mathcal{C}\mathrm{at}_\infty$ at all). However, the weak equivalences remain unchanged: the subcategory $\mathbf{W}_{\mathrm{Th}}^{\mathbb{C}\mathrm{at}} \subset \mathbb{C}\mathrm{at}$ is pulled back from the subcategory $\mathbf{W}_{\mathrm{Th}}^{\mathcal{C}\mathrm{at}_\infty} \subset \mathcal{C}\mathrm{at}_\infty$ along the composite functor $\mathbb{C}\mathrm{at} \to \mathcal{C}\mathrm{at} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$. To illustrate this, we recall the history of this classical model category.

To begin, we recall the main point: categories can individually be considered as "presentations of spaces" (via simplicial sets) via the nerve functor $\mathrm{N} : \mathbb{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$. Thus, it is natural to wonder whether this can be extended to a global presentation of the *homotopy category* of spaces (i.e. $\mathrm{ho}(\mathcal{S}) \simeq s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{KQ}}^{-1}]$) as some localization of the category $\mathbb{C}\mathrm{at}$ of categories. As a first step in this direction, it was proved in [Ill72, 3.3.1] (but attributed there to Quillen) that the nerve functor $\mathbb{C}\mathrm{at} \xrightarrow{\mathrm{N}} s\mathcal{S}\mathrm{et}$ does indeed induce an equivalence $\mathbb{C}\mathrm{at}[\mathbf{W}_{\mathrm{Th}}^{-1}] \xrightarrow{\sim} s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{KQ}}^{-1}]$ on (1-categorical) localizations. In other words, the relative category $(\mathbb{C}\mathrm{at}, \mathbf{W}_{\mathrm{Th}}) \in \mathfrak{R}\mathrm{el}\mathbb{C}\mathrm{at}$ has as its (1-categorical) localization the homotopy category $\mathrm{ho}(\mathcal{S})$ of spaces, as desired.

On the other hand, relative categories are not so easy to work with, and so one might then further wonder whether this relative category structure can be promoted

[14] Of course, we already knew that $s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{Joyal}}^{-1}]$ was equivalent to $\mathcal{CSS}$ (since it is equivalent to $\mathcal{C}\mathrm{at}_\infty$); the new information here is that this particular composite functor (of ∞-categories) *also* induces this equivalence.

to a model category structure. Now, the most obvious way that one might hope to obtain this would be to simply lift the classical Kan–Quillen model structure (as in Theorem 1.3.12, but of course really just using [Hir03, Theorem 11.3.2]) along the adjunction $\mathrm{L}_{\mathbb{cat}} : s\mathcal{S}\mathrm{et} \rightleftarrows \mathbb{cat} : \mathrm{N}$. However, it is easy to see that such a Quillen adjunction could not possibly be a Quillen equivalence: the fibrant objects of $\mathbb{cat}$ would be precisely the subcategory $\mathbb{gpd} \subset \mathbb{cat}$ of groupoids, but these (or rather their nerves) do not model all objects of $s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{KQ}}^{-1}]$, and so the derived right adjoint could not possibly be surjective.

But in [Tho80], Thomason showed that if we instead lift the Kan–Quillen model structure along the composite right adjoint

$$s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} \xleftarrow{\mathrm{Ex}} s\mathcal{S}\mathrm{et} \xleftarrow{\mathrm{Ex}} s\mathcal{S}\mathrm{et} \xleftarrow{\mathrm{N}} \mathbb{cat},$$

then the resulting Quillen adjunction *is* a Quillen equivalence.[15,16] This defines what is now called the *Thomason model structure* on $\mathbb{cat}$. From this description, it is clear that the model structure $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ does not extend the model structure $\mathbb{cat}_{\mathrm{Th}}$. For instance, it follows from Proposition 3.6.5 that all objects of $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ are cofibrant, whereas [Tho80, Proposition 5.7] asserts that all cofibrant objects of $\mathbb{cat}_{\mathrm{Th}}$ are in fact posets. Moreover, that same result also implies that their notions of fibrancy disagree: it follows from it that any bifibrant object of $\mathbb{cat}_{\mathrm{Th}}$ is a poset, whereas according to Remark 3.6.4 the fibrant objects of $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ are precisely the ∞-groupoids.[17]

Remark 3.6.7. One way to interpret Remark 3.6.6 is to say that the model ∞-category $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ entirely accounts for the quirky definition of the model category $\mathbb{cat}_{\mathrm{Th}}$: in this sense, the only "obstruction" to obtaining a model structure on $\mathbb{cat}$ presenting $\mathcal{S}$ by lifting directly along the nerve functor is the lack of would-be fibrant objects in $\mathbb{cat}$. While not every space is presented by a groupoid, certainly every space is presented by an ∞-groupoid (!), and so this obstruction vanishes when we pass from $\mathbb{cat}$ to $\mathcal{C}\mathrm{at}_\infty$.

[15] Heuristically, one might say that the Ex functor "makes more simplicial sets fibrant". Indeed, recall that it comes equipped with a natural transformation $\mathrm{id}_{s\mathcal{S}\mathrm{et}} \to \mathrm{Ex}$, and the resulting transfinite composition defines a fibrant replacement functor on $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ (see [GJ99, §III.4]).

[16] The Ex functor is not only a right Quillen equivalence from $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ to itself, but it is also a relative functor – indeed, it defines a weak equivalence in $\mathbb{relcat}_{\mathrm{BK}}$. Thus, even though here we are (crucially!) not only applying it to fibrant objects of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, in the end this composite still presents an equivalent map in $\mathcal{C}\mathrm{at}_\infty$ (namely $\mathcal{S} \xrightarrow{\mathrm{id}_{\mathcal{S}}} \mathcal{S}$).

[17] As posets are gaunt, the composites $\mathbb{cat} \to \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}_\infty} s\mathcal{S}$ and $\mathbb{cat} \xrightarrow{\mathrm{N}} s\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S}$ are equivalent on such objects.

Remark 3.6.8. In contrast with Remark 3.6.4, it is not so straightforward to characterize which maps in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ are fibrations.[18] By definition, this would be a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathcal{C}\mathrm{at}_\infty$ such that the corresponding map

$$\mathrm{N}_\infty(\mathcal{C}) \xrightarrow{\mathrm{N}_\infty(F)} \mathrm{N}_\infty(\mathcal{D})$$

on nerves in $\mathcal{CSS} \subset s\mathcal{S}$ has $\mathrm{rlp}(I_{\mathrm{KQ}})$. On the one hand, arguments similar to those of Remark 3.6.4 imply that these maps likewise admit *unique* lifts for the *inner* horn inclusions. On the other hand, the condition that this map have the right lifting property against the outer horn inclusions seems to be a good deal more subtle.

- At $n = 1$, the requirement that our map have the right lifting property against the outer horn inclusion $\Lambda^1_0 \to \Delta^1$ (resp. the outer horn inclusion $\Lambda^1_1 \to \Delta^1$) is equivalent to the condition that for all objects $c \in \mathcal{C}$, the functor $\mathcal{C}_{/c} \to \mathcal{D}_{/F(c)}$ (resp. the functor $\mathcal{C}_{c/} \to \mathcal{D}_{F(c)/}$) is surjective.

- At $n = 2$, even just the requirement that our map have the right lifting property against the outer horn inclusion $\Lambda^2_0 \to \Delta^2$ already implies that the equivalences in $\mathcal{C}$ are created by the functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$. However, this does not appear to be a sufficient condition. Of course, one can at least rephrase the condition as follows: for our map to have the right lifting property against $\Lambda^2_0 \to \Delta^2$ (resp. $\Lambda^2_2 \to \Delta^2$), it must be the case that, given any two maps φ and ψ in $\mathcal{C}$ whose sources (resp. targets) have been identified, then any factorization of one of the maps $F(\varphi)$ or $F(\psi)$ in $\mathcal{D}$ through the other must already exist in $\mathcal{C}$.

- Requiring that our map have the right lifting property against the higher outer horn inclusions appears to demand similar but even more exotic properties of our original functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$.

Remark 3.6.9. Combining Proposition 3.6.5 with the discussion of Example 1.2.12, we obtain that $(\mathbf{W} \cap \mathbf{F})_{\mathrm{Th}} = (\mathcal{C}\mathrm{at}_\infty)^{\simeq} \subset \mathcal{C}\mathrm{at}_\infty$; in other words, any fibration in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ which induces an equivalence on groupoid completions must in fact itself be an equivalence. In light of the discussion of Remark 3.6.8 regarding the complexities of the subcategory $\mathbf{F}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$, this appears to be a rather nontrivial fact.[19]

[18] Nor should this necessarily be expected to be straightforward: by the lifting axiom $\mathrm{M}_\infty 4$ for model ∞-categories and Proposition 3.6.5, we have $\mathbf{F}_{\mathrm{Th}} = \mathrm{rlp}((\mathbf{W} \cap \mathbf{C})_{\mathrm{Th}}) = \mathrm{rlp}(\mathbf{W}_{\mathrm{Th}})$.

[19] It is not so hard to see directly that if $\mathcal{D} \in \mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$, then a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is in $\mathbf{F}_{\mathrm{Th}}$ if and only if we also have $\mathcal{C} \in \mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$. (Thus, $\mathcal{S}_{\mathrm{triv}} \subset (\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ is a model subcategory.) From here, it is clear at least that $((\mathbf{W} \cap \mathbf{F})_{\mathrm{Th}} \cap \mathcal{S}) = \mathcal{S}^{\simeq} \subset \mathcal{C}\mathrm{at}_\infty$.

Remark 3.6.10. Both Thomason model structures $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ and $\mathbb{C}\mathrm{at}_{\mathrm{Th}}$ are rather quirky in their own ways. On the other hand, recalling Corollary 3.4.3, it appears that there should exist the structure of an "∞-category of fibrant objects" structure on $\mathcal{C}\mathrm{at}_\infty$ (or a "category of fibrant objects" structure on $\mathbb{C}\mathrm{at}$), in which the co/cartesian fibrations $\mathcal{D} \to \mathcal{C}$ classified by functors $\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ that have property Q are among the fibrations.[20] In some vague sense, this would appear to be a more "true" articulation of the role of $\mathcal{C}\mathrm{at}_\infty$ (or of $\mathbb{C}\mathrm{at}$) as a presentation of $\mathcal{S}$ than either of the corresponding Thomason model structures.

[20] With the model ∞-category $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ in hand, this should be obtainable from arguments along the lines of those contained in [Bro71].

Chapter 4

Hammocks and fractions in relative ∞-categories

In this chapter, we study the *homotopy theory* of ∞-categories enriched in the ∞-category $s\mathcal{S}$ of simplicial spaces. That is, we consider $s\mathcal{S}$-enriched ∞-categories as presentations of ordinary ∞-categories by means of a "local" geometric realization functor $\mathcal{C}\mathrm{at}_{s\mathcal{S}} \to \mathcal{C}\mathrm{at}_\infty$, and we prove that their homotopy theory presents the ∞-category of ∞-categories, i.e. that this functor induces an equivalence $\mathcal{C}\mathrm{at}_{s\mathcal{S}}[\![\mathbf{W}_{\mathrm{DK}}^{-1}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$ from a localization of the ∞-category of $s\mathcal{S}$-enriched ∞-categories.

Following Dwyer–Kan, we define a *hammock localization* functor from relative ∞-categories to $s\mathcal{S}$-enriched ∞-categories, thus providing a rich source of examples of $s\mathcal{S}$-enriched ∞-categories. Simultaneously unpacking and generalizing one of their key results, we prove that given a relative ∞-category admitting a *homotopical three-arrow calculus*, one can explicitly describe the hom-spaces in the ∞-category presented by its hammock localization in a much more explicit and accessible way.

As an application of this framework, we give sufficient conditions for the Rezk nerve of a relative ∞-category to be a (complete) Segal space, generalizing joint work with Low.

4.0 Introduction

4.0.1 Introducing (even more) homotopy theory

In their groundbreaking papers [DK80c] and [DK80a], Dwyer–Kan gave the first presentation of *the ∞-category of ∞-categories*, namely the category $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ of ***categories enriched in simplicial sets***: in modern language, every $s\mathcal{S}$et-enriched

category has an *underlying* ∞*-category*, and this association induces an equivalence

$$\mathfrak{cat}_{s\mathcal{S}\mathrm{et}}[\![\mathbf{W}_{\mathrm{DK}}^{-1}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_{\infty}$$

from the (∞-categorical) localization of the category $\mathfrak{cat}_{s\mathcal{S}\mathrm{et}}$ at the subcategory $\mathbf{W}_{\mathrm{DK}} \subset \mathfrak{cat}_{s\mathcal{S}\mathrm{et}}$ of *Dwyer–Kan weak equivalences* to the ∞-category $\mathcal{C}\mathrm{at}_{\infty}$ of ∞-categories. Moreover, Dwyer–Kan provided a method of ***"introducing homotopy theory"*** into a category $\mathcal{R}$ equipped with a subcategory $\mathbf{W} \subset \mathcal{R}$ of weak equivalences, namely their *hammock localization* functor $\mathscr{L}_{\delta}^{H} : \mathfrak{Rel}\mathfrak{cat} \to \mathfrak{cat}_{s\mathcal{S}\mathrm{et}}$ of [DK80a].

In this chapter, we set up an analogous framework in the setting of ∞*-categories*: we prove that the ∞-category $\mathcal{C}\mathrm{at}_{s\mathcal{S}}$ of ***∞-categories enriched in simplicial spaces*** likewise models the ∞-category of ∞-categories via an equivalence

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}}[\![\mathbf{W}_{\mathrm{DK}}^{-1}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_{\infty},$$

and we define a *hammock localization* functor $\mathscr{L}^{H} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty} \to \mathcal{C}\mathrm{at}_{s\mathcal{S}}$ which likewise provides a method of ***"introducing (even more) homotopy theory"*** into relative ∞-categories. We moreover prove the following two results – the first generalizing a theorem of Dwyer–Kan, the second generalizing joint work with Low (see [LMG15]).

Theorem (4.3.4)**.** *Given a relative ∞-category $(\mathcal{R}, \mathbf{W})$ admitting a* ***homotopical three-arrow calculus****, the hom-spaces in the underlying ∞-category of its hammock localization admit a canonical equivalence*

$$\underline{\mathbf{3}}(x, y)^{\mathrm{gpd}} \xrightarrow{\sim} \left|\underline{\hom}_{\mathscr{L}^{H}(\mathcal{R},\mathbf{W})}(x, y)\right|$$

from the groupoid completion of the ∞-category of three-arrow zigzags $x \xleftarrow{\approx} \bullet \to \bullet \xleftarrow{\approx} y$ in $(\mathcal{R}, \mathbf{W})$.

Theorem (4.5.1)**.** *Given a relative ∞-category $(\mathcal{R}, \mathbf{W})$, its* ***Rezk nerve***

$$\mathrm{N}_{\infty}^{\mathrm{R}}(\mathcal{R}, \mathbf{W}) \in s\mathcal{S}$$

- *is a* ***Segal space*** *if $(\mathcal{R}, \mathbf{W})$ admits a homotopical three-arrow calculus, and*
- *is moreover a* ***complete Segal space*** *if moreover $(\mathcal{R}, \mathbf{W})$ is saturated and satisfies the two-out-of-three property.*

(The notion of a *homotopical three-arrow calculus* is a minor variant on Dwyer–Kan's "homotopy calculus of fractions" (see Definition 4.3.1). Meanwhile, the *Rezk nerve* is a straightforward generalization of Rezk's "classification diagram" construction, which we introduced in Chapter 2 and proved computes the ∞-categorical localization (see Theorem 2.3.8 and Proposition 2.3.9).)

Remark 4.0.1. In Remark 4.1.21, we show how our notion of "$s\mathcal{S}$-enriched ∞-category" fits with the corresponding notion coming from Lurie's theory of *distributors*.

Remark 4.0.2. Many of the original Dwyer–Kan definitions and proofs are quite point-set in nature. However, when working ∞-categorically, it is essentially impossible to make such ad hoc constructions. Thus, we have no choice but to be both much more careful and much more precise in our generalization of their work.[1] We find Dwyer–Kan's facility with universal constructions (displayed in that proof and elsewhere) to be really quite impressive, and we hope that our elaboration on their techniques will be pedagogically useful. Broadly speaking, our main technique is to *corepresent* higher coherence data.

4.0.2 Outline

We now provide a more detailed outline of the contents of this chapter.

- In §4.1, we introduce the ∞-category $\mathcal{C}\mathrm{at}_{s\mathcal{S}}$ of ∞-categories enriched in simplicial spaces, as well as an auxiliary ∞-category $\mathcal{S}s\mathcal{S}$ of *Segal simplicial spaces*. We endow both of these with subcategories of *Dwyer–Kan weak equivalences*, and prove that the resulting relative ∞-categories both model the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories.

- In §4.2, we define the ∞-categories of zigzags in a relative ∞-category $(\mathcal{R}, \mathbf{W})$ between two objects $x, y \in \mathcal{R}$, and use these to define the *hammock simplicial spaces* $\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x, y)$, which will be the hom-simplicial spaces in the hammock localization $\mathscr{L}^H(\mathcal{R}, \mathbf{W})$.

- In §4.3, we define what it means for a relative ∞-category to admit a homotopical three arrow calculus, and we prove the first of the two results stated above.

- In §4.4, we finally construct the hammock localization functor on relative ∞-categories, and we explore some of its basic features.

- In §4.5, we prove the second of the two results stated above.

[1] For example, our proof of Theorem 4.3.4 spans nearly four pages whereas the proof of [DK80a, Proposition 6.2(i)] (which it generalizes) is just half a page long, and our proof of Proposition 4.4.8 is nearly three pages whereas the proof of [DK80a, Proposition 3.3] (which it generalizes) is not even provided.

4.0.3 Acknowledgments

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4.1 Segal spaces, Segal simplicial spaces, and $s\mathcal{S}$-enriched ∞-categories

In this section, we develop the theory – and the *homotopy* theory – of two closely related flavors of higher categories whose hom-objects lie in the symmetric monoidal ∞-category $(s\mathcal{S}, \times)$ of simplicial spaces equipped with the cartesian symmetric monoidal structure. By "homotopy theory", we mean that we will endow the ∞-categories of these objects with *relative ∞-category* structures, whose weak equivalences are created by "local" (i.e. hom-object-wise) geometric realization. These therefore constitute "many-object" elaborations on the Kan–Quillen relative ∞-category $(s\mathcal{S}, \mathbf{W}_{\mathrm{KQ}})$, whose weak equivalences are created by geometric realization (see Theorem 1.4.4). A key source of such objects will be the *hammock localization* functor, which we will introduce in §4.4.

This section is organized as follows.

- In §4.1.1, we recall some basic facts regarding *Segal spaces*.
- in §4.1.2, we introduce *Segal simplicial spaces* and define the essential notions for "doing (higher) category theory" with them.
- In §4.1.3, we introduce their full (in fact, coreflective) subcategory of *simplicio-spatially-enriched* (or simply *$s\mathcal{S}$-enriched*) *∞-categories*. These are useful since they can more directly be considered as "presentations of ∞-categories".
- In § 4.1.4, we prove that freely inverting the *Dwyer–Kan weak equivalences* among either the Segal simplicial spaces or the $s\mathcal{S}$-enriched ∞-categories yields an ∞-category which is canonically equivalent to $\mathcal{C}\mathrm{at}_\infty$ itself. We also contextualize both of these sorts of objects with respect to the theory of enriched ∞-categories based in the notion of a *distributor*, and provide some justification for our interest in them.

4.1.1 Segal spaces

We begin this section with the following recollections. This subsection exists mainly in order to set the stage for the remainder of the section; we refer the reader seeking a more thorough discussion either to the original paper [Rez01] (which uses model categories) or to [Lur09c, §1] (which uses ∞-categories).

Definition 4.1.1. The ∞-category of ***Segal spaces*** is the full subcategory $\mathcal{SS} \subset s\mathcal{S}$ of those simplicial spaces satisfying the *Segal condition*. These sit in a left localization adjunction

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{SS}}}{\overset{\mathrm{L}_{\mathcal{SS}}}{\rightleftarrows}} \mathcal{SS},$$

which factors the left localization adjunction $\mathrm{L}_{\mathcal{CSS}} \dashv \mathrm{U}_{\mathcal{CSS}}$ of Definition 2.2.1 in the sense that we obtain a pair of composable left localization adjunctions

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{SS}}}{\overset{\mathrm{L}_{\mathcal{SS}}}{\rightleftarrows}} \mathcal{SS} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\rightleftarrows}} \mathcal{CSS}.$$

(This follows easily from [Rez01, Theorems 7.1 and 7.2], or alternatively more-or-less follows from [Lur09c, Remark 1.2.11].)

In order to make a few basic observations, it will be convenient to first introduce the following.

Definition 4.1.2. Suppose that $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ admits finite products. Then, we define the $0^{\mathbf{th}}$ ***coskeleton*** of an object $c \in \mathcal{C}$ (or perhaps more standardly, of the corresponding constant simplicial object $\mathrm{const}(c) \in s\mathcal{C}$) to be the simplicial object selected by the composite

$$\mathbf{\Delta}^{op} \hookrightarrow (s\mathcal{S}\mathrm{et})^{op} \xrightarrow{((-)_0)^{op}} \mathcal{S}\mathrm{et}^{op} \hookrightarrow \mathcal{S}^{op} \xrightarrow{-\pitchfork c} \mathcal{C}.$$

This assembles to a functor

$$\mathcal{C} \xrightarrow{(-)^{\times(\bullet+1)}} s\mathcal{C}$$

which, as the notation suggests, is given in degree n by $c \mapsto c^{\times(n+1)}$. This sits in an adjunction

$$(-)_0 : s\mathcal{C} \rightleftarrows \mathcal{C} : (-)^{\times(\bullet+1)},$$

which we refer to as the $0^{\mathbf{th}}$ ***coskeleton adjunction*** for $\mathcal{C}$. Using this, given a simplicial object $Z \in s\mathcal{C}$ and a map $Y \xrightarrow{\varphi} Z_0$ in $\mathcal{C}$, we define the ***pullback*** of Z along

φ to be the fiber product

$$\varphi^*(Z) = \lim \left(\begin{array}{ccc} & & Z_\bullet \\ & & \downarrow \\ Y^{\times(\bullet+1)} & \xrightarrow[\varphi^{\times(\bullet+1)}]{} & (Z_0)^{\times(\bullet+1)} \end{array} \right)$$

in $s\mathcal{C}$, where the vertical map is the component at the object $Z \in s\mathcal{C}$ of the unit of the 0^{th} coskeleton adjunction. In particular, note that we have a canonical equivalence $(\varphi^*(Z))_0 \simeq Y$ in $\mathcal{C}$.

Remark 4.1.3. Suppose that $Y \in \mathcal{SS}$, and let us write $Y \xrightarrow{\lambda} \mathrm{L}_{\mathcal{CSS}}(Y)$ for its localization map. Then, the map $Y_0 \xrightarrow{\lambda_0} \mathrm{L}_{\mathcal{CSS}}(Y)_0$ is a surjection in $\mathcal{S}$, and moreover we have a canonical equivalence

$$Y \simeq (\lambda_0)^*(\mathrm{L}_{\mathcal{CSS}}(Y))$$

in $\mathcal{SS} \subset s\mathcal{S}$. (The first claim follows from [Rez01, Theorem 7.7 and Corollary 6.5], while the second claim follows from combining [Lur09c, Definition 1.2.12(b) and Theorem 1.2.13(2)] with the Segal condition for $Y \in s\mathcal{S}$.) From here, it follows easily that we have an equivalence

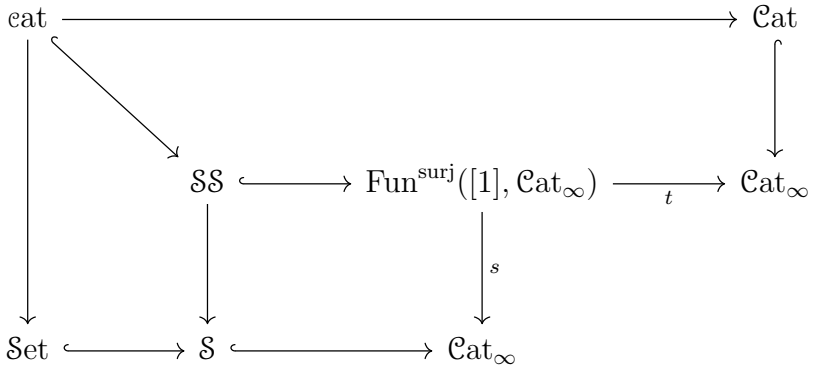
$$\mathcal{SS} \simeq \lim \left(\begin{array}{ccc} & & \mathrm{Fun}^{\mathrm{surj}}([1], \mathcal{C}\mathrm{at}_\infty) \\ & & \downarrow{\scriptstyle s} \\ \mathcal{S} & \hookrightarrow & \mathcal{C}\mathrm{at}_\infty \end{array} \right),$$

where $\mathrm{Fun}^{\mathrm{surj}}([1], \mathcal{C}\mathrm{at}_\infty) \subset \mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$ denotes the full subcategory on those functors $[1] \to \mathcal{C}\mathrm{at}_\infty$ that select surjective functors $\mathcal{C} \to \mathcal{D}$. From this viewpoint, the left localization $\mathrm{L}_{\mathcal{CSS}} : \mathcal{SS} \to \mathcal{CSS}$ is then just the composite functor

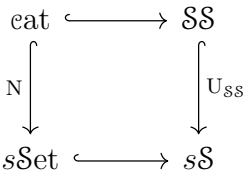
$$\mathcal{SS} \hookrightarrow \mathrm{Fun}^{\mathrm{surj}}([1], \mathcal{C}\mathrm{at}_\infty) \xrightarrow{t} \mathcal{C}\mathrm{at}_\infty \xrightarrow[\sim]{\mathrm{N}_\infty} \mathcal{CSS},$$

where $\mathrm{N}_\infty(-)_\bullet = \hom^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}_\infty}([\bullet], -)$ denotes the ∞-categorical nerve functor. Thus, one might think of $\mathcal{SS}$ as "the ∞-category of surjectively marked ∞-categories" (where by "surjectively marked" we mean "equipped with a surjective map from an ∞-groupoid").

Remark 4.1.4. Continuing with the observations of Remark 4.1.3, note that the category $\mathbb{e}\mathrm{at}$ of *strict* 1-categories can be recovered as a limit



in $\mathcal{C}\mathrm{at}_\infty$ (in which the square is already a pullback). (In fact, the inclusion $\mathbb{e}\mathrm{at} \hookrightarrow \mathcal{SS}$ itself fits into the defining pullback square



in $\mathcal{C}\mathrm{at}_\infty$.) We can therefore consider the ∞-category $\mathcal{SS}$ of Segal spaces as a close cousin of the 1-category $\mathbb{e}\mathrm{at}$ of strict categories, with the caveat that objects of $\mathbb{e}\mathrm{at}$ must be surjectively marked by a *discrete* space.

Remark 4.1.5. Suppose that $Y \in \mathcal{SS}$. Then, we can compute hom-spaces in the ∞-category

$$\mathcal{C} = \mathrm{N}_\infty^{-1}(\mathrm{L}_{\mathbb{e}\mathcal{SS}}(Y)) \in \mathcal{C}\mathrm{at}_\infty$$

as follows. Any pair of objects $x, y \in \mathcal{C}$ can be considered as defining a pair of points

$$x, y \in \mathcal{C}^{\simeq} \simeq \mathrm{N}_\infty(\mathcal{C})_0 \simeq \mathrm{L}_{\mathbb{e}\mathcal{SS}}(Y)_0.$$

Since the map $Y_0 \to \mathrm{L}_{\mathbb{e}\mathcal{SS}}(Y)_0$ is a surjection, these admit lifts $\tilde{x}, \tilde{y} \in Y_0$. Then, we have a composite equivalence

$$\hom_{\mathcal{C}}(x, y) \simeq \lim \left(\begin{array}{ccc} & & \mathrm{N}_\infty(\mathcal{C})_1 \\ & & \downarrow{\scriptstyle (s,t)} \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[(x,y)]{} & \mathrm{N}_\infty(\mathcal{C})_0 \times \mathrm{N}_\infty(\mathcal{C})_0 \end{array} \right) \simeq \lim \left(\begin{array}{ccc} & & Y_1 \\ & & \downarrow{\scriptstyle (s,t)} \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[(\tilde{x},\tilde{y})]{} & Y_0 \times Y_0 \end{array} \right)$$

by Remarks 2.2.2 and 4.1.3. (In particular, we can compute the hom-space $\hom_{\mathcal{C}}(x, y)$ using *any* choices of lifts $\tilde{x}, \tilde{y} \in Y_0$.)

4.1.2 Segal simplicial spaces

We now turn from the $\mathcal{S}$-enriched context to the $s\mathcal{S}$-enriched context.

Definition 4.1.6. We define the ∞-category of ***Segal simplicial spaces*** to be the full subcategory $\mathcal{S}s\mathcal{S} \subset s(s\mathcal{S})$ of those simplicial objects in $s\mathcal{S}$ which satisfy the Segal condition. These sit in a left localization adjunction $s(s\mathcal{S}) \rightleftarrows \mathcal{S}s\mathcal{S}$ by the adjoint functor theorem (Corollary T.5.5.2.9). To ease our discussion, we take the convention that these are organized according to the diagram

$$\begin{array}{ccc} \vdots & \vdots & \iddots \\ (\mathcal{C}_0)_1 \;\substack{\longleftarrow \\ \longrightarrow\\ \longleftarrow}\; (\mathcal{C}_1)_1 & & \cdots \\ \downarrow\uparrow\downarrow \quad\quad \downarrow\uparrow\downarrow & & \\ (\mathcal{C}_0)_0 \;\substack{\longleftarrow \\ \longrightarrow\\ \longleftarrow}\; (\mathcal{C}_1)_0 & & \cdots \end{array}$$

in $\mathcal{S}$: we think of the columns as the "internal" simplicial spaces, and denote them as $\mathcal{C}_n = (\mathcal{C}_n)_\bullet \in s\mathcal{S}$ (omitting the outer index if it's irrelevant for the discussion). Thus, the Segal condition applies in the horizontal direction, and asserts that the Segal map

$$\mathcal{C}_n \to \mathcal{C}_1 \underset{\mathcal{C}_0}{\times} \cdots \underset{\mathcal{C}_0}{\times} \mathcal{C}_1$$

is an equivalence in $s\mathcal{S}$.

Remark 4.1.7. In light of Remark 4.1.4, we can consider the ∞-category $\mathcal{S}s\mathcal{S}$ of Segal simplicial spaces as being a homotopical analog of the 1-category $s\mathcal{C}\mathrm{at} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C}\mathrm{at})$ of simplicial categories. The subcategory $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \subset s\mathcal{C}\mathrm{at}$ of $s\mathcal{S}$et-enriched categories then corresponds to the full subcategory on those Segal simplicial spaces $\mathcal{C}_\bullet \in \mathcal{S}s\mathcal{S}$ such that the object $\mathcal{C}_0 \in s\mathcal{S}$ is constant. We will restrict our attention to such objects in §4.1.3.

Definition 4.1.8. For any $\mathcal{C}_\bullet \in \mathcal{S}s\mathcal{S}$, we define the ***space of objects*** of $\mathcal{C}_\bullet$ to be the space

$$(\mathcal{C}_0)_0 \simeq \hom_{s\mathcal{S}}(\mathrm{pt}_{s\mathcal{S}}, \mathcal{C}_0) \in \mathcal{S},$$

and for any $x, y \in (\mathcal{C}_0)_0$, we define the ***hom-simplicial space*** from x to y in $\mathcal{C}_\bullet$ to be the pullback

$$\underline{\hom}_{\mathcal{C}_\bullet}(x,y) = \lim \left(\begin{array}{ccc} & & \mathcal{C}_1 \\ & & \downarrow{\scriptstyle (s,t)} \\ \mathrm{pt}_{s\mathcal{S}} & \xrightarrow[(x,y)]{} & \mathcal{C}_0 \times \mathcal{C}_0 \end{array} \right)$$

in $s\mathcal{S}$. We refer to the points of the space

$$\underline{\hom}_{\mathcal{C}_\bullet}(x,y)_0 \simeq \hom_{s\mathcal{S}}(\mathrm{pt}_{s\mathcal{S}}, \underline{\hom}_{\mathcal{C}_\bullet}(x,y))$$

simply as ***morphisms*** from x to y. The various hom-simplicial spaces of $\mathcal{C}_\bullet$ admit associative composition maps

$$\underline{\hom}_{\mathcal{C}_\bullet}(x_0,x_1) \times \cdots \times \underline{\hom}_{\mathcal{C}_\bullet}(x_{n-1},x_n) \xrightarrow{\chi^{\mathcal{C}_\bullet}_{x_0,\ldots,x_n}} \underline{\hom}_{\mathcal{C}_\bullet}(x_0,x_n)$$

in $s\mathcal{S}$, which are obtained as usual via the Segal conditions. For any $x \in (\mathcal{C}_0)_0$ there is an evident ***identity morphism*** from x to itself, denoted $\underline{\mathrm{id}}_x \in \underline{\hom}_{\mathcal{C}_\bullet}(x,x)_0$, which behaves as expected under these composition maps.

Definition 4.1.9. Given any $\mathcal{C}_\bullet \in \mathcal{S}s\mathcal{S}$ and any pair of objects $x, y \in (\mathcal{C}_0)_0$, we say that two morphisms

$$\mathrm{pt}_{s\mathcal{S}} \rightrightarrows \underline{\hom}_{\mathcal{C}_\bullet}(x,y)$$

are ***simplicially homotopic*** if the induced maps

$$\mathrm{pt}_{\mathcal{S}} \rightrightarrows |\underline{\hom}_{\mathcal{C}_\bullet}(x,y)|$$

are equivalent (i.e. select points in the same path component of the target). We then say that a morphism $f \in \underline{\hom}_{\mathcal{C}_\bullet}(x,y)_0$ is a ***simplicial homotopy equivalence*** if there exists a morphism $g \in \underline{\hom}_{\mathcal{C}_\bullet}(y,x)_0$ such that the composite morphisms

$$\chi^{\mathcal{C}_\bullet}_{x,y,x}(f,g) \in \underline{\hom}_{\mathcal{C}_\bullet}(x,x)$$

and

$$\chi^{\mathcal{C}_\bullet}_{y,x,y}(g,f) \in \underline{\hom}_{\mathcal{C}_\bullet}(y,y)$$

are simplicially homotopic to the respective identity morphisms.

Now, the objects of $\mathcal{S}s\mathcal{S}$ will indeed be "presentations of ∞-categories", but maps between them which are not equivalences may nevertheless induce equivalences between the ∞-categories that they present. We therefore introduce the following notion.

Definition 4.1.10. A map $\mathcal{C}_\bullet \xrightarrow{\varphi_\bullet} \mathcal{D}_\bullet$ in $\mathcal{S}s\mathcal{S}$ is called a ***Dwyer–Kan weak equivalence*** if

- it is ***weakly fully faithful***, i.e. for all pairs of objects $x, y \in (\mathcal{C}_0)_0$ the induced map
$$\left|\underline{\hom}_{\mathcal{C}_\bullet}(x,y)\right| \to \left|\underline{\hom}_{\mathcal{D}_\bullet}(\varphi(x),\varphi(y))\right|$$
is an equivalence in $\mathcal{S}$, and
- it is ***weakly surjective***, i.e. the map
$$\pi_0((\mathcal{C}_0)_0) \xrightarrow{\pi_0((\varphi_0)_0)} \pi_0((\mathcal{D}_0)_0)$$
is surjective up to the equivalence relation on $\pi_0((\mathcal{D}_0)_0)$ generated by simplicial homotopy equivalence.

Such morphisms define a subcategory $\mathbf{W}_{\mathrm{DK}} \subset \mathcal{S}s\mathcal{S}$ containing all the equivalences and satisfying the two-out-of-three property, and we denote the resulting relative ∞-category by $\mathcal{S}s\mathcal{S}_{\mathrm{DK}} = (\mathcal{S}s\mathcal{S}, \mathbf{W}_{\mathrm{DK}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.

Remark 4.1.11. Via the evident functor $\mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to \mathcal{S}s\mathcal{S}$ (recall Remark 4.1.7), the subcategory of Dwyer–Kan weak equivalences $\mathbf{W}_{\mathrm{DK}}^{\mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}} \subset \mathbb{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ of §4.0.1 (i.e. the subcategory of weak equivalences for the Bergner model structure) is pulled back from the subcategory $\mathbf{W}_{\mathrm{DK}}^{\mathcal{S}s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$.

4.1.3 $s\mathcal{S}$-enriched ∞-categories

In light of the discussion of §4.1.2, the natural guess for the sense in which a Segal simplicial space should be considered as a "presentation of an ∞-category" is via the levelwise geometric realization functor

$$s(s\mathcal{S}) \xrightarrow{s(|-|)} s\mathcal{S}.$$

However, this operation does not preserve Segal objects: taking fiber products of simplicial spaces does not generally commute with taking their geometric realizations. On the other hand, these two operations *do* commute when the common target of the cospan is constant. Hence, it will be convenient to restrict our attention to the following special class of objects.

Definition 4.1.12. We define the ∞-category of ***simplicio-spatially-enriched ∞-categories***, or simply of ***$s\mathcal{S}$-enriched ∞-categories***, to be the full subcategory

$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$ on those objects $\mathcal{C}_\bullet \in \mathcal{S}s\mathcal{S} \subset s(s\mathcal{S})$ such that $\mathcal{C}_0 \in s\mathcal{S}$ is constant. We write

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \xhookrightarrow{\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}} \mathcal{S}s\mathcal{S}$$

for the defining inclusion. Restricting the subcategory $\mathbf{W}_{\mathrm{DK}}^{\mathcal{S}s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$ of Dwyer–Kan weak equivalences along this inclusion, we obtain a relative ∞-category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}})_{\mathrm{DK}} = (\mathcal{C}\mathrm{at}_{s\mathcal{S}}, \mathbf{W}_{\mathrm{DK}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ (which also has the two-out-of-three property).

Lemma 4.1.13. *There is a canonical factorization*

$$\begin{array}{ccccccc} \mathcal{C}\mathrm{at}_{s\mathcal{S}} & \xhookrightarrow{\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}} & \mathcal{S}s\mathcal{S} & \hookrightarrow & s(s\mathcal{S}) & \xrightarrow{s(|-|)} & s\mathcal{S} \\ & \searrow & & & & & \uparrow \\ & & & & & & \mathcal{S}\mathcal{S} \end{array}$$

of the restriction of the levelwise geometric realization functor

$$s(s\mathcal{S}) \xrightarrow{s(|-|)} s\mathcal{S}$$

to the subcategory $\mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset s(s\mathcal{S})$ of $s\mathcal{S}$-enriched ∞-categories.

Proof. Choose any $\mathcal{C}_\bullet \in \mathcal{C}\mathrm{at}_{s\mathcal{S}}$. Since the functor $\mathcal{S}\mathcal{S} \hookrightarrow s\mathcal{S}$ is the inclusion of a full subcategory, it suffices to show that $s(|\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}(\mathcal{C}_\bullet)|) \in \mathcal{S}\mathcal{S}$, for which in turn it suffices to show that the evident map

$$|\mathcal{C}_n| \to |\mathcal{C}_1| \underset{|\mathcal{C}_0|}{\times} |\mathcal{C}_{n-1}|$$

is an equivalence. Towards this aim, write

$$|\mathcal{C}_0| \simeq \coprod_i |\mathcal{C}_0|_i$$

for the decomposition of $|\mathcal{C}_0| \in \mathcal{S}$ into its connected components; since by assumption $\mathcal{C}_0 \simeq \mathrm{const}(|\mathcal{C}_0|)$, this induces a decomposition

$$\mathcal{C}_0 \simeq \mathrm{const}\left(\coprod_i |\mathcal{C}_0|_i\right) \simeq \coprod_i \mathrm{const}(|\mathcal{C}_0|_i)$$

of $\mathcal{C}_0 \in s\mathcal{S}$. Let us similarly write

$$\mathcal{C}_1 \simeq \coprod_i (\mathcal{C}_1)_i$$

and

$$\mathcal{C}_{n-1} \simeq \coprod_i (\mathcal{C}_{n-1})_i$$

for the resulting pulled back decompositions. Then, using Lemma A.5.5.6.17 (applied to the ∞-topos $\mathcal{S}$) and the fact that coproducts commute with connected limits, we can identify the target of the above map as

$$\begin{aligned} |\mathcal{C}_1| \underset{|\mathcal{C}_0|}{\times} |\mathcal{C}_{n-1}| &\simeq \coprod_i \left(|(\mathcal{C}_1)_i| \underset{|\mathcal{C}_0|_i}{\times} |(\mathcal{C}_{n-1})_i| \right) \\ &\simeq \coprod_i \left| (\mathcal{C}_1)_i \underset{\mathrm{const}(|\mathcal{C}_0|_i)}{\times} (\mathcal{C}_{n-1})_i \right| \\ &\simeq \left| \coprod_i \left((\mathcal{C}_1)_i \underset{\mathrm{const}(|\mathcal{C}_0|_i)}{\times} (\mathcal{C}_{n-1})_i \right) \right| \\ &\simeq \left| \mathcal{C}_1 \underset{\mathcal{C}_0}{\times} \mathcal{C}_{n-1} \right| . \end{aligned}$$

As $\mathcal{C}_\bullet$ satisfies the Segal condition by assumption, this proves the claim. □

Remark 4.1.14. could also just ask levelwise pi-0 to be constant The proof of Lemma 4.1.13 shows that it would suffice to make the weaker assumption that the object $\pi_0^{\mathrm{lw}}(\mathcal{C}_0) \in s\mathcal{S}\mathrm{et}$ is constant in order to conclude that $s(|\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}(\mathcal{C}_\bullet)|) \in \mathcal{S}\mathcal{S}$.

Definition 4.1.15. We denote simply by

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \xrightarrow{|-|} \mathcal{S}\mathcal{S}$$

the factorization of Lemma 4.1.13, and refer to it as the ***geometric realization*** functor on $s\mathcal{S}$-enriched ∞-categories.

Definition 4.1.16. The composite inclusion

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow[\sim]{\mathrm{N}_\infty} \mathcal{C}\mathcal{S}\mathcal{S} \xhookrightarrow{\mathrm{U}_{\mathcal{C}\mathcal{S}\mathcal{S}}} s(\mathcal{S}) \xhookrightarrow{s(\mathrm{const})} s(s\mathcal{S})$$

clearly factors through the subcategory $\mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset \mathcal{S}s\mathcal{S} \subset s(s\mathcal{S})$. We simply write

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{const}} \mathcal{C}\mathrm{at}_{s\mathcal{S}}$$

for this factorization, and refer to it as the ***constant*** $s\mathcal{S}$***-enriched*** ∞***-category*** functor. Thus, for an ∞-category $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, the simplicial object

$$\mathrm{const}(\mathcal{C})_\bullet \in \mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset s(s\mathcal{S})$$

is given in degree n by

$$\mathrm{const}(\mathrm{N}_\infty(\mathcal{C})_n) \in s\mathcal{S},$$

the constant simplicial space on the object

$$\mathrm{N}_\infty(\mathcal{C})_n = \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C}) \in \mathcal{S}.$$

This functor clearly participates in a commutative diagram

$$\begin{array}{ccccc} \mathcal{C}\mathrm{at}_\infty & \xrightarrow{\mathrm{const}} & \mathcal{C}\mathrm{at}_{s\mathcal{S}} & \xrightarrow{|-|} & \mathcal{S}\mathcal{S} \\ & \searrow^{\sim}_{N_\infty} & & & \uparrow{\scriptstyle \mathrm{U}_{\mathrm{ess}}} \\ & & & & \mathcal{C}\mathcal{S}\mathcal{S} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$.

Remark 4.1.17. Suppose we are given a Segal simplicial space $\mathcal{C}_\bullet \in \mathcal{S}s\mathcal{S}$ and a map $Z \to (\mathcal{C}_0)_0$ in $\mathcal{S}$ to its space of objects. Write $\mathrm{const}(Z) \xrightarrow{\varphi} \mathcal{C}_0$ for the corresponding map in $s\mathcal{S}$. Then, the canonical map

$$\varphi^*(\mathcal{C}_\bullet) \to \mathcal{C}_\bullet$$

is *fully faithful* (in the $s\mathcal{S}$-*enriched* sense): for any objects $x, y \in Z \simeq (\varphi^*(\mathcal{C}_\bullet)_0)_0$, the induced map

$$\underline{\hom}_{\varphi^*(\mathcal{C}_\bullet)}(x, y) \to \underline{\hom}_{\mathcal{C}_\bullet}(\varphi(x), \varphi(y))$$

is already an equivalence in $s\mathcal{S}$ (instead of just being an equivalence upon geometric realization). Of course, the map $\varphi^*(\mathcal{C}_\bullet) \to \mathcal{C}_\bullet$ is therefore in particular weakly fully faithful as well. As we can always choose our original map $Z \xrightarrow{\varphi} (\mathcal{C}_0)_0$ so that the induced map $\varphi^*(\mathcal{C}_\bullet) \to \mathcal{C}_\bullet$ is additionally weakly surjective (e.g. by taking φ to be a surjection), it follows that any Segal simplicial space admits a Dwyer–Kan weak equivalence from a $s\mathcal{S}$-enriched category; indeed, we can even arrange to have $Z \in \mathcal{S}\mathrm{et} \subset \mathcal{S}$.

Improving on Remark 4.1.17, we now describe a *universal* way of extracting a $s\mathcal{S}$-enriched ∞-category from a Segal simplicial space.

Definition 4.1.18. We define the ***spatialization*** functor $\mathrm{sp} : \mathcal{S}s\mathcal{S} \to \mathcal{C}\mathrm{at}_{s\mathcal{S}}$ as follows.[2] Any $\mathcal{C}_\bullet \in \mathcal{S}s\mathcal{S}$ gives rise to a natural map

$$\mathrm{const}((\mathcal{C}_0)_0) \xrightarrow{\varepsilon} \mathcal{C}_0$$

[2]The word "spatialization" is meant to indicate that the 0^{th} object of its output will lie in the subcategory $\mathcal{S} \subset s\mathcal{S}$ of constant simplicial spaces.

in $s\mathcal{S}$, the component at $\mathcal{C}_0 \in s\mathcal{S}$ of the counit of the right localization adjunction $\mathrm{const} : \mathcal{S} \rightleftarrows s\mathcal{S} : \lim$. The spatialization of $\mathcal{C}_\bullet$ is then the pullback

$$\mathrm{sp}(\mathcal{C}_\bullet) = \varepsilon^*(\mathcal{C}_\bullet).$$

(Note that the fiber product of Definition 4.1.2 that yields this pullback may be equivalently taken either in $\mathcal{S}s\mathcal{S}$ or in $s(s\mathcal{S})$, in light of the left localization adjunction of Definition 4.1.6.) This clearly assembles to a functor, and in fact it is not hard to see that this participates in a right localization adjunction

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \underset{\mathrm{sp}}{\overset{\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}}{\rightleftarrows}} \mathcal{S}s\mathcal{S},$$

whose counit components $\mathrm{sp}(\mathcal{C}_\bullet) \to \mathcal{C}_\bullet$ are Dwyer–Kan weak equivalences (which are even fully faithful as in Remark 4.1.17).

4.1.4 $\mathcal{S}s\mathcal{S}$ and $\mathcal{C}\mathrm{at}_{s\mathcal{S}}$ as presentations of $\mathcal{C}\mathrm{at}_\infty$

The following pair of results asserts that both $s\mathcal{S}$-enriched ∞-categories and Segal simplicial spaces, equipped with their respective subcategories of Dwyer–Kan weak equivalences, present the ∞-category of ∞-categories.

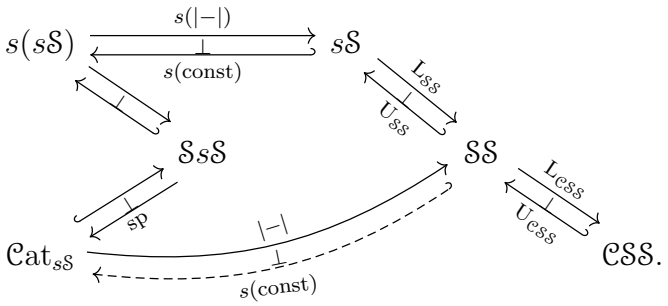
Proposition 4.1.19. *The composite functor*

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \xrightarrow{|-|} \mathcal{S}\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{C}\mathcal{S}\mathcal{S}}} \mathcal{C}\mathcal{S}\mathcal{S} \simeq \mathcal{C}\mathrm{at}_\infty$$

induces an equivalence

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}}[\mathbf{W}_{\mathrm{DK}}^{-1}] \xrightarrow{\sim} \mathcal{C}\mathcal{S}\mathcal{S} \simeq \mathcal{C}\mathrm{at}_\infty.$$

Proof. So far, we have obtained the solid diagram



The right adjoint of the composite left localization adjunction

$$s(s\mathcal{S}) \underset{s(\mathrm{const})}{\overset{s(|-|)}{\rightleftarrows}} s\mathcal{S} \underset{\mathrm{U}_{\mathcal{S}\mathcal{S}}}{\overset{\mathrm{L}_{\mathcal{S}\mathcal{S}}}{\rightleftarrows}} \mathcal{S}\mathcal{S}$$

clearly lands in the full subcategory $\mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset s(s\mathcal{S})$, and hence restricts to give the right adjoint of a left localization adjunction as indicated by the dotted arrow above. This composes to a left localization adjunction

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \underset{s(\mathrm{const})}{\overset{|-|}{\rightleftarrows}} \mathcal{SS} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\rightleftarrows}} \mathcal{CSS}.$$

Moreover, the definition of Dwyer–Kan weak equivalence is precisely chosen so that the composite left adjoint creates the subcategory $\mathbf{W}_{\mathrm{DK}} \subset \mathcal{C}\mathrm{at}_{s\mathcal{S}}$ (i.e. it is pulled back from the subcategory of equivalences (see Definition 2.1.5)). Hence, by Example 2.1.13, it does indeed induce an equivalence

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}}[\![\mathbf{W}_{\mathrm{DK}}^{-1}]\!] \xrightarrow{\sim} \mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_{\infty},$$

as desired. □

Proposition 4.1.20. *Both adjoints in the right localization adjunction*

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \underset{\mathrm{sp}}{\overset{\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}}{\rightleftarrows}} \mathcal{S}s\mathcal{S}$$

are functors of relative ∞-categories (with respect to their respective Dwyer–Kan relative structures), and moreover they induce inverse equivalences

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}}[\![(\mathbf{W}_{\mathrm{DK}}^{\mathcal{C}\mathrm{at}_{s\mathcal{S}}})^{-1}]\!] \simeq \mathcal{S}s\mathcal{S}[\![(\mathbf{W}_{\mathrm{DK}}^{\mathcal{S}s\mathcal{S}})^{-1}]\!]$$

in $\mathcal{C}\mathrm{at}_{\infty}$ on localizations.

Proof. The left adjoint inclusion is a functor of relative ∞-categories by definition. On the other hand, suppose that $\mathcal{C}_\bullet \xrightarrow{\approx} \mathcal{D}_\bullet$ is a map in $\mathbf{W}_{\mathrm{DK}}^{\mathcal{S}s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$. Via the right localization adjunction, its spatialization fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{sp}(\mathcal{C}_\bullet) & \xrightarrow{\approx} & \mathcal{C}_\bullet \\ \downarrow & & \downarrow \wr \\ \mathrm{sp}(\mathcal{D}_\bullet) & \xrightarrow[\approx]{} & \mathcal{D}_\bullet \end{array}$$

in $\mathcal{S}s\mathcal{S}_{\mathrm{DK}}$, and hence is also in $\mathbf{W}_{\mathrm{DK}}^{\mathcal{S}s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$ by the two-out-of-three property. This shows that the right adjoint is also a functor of relative ∞-categories.

To see that these adjoints induce inverse equivalences on localizations, note that the composite

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \xhookrightarrow{\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}} \mathcal{S}s\mathcal{S} \xrightarrow{\mathrm{sp}} \mathcal{C}\mathrm{at}_{s\mathcal{S}}$$

is the identity, while the composite

$$\mathcal{S}s\mathcal{S} \xrightarrow{\mathrm{sp}} \mathcal{C}\mathrm{at}_{s\mathcal{S}} \xrightarrow{\mathrm{U}_{\mathcal{C}\mathrm{at}_{s\mathcal{S}}}} \mathcal{S}s\mathcal{S}$$

admits a natural weak equivalence in $\mathcal{S}s\mathcal{S}_{\mathrm{DK}}$ to the identity functor (namely, the counit of the adjunction). Hence, the claim follows from Lemma 2.1.24. □

To conclude this section, we make a pair of general remarks regarding $\mathcal{S}s\mathcal{S}$ and $\mathcal{C}\mathrm{at}_{s\mathcal{S}}$. We begin by contextualizing these ∞-categories with respect to Lurie's theory of enriched ∞-categories, which is described in [Lur09c, §1].

Remark 4.1.21. Lurie's theory of enriched ∞-categories – which provides a satisfactory, compelling, and apparently complete picture (at least when the enriching ∞-category is equipped with the *cartesian* symmetric monoidal structure) – is premised on the notion of a *distributor*, the data of which is simply an ∞-category $\mathcal{Y}$ equipped with a full subcategory $\mathcal{X} \subset \mathcal{Y}$ (see [Lur09c, Definition 1.2.1]).[3] Given such a distributor, one can then define ∞-categories $\mathcal{SS}_{\mathcal{X}\subset\mathcal{Y}}$ and $\mathcal{CSS}_{\mathcal{X}\subset\mathcal{Y}}$ of *Segal space objects* and of *complete Segal space objects* with respect to it: these sit as full (in fact, reflective) subcategories

$$\mathcal{CSS}_{\mathcal{X}\subset\mathcal{Y}} \subset \mathcal{SS}_{\mathcal{X}\subset\mathcal{Y}} \subset s\mathcal{Y},$$

in which

- the subcategory $\mathcal{SS}_{\mathcal{X}\subset\mathcal{Y}} \subset s\mathcal{Y}$ consists of those simplicial objects $Y_\bullet \in s\mathcal{Y}$ such that
  - $Y_\bullet$ satisfies the Segal condition and
  - $Y_0 \in \mathcal{X}$

  (see [Lur09c, Definition 1.2.7]), while
- the subcategory $\mathcal{CSS}_{\mathcal{X}\subset\mathcal{Y}} \subset \mathcal{SS}_{\mathcal{X}\subset\mathcal{Y}}$ consists of those objects which additionally satisfy a certain *completeness* condition (see [Lur09c, Definition 1.2.10]).

Thus, $\mathcal{Y}$ plays the role of the "enriching ∞-category", i.e. the ∞-category containing the hom-objects in our enriched ∞-category, while its subcategory $\mathcal{X} \subset \mathcal{Y}$ provides a home for the "object of objects" of the enriched ∞-category. As in the classical case – indeed, the identity distributor $\mathcal{S} \subset \mathcal{S}$ simply has $\mathcal{SS}_{\mathcal{S}\subset\mathcal{S}} \simeq \mathcal{SS}$ and $\mathcal{CSS}_{\mathcal{S}\subset\mathcal{S}} \simeq \mathcal{CSS}$ –, one can already meaningfully extract an enriched ∞-category from a Segal space object, but it is only by restricting to the complete ones that one obtains the desired ∞-category of such.

[3] Note that there is a typo in [Lur09c, Definition 1.2.1]: condition (4) should say that the functor $\mathcal{X} \to (\mathcal{C}\mathrm{at}_\infty)^{op}$ preserves *colimits*, not limits. This is clear from [Lur09c, Example 1.2.3] (see Lemma T.6.1.3.7 and Definition T.6.1.3.8).

Now, obviously we have

$$\mathcal{S}s\mathcal{S} \simeq \mathcal{S}\mathcal{S}_{s\mathcal{S}\subset s\mathcal{S}},$$

as Segal simplicial spaces are nothing but Segal space objects with respect to the identity distributor $s\mathcal{S} \subset s\mathcal{S}$ on the ∞-category $s\mathcal{S}$ of simplicial spaces. We can clearly also identify the ∞-category of $s\mathcal{S}$-enriched ∞-categories as

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}} \simeq \mathcal{S}\mathcal{S}_{\mathcal{S}\subset s\mathcal{S}},$$

the Segal space objects with respect to the distributor $\mathcal{S} \subset s\mathcal{S}$ (the embedding of spaces as the constant simplicial spaces).[4,5] On the other hand, the subcategory of *complete* Segal space objects can be identified as the pullback

$$\begin{array}{ccc} \mathcal{CSS}_{\mathcal{S}\subset s\mathcal{S}} & \hookrightarrow & \mathcal{S}\mathcal{S}_{\mathcal{S}\subset s\mathcal{S}} \\ \downarrow & & \downarrow \\ \mathcal{CSS} & \hookrightarrow & \mathcal{S}\mathcal{S} \end{array}$$

in which the right vertical functor takes an $s\mathcal{S}$-enriched ∞-category $\mathcal{C}_\bullet \in \mathcal{S}\mathcal{S}_{\mathcal{S}\subset s\mathcal{S}} \simeq \mathcal{C}\mathrm{at}_{s\mathcal{S}}$ to its "levelwise 0^{th} space" object $(\mathcal{C}_\bullet)_0 \in \mathcal{S}\mathcal{S}$.

We now explain the source of our interest in the ∞-categories $\mathcal{S}s\mathcal{S}$ and $\mathcal{C}\mathrm{at}_{s\mathcal{S}}$.

Remark 4.1.22. First and foremost, the reason we are interested in $\mathcal{S}s\mathcal{S}$ is because this is the natural target of the "pre-hammock localization" functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}^H_{\mathrm{pre}}} \mathcal{S}s\mathcal{S},$$

whose construction constitutes the main ingredient of the construction of the hammock localization functor itself (see §4.4). On the other hand, we then restrict to the (coreflective) subcategory $\mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$ since this is a convenient full subcategory of $\mathcal{S}s\mathcal{S} \subset s(s\mathcal{S})$ on which the levelwise geometric realization functor

$$s(s\mathcal{S}) \xrightarrow{s(|-|)} s\mathcal{S}$$

[4]To see that the inclusion $\mathcal{S} \subset s\mathcal{S}$ of the full subcategory of constant objects is a distributor, note that if $\mathcal{Y}$ is an ∞-topos and $\mathcal{X} \subset \mathcal{Y}$ is a full subcategory which is stable under limits and colimits, then $\mathcal{X} \subset \mathcal{Y}$ is automatically a distributor. The only remaining point is to verify condition (4) of [Lur09c, Definition 1.2.1]. The functor $\mathcal{X} \to (\mathcal{C}\mathrm{at}_\infty)^{op}$ is given on objects by $x \mapsto (\mathcal{Y}_{/x})^\circ$, with functoriality given by pullback in $\mathcal{Y}$. This clearly factors as the composite $\mathcal{X} \hookrightarrow \mathcal{Y} \to (\mathcal{C}\mathrm{at}_\infty)^{op}$, in which the latter functor is similarly given by $y \mapsto (\mathcal{Y}_{/y})^\circ$, which then preserves colimits by Proposition T.6.1.3.10 and Theorem T.6.1.3.9.

[5]In contrast with Remark 4.1.7, $s\mathcal{S}$-enriched ∞-categories do not quite have an analog in ordinary category theory, only in enriched category theory. (It is only a coincidence of the special case presently under study that the two ∞-categories $\mathcal{S}$ and $s\mathcal{S}$ participating in the distributor appear to be so closely related.)

(which is a colimit) preserves the Segal condition (which is defined in terms of limits) (recall (the proof of) Lemma 4.1.13).[6] Indeed, if our "local geometric realization" functor failed to preserve the Segal condition, it would necessarily destroy all "category-ness" inherent in our objects of study. In turn, this would effectively invalidate our right to declare the hammock simplicial spaces

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x,y) \in s\mathcal{S}$$

(see Definition 4.2.17) – which will of course be the hom-simplicial spaces in the hammock localization $\mathscr{L}^H(\mathcal{R},\mathbf{W}) \in \mathcal{C}\mathrm{at}_{s\mathcal{S}}$ – as "presentations of hom-spaces" in any reasonable sense.

For these reasons, Segal simplicial spaces are therefore not really our primary interest. However, since for a Segal simplicial space $\mathcal{C}_\bullet \in \mathcal{S}s\mathcal{S}$, the counit $\mathrm{sp}(\mathcal{C}_\bullet) \to \mathcal{C}_\bullet$ of the spatialization right localization adjunction is actually fully faithful in the $s\mathcal{S}$-enriched sense, the hammock localization

$$\mathscr{L}^H(\mathcal{R},\mathbf{W}) = \mathrm{sp}(\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})) \in \mathcal{C}\mathrm{at}_{s\mathcal{S}}$$

will then simultaneously

- have the hammock simplicial spaces as its hom-simplicial spaces, and
- have composition maps which both
  - directly present composition in its geometric realization, and
  - manifestly encode the notion of "concatenation of zigzags".

Of course, it would also be possible to restrict further to the (reflective) subcategory

$$\mathcal{CSS}_{\mathcal{S}\subset s\mathcal{S}} \subset \mathcal{SS}_{\mathcal{S}\subset s\mathcal{S}} \simeq \mathcal{C}\mathrm{at}_{s\mathcal{S}}$$

of complete Segal space objects (recall Remark 4.1.21). However, this is unnecessary for our purposes, since both the pre-hammock localization functor and the hammock localization functor will land in ∞-categories (namely $\mathcal{S}s\mathcal{S}$ and $\mathcal{C}\mathrm{at}_{s\mathcal{S}}$, respectively) which admit canonical relative structures via which they present the ∞-category $\mathcal{C}\mathrm{at}_\infty$, thus endowing these constructions with external meaning (which are of course compatible with each other in light of Proposition 4.1.20). Moreover, as the successive inclusions

$$\mathcal{CSS}_{\mathcal{S}\subset s\mathcal{S}} \subset \mathcal{SS}_{\mathcal{S}\subset s\mathcal{S}} \simeq \mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$$

respectively admit a *left* adjoint and a *right* adjoint, this further restriction would in all probability make for a somewhat messier story.

[6] In light of Proposition 4.1.19, it seems unnecessary to use the larger subcategory of $\mathcal{S}s\mathcal{S}$ afforded by **??**.

4.2 Zigzags and hammocks in relative ∞-categories

In studying relative 1-categories and their 1-categorical localizations, one is naturally led to study *zigzags*. Given a relative category $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ and a pair of objects $x, y \in \mathcal{R}$, a zigzag from x to y is a diagram of the form

$$x \overset{\approx}{\leftarrow} \cdots \to \cdots \overset{\approx}{\leftarrow} \cdots \to \cdots \overset{\approx}{\leftarrow} y,$$

i.e. a sequence of both forwards and backwards morphisms in $\mathcal{R}$ (in arbitrary (finite) quantities and in any order) such that all backwards morphisms lie in $\mathbf{W} \subset \mathcal{R}$. Under the 1-categorical localization $\mathcal{R} \to \mathcal{R}[\mathbf{W}^{-1}]$, such a diagram is taken to a sequence of morphisms such that all backwards maps are *isomorphisms*, so that it is in effect just a sequence of composable (forwards) arrows. Taking their composite, we obtain a single morphism $x \to y$ in $\mathcal{R}[\mathbf{W}^{-1}]$. In fact, one can explicitly construct $\mathcal{R}[\mathbf{W}^{-1}]$ in such a way that *all* of its morphisms arise from this procedure.

It is a good deal more subtle to show, but in fact the same is true of relative ∞-*categories* and their (∞-categorical) localizations: given a relative ∞-category $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, it turns out that every morphism in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ can likewise be presented by a zigzag in $(\mathcal{R}, \mathbf{W})$ itself. (We prove a precise statement of this assertion as Proposition 4.2.11.)

The representation of a morphism in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ by a zigzag in $(\mathcal{R}, \mathbf{W})$ is quite clearly overkill: many different zigzags in $(\mathcal{R}, \mathbf{W})$ will present the same morphism in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$. For example, we can consider a zigzag as being selected by a morphism $\underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W})$ of relative ∞-categories, where $\underline{\mathbf{m}} \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ is a *zigzag type* which is determined by the shape of the zigzag in question; then, precomposition with a suitable morphism $\underline{\mathbf{m}}' \to \underline{\mathbf{m}}$ of zigzag types will yield a composite $\underline{\mathbf{m}}' \to \underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W})$ which presents a canonically equivalent morphism in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$. Thus, in order to obtain a closer approximation to $\hom_{\mathcal{R}[\![\mathbf{W}^{-1}]\!]}(x, y)$, we should take a *colimit* of the various spaces of zigzags from x to y indexed over the category of zigzag types.

However, this colimit alone will still not generally capture all the redundancy inherent in the representation of morphisms in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ by zigzags in $(\mathcal{R}, \mathbf{W})$. Namely, a *natural weak equivalence* between two zigzags of the same type (which fixes the endpoints) will, upon postcomposing to the localization $\mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$, yield a *homotopy* between the morphisms presented by the respective zigzags. Pursuing this observation, we are thus led to consider certain ∞-*categories*, denoted $\underline{\mathbf{m}}(x, y)$ (for varying zigzag types $\underline{\mathbf{m}}$), whose objects are the $\underline{\mathbf{m}}$-shaped zigzags from x to y and whose morphisms are the natural weak equivalences (fixing x and y) between them.

Finally, putting these two observations of redundancy together, we see that in order to approximate the hom-space $\hom_{\mathcal{R}[\mathbf{W}^{-1}]}(x, y)$, we should be taking a colimit of the various ∞-categories $\underline{\mathbf{m}}(x, y)$ over the category of zigzag types. In fact, rather than taking a colimit of these ∞-categories, we will take a colimit of their corresponding *complete Segal spaces* (see §2.2), not within the ∞-category $\mathcal{CSS}$ of such but rather within the larger ambient ∞-category $s\mathcal{S}$ in which it is definitionally contained; this, finally, will yield the *hammock simplicial space* $\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x, y) \in s\mathcal{S}$, which (as the notation suggests) will be the hom-simplicial space in the hammock localization $\mathscr{L}^H(\mathcal{R}, \mathbf{W}) \in \mathcal{C}\mathrm{at}_{s\mathcal{S}}$.[7]

This section is organized as follows.

- In §4.2.1, we lay some groundwork regarding *doubly-pointed* relative ∞-categories, which will allow us to efficiently corepresent our ∞-categories of zigzags.
- In §4.2.2, we use this to define ∞-categories of zigzags in a relative ∞-category.
- In §4.2.3, we prove a precise articulation of the assertion made above, that all morphisms in the localization $\mathcal{R}[\mathbf{W}^{-1}]$ are represented by zigzags in $(\mathcal{R}, \mathbf{W})$.
- In §4.2.4, we finally define our hammock simplicial spaces and compare them with the hammock simplicial sets of Dwyer–Kan (in the special case of a relative 1-category).
- In §4.2.5, we assemble some technical results regarding zigzags in relative ∞-categories which will be useful later; notably, we prove that for a concatenation $[\underline{\mathbf{m}}; \underline{\mathbf{m}}']$ of zigzag types, we can recover the ∞-category $[\underline{\mathbf{m}}; \underline{\mathbf{m}}'](x, y)$ via the *two-sided Grothendieck construction* (see Definition 3.3.3).

4.2.1 Doubly-pointed relative ∞-categories

In this subsection, we make a number of auxiliary definitions which will streamline our discussion throughout the remainder of this chapter.

[7] As the functor $\mathrm{L}_{\mathcal{CSS}} : s\mathcal{S} \to \mathcal{CSS}$ is left adjoint to the inclusion $\mathcal{CSS} \subset s\mathcal{S}$ and hence in particular commutes with colimits, its application to the hammock simplicial space will yield the aforementioned colimit of ∞-categories. Moreover, since we are ultimately interested in hammock simplicial spaces for their geometric realizations, in view of Proposition 2.2.4 we can consider this shift in ambient ∞-category merely as a technical convenience. For instance, there is an evident explicit description of the constituent spaces in the hammock simplicial space (analogous to the 1-categorical case (see [DK80a, 2.1])).

Definition 4.2.1. A ***doubly-pointed relative ∞-category*** is a relative ∞-category $(\mathcal{R}, \mathbf{W})$ equipped with a map $\mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \sqcup \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \to \mathcal{R}$. The two inclusions $\mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \hookrightarrow \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \sqcup \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty}$ select objects $s, t \in \mathcal{R}$, which we call the ***source*** and the ***target***; we will sometimes subscript these to remove ambiguity, e.g. as $s_{\mathcal{R}}$ and $t_{\mathcal{R}}$. These assemble into the evident ∞-category, which we denote by

$$(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**} = (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{(\mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \sqcup \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty})/}.$$

Of course, there is a forgetful functor $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. We will often implicitly consider a relative ∞-category $(\mathcal{R}, \mathbf{W})$ equipped with two chosen objects $x, y \in \mathcal{R}$ as a doubly-pointed relative ∞-category; on the other hand, we may also write $((\mathcal{R}, \mathbf{W}), x, y) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$ to be more explicit. We write $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \subset (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$ for the full subcategory of ***doubly-pointed relative categories***, i.e. of those doubly-pointed relative ∞-categories whose underlying ∞-category is a 1-category.

Notation 4.2.2. Recall from Notation 2.1.6 that $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ is a cartesian closed symmetric monoidal ∞-category. With respect to this structure, $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$ is enriched and tensored over $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. As for the enrichment, for any $(\mathcal{R}_1\mathbf{W}_1), (\mathcal{R}_2, \mathbf{W}_2) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$, we define the object

$$\left(\mathrm{Fun}_{**}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}_{**}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\right) = \lim \left(\begin{array}{ccc} & & \left(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\right) \\ & & \downarrow{\scriptstyle (\mathrm{ev}_{s_1}, \mathrm{ev}_{t_1})} \\ \{(s_2, t_2)\} & \hookrightarrow & (\mathcal{R}_2, \mathbf{W}_2) \times (\mathcal{R}_2, \mathbf{W}_2) \end{array} \right)$$

of $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ (where we write $s_1, t_1 \in \mathcal{R}_1$ and $s_2, t_2 \in \mathcal{R}_2$ to distinguish between the source and target objects); informally, this should be thought of as the relative ∞-category whose objects are the doubly-pointed relative functors from $(\mathcal{R}_1, \mathbf{W}_1)$ to $(\mathcal{R}_2, \mathbf{W}_2)$, whose morphisms are the doubly-pointed natural transformations between these (i.e. those natural transformations whose components at s_1 and t_1 are id_{s_2} and id_{t_2}, resp.), and whose weak equivalences are the doubly-pointed natural weak equivalences. Then, the tensoring is obtained by taking $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ and $(\mathcal{R}_1, \mathbf{W}_1) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$ to the pushout

$$\mathrm{colim} \left(\begin{array}{ccc} \mathcal{R} \times \{s, t\} & \longrightarrow & \mathcal{R} \times \mathcal{R}_1 \\ \downarrow & & \\ \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \times \{s, t\} & & \end{array} \right)$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, with its double-pointing given by the natural map from $\mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \sqcup \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \simeq \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \times \{s,t\}$. We will write

$$(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**} \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{-\odot-} (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$$

to denote this tensoring.

Notation 4.2.3. In order to simultaneously refer to the situations of unpointed and doubly-pointed relative ∞-categories, we will use the notation $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{(**)}$ (and similarly for other related notations). When we use this notation, we will mean for the entire statement to be interpreted either in the unpointed context or the doubly-pointed context.

Notation 4.2.4. We will write

$$(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{(**)} \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{-\odot-} (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{(**)}$$

to denote either the tensoring of Notation 4.2.2 in the doubly-pointed case or else simply the cartesian product in the unpointed case.

4.2.2 Zigzags in relative ∞-categories

In this subsection we introduce the first of the two key concepts of this section, namely the ∞-*categories of zigzags* in a relative ∞-category between two given objects.

We begin by defining the objects which will corepresent our ∞-categories of zigzags.

Definition 4.2.5. We define a ***relative word*** to be a (possibly empty) word $\underline{\mathbf{m}}$ in the symbols $\mathbf{A}$ (for "any arbitrary arrow") and $\mathbf{W}^{-1}$. We will write $\mathbf{A}^{\circ n}$ to denote n consecutive copies of the symbol $\mathbf{A}$ (for any $n \geq 0$), and similarly for $(\mathbf{W}^{-1})^{\circ n}$. We can extract a doubly-pointed relative category from a relative word, which for our sanity we will carry out by reading *forwards*. So for instance, the relative word $\underline{\mathbf{m}} = [\mathbf{A}; (\mathbf{W}^{-1})^{\circ 2}; \mathbf{A}^{\circ 2}]$ defines the doubly-pointed relative category

$$s \longrightarrow \bullet \xleftarrow{\approx} \bullet \xleftarrow{\approx} \bullet \longrightarrow \bullet \longrightarrow t.$$

We denote this object by $\underline{\mathbf{m}} \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**}$. Thus, by convention, the empty relative word determines the terminal object $[\varnothing] \simeq \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**}} \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**}$ (which is the unique relative word determining a doubly-pointed relative category whose source and target objects are equivalent). Restricting to the *order-preserving* maps between relative words (with respect to the evident ordering on their objects, i.e. starting from s and

ending at t), we obtain a (non-full) subcategory $\mathcal{Z} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**}$ of ***zigzag types***.[8,9,10] We will occasionally also use this same relative word notation with the symbol $\mathbf{W}$, but the resulting doubly-pointed relative categories will not be objects of $\mathcal{Z}$.

Remark 4.2.6. Let $\underline{\mathbf{m}}, \underline{\mathbf{m}}' \in \mathcal{Z} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \subset (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$ be relative words. Then, their concatenation can be characterized as a pushout

$$\begin{array}{ccc} \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} & \xrightarrow{s} & \underline{\mathbf{m}}' \\ {\scriptstyle t}\downarrow & & \downarrow \\ \underline{\mathbf{m}} & \longrightarrow & [\underline{\mathbf{m}}; \underline{\mathbf{m}}'] \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ (as well as in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$).

Notation 4.2.7. For any $\underline{\mathbf{m}} \in \mathcal{Z}$, we will write $|\underline{\mathbf{m}}|_{\mathbf{A}} \in \mathbb{N}$ to denote the number of times that $\mathbf{A}$ appears in $\underline{\mathbf{m}}$, and we will write $|\underline{\mathbf{m}}|_{\mathbf{W}^{-1}} \in \mathbb{N}$ to denote the number of times that $\mathbf{W}^{-1}$ appears in $\underline{\mathbf{m}}$.

Remark 4.2.8. The localization functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$$

acts on the subcategory $\mathcal{Z} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ of zigzag types as

$$\mathscr{L}(\underline{\mathbf{m}}) \simeq [|\underline{\mathbf{m}}|_{\mathbf{A}}] \in \mathbf{\Delta} \subset \mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty :$$

in effect, it collapses all the copies of $[\mathbf{W}^{-1}]$ and leaves the copies of $[\mathbf{A}]$ untouched.

We now define the first of the two key concepts of this section, an analog of [DK80a, 5.1].

Definition 4.2.9. Given a relative ∞-category $(\mathcal{R}, \mathbf{W})$ equipped with two chosen objects $x, y \in \mathcal{R}$, and given a relative word $\underline{\mathbf{m}} \in \mathcal{Z}$, we define the ∞-category of ***zigzags*** in $(\mathcal{R}, \mathbf{W})$ from x to y of type $\underline{\mathbf{m}}$ to be

$$\underline{\mathbf{m}}_{(\mathcal{R},\mathbf{W})}(x, y) = \mathrm{Fun}_{**}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}}.$$

If the relative ∞-category $(\mathcal{R}, \mathbf{W})$ is clear from context, we will simply write $\underline{\mathbf{m}}(x, y)$.

[8] Note that the objects of $\mathcal{Z}$ can in fact be considered as *strict* doubly-pointed relative categories, and moreover $\mathcal{Z}$ itself can be considered as a *strict* category. However, as we will only use these objects in invariant manipulations, we will not need these observations.

[9] Omitting the terminal relative word from $\mathcal{Z}$ (and considering it as a strict category), we obtain the opposite of the indexing category $\mathbf{II}$ of [DK80a, 4.1]. We prefer to include this terminal object: it is the unit object for a monoidal structure on $\mathcal{Z}$ given by concatenation, which will play a key role in the definition of the hammock localization (see Construction 4.4.1).

[10] Note that an order-preserving map must lay each morphism $[\mathbf{A}]$ across some $[\mathbf{A}^{\circ m}]$ (for some $m \geq 0$), and must lay each morphism $[\mathbf{W}^{-1}]$ across some $[(\mathbf{W}^{-1})^{\circ n}]$ (for some $n \geq 0$). In particular, it cannot lay a morphism $[\mathbf{A}]$ across a morphism $[\mathbf{W}^{-1}]$ (or vice versa, of course).

4.2.3 Representing maps in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ by zigzags in $(\mathcal{R}, \mathbf{W})$

In this subsection, we take a digression to illustrate that our study of zigzags in relative ∞-categories is well-founded: roughly speaking, we show that any morphism in the localization of a relative ∞-category is represented by a zigzag in the relative ∞-category itself. We will give the precise assertion as Proposition 4.2.11. In order to state it, however, we first introduce the following terminology.

Definition 4.2.10. Let $(\mathcal{R}, \mathbf{W}_{\mathcal{R}})$ and $(\mathcal{D}, \mathbf{W}_{\mathcal{D}})$ be relative ∞-categories. We will say that a morphism

$$(\mathcal{D}, \mathbf{W}_{\mathcal{D}}) \to (\mathcal{R}, \mathbf{W}_{\mathcal{R}})$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ ***represents*** the morphism

$$\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!] \to \mathcal{R}[\![\mathbf{W}_{\mathcal{R}}^{-1}]\!]$$

in $\mathcal{C}\mathrm{at}_\infty$ induced by the localization functor. We will also say that it represents the morphism

$$\mathrm{ho}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!]) \to \mathrm{ho}(\mathcal{R}[\![\mathbf{W}_{\mathcal{R}}^{-1}]\!])$$

in $\mathcal{C}\mathrm{at}$ induced from the previous one by the homotopy category functor. In a slight abuse of terminology, we will moreover say that a zigzag

$$\underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W}_{\mathcal{R}})$$

represents the composite

$$[1] \to \mathscr{L}(\underline{\mathbf{m}}) \to \mathcal{R}[\![\mathbf{W}_{\mathcal{R}}^{-1}]\!]$$

in $\mathcal{C}\mathrm{at}_\infty$, where the map $[1] \to \mathscr{L}(\underline{\mathbf{m}}) \simeq [|\underline{\mathbf{m}}|_{\mathbf{A}}]$ is given by $0 \mapsto 0$ and $1 \mapsto |\underline{\mathbf{m}}|_{\mathbf{A}}$ (i.e. it corepresents the operation of composition), and likewise for the morphism in the homotopy category $\mathrm{ho}(\mathcal{R}[\![\mathbf{W}_{\mathcal{R}}^{-1}]\!])$ of the localization selected by either three-fold composite in the commutative diagram

$$\begin{array}{ccc}
[1] & & \\
\searrow & & \\
\mathscr{L}(\underline{\mathbf{m}}) & \longrightarrow & \mathcal{R}[\![\mathbf{W}_{\mathcal{R}}^{-1}]\!] \\
\downarrow \wr & & \downarrow \\
\mathrm{ho}(\mathscr{L}(\underline{\mathbf{m}})) & \longrightarrow & \mathrm{ho}(\mathcal{R}[\![\mathbf{W}_{\mathcal{R}}^{-1}]\!])
\end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proposition 4.2.11. *Let* $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *be a relative* ∞*-category, and let* $[1] \xrightarrow{F} \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ *be a functor selecting a morphism in its localization. Then, for some relative word* $\underline{\mathbf{m}} \in \mathcal{Z}$*, there exists a zigzag* $\underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W})$ *which represents* F*.*

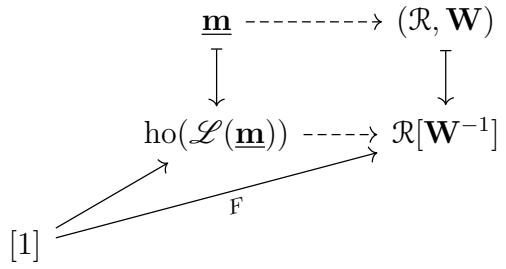
We will prove Proposition 4.2.11 in stages of increasing generality. We begin by recalling that any morphism in the *1-categorical* localization of a relative *1-category* is represented by a zigzag.

Lemma 4.2.12. *Let* $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ *be a relative 1-category, and let* $[1] \xrightarrow{F} \mathcal{R}[\mathbf{W}^{-1}]$ *be a functor selecting a morphism in its 1-categorical localization. Then, for some relative word* $\underline{\mathbf{m}} \in \mathcal{Z}$*, there exists a zigzag* $\underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W})$ *which represents* F*.*

Proof. This follows directly from the standard construction of the 1-categorical localization of a relative 1-category (see e.g. [DK80a, Proposition 3.1]). □

Remark 4.2.13. Lemma 4.2.12 accounts for the fundamental role that zigzags play in the theory of relative categories and their 1-categorical localizations. We can therefore view Proposition 4.2.11 as asserting that zigzags play an analogous fundamental role in the theory of relative ∞-categories and their (∞-categorical) localizations.

Remark 4.2.14. We can view Lemma 4.2.12 as guaranteeing the existence of a diagram



for some relative word $\underline{\mathbf{m}} \in \mathcal{Z}$, in which

- the upper dotted arrow is a morphism in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$,
- the lower dotted arrow is its image under the 1-categorical localization functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{ho}} \mathcal{C}\mathrm{at},$$

  and

- the map $[1] \to \mathrm{ho}(\mathscr{L}(\underline{\mathbf{m}})) \simeq \mathrm{ho}([|\underline{\mathbf{m}}|_{\mathbf{A}}]) \simeq [|\underline{\mathbf{m}}|_{\mathbf{A}}]$ is as in Definition 4.2.10.

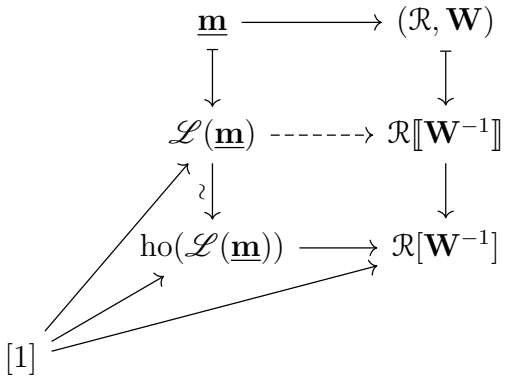
With Lemma 4.2.12 recalled, we now move on to the case of ∞*-categorical* localizations of relative 1-categories.

Lemma 4.2.15. *Let* $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ *be a relative 1-category, and let* $[1] \xrightarrow{F} \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ *be a functor selecting a morphism in its localization. Then, for some relative word* $\underline{\mathbf{m}} \in \mathcal{Z}$*, there exists a zigzag* $\underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W})$ *which represents* F*.*

Proof. Recall from Remark 2.1.29 that we have an equivalence $\mathrm{ho}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \xrightarrow{\sim} \mathcal{R}[\mathbf{W}^{-1}]$. The resulting postcomposition

$$[1] \xrightarrow{F} \mathcal{R}[\![\mathbf{W}^{-1}]\!] \to \mathrm{ho}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \xrightarrow{\sim} \mathcal{R}[\mathbf{W}^{-1}]$$

of F with the projection to the homotopy category selects a morphism in the 1-categorical localization $\mathcal{R}[\mathbf{W}^{-1}]$. Hence, by Lemma 4.2.12, we obtain a diagram



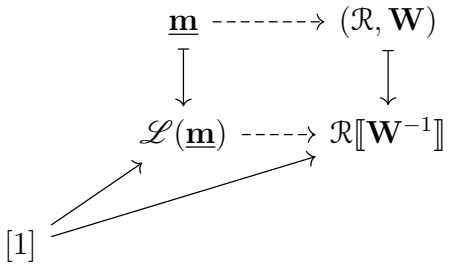
for some relative word $\underline{\mathbf{m}} \in \mathcal{Z}$, in which

- the solid horizontal arrows are as in Remark 4.2.14,
- the upper map in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ induces the dotted map under the functor $\mathscr{L} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$, so that
- the (lower) square in $\mathcal{C}\mathrm{at}_\infty$ commutes.

That the resulting composite

$$[1] \to \mathscr{L}(\underline{\mathbf{m}}) \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$$

is equivalent to the functor $[1] \xrightarrow{F} \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ follows from Lemma 4.2.16. Thus, in effect, we obtain a diagram



analogous to the one in Remark 4.2.14 (only with the 1-categorical localizations replaced by the ∞-categorical localizations), which proves the claim. □

Lemma 4.2.16. *For any ∞-category $\mathcal{C}$ and any map $[1] \to \mathrm{ho}(\mathcal{C})$, the space of lifts*

$$\begin{array}{ccc} & & \mathcal{C} \\ & \nearrow & \downarrow \\ [1] & \longrightarrow & \mathrm{ho}(\mathcal{C}) \end{array}$$

is connected.

Proof. Since the functor $\mathcal{C} \to \mathrm{ho}(\mathcal{C})$ creates the subcategory $\mathcal{C}^{\simeq} \subset \mathcal{C}$, there is a connected space of lifts of the maximal subgroupoid $\{0,1\} \simeq [1]^{\simeq} \subset [1]$. Then, in any solid commutative square

$$\begin{array}{ccc} [1]^{\simeq} & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ [1] & \longrightarrow & \mathrm{ho}(\mathcal{C}) \end{array}$$

there exists a connected space of dotted lifts by definition of the homotopy category. □

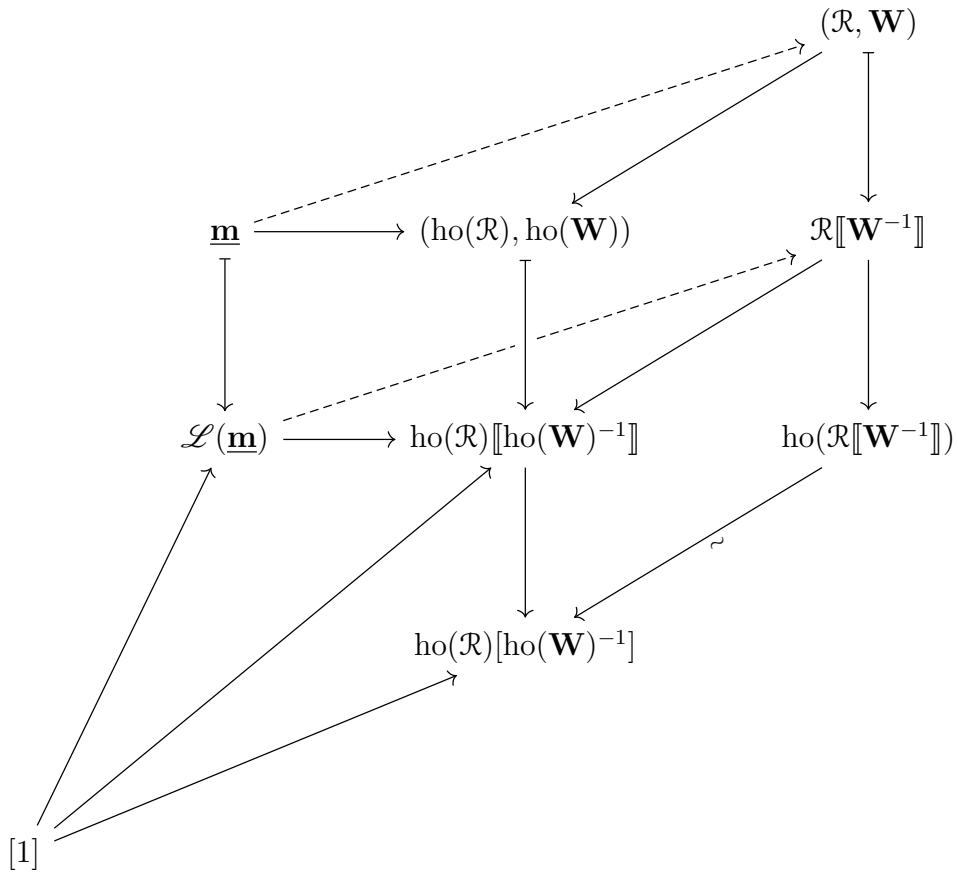
With Lemma 4.2.15 in hand, we now proceed to the fully general case of ∞-categorical localizations of relative ∞-*categories*.

Proof of Proposition 4.2.11. Observe that the morphism $(\mathcal{R}, \mathbf{W}) \to (\mathrm{ho}(\mathcal{R}), \mathrm{ho}(\mathbf{W}))$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ induces a postcomposition

$$[1] \xrightarrow{F} \mathcal{R}[\![\mathbf{W}^{-1}]\!] \to \mathrm{ho}(\mathcal{R})[\![\mathrm{ho}(\mathbf{W})^{-1}]\!]$$

selecting a morphism in the ∞-categorical localization of the relative 1-category

$(\mathrm{ho}(\mathcal{R}), \mathrm{ho}(\mathbf{W})) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$. Hence, by Lemma 4.2.15, we obtain a solid diagram



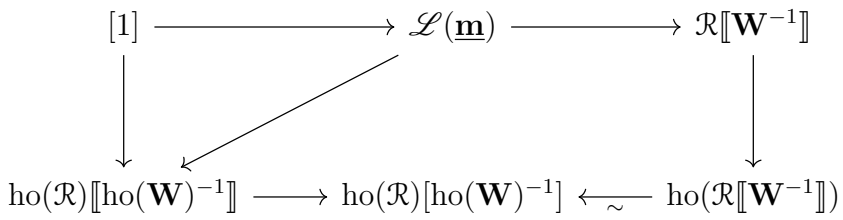
for some relative word $\underline{\mathbf{m}} \in \mathcal{Z}$, in which

- the lower right diagonal map is an equivalence by Remark 2.1.29,
- we moreover obtain the upper dotted arrow from Remark 4.2.6 by induction, and
- we define the lower dotted arrow to be its image under localization.

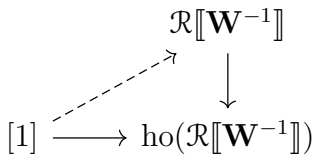
Now, the resulting composite

$$[1] \to \mathscr{L}(\underline{\mathbf{m}}) \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$$

fits into a commutative diagram



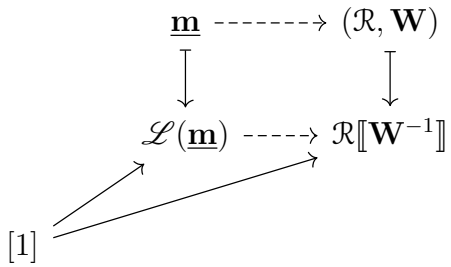
in $\mathcal{C}\mathrm{at}_\infty$. In particular, we have obtained a lift



of the composite

$$[1] \xrightarrow{F} \mathcal{R}[\![\mathbf{W}^{-1}]\!] \to \mathrm{ho}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]),$$

which must therefore be equivalent to F itself by Lemma 4.2.16. Thus, we obtain a diagram



as in the proof of Lemma 4.2.15, which proves the claim. □

Thus, zigzags play an important role not just in the theory of relative 1-categories and their 1-categorical localizations, but more generally in the theory of relative ∞-categories and their ∞-categorical localizations.

4.2.4 Hammocks in relative ∞-categories

For a general relative ∞-category $(\mathcal{R}, \mathbf{W})$, the representation of a morphism in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$ by a zigzag $\underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W})$ guaranteed by Proposition 4.2.11 is clearly far from unique. Indeed, any morphism $\underline{\mathbf{m}}' \to \underline{\mathbf{m}}$ in $\mathcal{Z}$ gives rise to a composite $\underline{\mathbf{m}}' \to \underline{\mathbf{m}} \to (\mathcal{R}, \mathbf{W})$ which presents the same morphism in $\mathcal{R}[\![\mathbf{W}^{-1}]\!]$: in other words,

the morphisms in $\mathcal{Z}$ corepresent *universal equivalence relations* between zigzags in relative ∞-categories (with respect to the morphisms that they represent upon localization).

In order to account for this over-representation, we are led to the following definition, the second of the two key concepts of this section, an analog of [DK80a, 2.1].

Definition 4.2.17. Suppose $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, and suppose $x, y \in \mathcal{R}$. We define the ***simplicial space of hammocks*** (or alternatively the ***hammock simplicial space***) in $(\mathcal{R}, \mathbf{W})$ from x to y to be the colimit

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x, y) = \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(\underline{\mathbf{m}}(x, y)) \in s\mathcal{S}.$$

We will extend the hammock simplicial space construction further – and in particular, justify its notation – by constructing the *hammock localization*

$$\mathscr{L}^H(\mathcal{R}, \mathbf{W}) \in \mathcal{C}\mathrm{at}_{s\mathcal{S}}$$

of $(\mathcal{R}, \mathbf{W})$ in §4.4 (see Remark 4.4.5).

We now compare our hammock simplicial spaces of Definition 4.2.17 with Dwyer–Kan's classical hammock simplicial *sets* (in relative 1-categories).

Remark 4.2.18. Suppose that $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ is a relative category. Then, by [DK80a, Proposition 5.5], we have an identification

$$\underline{\hom}_{\mathscr{L}^H_\delta(\mathcal{R},\mathbf{W})}(x, y) \cong \mathrm{colim}^{s\mathcal{S}\mathrm{et}}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}(\underline{\mathbf{m}}(x, y))$$

of the classical simplicial *set* of hammocks defined in [DK80a, 2.1] as an analogous colimit over the *1-categorical* nerves of the (strict) categories of zigzags in $(\mathcal{R}, \mathbf{W})$ from x to y.[11] However, there are two reasons that this does not coincide with Definition 4.2.17.

- The colimit computing $\underline{\hom}_{\mathscr{L}^H_\delta(\mathcal{R},\mathbf{W})}(x, y)$ is taken in the subcategory $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$. This inclusion (being a right adjoint) does not generally commute with colimits.

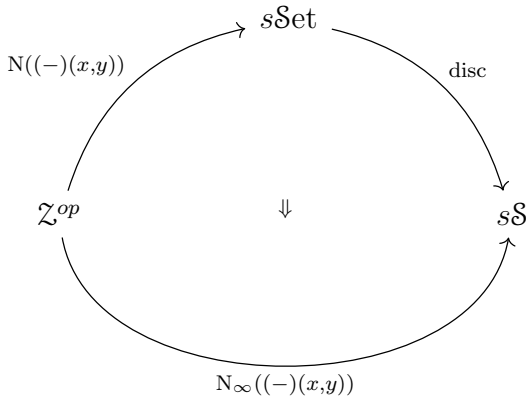
- The functors $\mathcal{C}\mathrm{at} \xrightarrow{\mathrm{N}} s\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S}$ and $\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}_\infty} s\mathcal{S}$ do not generally agree, but are only related by a natural transformation

$$\begin{array}{ccc} \mathcal{C}\mathrm{at} & \xrightarrow{\mathrm{N}} & s\mathcal{S}\mathrm{et} \\ \downarrow & \swarrow & \downarrow{\scriptstyle \mathrm{disc}} \\ \mathcal{C}\mathrm{at}_\infty & \xrightarrow[\mathrm{N}_\infty]{} & s\mathcal{S} \end{array}$$

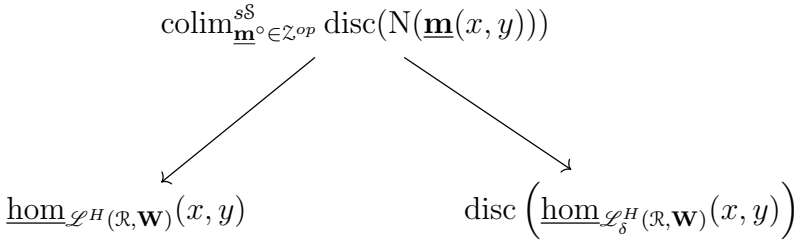
[11] It is not hard to see that the presence of the initial object $[\varnothing]^\circ \in \mathcal{Z}^{op}$ (which is what distinguishes this indexing category from $\mathbf{\Pi}$) does not change this colimit.

in $\mathrm{Fun}(\mathcal{C}at, s\mathcal{S})$ (see Remark 2.2.6).

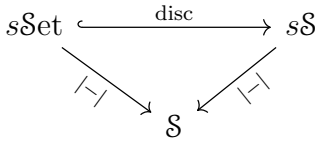
On the other hand, these two constructions do at least participate in a diagram



in $\mathcal{C}\mathrm{at}_\infty$, which induces a span



in $s\mathcal{S}$. We claim that this span lies in the subcategory $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$, i.e. that it becomes an equivalence upon geometric realization; as we have a commutative triangle



in $\mathcal{C}\mathrm{at}_\infty$, this will imply that we have a canonical equivalence

$$\left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x,y)\right| \simeq \left|\underline{\hom}_{\mathscr{L}^H_\delta(\mathcal{R},\mathbf{W})}(x,y)\right|$$

in $\mathcal{S}$. We view this as a satisfactory state of affairs, since we are only ultimately interested in simplicial sets/spaces of hammocks as presentations of hom-spaces, anyways.

To see the claim, note first that since $|-| : s\mathcal{S} \to \mathcal{S}$ is a left adjoint, it commutes with colimits, and so the left leg of the span lies in $\mathbf{W}_{\mathrm{KQ}}$ by the fact that upon postcomposition with the geometric realization functor $|-| : s\mathcal{S} \to \mathcal{S}$, the natural transformation

$$\mathrm{disc} \circ \mathrm{N} \to \mathrm{N}_\infty$$

in $\mathrm{Fun}(\mathbb{C}\mathrm{at}, s\mathcal{S})$ becomes a natural equivalence

$$|-| \circ \mathrm{disc} \circ \mathrm{N} \xrightarrow{\sim} |-| \circ \mathrm{N}_\infty$$

in $\mathrm{Fun}(\mathbb{C}\mathrm{at}, \mathcal{S})$ (again see Remark 2.2.6). By Proposition 2.2.4, these geometric realizations of colimits in $s\mathcal{S}$ both evaluate to

$$\mathrm{colim}^{\mathcal{S}}_{\underline{\mathbf{m}}^\circ \in \mathcal{Z}^{op}} \underline{\mathbf{m}}(x,y)^{\mathrm{gpd}}.$$

Now, in order to compute the geometric realization

$$\left|\mathrm{disc}\left(\underline{\hom}_{\mathscr{L}^H_\delta(\mathcal{R},\mathbf{W})}(x,y)\right)\right| \simeq \left|\underline{\hom}_{\mathscr{L}^H_\delta(\mathcal{R},\mathbf{W})}(x,y)\right|,$$

we begin by observing that that the category $\mathcal{Z}$ has an evident Reedy structure, which one can verify has cofibrant constants, so that the dual Reedy structure on $\mathcal{Z}^{op}$ has fibrant constants. Moreover, it is not hard to verify that the functor

$$\mathcal{Z}^{op} \xrightarrow{\mathrm{N}((-)(x,y))} s\mathcal{S}\mathrm{et}$$

defines a cofibrant object of $\mathrm{Fun}(\mathcal{Z}^{op}, s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. Hence, the colimit

$$\underline{\hom}_{\mathscr{L}^H_\delta(\mathcal{R},\mathbf{W})}(x,y) \cong \mathrm{colim}^{s\mathcal{S}\mathrm{et}}_{\underline{\mathbf{m}}^\circ \in \mathcal{Z}^{op}} \mathrm{N}(\underline{\mathbf{m}}(x,y))$$

computes the homotopy colimit in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, i.e. the colimit of the composite

$$\mathcal{Z}^{op} \xrightarrow{\mathrm{N}((-)(x,y))} s\mathcal{S}\mathrm{et} \xrightarrow{|-|} s\mathcal{S}\mathrm{et}[\![\mathbf{W}^{-1}_{\mathrm{KQ}}]\!] \simeq \mathcal{S}.$$

The claim then follows from the string of equivalences

$$|-| \circ \mathrm{N} \simeq |-| \circ \mathrm{disc} \circ \mathrm{N} \simeq |-| \circ \mathrm{N}_\infty \simeq (-)^{\mathrm{gpd}}$$

in $\mathrm{Fun}(\mathbb{C}\mathrm{at}, \mathcal{S})$ (again appealing to Proposition 2.2.4).

Remark 4.2.19. Dwyer–Kan give a point-set definition of the hammock simplicial *set* in [DK80a, 2.1], and then prove it is isomorphic to the colimit indicated in Remark 4.2.18. However, working ∞-categorically, it is essentially impossible to make such an ad hoc definition. Thus, we have simply *defined* our hammock simplicial space as the colimit to which we would like it to be equivalent anyways.

4.2.5 Functoriality and gluing for zigzags

In this subsection, we prove that ∞-categories of zigzags are suitably functorial for weak equivalences among source and target objects (see Notation 4.2.23), and we use this to give a formula for an ∞-category of zigzags of type $[\underline{\mathbf{m}}; \underline{\mathbf{m}}']$, the concatenation of two arbitrary relative words $\underline{\mathbf{m}}, \underline{\mathbf{m}}' \in \mathcal{Z}$ (see Lemma 4.2.24).

Recall from Remark 4.2.6 that concatenations of relative words compute pushouts in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. This allows for inductive arguments, in which at each stage we freely adjoin a new morphism along either its source or its target. For these, we will want to have a certain functoriality property for diagrams of this shape. To describe it, let us first work in the special case of $\mathcal{C}\mathrm{at}_\infty$ (instead of $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$). There, if for instance we have an ∞-category $\mathcal{D}'$ with a chosen object $d \in \mathcal{D}'$ and we use this to define a new ∞-category $\mathcal{D}$ as the pushout

$$\begin{array}{ccc} \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow{t} & [1] \\ {\scriptstyle d}\downarrow & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{D}, \end{array}$$

then for any target ∞-category $\mathcal{C}$, the evaluation

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}([1], \mathcal{C}) \xrightarrow{s} \mathcal{C}$$

will be a cartesian fibration by Corollary T.2.4.7.12 (applied to the functor $\mathrm{Fun}(\mathcal{D}', \mathcal{C}) \xrightarrow{\mathrm{ev}_d} \mathcal{C}$). The following result is then an analog of this observation for *relative* ∞-categories; note that there are now two types of "freely adjoined morphisms" we must consider.

Lemma 4.2.20. *Let $(\mathcal{I}', \mathbf{W}_{\mathcal{I}'}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, choose any $i \in \mathcal{I}'$, and suppose we are given any $(\mathcal{R}, \mathbf{W}_{\mathcal{R}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.*

(1) (a) If we form the pushout

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{s} & [\mathbf{W}] \\ {\scriptstyle i}\downarrow & & \downarrow \\ (\mathcal{I}', \mathbf{W}_{\mathcal{I}'}) & \longrightarrow & (\mathcal{I}, \mathbf{W}_{\mathcal{I}}) \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, then the composite restriction

$$\mathrm{Fun}(\mathcal{I}, \mathcal{R})^{\mathbf{W}} \to \mathrm{Fun}([\mathbf{W}], \mathcal{R})^{\mathbf{W}} \xrightarrow{t} \mathbf{W}_{\mathcal{R}}$$

is a cocartesian fibration.

(b) Dually, if we form the pushout

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{t} & [\mathbf{W}] \\ {\scriptstyle i}\downarrow & & \downarrow \\ (\mathcal{I}', \mathbf{W}_{\mathcal{I}'}) & \longrightarrow & (\mathcal{I}, \mathbf{W}_{\mathcal{I}}) \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$*, then the composite restriction*

$$\mathrm{Fun}(\mathcal{I}, \mathcal{R})^{\mathbf{W}} \to \mathrm{Fun}([\mathbf{W}], \mathcal{R})^{\mathbf{W}} \xrightarrow{s} \mathbf{W}_{\mathcal{R}}$$

is a cartesian fibration.

(2) (a) If we form the pushout

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{s} & [\mathbf{A}] \\ {\scriptstyle i}\downarrow & & \downarrow \\ (\mathcal{I}', \mathbf{W}_{\mathcal{I}'}) & \longrightarrow & (\mathcal{I}, \mathbf{W}_{\mathcal{I}}) \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$*, then the composite restriction*

$$\mathrm{Fun}(\mathcal{I}, \mathcal{R})^{\mathbf{W}} \to \mathrm{Fun}([\mathbf{A}], \mathcal{R})^{\mathbf{W}} \xrightarrow{t} \mathbf{W}_{\mathcal{R}}$$

is a cocartesian fibration.

(b) Dually, if we form the pushout

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{t} & [\mathbf{A}] \\ {\scriptstyle i}\downarrow & & \downarrow \\ (\mathcal{I}', \mathbf{W}_{\mathcal{I}'}) & \longrightarrow & (\mathcal{I}, \mathbf{W}_{\mathcal{I}}) \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$*, then the composite restriction*

$$\mathrm{Fun}(\mathcal{I}, \mathcal{R})^{\mathbf{W}} \to \mathrm{Fun}([\mathbf{A}], \mathcal{R})^{\mathbf{W}} \xrightarrow{s} \mathbf{W}_{\mathcal{R}}$$

is a cartesian fibration.

Proof. We first prove item (1)(b). Applying Corollary T.2.4.7.12 to the functor

$$\mathrm{Fun}(\mathcal{I}', \mathcal{R})^{\mathbf{W}} \xrightarrow{i} \mathbf{W}_{\mathcal{R}}$$

and noting that $\mathrm{Fun}([\mathbf{W}],\mathcal{R})^{\mathbf{W}} \simeq \mathrm{Fun}([1],\mathbf{W}_{\mathcal{R}})$ (in a way compatible with the evaluation maps), we obtain that the composite restriction

$$\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}} \simeq \lim \left(\begin{array}{ccc} & & \mathrm{Fun}([\mathbf{W}],\mathcal{R})^{\mathbf{W}} \\ & & \downarrow{\scriptstyle t} \\ \mathrm{Fun}(\mathcal{I}',\mathcal{R})^{\mathbf{W}} & \xrightarrow[i]{} & \mathbf{W}_{\mathcal{R}} \end{array} \right) \to \mathrm{Fun}([\mathbf{W}],\mathcal{R})^{\mathbf{W}} \xrightarrow{s} \mathbf{W}_{\mathcal{R}}$$

is a cartesian fibration, as desired. The proof of item (1)(a) is completely dual.

We now prove item (2)(b). For this, consider the diagram

$$\begin{array}{ccccc} \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}} & & \longrightarrow & & \mathrm{Fun}(\mathcal{I}',\mathcal{R})^{\mathbf{W}} \\ \downarrow & & & & \downarrow \\ \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s} & \longrightarrow & \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}} & \longrightarrow & \mathrm{Fun}(\mathcal{I}',\mathcal{R})^{\mathcal{R}\mathrm{el}} \\ \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle i} \\ & & \mathrm{Fun}([\mathbf{A}],\mathcal{R})^{\mathcal{R}\mathrm{el}} & \xrightarrow[t]{} & \mathcal{R} \\ & & \downarrow{\scriptstyle s} & & \\ \mathbf{W}_{\mathcal{R}} & \longrightarrow & \mathcal{R} & & \end{array}$$

in which all small rectangles are pullbacks and in which we have introduced the ad hoc notation

$$\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s} \subset \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}}$$

for the wide subcategory whose morphisms are those natural transformations whose component at $s \in [\mathbf{A}] \subset \mathcal{I}$ lies in $\mathbf{W}_{\mathcal{R}} \subset \mathcal{R}$. Observing that $\mathrm{Fun}([\mathbf{A}],\mathcal{R})^{\mathcal{R}\mathrm{el}} \simeq \mathrm{Fun}([1],\mathcal{R})$ (in a way compatible with the evaluation maps), it follows from applying Corollary T.2.4.7.12 to the functor

$$\mathrm{Fun}(\mathcal{I}',\mathcal{R})^{\mathcal{R}\mathrm{el}} \xrightarrow{i} \mathcal{R}$$

that the composite

$$\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}} \to \mathrm{Fun}([\mathbf{A}],\mathcal{R})^{\mathcal{R}\mathrm{el}} \xrightarrow{s} \mathcal{R}$$

is a cartesian fibration, for which the cartesian morphisms are precisely those that are sent to equivalences under the restriction functor

$$\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}} \to \mathrm{Fun}(\mathcal{I}',\mathcal{R})^{\mathcal{R}\mathrm{el}}.$$

Then, by Propositions T.2.4.2.2.3(2) and T.2.4.1.3(2), the functor

$$\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s} \xrightarrow{s} \mathbf{W}_{\mathcal{R}}$$

is also a cartesian fibration, for which any morphism that is sent to an equivalence under the composite

$$\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s} \to \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}} \to \mathrm{Fun}(\mathcal{I}',\mathcal{R})^{\mathcal{R}\mathrm{el}}$$

is cartesian. Now, for any map $x' \xrightarrow{\varphi} x$ in $\mathbf{W}_{\mathcal{R}}$ and any object

$$G \in \left(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \underset{x,\mathbf{W}_{\mathcal{R}},s}{\times} \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s}\right),$$

choose such a cartesian morphism

$$(F \xrightarrow{\tilde{\varphi}} G) \in \left(\mathrm{Fun}\left([1],\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s}\right) \underset{\mathrm{Fun}([1],s),\mathrm{Fun}([1],\mathbf{W}_{\mathcal{R}}),\varphi}{\times} \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}\right).$$

Since by definition $\mathcal{R}^{\simeq} \subset \mathbf{W}_{\mathcal{R}}$, it follows that this is in fact a morphism in the (wide) subcategory $\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s}$. Hence, we obtain a diagram

$$\begin{array}{ccccc}
\left(\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}}\right)_{/\tilde{\varphi}} & \longrightarrow & \left(\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s}\right)_{/\tilde{\varphi}} & \longrightarrow & (\mathbf{W}_{\mathcal{R}})_{/\varphi} \\
\downarrow & & \downarrow & & \downarrow \\
\left(\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}}\right)_{/G} & \longrightarrow & \left(\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s}\right)_{/G} & \longrightarrow & (\mathbf{W}_{\mathcal{R}})_{/x}
\end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$, in which the right square is a pullback since $\tilde{\varphi}$ is a cartesian morphism. Moreover, again using the fact that $\mathcal{R}^{\simeq} \subset \mathbf{W}_{\mathcal{R}}$, it is easy to check that the left square is also a pullback. So the entire rectangle is a pullback, and hence $\tilde{\varphi}$ is also a cartesian morphism for the functor

$$\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}} \xrightarrow{s} \mathbf{W}_{\mathcal{R}}.$$

From here, it follows from the fact that $\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}@s}$ is a subcategory that this functor is indeed a cartesian fibration. The proof of item (2)(a) is completely dual. □

Given an arbitrary doubly-pointed relative ∞-category $(\mathcal{I},\mathbf{W}_{\mathcal{I}}) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$ and some relative ∞-category $(\mathcal{R},\mathbf{W}_{\mathcal{R}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ which we consider to be doubly-pointed via some choice $x,y \in \mathcal{R}$ of a pair of objects, we will be interested in the functoriality of the construction

$$\mathrm{Fun}_{**}((\mathcal{I},\mathbf{W}_{\mathcal{I}}),((\mathcal{R},\mathbf{W}_{\mathcal{R}}),x,y))^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$$

in the variable $x \in \mathbf{W}$ but for a fixed choice of $y \in \mathbf{W}$ (or vice versa). This functoriality will be expressed by a variant of Lemma 4.2.20. However, in order to accommodate the fixing of just one of the two chosen objects, we must first introduce the following notation.

Notation 4.2.21. Let $\mathcal{I} \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$, let $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, and let $x, y \in \mathcal{R}$. Then, we write

$$\big(\mathrm{Fun}_{*\circ}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}_{*\circ}(\mathcal{I},\mathcal{R})^{\mathbf{W}}\big) = \lim \left(\begin{array}{ccc} & & \big(\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}}\big) \\ & & \big\downarrow{\scriptstyle s} \\ \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[x]{} & (\mathcal{R}, \mathbf{W}) \end{array} \right)$$

and

$$\big(\mathrm{Fun}_{\circ *}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}_{\circ *}(\mathcal{I},\mathcal{R})^{\mathbf{W}}\big) = \lim \left(\begin{array}{ccc} & & \big(\mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{I},\mathcal{R})^{\mathbf{W}}\big) \\ & & \big\downarrow{\scriptstyle t} \\ \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[y]{} & (\mathcal{R}, \mathbf{W}) \end{array} \right).$$

We now give a "half-doubly-pointed" variant of Lemma 4.2.20, but stated only in the special case that we will need.

Lemma 4.2.22. *Let $\underline{\mathbf{m}} \in \mathcal{Z}$, let $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, and let $x, y \in \mathcal{R}$.*

(1) The functor $\mathrm{Fun}_{\circ *}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}} \xrightarrow{s} \mathbf{W}$

  (a) is a cocartesian fibration if $\underline{\mathbf{m}}$ begins with $\mathbf{W}^{-1}$, and

  (b) is a cartesian fibration if $\underline{\mathbf{m}}$ begins with $\mathbf{A}$.

(2) The functor $\mathrm{Fun}_{*\circ}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}} \xrightarrow{t} \mathbf{W}$

  (a) is a cartesian fibration if $\underline{\mathbf{m}}$ ends with $\mathbf{W}^{-1}$, and

  (b) is a cocartesian fibration if $\underline{\mathbf{m}}$ ends with $\mathbf{A}$.

Proof. If we simply have $\underline{\mathbf{m}} = [\mathbf{A}]$ or $\underline{\mathbf{m}} = [\mathbf{W}^{-1}]$ then these statements follow trivially from Lemma 4.2.20, so let us assume that the relative word $\underline{\mathbf{m}}$ has length greater than 1.

To prove item (2)(a), suppose that $\underline{\mathbf{m}} = [\underline{\mathbf{m}}'; \mathbf{W}^{-1}]$. Then we have a pullback square

$$\begin{array}{ccc} \mathrm{Fun}_{*\circ}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}} & \longrightarrow & \mathrm{Fun}([\mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}} \\ \downarrow & & \downarrow{\scriptstyle s_{[\mathbf{W}^{-1}]}} \\ \mathrm{Fun}_{*\circ}(\underline{\mathbf{m}}', \mathcal{R})^{\mathbf{W}} & \xrightarrow[t_{\underline{\mathbf{m}}'}]{} & \mathbf{W} \end{array}$$

which, making the identification of $[\mathbf{W}^{-1}]$ with $[\mathbf{W}]$ in a way which switches the source and target objects, is equivalently a pullback square

$$\begin{array}{ccc} \mathrm{Fun}_{*\circ}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}} & \longrightarrow & \mathrm{Fun}([\mathbf{W}], \mathcal{R})^{\mathbf{W}} \\ \downarrow & & \downarrow{\scriptstyle t_{[\mathbf{W}]}} \\ \mathrm{Fun}_{*\circ}(\underline{\mathbf{m}}', \mathcal{R})^{\mathbf{W}} & \xrightarrow[t_{\underline{\mathbf{m}}'}]{} & \mathbf{W}. \end{array}$$

From here, the proof parallels that of Lemma 4.2.20(1)(b), only now we apply Corollary T.2.4.7.12 to the functor

$$\mathrm{Fun}_{*\circ}(\underline{\mathbf{m}}', \mathcal{R})^{\mathbf{W}} \xrightarrow{t_{\underline{\mathbf{m}}'}} \mathbf{W}.$$

The proof of item (1)(a) is completely dual.

To prove item (1)(b), let us now suppose that $\underline{\mathbf{m}} = [\mathbf{A}; \underline{\mathbf{m}}']$. Then we have a diagram

$$\begin{array}{ccccc} \mathrm{Fun}_{\circ *}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}} & & \longrightarrow & & \mathrm{Fun}_{\circ *}(\underline{\mathbf{m}}', \mathcal{R})^{\mathbf{W}} \\ \downarrow & & & & \downarrow \\ \mathrm{Fun}_{\circ *}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}@s} & \longrightarrow & \mathrm{Fun}_{\circ *}(\underline{\mathbf{m}}, \mathcal{R})^{\mathcal{R}\mathrm{el}} & \longrightarrow & \mathrm{Fun}_{\circ *}(\underline{\mathbf{m}}', \mathcal{R})^{\mathcal{R}\mathrm{el}} \\ \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle s_{\underline{\mathbf{m}}'}} \\ & & \mathrm{Fun}([\mathbf{A}], \mathcal{R})^{\mathbf{W}} & \xrightarrow[t_{[\mathbf{A}]}]{} & \mathcal{R} \\ & & \downarrow{\scriptstyle s_{[\mathbf{A}]}} & & \\ \mathbf{W} & \longrightarrow & \mathcal{R} & & \end{array}$$

in which all small rectangles are pullbacks, almost identical to that of the proof of Lemma 4.2.20(2)(b). From here, the proof proceeds in a completely analogous way to that one. The proof of item (2)(b) is completely dual. □

Lemma 4.2.22, in turn, enables us to make the following definitions.

Notation 4.2.23. Let $\underline{\mathbf{m}} \in \mathcal{Z}$, let $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, and let $x, y \in \mathcal{R}$.

- If $\underline{\mathbf{m}}$ begins with $\mathbf{W}^{-1}$, we write

  $$\mathbf{W} \xrightarrow{\underline{\mathbf{m}}(-,y)} \mathcal{C}\mathrm{at}_\infty$$

  for the functor classifying the cocartesian fibration of Lemma 4.2.22(1)(a). On the other hand, if $\underline{\mathbf{m}}$ begins with $\mathbf{A}$, we write

  $$\mathbf{W}^{op} \xrightarrow{\underline{\mathbf{m}}(-,y)} \mathcal{C}\mathrm{at}_\infty$$

  for the functor classifying the cartesian fibration of Lemma 4.2.22(1)(b).

- If $\underline{\mathbf{m}}$ ends with $\mathbf{W}^{-1}$, we write

  $$\mathbf{W}^{op} \xrightarrow{\underline{\mathbf{m}}(x,-)} \mathcal{C}\mathrm{at}_\infty$$

  for the functor classifying the cartesian fibration of Lemma 4.2.22(2)(a). On the other hand, if $\underline{\mathbf{m}}$ ends with $\mathbf{A}$, we write

  $$\mathbf{W} \xrightarrow{\underline{\mathbf{m}}(x,-)} \mathcal{C}\mathrm{at}_\infty$$

  for the functor classifying the cocartesian fibration of Lemma 4.2.22(2)(b).

- By convention and for convenience, if $\underline{\mathbf{m}} = [\varnothing] \in \mathcal{Z}$ is the empty relative word (which defines the terminal relative ∞-category), we let both $\underline{\mathbf{m}}(x, -)$ and $\underline{\mathbf{m}}(-, y)$ denote either functor

  $$\mathbf{W} \xrightarrow{\mathrm{const}(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})} \mathcal{C}\mathrm{at}_\infty$$

  or

  $$\mathbf{W}^{op} \xrightarrow{\mathrm{const}(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})} \mathcal{C}\mathrm{at}_\infty.$$

Using Notation 4.2.23, we now express the ∞-category $[\underline{\mathbf{m}}; \underline{\mathbf{m}}']_{(\mathcal{R},\mathbf{W})}(x, y)$ of zigzags in $(\mathcal{R}, \mathbf{W})$ from x to y of the concatenated zigzag type $[\underline{\mathbf{m}}; \underline{\mathbf{m}}']$ in terms of the two-sided Grothendieck construction (see Definition 3.3.3). This is an analog of [DK80a, 9.4].[12]

[12] In the statement of [DK80a, 9.4], the third appearance of $\underline{\mathbf{m}}$ should actually be $\underline{\mathbf{m}}'$.

Lemma 4.2.24. *Let* $\underline{\mathbf{m}}, \underline{\mathbf{m}}' \in \mathcal{Z}$. *Then for any* $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *and any* $x, y \in \mathcal{R}$, *we have an equivalence*

$$[\underline{\mathbf{m}}; \underline{\mathbf{m}}'](x,y) \simeq \begin{cases} \mathrm{Gr}\left(\underline{\mathbf{m}}'(-,y), \mathbf{W}, \underline{\mathbf{m}}(x,-)\right), & \underline{\mathbf{m}} \text{ ends with } \mathbf{A} \text{ and } \underline{\mathbf{m}}' \text{ begins with } \mathbf{A} \\ \mathrm{Gr}\left(\underline{\mathbf{m}}(x,-), \mathbf{W}, \underline{\mathbf{m}}'(-,y)\right), & \underline{\mathbf{m}} \text{ ends with } \mathbf{W}^{-1} \text{ and } \underline{\mathbf{m}}' \text{ begins with } \mathbf{W}^{-1} \\ \mathrm{Gr}\left(\mathrm{const}(\mathrm{pt}), \mathbf{W}, (\underline{\mathbf{m}}(x,-) \times \underline{\mathbf{m}}'(-,y))\right), & \underline{\mathbf{m}} \text{ ends with } \mathbf{A} \text{ and } \underline{\mathbf{m}}' \text{ begins with } \mathbf{W}^{-1} \\ \mathrm{Gr}\left((\underline{\mathbf{m}}(x,-) \times \underline{\mathbf{m}}'(-,y)), \mathbf{W}, \mathrm{const}(\mathrm{pt})\right), & \underline{\mathbf{m}} \text{ ends with } \mathbf{W}^{-1} \text{ and } \underline{\mathbf{m}}' \text{ begins with } \mathbf{A} \end{cases}$$

which is natural in $((\mathcal{R}, \mathbf{W}), x, y) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$.

Proof. Recall from Remark 4.2.6 that we have a pushout square

$$\begin{array}{ccc} \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} & \xrightarrow{s} & \underline{\mathbf{m}}' \\ {\scriptstyle t}\downarrow & & \downarrow \\ \underline{\mathbf{m}} & \longrightarrow & [\underline{\mathbf{m}}; \underline{\mathbf{m}}'] \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, through which $[\underline{\mathbf{m}}; \underline{\mathbf{m}}']$ acquires its source object from $\underline{\mathbf{m}}$ and its target object from $\underline{\mathbf{m}}'$. This gives rise to a string of equivalences

$$[\underline{\mathbf{m}}; \underline{\mathbf{m}}'](x,y) = \mathrm{Fun}_{**}([\underline{\mathbf{m}}; \underline{\mathbf{m}}'], \mathcal{R})^{\mathbf{W}}$$

$$\simeq \lim \left(\begin{array}{ccccc} & & & & \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \\ & & & & \downarrow{\scriptstyle y} \\ & & & \mathrm{Fun}(\underline{\mathbf{m}}', \mathcal{R})^{\mathbf{W}} & \xrightarrow[t]{} \mathbf{W} \\ & & & \downarrow{\scriptstyle s} & \\ & & \mathrm{Fun}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}} & \xrightarrow[t]{} \mathbf{W} & \\ & & \downarrow{\scriptstyle s} & & \\ \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[x]{} \mathbf{W} & & & \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{ccc} & & \mathrm{Fun}_{\circ *}(\underline{\mathbf{m}}', \mathcal{R})^{\mathbf{W}} \\ & & \downarrow{\scriptstyle s} \\ \mathrm{Fun}_{* \circ}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}} & \xrightarrow[t]{} & \mathbf{W} \end{array} \right)$$

in $\mathcal{C}\mathrm{at}_\infty$. From here, the first and second cases follow from Lemma 4.2.22, Notation 4.2.23, and Definition 3.3.3, while the third and fourth cases follow by additionally appealing to Example 3.1.9 and Example 3.1.10. □

4.3 Homotopical three-arrow calculi in relative ∞-categories

In the previous section, given a relative ∞-category $(\mathcal{R}, \mathbf{W})$, we introduced the *hammock simplicial space*

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x, y) \in s\mathcal{S}$$

for two objects $x, y \in \mathcal{R}$. The definition of this simplicial space is fairly explicit, but it is nevertheless quite large. In this section, we show that under a certain condition – namely, that $(\mathcal{R}, \mathbf{W})$ admits a *homotopical three-arrow calculus* – we can at least recover this simplicial space up to weak equivalence in $s\mathcal{S}_{\mathrm{KQ}}$ (i.e. we can recover its geometric realization) from a much smaller simplicial space, in fact from one of the constituent simplicial spaces in its defining colimit. This condition is often satisfied in practice; for example, it holds when $(\mathcal{R}, \mathbf{W})$ admits the additional structure of a *model ∞-category* (see Lemma 6.8.2).

This section is organized as follows.

- In §4.3.1, we define what it means for a relative ∞-category to admit a homotopical three-arrow calculus, and we state the *fundamental theorem of homotopical three-arrow calculi* (4.3.4) described above.
- In §4.3.2, in preparation for the proof of Theorem 4.3.4, we assemble some auxiliary results regarding relative ∞-categories.
- In §4.3.3, in preparation for the proof of Theorem 4.3.4, we assemble some auxiliary results regarding ends and coends.
- In §4.3.4, we give the proof of Theorem 4.3.4.

4.3.1 The fundamental theorem of homotopical three-arrow calculi

We begin with the main definition of this section, whose terminology will be justified by Theorem 4.3.4; it is a straightforward generalization of [LMG15, Definition 4.1], which is itself a minor variant of [DK80a, 6.1(i)].

Definition 4.3.1. Let $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. We say that $(\mathcal{R}, \mathbf{W})$ admits a ***homotopical three-arrow calculus*** if for all $x, y \in \mathcal{R}$ and for all $i, j \geq 1$, the map

$$[\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}] \to [\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}]$$

in $\mathcal{Z} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**}$ obtained by collapsing the middle weak equivalence induces a map

$$\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}} \to \mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}}$$

in $\mathbf{W}_{\mathrm{Th}}^{\mathcal{C}\mathrm{at}_\infty} \subset \mathcal{C}\mathrm{at}_\infty$ (i.e. it becomes an equivalence upon applying the groupoid completion functor $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$).

Notation 4.3.2. Since it will appear repeatedly, we make the abbreviation $\underline{\mathbf{3}} = [\mathbf{W}^{-1}; \mathbf{A}; \mathbf{W}^{-1}]$ for the relative word

$$s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} t.$$

Definition 4.3.3. For any relative ∞-category $(\mathcal{R}, \mathbf{W})$ and any objects $x, y \in \mathcal{R}$, we will refer to

$$\underline{\mathbf{3}}(x, y) = \mathrm{Fun}_{**}(\underline{\mathbf{3}}, \mathcal{R})^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$$

as the ∞-category of ***three-arrow zigzags*** in $\mathcal{R}$ from x to y.

We now state the ***fundamental theorem of homotopical three-arrow calculi***, an analog of [DK80a, Proposition 6.2(i)]; we will give its proof in §4.3.4.

Theorem 4.3.4. *If $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ admits a homotopical three-arrow calculus, then for any $x, y \in \mathcal{R}$, the natural map*

$$\mathrm{N}_\infty(\underline{\mathbf{3}}(x, y)) \to \underline{\mathrm{hom}}_{\mathscr{L}^H(\mathcal{R}, \mathbf{W})}(x, y)$$

in $s\mathcal{S}$ becomes an equivalence under the geometric realization functor $|-| : s\mathcal{S} \to \mathcal{S}$.

4.3.2 Supporting material: relative ∞-categories

In this subsection, we give two results regarding relative ∞-categories which will be used in the proof of Theorem 4.3.4. Both concern *corepresentation*, namely the effect of the functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{(**)} \xrightarrow{\mathrm{Fun}(-,\mathcal{R})^{\mathbf{W}}} \mathcal{C}\mathrm{at}_{\infty}$$

on certain data in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{(**)}$ (for a given relative ∞-category $(\mathcal{R}, \mathbf{W})$).

Lemma 4.3.5. *Given a pair of maps $\mathcal{I} \rightrightarrows \mathcal{J}$ in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty})_{(**)}$, a morphism between them in $\mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}$ induces, for any $(\mathcal{R}, \mathbf{W}) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty})_{(**)}$, a natural transformation between the two induced functors*

$$\mathrm{Fun}_{(**)}(\mathcal{J}, \mathcal{R})^{\mathbf{W}} \rightrightarrows \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathbf{W}}.$$

Proof. First of all, the morphism in $\mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}$ is selected by a map $[1] \to \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}$; this is equivalent to a map

$$[1]_{\mathbf{W}} \to \left(\mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{J})^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}\right)$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty}$, which is adjoint to a map

$$\mathcal{I} \odot [1]_{\mathbf{W}} \to \mathcal{J}$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty})_{(**)}$. Then, for any $(\mathcal{R}, \mathbf{W}) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty})_{(**)}$, composing with this map yields a functor

$$\begin{aligned}
\mathrm{Fun}_{(**)}(\mathcal{J}, \mathcal{R})^{\mathbf{W}} &\to \mathrm{Fun}_{(**)}(\mathcal{I} \odot [1]_{\mathbf{W}}, \mathcal{R})^{\mathbf{W}} \\
&\simeq \mathrm{Fun}\left([1]_{\mathbf{W}}, \left(\mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathbf{W}}\right)\right) \\
&\simeq \mathrm{Fun}\left([1], \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathbf{W}}\right),
\end{aligned}$$

which is adjoint to a map

$$[1] \times \mathrm{Fun}_{(**)}(\mathcal{J}, \mathcal{R})^{\mathbf{W}} \to \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathbf{W}},$$

which selects a natural transformation between the two induced functors

$$\mathrm{Fun}_{(**)}(\mathcal{J}, \mathcal{R})^{\mathbf{W}} \rightrightarrows \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathbf{W}},$$

as desired. □

Lemma 4.3.6. *Let* $(\mathcal{I}, \mathbf{W}_{\mathcal{I}}) \in (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{(**)}$*, and form any pushout diagram*

$$\begin{array}{ccc} [\mathbf{W}] & \longrightarrow & (\mathcal{I}, \mathbf{W}_{\mathcal{I}}) \\ \downarrow & & \downarrow \\ [\mathbf{W}^{\circ 2}] & \longrightarrow & (\mathcal{J}, \mathbf{W}_{\mathcal{J}}) \end{array}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{(**)}$*, where the left map is the unique map in* $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**}$*. Note that the two possible retractions* $[\mathbf{W}^{\circ 2}] \rightrightarrows [\mathbf{W}]$ *in* $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**}$ *of the given map induce retractions* $(\mathcal{J}, \mathbf{W}_{\mathcal{J}}) \rightrightarrows (\mathcal{I}, \mathbf{W}_{\mathcal{I}})$ *in* $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{(**)}$*. Then, for any* $(\mathcal{R}, \mathbf{W}_{\mathcal{R}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{(**)}$*, the induced map*

$$\mathrm{Fun}_{(**)}(\mathcal{J}, \mathcal{R})^{\mathbf{W}} \to \mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathbf{W}}$$

becomes an equivalence under the functor $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$*, with inverse given by either map*

$$\big(\mathrm{Fun}_{(**)}(\mathcal{I}, \mathcal{R})^{\mathbf{W}}\big)^{\mathrm{gpd}} \rightrightarrows \big(\mathrm{Fun}_{(**)}(\mathcal{J}, \mathcal{R})^{\mathbf{W}}\big)^{\mathrm{gpd}}$$

in $\mathcal{S}$ *induced by one of the given retractions.*

Proof. Note that both composites

$$[\mathbf{W}^{\circ 2}] \rightrightarrows [\mathbf{W}] \to [\mathbf{W}^{\circ 2}]$$

(of one of the two possible retractions followed by the given map) are connected to $\mathrm{id}_{[\mathbf{W}^{\circ 2}]}$ by a map in

$$\mathrm{Fun}_{**}([\mathbf{W}^{\circ 2}], [\mathbf{W}^{\circ 2}])^{\mathbf{W}}.$$

In turn, both composites

$$(\mathcal{J}, \mathbf{W}_{\mathcal{J}}) \rightrightarrows (\mathcal{I}, \mathbf{W}_{\mathcal{I}}) \to (\mathcal{J}, \mathbf{W}_{\mathcal{J}})$$

are connected to $\mathrm{id}_{(\mathcal{J}, \mathbf{W}_{\mathcal{J}})}$ by a map in $\mathrm{Fun}_{(**)}(\mathcal{J}, \mathcal{J})^{\mathbf{W}}$. Hence, the result follows from Lemmas 4.3.5 and 2.1.26. □

4.3.3 Supporting material: co/ends

In this subsection, we give a few results regarding ends and coends which will be used in the proof of Theorem 4.3.4. For a brief review of these universal constructions in the ∞-categorical setting, we refer the reader to [GHN, §2].

We begin by recalling a formula for the space of natural transformations between two functors.

Lemma 4.3.7. *Given any $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ and any $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we have a canonical equivalence*

$$\hom_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \simeq \int_{c \in \mathcal{C}} \hom_{\mathcal{D}}(F(c), G(c)).$$

Proof. This appears as [Gla, Proposition 2.3] (and as [GHN, Proposition 5.1]). □

We now prove a "ninja Yoneda lemma".[13]

Lemma 4.3.8. *If $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ is an ∞-category equipped with a tensoring $- \odot - : \mathcal{C} \times \mathcal{S} \to \mathcal{C}$, then for any functor $\mathcal{I}^{op} \xrightarrow{F} \mathcal{C}$, we have an equivalence*

$$F(-) \simeq \int^{i \in \mathcal{I}} F(i) \odot \hom_{\mathcal{I}}(-, i)$$

in $\mathrm{Fun}(\mathcal{I}^{op}, \mathcal{C})$.

Proof. For any test objects $j \in \mathcal{I}^{op}$ and $Y \in \mathcal{C}$, we have a string of natural equivalences

$$\begin{aligned}
\hom_{\mathcal{C}}\left(\int^{i \in \mathcal{I}} F(i) \odot \hom_{\mathcal{I}}(j, i), Y\right) &\simeq \int_{i \in \mathcal{I}} \hom_{\mathcal{C}}(F(i) \odot \hom_{\mathcal{I}}(j, i), Y) \\
&\simeq \int_{i \in \mathcal{I}} \hom_{\mathcal{S}}(\hom_{\mathcal{I}}(j, i), \hom_{\mathcal{C}}(F(i), Y)) \\
&\simeq \hom_{\mathrm{Fun}(\mathcal{I},\mathcal{S})}(\hom_{\mathcal{I}}(j, -), \hom_{\mathcal{C}}(F(-), Y)) \\
&\simeq \hom_{\mathcal{C}}(F(j), Y),
\end{aligned}$$

where the first line follows from the definition of a coend as a colimit (see e.g. [GHN, Definition 2.5]), the second line uses the tensoring, the third line follows from Lemma 4.3.7, and the last line follows from the usual Yoneda lemma (Proposition T.5.1.3.1). Hence, again by the Yoneda lemma, we obtain an equivalence

$$F(j) \simeq \int^{i \in \mathcal{I}} F(i) \odot \hom_{\mathcal{I}}(j, i)$$

which is natural in $j \in \mathcal{I}^{op}$. □

Then, we have the following result on the preservation of colimits.[14]

[13] The name is apparently due to Leinster (see [Lor, Remark 2.2]).

[14] Lemma 4.3.9 is actually implicitly about weighted colimits (see [GHN, Definition 2.7]).

Lemma 4.3.9. *If $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ is an ∞-category equipped with a tensoring $- \odot - : \mathcal{C} \times \mathcal{S} \to \mathcal{C}$, then for any functor $\mathcal{I}^{op} \xrightarrow{F} \mathcal{C}$, the functor*

$$\mathrm{Fun}(\mathcal{I}, \mathcal{S}) \xrightarrow{\int^{i\in\mathcal{I}} F(i)\odot(-)(i)} \mathcal{C}$$

is a left adjoint.

Proof. It suffices to check that for every $c \in \mathcal{C}$, the functor

$$\mathrm{Fun}(\mathcal{I}, \mathcal{S})^{op} \xrightarrow{\hom_{\mathcal{C}}\left(\int^{i\in\mathcal{I}} F(i)\odot(-)(i), c\right)} \mathcal{S}$$

is representable. For this, given any $W \in \mathrm{Fun}(\mathcal{I}, \mathcal{S})$ we compute that

$$\begin{aligned}
\hom_{\mathcal{C}}\left(\int^{i\in\mathcal{I}} F(i) \odot W(i), c\right) &\simeq \int_{i\in\mathcal{I}} \hom_{\mathcal{C}}(F(i) \odot W(i), c) \\
&\simeq \int_{i\in\mathcal{I}} \hom_{\mathcal{S}}(W(i), \hom_{\mathcal{C}}(F(i), c)) \\
&\simeq \hom_{\mathrm{Fun}(\mathcal{I},\mathcal{S})}(W, \hom_{\mathcal{C}}(F(-), c)),
\end{aligned}$$

where the first line follows from the definition of a co/end as a co/limit (again see e.g. [GHN, Definition 2.5]), the second line uses the tensoring, and the last line follows from Lemma 4.3.7. □

4.3.4 The proof of Theorem 4.3.4

Having laid out the necessary supporting material in the previous two subsection, we now proceed to prove the fundamental theorem of homotopical three-arrow calculi (4.3.4). This proof is based closely on that of [DK80a, Proposition 6.2(i)], although we give many more details (recall Remark 4.0.2).

Proof of Theorem 4.3.4. We will construct a commutative diagram

$$\begin{array}{ccc}
|\mathrm{N}_\infty(\underline{\mathbf{3}}(x,y))| & \xrightarrow[\sim]{|\beta|} & |\mathrm{colim}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(G(\underline{\mathbf{m}})(x,y))| \\
{\scriptstyle|\alpha|}\downarrow & & \wr\downarrow{\scriptstyle|\underline{\psi}|} \\
|\mathrm{colim}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(\underline{\mathbf{m}}(x,y))| & \xrightarrow{|\underline{\varphi}|} & |\mathrm{colim}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y))|
\end{array}$$

$$|\rho| : |\mathrm{colim}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y))| \to |\mathrm{colim}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(\underline{\mathbf{m}}(x,y))|$$

in $\mathcal{S}$, i.e. a commutative square in which the bottom arrow is equipped with a retraction and in which moreover the top and right map are equivalences. Note that by definition, the object on the bottom left is precisely $\left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x,y)\right|$; the left map will be the natural map referred to in the statement of the result. The equivalences in $\mathcal{S}$ satisfy the two-out-of-six property, and applying this to the composable sequence of arrows $[|\alpha|; |\underline{\varphi}|; |\rho|]$, we deduce that $|\alpha|$ is also an equivalence, proving the claim.

We will accomplish this by running through the following sequence of tasks.

(1) Define the two objects on the right.

(2) Define the maps in the diagram.

(3) Explain why the square commutes.

(4) Explain why $|\rho|$ gives a retraction of $|\underline{\varphi}|$.

(5) Explain why the map $|\beta|$ is an equivalence.

(6) Explain why the map $|\underline{\psi}|$ is an equivalence.

We now proceed to accomplish these tasks in order.

(1) We define endofunctors $F, G \in \mathrm{Fun}(\mathcal{Z}, \mathcal{Z})$ by the formulas

$$F(\underline{\mathbf{m}}) = [\mathbf{W}^{-1}; \underline{\mathbf{m}}; \mathbf{W}^{-1}]$$

and

$$G(\underline{\mathbf{m}}) = [\mathbf{W}^{-1}; \mathbf{A}^{\circ|\underline{\mathbf{m}}|_{\mathbf{A}}}; \mathbf{W}^{-1}].$$

Then, the object in the upper right is given by

$$\left|\mathrm{colim}\left(\mathcal{Z}^{op} \xrightarrow{G^{op}} \mathcal{Z}^{op} \xrightarrow{\mathrm{N}_\infty((-)(x,y))} s\mathcal{S}\right)\right|,$$

and the object in the bottom right is given by

$$\left|\mathrm{colim}\left(\mathcal{Z}^{op} \xrightarrow{F^{op}} \mathcal{Z}^{op} \xrightarrow{\mathrm{N}_\infty((-)(x,y))} s\mathcal{S}\right)\right|.$$

(2) We define the two evident natural transformations $F \xrightarrow{\varphi} \mathrm{id}_{\mathcal{Z}}$ (given by collapsing the two newly added copies of $[\mathbf{W}^{-1}]$) and $F \xrightarrow{\psi} G$ (given by collapsing all internal copies of $[\mathbf{W}^{-1}]$) in $\mathrm{Fun}(\mathcal{Z}, \mathcal{Z})$; these induce natural transformations

$\mathrm{id}_{\mathcal{Z}^{op}} \xrightarrow{\varphi^{op}} F^{op}$ and $G^{op} \xrightarrow{\psi^{op}} F^{op}$ in $\mathrm{Fun}(\mathcal{Z}^{op}, \mathcal{Z}^{op})$.[15] We then define the maps in the diagram as follows.

- The left map is obtained by taking the geometric realization of the inclusion

$$\mathrm{N}_\infty(\underline{\mathbf{3}}(x,y)) \xrightarrow{\alpha} \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x,y) = \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(\underline{\mathbf{m}}(x,y))$$

  into the colimit at the object $\underline{\mathbf{3}} \in \mathcal{Z}^{op}$.

- The top map is obtained by taking the geometric realization of the inclusion

$$\mathrm{N}_\infty(\underline{\mathbf{3}}(x,y)) \simeq \mathrm{N}_\infty(G([\mathbf{A}])(x,y)) \xrightarrow{\beta} \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(G(\underline{\mathbf{m}})(x,y))$$

  into the colimit at the object $[\mathbf{A}] \in \mathcal{Z}^{op}$. (Note that $\underline{\mathbf{3}} \cong G([\mathbf{A}])$ in $\mathcal{Z}^{op}$.)

- The right map is obtained by taking the geometric realization of the map

$$\mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(G(\underline{\mathbf{m}})(x,y)) \xrightarrow{\psi} \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y))$$

  on colimits induced by the natural transformation $\mathrm{id}_{\mathrm{N}_\infty((-)(x,y))} \circ \psi^{op}$ in $\mathrm{Fun}(\mathcal{Z}^{op}, s\mathcal{S})$.

- The bottom map in the square (i.e. the straight bottom map) is obtained by taking the geometric realization of the map

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x,y) = \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(\underline{\mathbf{m}}(x,y)) \xrightarrow{\varphi} \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y))$$

  on colimits induced by the natural transformation $\mathrm{id}_{\mathrm{N}_\infty((-)(x,y))} \circ \varphi^{op}$ in $\mathrm{Fun}(\mathcal{Z}^{op}, s\mathcal{S})$.

- The curved map is obtained by taking the geometric realization of the map

$$\mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y)) \xrightarrow{\rho} \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(\underline{\mathbf{m}}(x,y)) = \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x,y)$$

  on colimits induced by the functor

$$\mathrm{Fun}(\mathcal{Z}^{op}, s\mathcal{S}) \xleftarrow{-\circ F^{op}} \mathrm{Fun}(\mathcal{Z}^{op}, s\mathcal{S}).$$

[15] Recall that the involution $(-)^{op} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is *contravariant* on 2-morphisms.

(3) The upper composite in the square is given by the geometric realization of the composite

$$\begin{array}{ccc} \mathrm{N}(\underline{\mathbf{3}}(x,y)) \simeq \mathrm{N}_\infty(G([\mathbf{A}])(x,y)) & \xrightarrow[\sim]{\mathrm{N}_\infty((\psi^{op}_{[\mathbf{A}]})(x,y))} & \mathrm{N}_\infty(F([\mathbf{A}])(x,y)) \\ & & \downarrow \\ & & \mathrm{colim}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y)) \end{array}$$

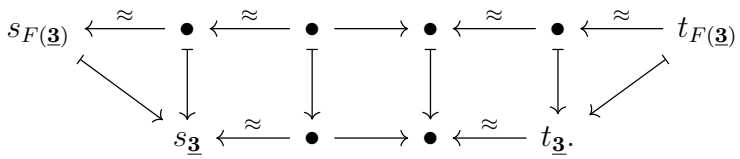
of the equivalence induced by the component of ψ^{op} at the object $[\mathbf{A}] \in \mathcal{Z}^{op}$ (which is an isomorphism in $\mathcal{Z}^{op}$) followed by the inclusion into the colimit at $[\mathbf{A}]$. So, via the (unique) identification $\underline{\mathbf{3}} \cong F([\mathbf{A}])$, we can identify this composite with the inclusion into the colimit at $[\mathbf{A}] \in \mathcal{Z}^{op}$.

Meanwhile, the lower composite in the square is given by the geometric realization of the composite

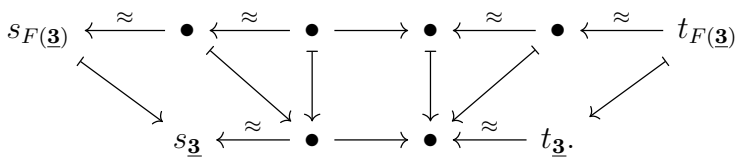
$$\mathrm{N}_\infty(\underline{\mathbf{3}}(x,y)) \xrightarrow{\mathrm{N}_\infty((\varphi^{op}_{\underline{\mathbf{3}}})(x,y))} \mathrm{N}_\infty(F(\underline{\mathbf{3}})(x,y)) \to \mathrm{colim}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y))$$

of the map induced by the component of φ^{op} at $\underline{\mathbf{3}}$ followed by the inclusion into the colimit at $\underline{\mathbf{3}}$.

Now, the map $F(\underline{\mathbf{3}}) \xrightarrow{\varphi_{\underline{\mathbf{3}}}} \underline{\mathbf{3}}$ in $\mathcal{Z}$ is given by



On the other hand, applying F to the unique map $\underline{\mathbf{3}} \xrightarrow{\gamma} [\mathbf{A}]$ in $\mathcal{Z}$, we obtain a map $F(\underline{\mathbf{3}}) \xrightarrow{F(\gamma)} F([\mathbf{A}]) \cong \underline{\mathbf{3}}$ in $\mathcal{Z}$ given by



which corepresents a map

$$\mathrm{N}_\infty(\underline{\mathbf{3}}(x,y)) \simeq \mathrm{N}_\infty(F([\mathbf{A}])(x,y)) \xrightarrow{\mathrm{N}_\infty((F(\gamma))(x,y))} \mathrm{N}_\infty(F(\underline{\mathbf{3}})(x,y))$$

in $s\mathcal{S}$ which participates in the diagram

$$\mathcal{Z}^{op} \xrightarrow{F^{op}} \mathcal{Z}^{op} \xrightarrow{\mathrm{N}_\infty((-)(x,y))} s\mathcal{S}$$

defining $\mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y))$. So, in order to witness the commutativity of the square, it suffices to obtain an equivalence between the two maps

$$\left|\mathrm{N}_\infty((\varphi^{op}_{\underline{\mathbf{3}}})(x,y))\right|, |\mathrm{N}_\infty((F(\gamma))(x,y))| \in \hom_{\mathcal{S}}\left(|\mathrm{N}_\infty(\underline{\mathbf{3}}(x,y))|, |\mathrm{N}_\infty(F(\underline{\mathbf{3}})(x,y))|\right).$$

But there is an evident cospan in $\mathrm{Fun}_{**}(F(\underline{\mathbf{3}}), \underline{\mathbf{3}})^{\mathbf{W}}$ between the two maps $\varphi_{\underline{\mathbf{3}}}$ and $F(\gamma)$, so this follows from Lemma 4.3.5, Lemma 2.1.26, and Proposition 2.2.4.

(4) The fact that $|\rho| \circ |\underline{\varphi}| \simeq \mathrm{id}_{\left|\mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(\underline{\mathbf{m}}(x,y))\right|}$ follows from applying Proposition 3.3.5 to the diagram

$$\mathcal{Z}^{op} \underset{F^{op}}{\overset{\mathrm{id}_{\mathcal{Z}^{op}}}{\rightrightarrows}} \mathcal{Z}^{op} \xrightarrow{((-)(x,y))^{\mathrm{gpd}}} \mathcal{S} \qquad (\varphi^{op} \Downarrow)$$

and invoking Proposition 2.2.4 to obtain a retraction diagram

$$\begin{array}{ccc} \mathrm{colim}((-)^{\mathrm{gpd}} \circ \mathrm{N}_\infty((-)(x,y)) \circ \mathrm{id}_{\mathcal{Z}^{op}}) & & \\ \Big\downarrow{\scriptstyle |\underline{\varphi}|} & \overset{\sim}{\searrow} & \\ & & \mathrm{colim}((-)^{\mathrm{gpd}} \circ \mathrm{N}_\infty((-)(x,y))). \\ & \overset{|\rho|}{\nearrow} & \\ \mathrm{colim}((-)^{\mathrm{gpd}} \circ \mathrm{N}_\infty((-)(x,y)) \circ F^{op}) & & \end{array}$$

(5) We first claim that for any $\underline{\mathbf{m}}' \in \mathcal{Z}$, the map

$$\hom_{\mathcal{Z}}(\underline{\mathbf{3}}, \underline{\mathbf{m}}') \simeq \hom_{\mathcal{Z}}(G([\mathbf{A}]), \underline{\mathbf{m}}') \to \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \hom_{\mathcal{Z}}(G(\underline{\mathbf{m}}), \underline{\mathbf{m}}')$$

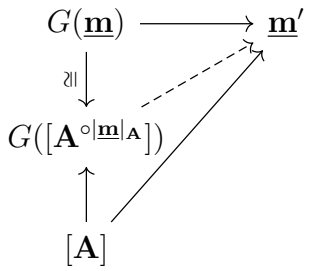
is an isomorphism. Indeed, note that by Proposition 3.3.1, we have an equivalence

$$\mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \hom_{\mathcal{Z}}(G(\underline{\mathbf{m}}), \underline{\mathbf{m}}') \simeq \mathrm{Gr}\left(\mathcal{Z}^{op} \xrightarrow{\hom_{\mathcal{Z}}(G(-), \underline{\mathbf{m}}')} \mathcal{S}\mathrm{et}\right)^{\mathrm{gpd}}.$$

The category

$$\mathrm{Gr}\left(\mathcal{Z}^{op} \xrightarrow{\hom_{\mathcal{Z}}(G(-),\underline{\mathbf{m}}')} \mathcal{S}\mathrm{et}\right)$$

admits a span of natural transformations from the identity functor to its fiber over the object $[\mathbf{A}] \in \mathcal{Z}^{op}$, whose component at an object $(\underline{\mathbf{m}} \in \mathcal{Z}^{op}, G(\underline{\mathbf{m}}) \to \underline{\mathbf{m}}')$ is indicated by the natural commutative diagram



in $\mathcal{Z}$ (in which the dotted arrow is simply the extension of the upper map over an isomorphism).[16] Hence, by Lemma 2.1.26 the inclusion of the fiber over $[\mathbf{A}] \in \mathcal{Z}^{op}$ induces an equivalence upon groupoid completions. But this fiber is precisely $\hom_{\mathcal{Z}}(G([\mathbf{A}]), \underline{\mathbf{m}}') \simeq \hom_{\mathcal{Z}}(\underline{\mathbf{3}}, \underline{\mathbf{m}}')$.

Now, assembling the above observation over all $\underline{\mathbf{m}}' \in \mathcal{Z}$, we see that the map

$$\hom_{\mathcal{Z}}(\underline{\mathbf{3}}, -) \to \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \hom_{\mathcal{Z}}(G(\underline{\mathbf{m}}), -)$$

is an equivalence in $\mathrm{Fun}(\mathcal{Z}, \mathcal{S}\mathrm{et}) \subset \mathrm{Fun}(\mathcal{Z}, \mathcal{S})$. Using this, and denoting by $- \odot - : s\mathcal{S} \times \mathcal{S} \to s\mathcal{S}$ the evident tensoring

$$s\mathcal{S} \times \mathcal{S} \xrightarrow{\mathrm{id}_{s\mathcal{S}} \times \mathrm{const}} s\mathcal{S} \times s\mathcal{S} \xrightarrow{- \times -} s\mathcal{S},$$

we obtain the map

$$\mathrm{N}_\infty(\underline{\mathbf{3}}(x, y)) \xrightarrow{\beta} \mathrm{colim}_{\underline{\mathbf{m}} \in \mathcal{Z}^{op}} \mathrm{N}_\infty(G(\underline{\mathbf{m}})(x, y))$$

as string of equivalences

$$\begin{aligned} \mathrm{N}_\infty(\underline{\mathbf{3}}(x, y)) &\simeq \int^{\underline{\mathbf{m}}' \in \mathcal{Z}} \mathrm{N}_\infty(\underline{\mathbf{m}}'(x, y)) \odot \hom_{\mathcal{Z}}(\underline{\mathbf{3}}, \underline{\mathbf{m}}') \\ &= \int^{\mathcal{Z}} \mathrm{N}_\infty((-)(x, y)) \odot \hom_{\mathcal{Z}}(\underline{\mathbf{3}}, -) \end{aligned}$$

[16] Each path component of this category contains exactly one object lying over $[\mathbf{A}] \in \mathcal{Z}^{op}$.

$$\begin{aligned}
&\xrightarrow{\sim} \int^{\mathcal{Z}} \mathrm{N}_\infty((-)(x,y)) \odot \left(\mathrm{colim}^{\mathrm{Fun}(\mathcal{Z},\mathcal{S})}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \hom_{\mathcal{Z}}(G(\underline{\mathbf{m}}),-)\right)\\
&\simeq \mathrm{colim}^{s\mathcal{S}}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \left(\int^{\mathcal{Z}} \mathrm{N}_\infty((-)(x,y)) \odot \hom_{\mathcal{Z}}(G(\underline{\mathbf{m}}),-)\right)\\
&= \mathrm{colim}^{s\mathcal{S}}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \left(\int^{\underline{\mathbf{m}}'\in\mathcal{Z}} \mathrm{N}_\infty(\underline{\mathbf{m}}'(x,y)) \odot \hom_{\mathcal{Z}}(G(\underline{\mathbf{m}}),\underline{\mathbf{m}}')\right)\\
&\simeq \mathrm{colim}^{s\mathcal{S}}_{\underline{\mathbf{m}}\in\mathcal{Z}^{op}} \mathrm{N}_\infty(G(\underline{\mathbf{m}})(x,y))
\end{aligned}$$

in $s\mathcal{S}$, in which

- the second and fifth lines are purely for notational convenience,
- we apply to the functor

$$\mathcal{Z}^{op} \xrightarrow{\mathrm{N}_\infty((-)(x,y))} s\mathcal{S}$$

  - Lemma 4.3.8 to obtain the first line,
  - Lemma 4.3.9 to obtain the fourth line, and
  - Lemma 4.3.8 again to obtain the last line,

  and
- the third line follows from the equivalence in $\mathrm{Fun}(\mathcal{Z},\mathcal{S})$ obtained above.

(So in fact, the map β itself is already an equivalence in $s\mathcal{S}$ (i.e. before geometric realization).)

(6) We claim that for every $\underline{\mathbf{m}} \in \mathcal{Z}^{op}$ the map

$$\mathrm{N}_\infty(G(\underline{\mathbf{m}})(x,y)) \xrightarrow{\mathrm{N}_\infty((\psi^{op}_{\underline{\mathbf{m}}})(x,y))} \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y))$$

in $s\mathcal{S}$ becomes an equivalence after geometric realization. This follows from an analysis of the corepresenting map $F(\underline{\mathbf{m}}) \xrightarrow{\psi_{\underline{\mathbf{m}}}} G(\underline{\mathbf{m}})$ in $\mathcal{Z} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$: it can be obtained as a composite

$$F(\underline{\mathbf{m}}) = \underline{\mathbf{m}}'_0 \to \underline{\mathbf{m}}'_1 \to \cdots \to \underline{\mathbf{m}}'_{|\underline{\mathbf{m}}|_{\mathbf{W}^{-1}}-1} \to \underline{\mathbf{m}}'_{|\underline{\mathbf{m}}|_{\mathbf{W}^{-1}}} = G(\underline{\mathbf{m}})$$

in $\mathcal{Z}$, in which each $\underline{\mathbf{m}}'_i$ is obtained from $\underline{\mathbf{m}}'_{i-1}$ by omitting one of the internal appearances of $\mathbf{W}^{-1}$ in $F(\underline{\mathbf{m}})$, and the corresponding map $\underline{\mathbf{m}}'_i \to \underline{\mathbf{m}}'_{i+1}$ is obtained by collapsing this copy of $\mathbf{W}^{-1}$ to an identity map. Each map

$$\mathrm{N}_\infty(\underline{\mathbf{m}}'_i(x,y)) \to \mathrm{N}_\infty(\underline{\mathbf{m}}'_{i-1}(x,y))$$

in $s\mathcal{S}$ becomes an equivalence after geometric realization, by Lemma 4.3.6 when the about-to-be-omitted appearance of $\mathbf{W}^{-1}$ in $\underline{\mathbf{m}}'_{i-1}$ is adjacent to another appearance of $\mathbf{W}^{-1}$, and by applying the definition of $(\mathcal{R}, \mathbf{W})$ admitting a homotopical three-arrow calculus (Definition 4.3.1) to (either one or two iterations, depending on the shape of $\underline{\mathbf{m}}'_{i-1}$, of) the combination of Lemma 4.2.24 and Proposition 3.3.4. Hence, the composite map

$$\mathrm{N}_\infty(G(\underline{\mathbf{m}})(x,y)) = \mathrm{N}_\infty(\underline{\mathbf{m}}'_{|\underline{\mathbf{m}}|_{\mathbf{W}^{-1}}}(x,y)) \to \cdots \to \mathrm{N}_\infty(\underline{\mathbf{m}}'_0(x,y)) = \mathrm{N}_\infty(F(\underline{\mathbf{m}})(x,y)),$$

which is precisely the map $\mathrm{N}_\infty((\psi^{op}_{\underline{\mathbf{m}}})(x,y))$, does indeed become an equivalence upon geometric realization as well. Then, since colimits commute, it follows that the induced map

$$|\mathrm{colim}_{\underline{\mathbf{m}}' \in \mathcal{Z}^{op}} \mathrm{N}_\infty(G(\underline{\mathbf{m}}')(x,y))| \xrightarrow{|\psi|} |\mathrm{colim}_{\underline{\mathbf{m}}' \in \mathcal{Z}^{op}} \mathrm{N}_\infty(F(\underline{\mathbf{m}}')(x,y))|$$

is an equivalence in $\mathcal{S}$. □

4.4 Hammock localizations of relative ∞-categories

In §4.2, given a relative ∞-category $(\mathcal{R}, \mathbf{W})$ and a pair of objects $x, y \in \mathcal{R}$, we defined the corresponding hammock simplicial space

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(x,y) \in s\mathcal{S}$$

(see Definition 4.2.17). In this section, we proceed to *globalize* this construction, assembling the various hammock simplicial spaces of $(\mathcal{R}, \mathbf{W})$ into a Segal simplicial space – and thence a $s\mathcal{S}$-enriched ∞-category – whose compositions encode the *concatenation* of zigzags in $(\mathcal{R}, \mathbf{W})$.

The bulk of the construction of the hammock localization consists in constructing the *pre*-hammock localization: this will be a Segal simplicial space

$$\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W}) \in \mathcal{S}s\mathcal{S} \subset s(s\mathcal{S}),$$

whose n^{th} level is given by the colimit

$$\mathrm{colim}^{s\mathcal{S}}_{(\underline{\mathbf{m}}_1,\dots,\underline{\mathbf{m}}_n)\in(\mathcal{Z}^{op})^{\times n}} \mathrm{N}_\infty\left(\mathrm{Fun}([\underline{\mathbf{m}}_1;\dots;\underline{\mathbf{m}}_n],\mathcal{R})^{\mathbf{W}}\right).$$

For clarity, we proceed in stages.

First, we build an object which simultaneously corepresents

- all possible sequences (of any length) of composable zigzags, and
- all possible concatenations among these sequences.

Construction 4.4.1. Observe that $\mathcal{Z} \in \mathcal{C}\mathrm{at}$ is a monoid object, i.e. a monoidal category: its multiplication is given by the concatenation functor

$$\mathcal{Z} \times \mathcal{Z} \xrightarrow{[-;-]} \mathcal{Z},$$

and the unit map $\mathrm{pt}_{\mathcal{C}\mathrm{at}} \to \mathcal{Z}$ selects the terminal object $[\varnothing] \in \mathcal{Z}$.[17] We can thus define its bar construction

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{Bar}(\mathcal{Z})_\bullet} \mathcal{C}\mathrm{at},$$

which has $\mathrm{Bar}(\mathcal{Z})_n = \mathcal{Z}^{\times n}$ (so that $\mathrm{Bar}(\mathcal{Z})_0 = \mathcal{Z}^{\times 0} = \mathrm{pt}_{\mathcal{C}\mathrm{at}}$), with face maps given by concatenation and with degeneracy maps given by the unit. This admits an *oplax* natural transformation to the functor

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{const}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at})} \mathcal{C}\mathrm{at},$$

which we encode as a commutative triangle

$$\begin{array}{ccc} \mathrm{Gr}^-(\mathrm{Bar}(\mathcal{Z})_\bullet) & \longrightarrow & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \times \mathbf{\Delta} \\ & \searrow \quad \swarrow & \\ & \mathbf{\Delta} & \end{array}$$

in $\mathcal{C}\mathrm{at}$ (recall Definition 3.2.1 and Example 3.1.15): in simplicial degree n, this is given by the iterated concatenation functor

$$\mathrm{Bar}(\mathcal{Z})_n = \mathcal{Z}^{\times n} \xrightarrow{[-;\cdots;-]} \mathcal{Z} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$$

(which in degree 0 is simply the composite

$$\{[\varnothing]\} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at},$$

[17] In fact, we can even consider $\mathcal{Z}$ as a monoid object in $\mathrm{e}\mathrm{at}$ (i.e. a *strict* monoidal category), but this is unnecessary for our purposes.

i.e. the inclusion of the terminal object $\{\mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}}\} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$).[18,19] Taking opposites, we obtain a commutative triangle

$$\begin{array}{ccc} \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) & \longrightarrow & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}^{op} \times \boldsymbol{\Delta}^{op} \\ & \searrow \quad \swarrow & \\ & \boldsymbol{\Delta}^{op} & \end{array}$$

in $\mathcal{C}\mathrm{at}$, which now encodes a *lax* natural transformation from the bar construction

$$\boldsymbol{\Delta}^{op} \xrightarrow{\mathrm{Bar}(\mathcal{Z}^{op})_\bullet} \mathcal{C}\mathrm{at}$$

on the monoid object $\mathcal{Z}^{op} \in \mathcal{C}\mathrm{at}$ (note that the involution $(-)^{op} : \mathcal{C}\mathrm{at} \xrightarrow{\sim} \mathcal{C}\mathrm{at}$ is covariant) to the functor

$$\boldsymbol{\Delta}^{op} \xrightarrow{\mathrm{const}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}^{op})} \mathcal{C}\mathrm{at}.$$

We now map into an arbitrary relative ∞-category and extract the indicated colimits, all in a functorial way.

Construction 4.4.2. A relative ∞-category $(\mathcal{R}, \mathbf{W})$ represents a composite functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{Fun}(-,\mathcal{R})^{\mathbf{W}}} \mathcal{C}\mathrm{at}_\infty \xrightarrow[\sim]{\mathrm{N}_\infty} \mathcal{CSS} \hookrightarrow s\mathcal{S}.$$

Considering this as a natural transformation $\mathrm{const}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}^{op}) \to \mathrm{const}(s\mathcal{S})$ in $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{C}\mathrm{at}_\infty)$, we can postcompose it with the lax natural transformation obtained in Construction 4.4.1, yielding a composite lax natural transformation encoded by the diagram

$$\begin{array}{ccccc} \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) & \longrightarrow & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}^{op} \times \boldsymbol{\Delta}^{op} & \xrightarrow{\mathrm{N}_\infty\left(\mathrm{Fun}(-,\mathcal{R})^{\mathbf{W}}\right)\times \mathrm{id}_{\boldsymbol{\Delta}^{op}}} & s\mathcal{S} \times \boldsymbol{\Delta}^{op} \\ & \searrow & \downarrow & \swarrow & \\ & & \boldsymbol{\Delta}^{op} & & \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$. Then, by Proposition T.4.2.2.7, there is a unique "fiberwise colimit" lift in the diagram

$$\begin{array}{ccc} \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) & \longrightarrow & s\mathcal{S} \times \boldsymbol{\Delta}^{op} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) \underset{\boldsymbol{\Delta}^{op}}{\diamond} \boldsymbol{\Delta}^{op} & \longrightarrow & \boldsymbol{\Delta}^{op} \end{array}$$

[18] The reason that we must compose with the forgetful functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ is that the oplax structure maps (e.g. the inclusion $\underline{\mathbf{m}}_1 \hookrightarrow [\underline{\mathbf{m}}_1; \underline{\mathbf{m}}_2]$) do not respect the double-pointings.

[19] It is also true that for a monoidal (∞-)category $\mathcal{C}$ whose unit object is terminal, the bar construction $\mathrm{Bar}(\mathcal{C})_\bullet$ admits a canonical *lax* natural transformation to $\mathrm{const}(\mathcal{C})$, whose components are again given by the iterated monoidal product. But this is distinct from what we seek here.

in $\mathcal{C}\mathrm{at}_\infty$.[20] Thus, the resulting composite

$$\mathbf{\Delta}^{op} \to \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \to s\mathcal{S} \times \mathbf{\Delta}^{op} \to s\mathcal{S}$$

takes each object $[n]^\circ \in \mathbf{\Delta}^{op}$ to the colimit of the composite

$$\mathrm{Bar}(\mathcal{Z}^{op})_n = (\mathcal{Z}^{op})^{\times n} \xrightarrow{[-;\cdots;-]^{op}} \mathcal{Z}^{op} \hookrightarrow (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**})^{op} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}^{op} \xrightarrow{\mathrm{N}_\infty\left(\mathrm{Fun}(-,\mathcal{R})^{\mathbf{W}}\right)} s\mathcal{S}.$$

We denote this simplicial object in simplicial spaces by

$$\mathbf{\Delta}^{op} \xrightarrow{\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})} s\mathcal{S}.$$

Allowing $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ to vary, this assembles into a functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}^H_{\mathrm{pre}}} s(s\mathcal{S}).$$

We now show that the bisimplicial spaces of Construction 4.4.2 are in fact Segal simplicial spaces.

Lemma 4.4.3. *For any $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, the object $\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W}) \in s(s\mathcal{S})$ satisfies the Segal condition.*

Proof. We must show that for every $n \geq 2$, the n^{th} Segal map

$$\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_n \to \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_1 \underset{t,\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})_0,s}{\times} \cdots \underset{t,\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})_0,s}{\times} \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_1$$

(to the n-fold fiber product) is an equivalence in $s\mathcal{S}$. As $s\mathcal{S}$ is an ∞-topos, colimits therein are universal, i.e. they commute with pullbacks (see Definition T.6.1.0.4 and Theorem T.6.1.0.6 (and the discussion at the beginning of §T.6.1.1)). Moreover, note that we have a canonical equivalence $\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_0 \simeq \mathrm{N}_\infty(\mathbf{W})$ in $s\mathcal{S}$. Hence, by induction, we have a string of equivalences

$$\begin{aligned}
&\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_1 \underset{t,\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})_0,s}{\times} \cdots \underset{t,\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})_0,s}{\times} \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_1 \\
&\simeq \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_1 \underset{\{1\},\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})_0,\{0\}}{\times} \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_{n-1}
\end{aligned}$$

[20] The object in the bottom left of this diagram is a "relative join" (see Definition T.4.2.2.1), which in this case actually simply reduces to a "directed mapping cylinder" (see Example 3.1.8).

$$
\begin{aligned}
&= \lim \left(\begin{array}{ccc} & & \operatorname{colim}_{(\underline{\mathbf{m}}_2,\dots,\underline{\mathbf{m}}_n)\in(\mathcal{Z}^{op})^{\times(n-1)}} \mathrm{N}_\infty\left(\mathrm{Fun}([\underline{\mathbf{m}}_2;\dots;\underline{\mathbf{m}}_n],\mathcal{R})^{\mathbf{W}}\right) \\ & & \downarrow \\ \operatorname{colim}_{\underline{\mathbf{m}}_1\in\mathcal{Z}^{op}} \mathrm{N}_\infty\left(\mathrm{Fun}(\underline{\mathbf{m}}_1,\mathcal{R})^{\mathbf{W}}\right) & \longrightarrow & \mathrm{N}_\infty(\mathbf{W}) \end{array} \right) \\
&\simeq \operatorname{colim}_{\underline{\mathbf{m}}_1\in\mathcal{Z}^{op}} \left(\lim \left(\begin{array}{ccc} & & \operatorname{colim}_{(\underline{\mathbf{m}}_2,\dots,\underline{\mathbf{m}}_n)\in(\mathcal{Z}^{op})^{\times(n-1)}} \mathrm{N}_\infty\left(\mathrm{Fun}([\underline{\mathbf{m}}_2;\dots;\underline{\mathbf{m}}_n],\mathcal{R})^{\mathbf{W}}\right) \\ & & \downarrow \\ \mathrm{N}_\infty\left(\mathrm{Fun}(\underline{\mathbf{m}}_1,\mathcal{R})^{\mathbf{W}}\right) & \longrightarrow & \mathrm{N}_\infty(\mathbf{W}) \end{array} \right) \right) \\
&\simeq \operatorname{colim}_{\underline{\mathbf{m}}_1\in\mathcal{Z}^{op}} \left(\operatorname{colim}_{(\underline{\mathbf{m}}_2,\dots,\underline{\mathbf{m}}_n)\in(\mathcal{Z}^{op})^{\times(n-1)}} \left(\lim \left(\begin{array}{ccc} & & \mathrm{N}_\infty\left(\mathrm{Fun}([\underline{\mathbf{m}}_2;\dots;\underline{\mathbf{m}}_n],\mathcal{R})^{\mathbf{W}}\right) \\ & & \downarrow \\ \mathrm{N}_\infty\left(\mathrm{Fun}(\underline{\mathbf{m}}_1,\mathcal{R})^{\mathbf{W}}\right) & \longrightarrow & \mathrm{N}_\infty(\mathbf{W}) \end{array} \right) \right) \right) \\
&\simeq \operatorname{colim}_{(\underline{\mathbf{m}}_1,\dots,\underline{\mathbf{m}}_n)\in(\mathcal{Z}^{op})^{\times n}} \mathrm{N}_\infty\left(\mathrm{Fun}([\underline{\mathbf{m}}_1;\dots;\underline{\mathbf{m}}_n],\mathcal{R})^{\mathbf{W}}\right) \\
&= \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R},\mathbf{W})_n
\end{aligned}
$$

(where in the penultimate line we appeal to Fubini's theorem for colimits) which, chasing through the definitions, visibly coincides with the n^{th} Segal map. This proves the claim. □

We finally come to the main point of this section.

Definition 4.4.4. By Lemma 4.4.3, the functor given in Construction 4.4.2 admits a factorization

$$
\begin{array}{ccc} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty & \xrightarrow{\mathscr{L}^H_{\mathrm{pre}}} & s(s\mathcal{S}) \\ & \searrow & \uparrow \\ & & \mathcal{S}s\mathcal{S} \end{array}
$$

through the ∞-category of Segal simplicial spaces. We again denote this factorization by

$$
\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}^H_{\mathrm{pre}}} \mathcal{S}s\mathcal{S},
$$

and refer to it as the ***pre-hammock localization*** functor.[21] Then, we define the ***hammock localization*** functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}^H} \mathcal{C}\mathrm{at}_{s\mathcal{S}}$$

to be the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}^H_{\mathrm{pre}}} \mathcal{S}s\mathcal{S} \xrightarrow{\mathrm{sp}(-)} \mathcal{C}\mathrm{at}_{s\mathcal{S}}.$$

Remark 4.4.5. Given a relative ∞-category $(\mathcal{R}, \mathbf{W})$, the 0^{th} level of its pre-hammock localization

$$\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W}) \in \mathcal{S}s\mathcal{S} \subset s(s\mathcal{S})$$

is given by

$$\mathrm{colim}\left(\{[\varnothing]\}^\circ \hookrightarrow (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**})^{op} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}^{op} \xrightarrow{\mathrm{N}_\infty\left(\mathrm{Fun}(-\mathcal{R})^{\mathbf{W}}\right)} s\mathcal{S}\right),$$

which is simply the nerve $\mathrm{N}_\infty(\mathbf{W}) \in s\mathcal{S}$ of the subcategory $\mathbf{W} \subset \mathcal{R}$ of weak equivalences. Thus, its space of objects is simply

$$\mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_0 \simeq \mathrm{N}_\infty(\mathbf{W})_0 \simeq \mathbf{W}^{\simeq} \simeq \mathcal{R}^{\simeq}.$$

Moreover, unwinding the definitions, it is manifestly clear that

- its hom-simplicial spaces are precisely the hammock simplicial spaces of $(\mathcal{R}, \mathbf{W})$ (recall Definitions 4.1.8 and 4.2.17), and
- its compositions correspond to concatenation of zigzags (with identity morphisms corresponding to zigzags of type $[\varnothing] \in \mathcal{Z}$).

Of course, we have a canonical counit weak equivalence

$$\mathscr{L}^H(\mathcal{R}, \mathbf{W}) \xrightarrow{\simeq} \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})$$

in $\mathcal{S}s\mathcal{S}_{\mathrm{DK}}$ which is even fully faithful in the $s\mathcal{S}$-enriched sense, so that the hammock localization enjoys all these same properties.

Just as in the 1-categorical case, the hammock localization of $(\mathcal{R}, \mathbf{W})$ admits a natural map from $\mathcal{R}$.

[21] The terminology "pre-hammock localization" should be parsed as "pre-(hammock localization)": it already contains the hammock simplicial spaces (see Remark 4.4.5), it is just not itself the hammock localization.

Construction 4.4.6. Returning to Construction 4.4.1, observe that there is a tautological section

$$\begin{array}{c} \mathrm{Gr}^{-}(\mathrm{Bar}(\mathcal{Z})_\bullet) \\ \uparrow\downarrow \\ \mathbf{\Delta} \end{array}$$

which takes $[n] \in \mathbf{\Delta}$ to $([\mathbf{A}], \dots, [\mathbf{A}]) \in \mathcal{Z}^{\times n} = \mathrm{Bar}(\mathcal{Z})_n$, and which takes a map $[m] \xrightarrow{\varphi} [n]$ in $\mathbf{\Delta}$ to the map corresponding to the fiber map which, in the i^{th} factor of $\mathcal{Z}^{\times m}$, is given by the unique map

$$[\mathbf{A}] \to [\mathbf{A}^{\circ(\varphi(i)-\varphi(i-1))}]$$

in $\mathcal{Z}$. This is opposite to a tautological section

$$\begin{array}{c} \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) \\ \uparrow\downarrow \\ \mathbf{\Delta}^{op} \end{array}$$

which gives rise to a composite map

$$\mathbf{\Delta}^{op} \to \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) \to \mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}$$

admitting a natural transformation to the standard inclusion (as the "target" factor, i.e. the fiber over $1 \in [1]$). This postcomposes with the composite

$$\mathrm{Gr}(\mathrm{Bar}(\mathcal{Z}^{op})_\bullet) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \to s\mathcal{S} \times \mathbf{\Delta}^{op} \to s\mathcal{S}$$

appearing in Construction 4.4.2 to give a natural transformation

$$\mathrm{N}^{\mathrm{lw}}_{\infty}\left(\mathrm{Fun}([\bullet], \mathcal{R})^{\mathbf{W}}\right) \to \mathscr{L}^{H}_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_\bullet$$

in $\mathrm{Fun}(\mathbf{\Delta}^{op}, s\mathcal{S})$.[22] Thus, in simplicial degree n, this map is simply the inclusion into the colimit defining $\mathscr{L}^{H}_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_n \in s\mathcal{S}$ at the object

$$([\mathbf{A}]^\circ, \dots, [\mathbf{A}]^\circ) \in (\mathcal{Z}^{op})^{\times n}.$$

[22]Note that this source is just the image of the Rezk pre-nerve $\mathrm{preN}^{\mathrm{R}}_{\infty}(\mathcal{R}, \mathbf{W})_\bullet \in s\mathcal{C}\mathrm{at}_\infty$ under the inclusion $s\mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} s\mathcal{C}\mathcal{S}\mathcal{S} \hookrightarrow s(s\mathcal{S})$ (recall Definition 2.3.1).

Restricting levelwise to (the nerve of) the maximal subgroupoid, we obtain a composite

$$\begin{aligned}
\mathrm{const}(\mathcal{R})_\bullet &= \mathrm{const}^{\mathrm{lw}}(\mathrm{U}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R})))_\bullet \\
&= \mathrm{const}^{\mathrm{lw}}(\hom_{\mathcal{C}\mathrm{at}_\infty}([\bullet], \mathcal{R})) \\
&\simeq \mathrm{const}^{\mathrm{lw}}\left(\mathrm{Fun}([\bullet], \mathcal{R})^{\simeq}\right) \\
&\simeq \mathrm{N}^{\mathrm{lw}}_\infty\left(\mathrm{Fun}([\bullet], \mathcal{R})^{\simeq}\right) \\
&\hookrightarrow \mathrm{N}^{\mathrm{lw}}_\infty\left(\mathrm{Fun}([\bullet], \mathcal{R})^{\mathbf{W}}\right) \\
&\to \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W})_\bullet.
\end{aligned}$$

As this source lies in $\mathcal{C}\mathrm{at}_{s\mathcal{S}} \subset \mathcal{S}s\mathcal{S}$, we obtain a canonical factorization

$$\begin{array}{ccc}
\mathrm{const}(\mathcal{R}) & \longrightarrow & \mathscr{L}^H_{\mathrm{pre}}(\mathcal{R}, \mathbf{W}) \\
& \searrow & \uparrow \wr \\
& & \mathscr{L}^H(\mathcal{R}, \mathbf{W})
\end{array}$$

in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}})_{\mathrm{DK}}$. This clearly assembles into a natural transformation

$$\mathrm{const} \to \mathscr{L}^H$$

in $\mathrm{Fun}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathcal{C}\mathrm{at}_{s\mathcal{S}})$.

Definition 4.4.7. For a relative ∞-category $(\mathcal{R}, \mathbf{W})$, we refer to the map

$$\mathrm{const}(\mathcal{R}) \to \mathscr{L}^H(\mathcal{R}, \mathbf{W})$$

in $\mathcal{C}\mathrm{at}_{s\mathcal{S}}$ of Construction 4.4.6 as its ***tautological inclusion***.

We end this section with the following fundamental result, an analog of [DK80a, Proposition 3.3]. In essence, it shows that when considered as morphisms in the hammock localization, weak equivalences in $\mathcal{R}$ both represent and corepresent equivalences in the underlying ∞-category. Just as with the fundamental theorem of homotopical three-arrow calculi (4.3.4), its proof will be substantially more involved than that of its 1-categorical analog (recall Remark 4.0.2).

Proposition 4.4.8. *Let $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, and let $r, y, z \in \mathcal{R}$. Suppose we are given a weak equivalence*

$$w \in \hom_{\mathbf{W}}(y, z) \subset \hom_{\mathcal{R}}(y, z),$$

and let us also denote by $w \in \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(y,z)_0$ *the resulting composite morphism*

$$\mathrm{pt}_{s\mathcal{S}} \to \mathrm{N}_\infty([\mathbf{A}](y,z)) \to \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(y,z).$$

Then, the induced "composition with w*" maps*

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,y) \xrightarrow{\chi^{\mathscr{L}^H(R,\mathbf{W})}_{r,y,z}(-,w)} \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,z)$$

and

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(z,r) \xrightarrow{\chi^{\mathscr{L}^H(R,\mathbf{W})}_{y,r,z}(w,-)} \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(y,r)$$

in $s\mathcal{S}$ *become equivalences in* $\mathcal{S}$ *upon geometric realization. Moreover, if we denote by* $w^{-1} \in \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(z,y)_0$ *the composite morphism*

$$\mathrm{pt}_{s\mathcal{S}} \to \mathrm{N}_\infty([\mathbf{W}^{-1}](z,y)) \to \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(y,z),$$

then their inverses are respectively given by the geometric realizations of the induced "composition with w^{-1}*" maps*

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,z) \xrightarrow{\chi^{\mathscr{L}^H(\mathcal{R},\mathbf{W})}_{r,z,y}(-,w^{-1})} \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,y)$$

and

$$\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(y,r) \xrightarrow{\chi^{\mathscr{L}^H(\mathcal{R},\mathbf{W})}_{z,y,r}(w^{-1},-)} \underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(z,r).$$

in $s\mathcal{S}$.

Proof. We prove the first statement; the second statement follows by a nearly identical argument. Moreover, we will only show that the composite map

$$\left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,y)\right| \to \left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,z)\right| \to \left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,y)\right|$$

is an equivalence; that the composite

$$\left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,z)\right| \to \left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,y)\right| \to \left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,z)\right|$$

is an equivalence will follow from a very similar argument.

For each $\underline{\mathbf{m}} \in \mathcal{Z}^{op}$, let us define a functor

$$\underline{\mathbf{m}}(r,y) \xrightarrow{\varphi_{\underline{\mathbf{m}}}} [\underline{\mathbf{m}};\mathbf{A};\mathbf{W}^{-1}](r,y)$$

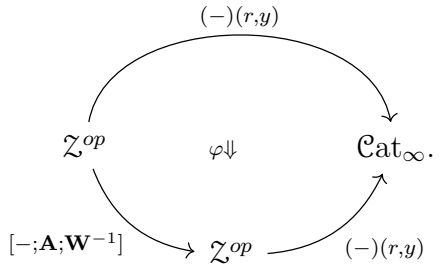
given informally by taking a zigzag

$$r \xrightarrow{\underline{\mathbf{m}}} y$$

in $(\mathcal{R}, \mathbf{W})$ to the zigzag

$$r \xrightarrow{\underline{\mathbf{m}}} y \longrightarrow z \xleftarrow{\approx} y$$

in $(\mathcal{R}, \mathbf{W})$, in which both new maps are the chosen weak equivalence w.[23] This operation is clearly natural in $\underline{\mathbf{m}} \in \mathcal{Z}^{op}$, i.e. it assembles into a natural transformation



Then, using Proposition 2.2.4 and the fact that the geometric realization functor $s\mathcal{S} \xrightarrow{|-|} \mathcal{S}$ commutes with colimits (being a left adjoint), we see that the composite

$$\left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,y)\right| \to \left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,z)\right| \to \left|\underline{\hom}_{\mathscr{L}^H(\mathcal{R},\mathbf{W})}(r,y)\right|$$

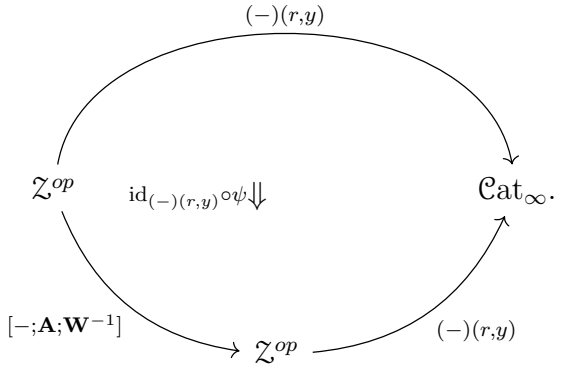
is obtained as the composite

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{Z}^{op}}\left((-)^{\mathrm{gpd}} \circ (-)(r,y)\right) & & \\ \Big\downarrow{\scriptstyle \mathrm{colim}_{\mathcal{Z}^{op}}(\mathrm{id}_{(-)^{\mathrm{gpd}}} \circ \varphi)} & & \\ \mathrm{colim}_{\mathcal{Z}^{op}}\left((-)^{\mathrm{gpd}} \circ (-)(r,y) \circ [-;\mathbf{A};\mathbf{W}^{-1}]\right) & \xrightarrow[\mathrm{colim}^{\mathcal{S}}([-;\mathbf{A};\mathbf{W}^{-1}])]{} & \mathrm{colim}_{\mathcal{Z}^{op}}\left((-)^{\mathrm{gpd}} \circ (-)(r,y)\right). \end{array}$$

To see that this is an equivalence, for each $\underline{\mathbf{m}} \in \mathcal{Z}^{op}$ let us define a map $\underline{\mathbf{m}} \xrightarrow{\psi_{\underline{\mathbf{m}}}} [\underline{\mathbf{m}}; \mathbf{A}; \mathbf{W}^{-1}]$ in $\mathcal{Z}^{op}$ to be opposite the map $[\underline{\mathbf{m}}; \mathbf{A}; \mathbf{W}^{-1}] \to \underline{\mathbf{m}}$ in $\mathcal{Z}$ which collapses the newly concatenated copy of $[\mathbf{A}; \mathbf{W}^{-1}]$ to the map $\mathrm{id}_{t\underline{\mathbf{m}}}$. These assemble into a

[23] This (and subsequent constructions) can easily be made precise by defining a suitable notion of a map in a relative word being forced to land at w; we will leave such a precise construction to the interested reader.

natural transformation $\mathrm{id}_{\mathcal{Z}^{op}} \xrightarrow{\psi} [-;\mathbf{A};\mathbf{W}^{-1}]$ in $\mathrm{Fun}(\mathcal{Z}^{op},\mathcal{Z}^{op})$, and hence we obtain a natural transformation



Moreover, For each $\underline{\mathbf{m}} \in \mathcal{Z}^{op}$ we have a functor

$$[1] \times \underline{\mathbf{m}}(r,y) \xrightarrow{\mu_{\underline{\mathbf{m}}}} [\underline{\mathbf{m}};\mathbf{A};\mathbf{W}^{-1}](r,y),$$

adjoint to a functor

$$\underline{\mathbf{m}}(r,y) \to \mathrm{Fun}([1],[\underline{\mathbf{m}};\mathbf{A};\mathbf{W}^{-1}](r,y)),$$

given informally by taking a zigzag

$$r \xrightarrow{\ \underline{\mathbf{m}}\ } y$$

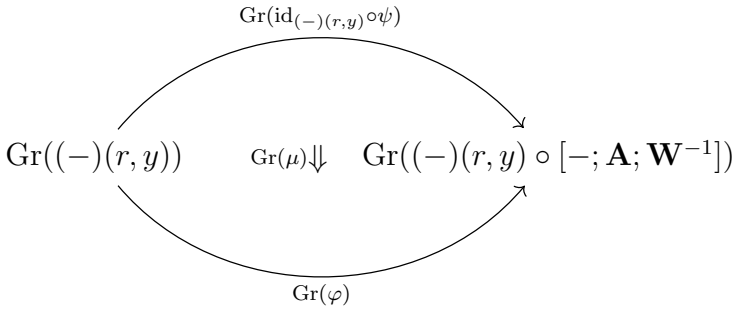
in $(\mathcal{R},\mathbf{W})$ to the diagram

$$\begin{array}{ccccccc}
r & \xrightarrow{\ \underline{\mathbf{m}}\ } & y & \longrightarrow & y & \xleftarrow{\approx} & y \\
\| & & \| & & \downarrow\wr & & \| \\
r & \xrightarrow[\ \underline{\mathbf{m}}\ ]{} & y & \longrightarrow & z & \xleftarrow[\approx]{} & y
\end{array}$$

in $(\mathcal{R},\mathbf{W})$ representing a morphism in $[\underline{\mathbf{m}};\mathbf{A};\mathbf{W}^{-1}](r,y)$, where the maps in the right two squares are all either the chosen weak equivalence $y \xrightarrow{\approx} z$ or are id_y. These assemble into a morphism

$$\mathrm{const}([1]) \times (-)(r,y) \xrightarrow{\mu} (-)(r,y) \circ [-;\mathbf{A};\mathbf{W}^{-1}]$$

in $\mathrm{Fun}(\mathcal{Z}^{op}, \mathcal{C}\mathrm{at}_\infty)$, i.e. a modification from $\mathrm{id}_{(-)(r,y)} \circ \psi$ to φ. By Proposition 3.3.8, this induces a natural transformation



which, by Lemma 2.1.26 and Proposition 3.3.1, gives a homotopy between the maps

$$\mathrm{colim}_{\mathcal{Z}^{op}} \left((-)^{\mathrm{gpd}} \circ (-)(r,y)\right) \xrightarrow{\mathrm{colim}_{\mathcal{Z}^{op}}(\mathrm{id}_{(-)^{\mathrm{gpd}}} \circ \mathrm{id}_{(-)(r,y)} \circ \psi)} \mathrm{colim}_{\mathcal{Z}^{op}} \left((-)^{\mathrm{gpd}} \circ (-)(r,y) \circ [-; \mathbf{A}; \mathbf{W}^{-1}]\right)$$

and

$$\mathrm{colim}_{\mathcal{Z}^{op}} \left((-)^{\mathrm{gpd}} \circ (-)(r,y)\right) \xrightarrow{\mathrm{colim}_{\mathcal{Z}^{op}}(\mathrm{id}_{(-)^{\mathrm{gpd}}} \circ \varphi)} \mathrm{colim}_{\mathcal{Z}^{op}} \left((-)^{\mathrm{gpd}} \circ (-)(r,y) \circ [-; \mathbf{A}; \mathbf{W}^{-1}]\right)$$

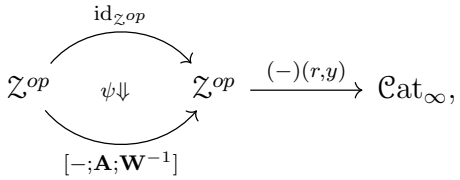
in $\mathcal{S}$. Hence, to show that the above composite is an equivalence, it suffices to show that the composite

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{Z}^{op}} \left((-)^{\mathrm{gpd}} \circ (-)(r,y)\right) & & \\ \Big\downarrow{\scriptstyle \mathrm{colim}_{\mathcal{Z}^{op}}(\mathrm{id}_{(-)^{\mathrm{gpd}}} \circ \mathrm{id}_{(-)(r,y)} \circ \psi)} & & \\ \mathrm{colim}_{\mathcal{Z}^{op}} \left((-)^{\mathrm{gpd}} \circ (-)(r,y) \circ [-; \mathbf{A}; \mathbf{W}^{-1}]\right) & \xrightarrow[\mathrm{colim}^{\mathcal{S}}([-;\mathbf{A};\mathbf{W}^{-1}])]{} & \mathrm{colim}_{\mathcal{Z}^{op}} \left((-)^{\mathrm{gpd}} \circ (-)(r,y)\right) \end{array}$$

is an equivalence. But this composite fits into a commutative triangle

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{Z}^{op}}((-)^{\mathrm{gpd}} \circ (-)(r,y) \circ \mathrm{id}_{\mathcal{Z}^{op}}) & & \\ \Big\downarrow & \overset{\sim}{\searrow} & \\ & & \mathrm{colim}_{\mathcal{Z}^{op}}((-)^{\mathrm{gpd}} \circ (-)(r,y)) \\ \mathrm{colim}_{\mathcal{Z}^{op}}((-)^{\mathrm{gpd}} \circ (-)(r,y) \circ [-; \mathbf{A}; \mathbf{W}^{-1}]) & \nearrow & \end{array}$$

obtained by applying Proposition 3.3.5 to the diagram



so it is an equivalence. This proves the claim. □

4.5 From fractions to complete Segal spaces, redux

As an application of the theory developed in this chapter, we now provide a *sufficient condition* for the Rezk nerve $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W}) \in s\mathcal{S}$ of a relative ∞-category $(\mathcal{R},\mathbf{W})$ to be either

- a Segal space or
- a complete Segal space,

thus giving a partial answer to our own Question 2.3.6, which we refer to as the ***calculus theorem***.[24] This result is itself a direct generalization of joint work with Low regarding relative 1-categories (see [LMG15, Theorem 4.11]). That result, in turn, generalizes work of Rezk, Bergner, and Barwick–Kan; we refer the reader to [LMG15, §1] for a more thorough history.

Theorem 4.5.1. *Suppose that* $(\mathcal{R},\mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *admits a homotopical three-arrow calculus.*

(1) $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W}) \in s\mathcal{S}$ *is a Segal space.*

(2) Suppose moreover that $\mathbf{W} \subset \mathcal{R}$ *satisfies the two-out-of-three property. Then* $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R},\mathbf{W}) \in s\mathcal{S}$ *is a complete Segal space if and only if* $(\mathcal{R},\mathbf{W})$ *is saturated.*

The proof of the calculus theorem (4.5.1) is very closely patterned on the proof of [LMG15, Theorem 4.11] (the main theorem of that paper), which is almost completely analogous but holds only for relative 1-categories.[25] We encourage any reader

[24] The Rezk nerve is a straightforward generalization of Rezk's "classification diagram" construction, which we introduced and studied in §2.3.

[25] The 1-categorical Rezk nerve and the Rezk nerve of a relative ∞-category are essentially equivalent (see Remark 2.3.2), which is why essentially the same proof can be applied in both cases.

who would like to understand it to first read that paper: there are no truly new ideas here, only generalizations from 1-categories to ∞-categories.

Proof of Theorem 4.5.1. For this proof, we give a detailed step-by-step explanation of what must be changed in the paper [LMG15] to generalize its main theorem from relative 1-categories to relative ∞-categories.

- For [LMG15, Definition 2.1], we replace the notion of a "weak homotopy equivalence" of categories by the notion of a map in $\mathcal{C}\mathrm{at}_\infty$ which becomes an equivalence under $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ (i.e. a Thomason weak equivalence (see Definition 3.6.2 and Remark 3.6.3)).

- The proof of [LMG15, Lemma 2.2] carries over easily using Lemma 2.1.26.

- For [LMG15, Definition 2.3], we replace the notion of a "homotopy pullback diagram" of categories by the notion of a commutative square in $\mathcal{C}\mathrm{at}_\infty$ which becomes a pullback square under $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ (i.e. a homotopy pullback diagram in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$).

- For [LMG15, Definition 2.4], we replace the notions of "Grothendieck fibrations" and "Grothendieck opfibrations" of categories by those of cartesian fibrations and cocartesian fibrations of ∞-categories (see §3.1 and [MG]).

- For [LMG15, Remark 2.5], as the entire theory of ∞-categories is in essence already only pseudofunctorial, there is no corresponding notion of a co/cartesian fibration being "split" (or rather, *every* co/cartesian fibration should be thought of as being "split").

- The evident generalization of [LMG15, Example 2.6] can be obtained by applying Corollary T.2.4.7.12 to an identity functor of ∞-categories.

- The evident generalization of (the first of the two dual statements of) [LMG15, Theorem 2.7] is proved as Corollary 3.4.3.

- The evident generalization of [LMG15, Corollary 2.8] again follows directly (or can alternatively be obtained by combining Example 2.1.12 and Lemma 2.1.20).

- For [LMG15, Definition 2.9], we use the definition of the "two-sided Grothendieck construction" given in Definition 3.3.3. (Note that the 1-categorical version is simply the corresponding (strict) fiber product.)

- The evident analog of [LMG15, Lemma 2.11] is proved as Proposition 3.3.4.

- For [LMG15, Definition 3.1], we replace the notion of a "relative category" by the notion of a "relative ∞-category" given in Definition 2.1.1; recall from Remark 2.1.2 that here we are actually working with a slightly weaker definition. We replace the notion of its "homotopy category" by that of its localization given in Definition 2.1.8. We have already defined the notion of a relative ∞-category being "saturated" in Definition 2.1.14.

- For [LMG15, Definition 3.2], we have already made the analogous definitions in Notation 2.1.6.

- For [LMG15, Definitions 3.3 and 3.6], we have already made the analogous definitions in Definitions 4.2.5 and 4.2.9.

- The evident analog of [LMG15, Remark 3.7] is now true by definition (recall Notation 4.2.2).

- For [LMG15, Proposition 3.8], the paper actually only uses part (ii), whose evident analog is provided by Lemma 4.2.20(1).

- For [LMG15, Lemma 3.10], note that the functors in the statement of the result as well as in its proof are all corepresented by maps in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \subset (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty})_{**}$; the proof of the analogous result thus carries over by Lemma 4.3.5.

- For [LMG15, Lemma 3.11], again everything in the statement of the result as well as in its proof are all corepresented; again the proof carries over by Lemma 4.3.5.

- For [LMG15, Definition 4.1], we have already defined a "homotopical three-arrow calculus" for a relative ∞-category in Definition 4.3.1.

- For [LMG15, Theorem 4.5], we use the more general but slightly different definition of hammocks given in Definition 4.2.17 (recall Remark 4.2.18); part (i) is proved as Theorem 4.3.4, while part (ii) follows immediately from the definitions, particularly Definitions 4.4.4 and 4.1.8. (Note that in the present framework, the "reduction map" is simply replaced by the canonical map to the colimit defining the simplicial space of hammocks.)

- For [LMG15, Corollary 4.7], the evident analog of [DK80a, Proposition 3.3] is proved as Proposition 4.4.8.

- For [LMG15, Proposition 4.8], the proof carries over essentially without change. (The functor considered there when proving that the rectangle (AC) is a homotopy pullback diagram is replaced by our functor $\mathbf{W}^{op} \xrightarrow{\underline{\mathbf{3}}(x,-)} \mathcal{C}\mathrm{at}_\infty$ of Notation 4.2.23.)

- For [LMG15, Lemma 4.9], the map itself in the statement of the result comes from the functoriality

$$\mathbf{W}^{op} \xrightarrow{[\mathbf{W}^{-1};\mathbf{A}^{\circ n};\mathbf{W}^{-1}](x,-)} \mathcal{C}\mathrm{at}_\infty$$

  and

$$\mathbf{W} \xrightarrow{[\mathbf{W}^{-1};\mathbf{A}^{\circ n};\mathbf{W}^{-1}](-,y)} \mathcal{C}\mathrm{at}_\infty$$

  of Notation 4.2.23, as do the vertical maps in the commutative square in the proof. The horizontal maps in that square are corepresented by maps in $\mathcal{Z} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \subset (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$, and it clearly commutes by construction. The evident analog of [DK80a, Proposition 9.4] is proved as Lemma 4.2.24.

- For [LMG15, Proposition 4.10], note that all morphisms in both the statement of the result and its proof are corepresented by maps in $\mathcal{Z} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \subset (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**}$; the proof itself carries over without change.

- For [LMG15, Theorem 4.11] (whose analog is Theorem 4.5.1 itself), note that we are now proving an ∞-categorical statement (instead of a model-categorical one), and so there are no issues with fibrant replacement.

  - The proof of part (1) of Theorem 4.5.1 is identical to the proof of part (i) there: it follows from our analog of [LMG15, Proposition 4.10].

  - We address the two halves of the proof of part (2) of Theorem 4.5.1 in turn.

    * The proof of the "only if" direction runs analogously to that of [LMG15, Theorem 4.11(ii)], only now we use that given two objects $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \rightrightarrows \mathcal{C}$ in an ∞-category $\mathcal{C}$, any path between their postcompositions $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \rightrightarrows \mathcal{C} \to \mathcal{C}^{\mathrm{gpd}}$ can be represented by a zigzag $\mathrm{N}^{-1}(\mathrm{sd}^i(\Delta^1)) \to \mathcal{C}$ connecting them (for some sufficiently large i).

∗ We must modify the proof of the "if" direction slightly, as follows. Assume that $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ is saturated. By the local universal property of the Rezk nerve (Theorem 2.3.8), we have an equivalence $\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W})) \simeq \mathrm{N}_\infty(\mathcal{R}[\![\mathbf{W}^{-1}]\!])$ in $\mathcal{CSS} \subset s\mathcal{S}$. Note also that by the two-out-of-three assumption, any two objects $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \rightrightarrows \mathrm{Fun}([1], \mathcal{R})^{\mathbf{W}}$ which select the same path component under the composite

$$\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \rightrightarrows \mathrm{Fun}([1], \mathcal{R})^{\mathbf{W}} \to \left(\mathrm{Fun}([1], \mathcal{R})^{\mathbf{W}}\right)^{\mathrm{gpd}} = \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \mathbf{W})_1$$

are either both weak equivalences or both not weak equivalences. Now, for any object of $\mathrm{Fun}([1], \mathcal{R})^{\mathbf{W}}$, recalling Remark 4.1.3 and invoking the saturation assumption, we see that the corresponding map $[1] \to \mathcal{R}$ selects an equivalence under the postcomposition $[1] \to \mathcal{R} \to \mathcal{R}[\![\mathbf{W}^{-1}]\!]$ if and only if it factors as $[1] \to \mathbf{W} \hookrightarrow \mathcal{R}$. From here, the proof proceeds identically. □

Remark 4.5.2. After establishing the necessary facts concerning model ∞-categories, we obtain an analog of [LMG15, Corollary 4.12] as Theorem 6.10.1.

Remark 4.5.3. In light of Remark 2.3.2, [LMG15, Remark 4.13] is strictly generalized by the local universal property of the Rezk nerve (Theorem 2.3.8).

Chapter 5

Model ∞-categories II: Quillen adjunctions

In this chapter, prove that various structures on model ∞-categories descend to corresponding structures on their localizations: (i) Quillen adjunctions; (ii) two-variable Quillen adjunctions; (iii) monoidal and symmetric monoidal model structures; and (iv) enriched model structures.

5.0 Introduction

5.0.1 Presenting structures on localizations of model ∞-categories

A *relative ∞-category* is a pair $(\mathcal{M}, \mathbf{W})$ of an ∞-category $\mathcal{M}$ and a subcategory $\mathbf{W} \subset \mathcal{M}$ containing all the equivalences, called the subcategory of *weak equivalences*. Freely inverting the weak equivalences, we obtain the *localization* of this relative ∞-category, namely the initial functor

$$\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$$

from $\mathcal{M}$ which sends all maps in $\mathbf{W}$ to equivalences. In general, it is extremely difficult to access the localization. In Chapter 1, we introduced the notion of a ***model structure*** extending the data of a relative ∞-category: just as in Quillen's classical theory of model structures on relative categories, this allows for much more control over manipulations within its localization.[1] For instance, in Chapter 6 we

[1]For the precise definition a model ∞-category, we refer the reader to §1.1. However, for the present discussion, it suffices to observe that it is simply a direct generalization of the standard

prove that a model structure provides an efficient and computable way of accessing the hom-spaces $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x, y)$.

However, we are not just interested in localizations of relative ∞-categories themselves. For example, adjunctions are an extremely useful structure, and we would therefore like a systematic way of presenting an adjunction on localizations via some structure on overlying relative ∞-categories. The purpose of this chapter is to show that model structures on relative ∞-categories are not only useful for computations *within* their localizations, but are in fact also useful for presenting structures *on* their localizations. More precisely, we prove the following sequence of results.[2]

Theorem (5.1.1 and 5.1.3)**.** *A* ***Quillen adjunction*** *between model ∞-categories induces a canonical adjunction on their localizations. If this is moreover a* ***Quillen equivalence****, then the resulting adjunction is an adjoint equivalence.*

Theorem (5.4.6)**.** *A* ***two-variable Quillen adjunction*** *between model ∞-categories induces a canonical two-variable adjunction on their localizations.*

Theorem (5.5.4 and 5.5.6)**.** *The localization of a (resp.* ***symmetric****)* ***monoidal model ∞-category*** *is canonically a closed (resp. symmetric) monoidal ∞-category.*

Theorem (5.6.7)**.** *The localization of an* ***enriched model ∞-category*** *is canonically enriched and bitensored over the localization of the enriching model ∞-category.*

Along the way, we also develop the foundations of the theory of ***homotopy co/limits*** in model ∞-categories.

Remark 5.0.1. Perhaps surprisingly, none of these results depends on the concrete identification of the hom-spaces $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x, y)$ in the localizations of model ∞-categories provided in Chapter 6. Rather, their proofs all rely on considerations involving subcategories of "nice" objects relative to the given structure, for instance the subcategory of cofibrant objects relative to a left Quillen functor. Such considerations are thus somewhat akin to the theory of "deformable" functors of Dwyer–Hirschhorn–Kan–Smith (see [DHKS04], as well as Shulman's excellent synthesis and contextualization [Shu]), but the philosophy can be traced back at least as far as Brown's "categories of fibrant objects" (see [Bro71]).

definition of a model category.

[2]The precise definitions of Quillen adjunctions and Quillen equivalences are also contained in §1.1, while the remaining relevant definitions are contained in the body of the present chapter. However, for the present discussion, it likewise suffices to observe that they are all direct generalizations of their classical counterparts.

Remark 5.0.2. In the special case of model categories and their *1-categorical* localizations, these results are all quite classical (and fairly easy to prove).[3] However, the study of ∞-*categorical* localizations – even just of model categories – is much more subtle, because it requires keeping track of a wealth of coherence data.

The following specializations of our results to model 1-categories (and their ∞-categorical localizations) appear in the literature.

- We proved this special case of the first of the results (regarding Quillen adjunctions) listed above as [MG16, Theorem 2.1]. (For a detailed history of partial results in this direction, we refer the reader to [MG16, §A].)

- Under a more restrictive definition of a (resp. symmetric) monoidal model category in which the unit object is required to be cofibrant (as opposed to unit axiom $\mathrm{MM}_\infty 2$ of Definition 5.5.1), Lurie proves that its localization admits a canonical (resp. symmetric) monoidal structure in §A.4.3.1 (see particularly Proposition A.4.1.3.4). Moreover, under an analogously more restrictive definition of a (resp. symmetric) monoidal model ∞-category, a canonical (resp. symmetric) monoidal structure on its localization likewise follows from this same result. (See Remark 5.5.7.)

Aside from these, the results of this chapter appear to be new, even in the special case of model 1-categories.

Remark 5.0.3. Our result [MG16, Theorem 2.1] is founded in point-set considerations, for instance making reference to an explicit "underlying quasicategory" functor from relative categories (e.g. model categories). By contrast, the proof of the generalization given here works invariantly, and relies on a crucial result of Gepner–Haugseng–Nikolaus identifying cocartesian fibrations as lax colimits, which appeared almost concurrently to our [MG16]. (Specifically, the proof of our "fiberwise localization" result Proposition 5.2.3 appeals multiple times to [GHN, Theorem 7.4].) Nevertheless, we hope that our model-specific proof will still carry some value: the techniques used therein seem fairly broadly applicable, and its point-set nature may someday prove useful as well.

[3] Given a relative category $(\mathcal{R}, \mathbf{W})$, its 1-categorical localization and its ∞-categorical localization are closely related: there is a natural functor $\mathcal{R}[\![\mathbf{W}^{-1}]\!] \to \mathrm{ho}(\mathcal{R}[\![\mathbf{W}^{-1}]\!]) \simeq \mathcal{R}[\mathbf{W}^{-1}]$ between them, namely the projection to the homotopy category (see Remark 2.1.29). Moreover, all of these structures – adjunctions, two-variable adjunctions, closed (symmetric) monoidal structures, and enrichments and bitensorings – descend canonically from ∞-categories to their homotopy categories.

5.0.2 Outline

We now provide a more detailed outline of the contents of this chapter.

- In §5.1, we begin by stating our results concerning Quillen adjunctions and Quillen equivalences (Theorem 5.1.1 and Corollary 5.1.3, resp.). We then develop the rudiments of the theory of homotopy co/limits in model ∞-categories, and provide a detailed study of Reedy model structures on functor ∞-categories.

- In § 5.2, we provide some auxiliary material on *relative co/cartesian fibrations* and on *bicartesian fibrations*. These two enhancements of the theory of co/cartesian fibrations are used in the proofs of the main results of the chapter.

- In §5.3, we prove Theorem 5.1.1 and Corollary 5.1.3.

- In § 5.4, we show that two-variable Quillen adjunctions between model ∞-categories present two-variable adjunctions between their localizations.

- In §5.5, we show that (resp. symmetric) monoidal model ∞-categories present closed (resp. symmetric) monoidal ∞-category.

- In §5.6, we show that enriched model ∞-categories present enriched and bitensored ∞-categories.

5.0.3 Acknowledgments

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5.1 Quillen adjunctions, homotopy co/limits, and Reedy model structures

Model structures on relative (1- and ∞-)categories are extremely useful for making computations *within* their localizations. However, it can also be quite useful to obtain relationships *between* their localizations. Perhaps the most important relationship

that two ∞-categories can share is that of being related by an *adjunction*. The central result of this section (Theorem 5.1.1) provides a systematic way of obtaining just such a relationship: a *Quillen adjunction* between model ∞-categories induces a canonical *derived adjunction* on their localizations. As a special case (Corollary 5.1.3), a *Quillen equivalence* induces a *derived equivalence* on localizations.[4]

This section is organized as follows. In §5.1.1, we state these fundamental theorems regarding Quillen adjunctions and Quillen equivalences. (However, their proofs will be postponed to §5.3, after we have developed some necessary scaffolding in §5.2.) Then, in §5.1.2 we study the important special case of *homotopy co/limits*, briefly introducing the *projective* and *injective* model structures. Finally, in §5.1.3, we pursue a more in-depth study of the *Reedy* model structure.

5.1.1 Quillen adjunctions and Quillen equivalences

The classical theory of *derived functors* arose out of a desire to "correct" functors between relative categories which do not respect weak equivalences to ones that do. There, one replaces a given object by a suitable *resolution* – the nature of which depends both on the context and on the sort of functor which one is attempting to correct – and then applies the original functor to this resolution, the point being that the functor *does* respect weak equivalences between such "nice" objects.

A Quillen adjunction

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

between model (1- or ∞-)categories is a prototypical and beautifully symmetric example of such a situation. In general, neither Quillen adjoint will preserve weak equivalences. However, in this case there are canonical choices for such subcategories of "nice" objects: left Quillen functors preserve weak equivalences between cofibrant objects, while right Quillen functors preserve weak equivalences between fibrant objects (see Kenny Brown's lemma (5.3.5)). Moreover, the inclusions $(\mathcal{M}^c, \mathbf{W}^c_{\mathcal{M}}) \hookrightarrow (\mathcal{M}, \mathbf{W}_{\mathcal{M}})$ and $(\mathcal{N}, \mathbf{W}_{\mathcal{N}}) \hookleftarrow (\mathcal{N}^f, \mathbf{W}^f_{\mathcal{N}})$ induce equivalences

$$\mathcal{M}^c[\![(\mathbf{W}^c_{\mathcal{M}})^{-1}]\!] \xrightarrow{\sim} \mathcal{M}[\![\mathbf{W}^{-1}_{\mathcal{M}}]\!]$$

and

$$\mathcal{N}[\![\mathbf{W}^{-1}_{\mathcal{N}}]\!] \xleftarrow{\sim} \mathcal{N}^f[\![(\mathbf{W}^f_{\mathcal{N}})^{-1}]\!]$$

on localizations (see Corollary 5.3.4). A perfect storm then ensues.

[4] Quillen adjunctions and Quillen equivalences are respectively given as Definitions 1.1.12 and 1.1.14. These are completely straightforward generalizations of the model 1-categorical counterparts, and so we do not feel the need to repeat them here.

Theorem 5.1.1. *A Quillen adjunction*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

of model ∞-categories induces a canonical adjunction

$$\mathbb{L}F : \mathcal{M}[\mathbf{W}_{\mathcal{M}}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}_{\mathcal{N}}^{-1}] : \mathbb{R}G$$

on localizations, whose left and right adjoints are respectively obtained by applying the localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ to the composites

$$\mathcal{M}^c \hookrightarrow \mathcal{M} \xrightarrow{F} \mathcal{N}$$

and

$$\mathcal{M} \xleftarrow{G} \mathcal{N} \hookleftarrow \mathcal{N}^f.$$

Definition 5.1.2. Given a Quillen adjunction $F \dashv G$, we refer to the the resulting adjunction $\mathbb{L}F \dashv \mathbb{R}G$ on localizations of Theorem 5.1.1 as its ***derived adjunction***. We refer to $\mathbb{L}F$ as the ***left derived functor*** of F, and to $\mathbb{R}G$ as the ***right derived functor*** of G.

Theorem 5.1.1 has the following easy consequence.

Corollary 5.1.3. *The derived adjunction*

$$\mathbb{L}F : \mathcal{M}[\mathbf{W}_{\mathcal{M}}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}_{\mathcal{N}}^{-1}] : \mathbb{R}G$$

of a Quillen equivalence

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

of model ∞-categories is an adjoint equivalence.

Remark 5.1.4. With Theorem 5.1.1 in hand, to prove Corollary 5.1.3 it would suffice to show that either one of the derived adjoint functors is an equivalence; this can be accomplished using the *fundamental theorem of model ∞-categories* (6.1.9), which provides an explicit description of the hom-spaces in the localization of a model ∞-category. However, our proofs of Theorem 5.1.1 and of Corollary 5.1.3 will not rely on that result (recall Remark 5.0.1).

Remark 5.1.5. A number of examples of Quillen adjunctions and Quillen equivalences are provided in §1.2.2.

5.1.2 Homotopy co/limits

Some of the most important operations one can perform within an ∞-category are the extraction of *limits* and *colimits*. However, co/limit functors on relative ∞-categories do not generally take natural weak equivalences to weak equivalences. In view of the theory of derived adjunctions laid out in §5.1.1, in the setting of model ∞-categories it is therefore important to determine sufficient conditions under which co/limit functors can be derived, i.e. under which they determine left/right Quillen functors.

We now codify this desired situation.

Notation 5.1.6. For a model ∞-category $\mathcal{M}$ and an ∞-category $\mathcal{C}$, we write $\mathbf{W}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})} \subset \mathrm{Fun}(\mathcal{C},\mathcal{M})$ for the subcategory of natural weak equivalences. Of course, considering $(\mathcal{M},\mathbf{W})$ as a relative ∞-category, via Notation 2.1.6 this identifies as

$$\begin{array}{ccc} \mathbf{W}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})} & \hookrightarrow & \mathrm{Fun}(\mathcal{C},\mathcal{M}) \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{Fun}(\mathrm{min}(\mathcal{C}),\mathcal{M})^{\mathbf{W}} & \hookrightarrow & \mathrm{Fun}(\mathrm{min}(\mathcal{C}),\mathcal{M})^{\mathfrak{Rel}}. \end{array}$$

Definition 5.1.7. Let $\mathcal{M}$ be a model ∞-category, and let $\mathcal{C}$ be an ∞-category. Suppose that $\mathcal{M}$ admits $\mathcal{C}$-shaped colimits, so that we obtain an adjunction

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C},\mathcal{M}) \rightleftarrows \mathcal{M} : \mathrm{const}.$$

If $(\mathrm{Fun}(\mathcal{C},\mathcal{M}), \mathbf{W}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})})$ admits a model structure such that this adjunction becomes a Quillen adjunction, we refer to its resulting left derived functor

$$\mathbb{L}\mathrm{colim} : \mathrm{Fun}(\mathcal{C},\mathcal{M})[\![\mathbf{W}^{-1}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}]\!] \to \mathcal{M}[\![\mathbf{W}^{-1}_{\mathcal{M}}]\!]$$

as a ***homotopy colimit*** functor. Dually, suppose that $\mathcal{M}$ admits $\mathcal{C}$-shaped limits, so that we obtain an adjunction

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{C},\mathcal{M}) : \mathrm{lim}.$$

If $(\mathrm{Fun}(\mathcal{C},\mathcal{M}), \mathbf{W}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})})$ admits a model structure such that this adjunction becomes a Quillen adjunction, we refer to its resulting right derived functor

$$\mathcal{M}[\![\mathbf{W}^{-1}_{\mathcal{M}}]\!] \leftarrow \mathrm{Fun}(\mathcal{C},\mathcal{M})[\![\mathbf{W}^{-1}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}]\!] : \mathbb{R}\mathrm{lim}$$

as a ***homotopy limit*** functor.

Now, to check that an adjunction between model ∞-categories is a Quillen adjunction, it suffices to show only that either its left adjoint is a left Quillen functor or that its right adjoint is a right Quillen functor. This leads us to define the following "absolute" model structures on functor ∞-categories.

Definition 5.1.8. Let $\mathcal{M}$ be a model ∞-category, and let $\mathcal{C}$ be an ∞-category. Suppose that there exists a model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ whose weak equivalences and fibrations are determined objectwise. In this case, we call this as the ***projective model structure***, and denote it by $\mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{proj}}$. Dually, suppose that there exists a model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ whose weak equivalences and cofibrations are determined objectwise. In this case, we call this the ***injective model structure***, and denote it by $\mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{inj}}$.

Remark 5.1.9. Definition 5.1.8 immediately implies

- that whenever $\mathcal{M}$ admits $\mathcal{C}$-shaped colimits and there exists a projective model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, then we obtain a Quillen adjunction

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathcal{M} : \mathrm{const},$$

  and

- that that whenever $\mathcal{M}$ admits $\mathcal{C}$-shaped limits and there exists an injective model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, then we obtain a Quillen adjunction

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{inj}} : \mathrm{lim}.$$

Remark 5.1.10. As in the classical case, the projective and injective model structures do not always exist. However, it appears that

- the projective model structure will exist whenever $\mathcal{M}$ is cofibrantly generated (see §1.3), while
- the injective model structure will exist whenever $\mathcal{M}$ is *combinatorial* (that is, its underlying ∞-category is presentable and its model structure is cofibrantly generated);

see [Hir03, Theorem 11.6.1] and Proposition T.A.2.8.2.[5]

[5] In the construction of the projective model structure, one can replace the appeal to the "set of objects" of $\mathcal{C}$ with an arbitrary surjective map $C \to \mathcal{C}$ from some $C \in \mathcal{S}\mathrm{et} \subset \mathcal{C}\mathrm{at}_\infty$; the necessary left Kan extension will exist as long as $\mathcal{M}$ is cocomplete, which seems to be generally true in practice. However, there is also some subtlety regarding whether the resulting sets of would-be generating cofibrations and generating acyclic cofibrations do indeed admit the small object argument: it

5.1.3 Reedy model structures

Whereas the projective and injective model structures of Definition 5.1.8 are not always known to exist (even for model 1-categories), there is a class of examples in which a model structure on $(\mathrm{Fun}(\mathcal{C}, \mathcal{M}), \mathbf{W}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})})$ is always guaranteed to exist: the *Reedy model structure*. This does not make any additional assumptions on the model ∞-category $\mathcal{M}$ (recall Remark 5.1.10), but instead it requires that $\mathcal{C}$ be a (strict) 1-category equipped with a certain additional structure.

The Reedy model structure will be useful in a number of settings: we'll use Example 5.1.18 a number of times in the proof of Theorem 5.1.1, it will be heavily involved in our development of "cylinder objects" and "path objects" in model ∞-categories in §6.1 (leading towards the fundamental theorem of model ∞-categories (6.1.9)), and it is also closely related to the resolution model structure (see e.g. §1.0.3).

We begin by fixing the following definition.

Definition 5.1.11. Let $\mathcal{C} \in \mathbb{cat}$ be a gaunt category equipped with a factorization system defined by two wide subcategories $\overrightarrow{\mathcal{C}}, \overleftarrow{\mathcal{C}} \subset \mathcal{C}$; that is, every morphism φ in $\mathcal{C}$ admits a unique factorization as a composite $\overrightarrow{\varphi} \circ \overleftarrow{\varphi}$, where $\overrightarrow{\varphi}$ is in $\overrightarrow{\mathcal{C}}$ and $\overleftarrow{\varphi}$ is in $\overleftarrow{\mathcal{C}}$. Suppose there do not exist any infinite "decreasing" zigzags of non-identity morphisms in $\mathcal{C}$, where by "decreasing" we mean that all forward-pointing arrows lie in $\overleftarrow{\mathcal{C}}$ and all backwards-pointing arrows lie in $\overrightarrow{\mathcal{C}}$. Then, we say that $\mathcal{C}$ is a ***Reedy category***, and we refer to the defining subcategories $\overrightarrow{\mathcal{C}}, \overleftarrow{\mathcal{C}} \subset \mathcal{C}$ respectively as its ***direct subcategory*** and its ***inverse subcategory***.

Remark 5.1.12. Definition 5.1.11 is lifted from Definition T.A.2.9.1. There is also a more restrictive definition in the literature, given for instance as [Hir03, Definition 15.1.2], in which one requires that $\mathcal{C}$ comes equipped with a "degree function" $\deg : \mathrm{N}(\mathcal{C})_0 \to \mathbb{N}$ such that all non-identity morphisms in $\overrightarrow{\mathcal{C}}$ raise degree while all non-identity morphisms in $\overleftarrow{\mathcal{C}}$ lower degree: the nonexistence of infinite decreasing zigzags then follows from the fact that $\mathbb{N}$ has a minimal element.

However, as pointed out in [Hir03, Remark 15.1.4], the results of [Hir03, Chapter 15] easily generalize to the case when the degree function takes values in ordinals rather than simply in nonnegative integers. Indeed, Notation T.A.2.9.11 introduces the notion of a "good filtration" on a Reedy category, which is a transfinite total

suffices that the set I (resp. J) of generating (resp. acyclic) cofibrations have that all the sources of its elements be small with respect to the *tensors* of its elements over the various hom-spaces of $\mathcal{C}$. However, it similarly seems that in practice these objects will in fact be small with respect to the entire ∞-category $\mathcal{M}$, so that this is not actually an issue.

ordering of its objects that effectively serves the same purpose as an ordinary degree function (although note that a degree function need not be injective in general), and Remark T.A.2.9.12 observes that good filtrations always exist.

In any case, these data (either degree functions or good filtrations) both reflect the most important feature of Reedy categories, namely their amenability to inductive manipulations. In practice, we will generally only use Reedy categories of the more restrictive sort, but it is no extra effort to work in the more general setting.

Definition 5.1.13. Given a Reedy category $\mathcal{C}$, we define its ***latching category*** at an object $c \in \mathcal{C}$ to be the full subcategory

$$\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right) \subset \overrightarrow{\mathcal{C}}_{/c}$$

on all objects besides id_c, and we define its ***matching category*** at an object $c \in \mathcal{C}$ to be the full subcategory

$$\partial\left(\overleftarrow{\mathcal{C}}_{c/}\right) \subset \overleftarrow{\mathcal{C}}_{c/}$$

on all objects besides id_c.

Remark 5.1.14. We will assume familiarity with the basic theory of Reedy categories. For further details, we refer the reader to [Hir03, Chapter 15] or to §T.A.2.9 (with the caveat that the latter source works in somewhat greater generality than the former, as explained in Remark 5.1.12). In particular, given a functor

$$\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right) \xrightarrow{F} \mathcal{M}$$

(e.g. the restriction of a functor $\mathcal{C} \xrightarrow{F} \mathcal{M}$) we will write

$$\mathrm{L}_c(F) = \mathrm{colim}_{\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)}(F),$$

and given a functor

$$\partial\left(\overleftarrow{\mathcal{C}}_{c/}\right) \xrightarrow{G} \mathcal{M}$$

(e.g. the restriction of a functor $\mathcal{C} \xrightarrow{G} \mathcal{M}$) we will write

$$\mathrm{M}_c(F) = \lim_{\partial\left(\overleftarrow{\mathcal{C}}_{c/}\right)}(G).$$

(This notation jibes with that of item A(29)).

Remark 5.1.15. In general, the usual constructions with Reedy categories go through equally well when the target is an ∞-category. In particular, we explicitly record here that given a bicomplete ∞-category $\mathcal{M}$ and a Reedy category $\mathcal{C}$, one can inductively construct both objects and morphisms of $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ in exactly the same manner as when $\mathcal{M}$ is merely a category, using latching/matching objects and (relative) latching/matching maps. For the construction of objects this is observed as Remark T.A.2.9.16, but both of these assertions follow easily from Proposition T.A.2.9.14.

As indicated at the beginning of this subsection, the primary reason for our interest in Reedy categories is the following result.

Theorem 5.1.16. *Let $\mathcal{M}$ be a model ∞-category, and let $\mathcal{C}$ be a Reedy category. Then there exists a model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, in which a map $F \to G$ is*

- *a weak equivalence if and only if the induced maps*

$$F(c) \to G(c)$$

  are in $\mathbf{W} \subset \mathcal{M}$ for all $c \in \mathcal{C}$,

- *a (resp. acyclic) cofibration if and only if the relative latching maps*

$$F(c) \coprod_{\mathrm{L}_c(F)} \mathrm{L}_c(G) \to G(c)$$

  are in $\mathbf{C} \subset \mathcal{M}$ (resp. $\mathbf{W} \cap \mathbf{C} \subset \mathcal{M}$) for all $c \in \mathcal{C}$,

- *a (resp. acyclic) fibration if and only if the relative matching maps*

$$F(c) \to \mathrm{M}_c(F) \times_{\mathrm{M}_c(G)} G(c)$$

  are in $\mathbf{F} \subset \mathcal{M}$ (resp. $\mathbf{W} \cap \mathbf{F} \subset \mathcal{M}$) for all $c \in \mathcal{C}$.

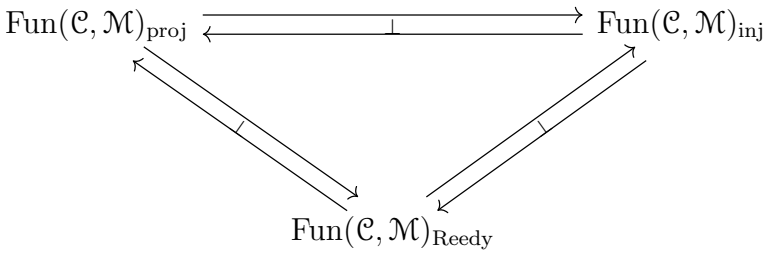
Proof. The proof is identical to that of Proposition T.A.2.9.19 (or to those of [Hir03, Theorems 15.3.4(1) and 15.3.5]). □

Definition 5.1.17. We refer to the model structure of Theorem 5.1.16 as the ***Reedy model structure*** on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, and we denote this model ∞-category by $\mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}}$.

Example 5.1.18. There is a Reedy category structure on $[n] \in \mathbf{\Delta} \subset \mathcal{C}\mathrm{at}$ determined by the degree function $\deg(i) = i$. As the inverse subcategory $\overleftarrow{[n]} \subset [n]$ associated to this Reedy category structure consists only of identity maps, the resulting Reedy model structure $\mathrm{Fun}([n], \mathcal{M})_{\mathrm{Reedy}}$ coincides with the projective model structure $\mathrm{Fun}([n], \mathcal{M})_{\mathrm{proj}}$ of Definition 5.1.8.

Remark 5.1.19. In particular, Example 5.1.18 shows that the projective model structure $\mathrm{Fun}([n], \mathcal{M})_{\mathrm{proj}}$ always exists (without any additional assumptions on $\mathcal{M}$). We will use this fact repeatedly without further comment.

Remark 5.1.20. It follows essentially directly from the definitions that whenever they all exist, the projective, injective, and Reedy model structures assemble into a commutative diagram



of Quillen equivalences. (If only two of them exist, then they still participate in the indicated Quillen equivalence.)

The Reedy model structure is also functorial in exactly the way one would hope.

Theorem 5.1.21. *For any Reedy category $\mathcal{C}$, if $\mathcal{M} \rightleftarrows \mathcal{N}$ is a Quillen adjunction (resp. Quillen equivalence) of model ∞-categories, then the induced adjunction*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{N})_{\mathrm{Reedy}}$$

is a Quillen adjunction (resp. Quillen equivalence) as well.

Proof. The proof is identical to that of [Hir03, Proposition 15.4.1]. □

Of course, much of our interest in functor ∞-categories stems from the fact that these are the source of co/limit functors. Thus, we will often want to know when a co/limit functor is a Quillen functor with respect to a given Reedy model structure. This will not always be the case. However, there does exist a class of "absolute" examples, as encoded by the following.

Definition 5.1.22. Let $\mathcal{C}$ be a Reedy category. We say that $\mathcal{C}$ has (***model ∞-categorical***) ***cofibrant constants*** if for every model ∞-category $\mathcal{M}$ admitting $\mathcal{C}$-shaped limits, the adjunction

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}} : \lim$$

is a Quillen adjunction. Dually, we say that $\mathcal{C}$ has (***model ∞-categorical***) ***fibrant constants*** if for every model ∞-category $\mathcal{M}$ admitting $\mathcal{C}$-shaped colimits, the adjunction

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const}$$

is a Quillen adjunction.

This notion differs slightly from the classical definition (see [Hir03, Definition 15.10.1]). We first provide a characterization, and then explain the difference in Remark 5.1.24.

Proposition 5.1.23. *Let $\mathcal{C}$ be a Reedy category. Then $\mathcal{C}$ has model ∞-categorical cofibrant constants if and only if for every $c \in \mathcal{C}$ the groupoid completion*

$$\left(\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)\right)^{\mathrm{gpd}}$$

of its latching category is either empty or contractible. Dually, $\mathcal{C}$ has model ∞-categorical fibrant constants if and only if for every $c \in \mathcal{C}$ the groupoid completion

$$\left(\partial\left(\overleftarrow{\mathcal{C}}_{c/}\right)\right)^{\mathrm{gpd}}$$

of its matching category is either empty or contractible.

Proof. Suppose that for every $c \in \mathcal{C}$ the latching category $\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)$ has either empty or contractible geometric realization, and suppose that $\mathcal{M}$ is a model ∞-category admitting $\mathcal{C}$-shaped limits. Fix an object $c \in \mathcal{C}$. Then, for any object $z \in \mathcal{M}$, the latching object

$$\mathrm{L}_c(\mathrm{const}(z)) = \mathrm{colim}_{\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)} \mathrm{const}(z)$$

is either

- always equivalent to $\varnothing_{\mathcal{M}}$, or
- always equivalent to z itself.

Hence, for any map $x \to y$ in $\mathcal{M}$, the relative latching map

$$(\mathrm{const}(x))(c) \coprod_{\mathrm{L}_c(\mathrm{const}(x))} \mathrm{L}_c(\mathrm{const}(y)) \to (\mathrm{const}(y))(c)$$

is either $x \to y$ or $y \to y$. It follows that $\mathrm{const} : \mathcal{M} \to \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}}$ is a left Quillen functor, so that the adjunction $\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}} : \mathrm{lim}$ is a Quillen adjunction, as desired.

Conversely, suppose that for some object $c \in \mathcal{C}$ the groupoid completion

$$\left(\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)\right)^{\mathrm{gpd}}$$

of the latching category at $c \in \mathcal{C}$ is not empty or contractible. Then for any nonempty object $x \in s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} \subset s\mathcal{S}_{\mathrm{KQ}}$ (i.e. considered in $s\mathcal{S}_{\mathrm{KQ}}$), the latching map at $c \in \mathcal{C}$ of the functor $\mathrm{const}(x)$ will not be a cofibration.

Of course, the dual claim follows from a dual argument. □

Remark 5.1.24. In the theory of ordinary model categories, according to [Hir03, Proposition 15.10.2(1)], a Reedy category has cofibrant constants if and only if the its latching categories are all either nonempty or *connected*. In light of the proof of Proposition 5.1.23, the reason for the difference should now be clear: it 1-category theory, in order for a constant diagram to have colimit isomorphic to its constant value, it suffices for the indexing category to merely be connected. By contrast, in ∞-category theory the colimit of a constant diagram recovers the *tensoring* of the value with the groupoid completion of the diagram ∞-category.

Example 5.1.25. For any simplicial set $K \in s\mathcal{S}\mathrm{et}$, its "category of simplices" (i.e. the category

$$\mathbf{\Delta}_{/K} = \mathbf{\Delta} \underset{s\mathcal{S}\mathrm{et}}{\times} s\mathcal{S}\mathrm{et}_{/K},$$

or equivalently the category $\mathrm{Gr}^{-}(K) \in \mathcal{CF}\mathrm{ib}(\mathbf{\Delta})$ obtained by considering $K \in \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}\mathrm{et})$) is a Reedy category with fibrant constants; this follows from the *proof* of [Hir03, Proposition 15.10.4]. In particular, the category $\mathrm{Gr}^{-}(\mathrm{pt}_{s\mathcal{S}\mathrm{et}}) \cong \mathbf{\Delta}$ itself has fibrant constants. By dualizing, we obtain that the category $\mathbf{\Delta}^{op}$ has cofibrant constants.

Remark 5.1.26. Note that in general, the observations of Example 5.1.25 only provides Quillen adjunctions

$$\mathrm{const} : \mathcal{M} \rightleftarrows s\mathcal{M}_{\mathrm{Reedy}} : \lim$$

and

$$\mathrm{colim} : c\mathcal{M}_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const},$$

which are rather useless in practice (since $\mathbf{\Delta}^{op}$ has an initial object and $\mathbf{\Delta}$ has a terminal object). To obtain a left Quillen functor $s\mathcal{M}_{\mathrm{Reedy}} \to \mathcal{M}$, we will generally

need to take a *resolution* of the object $\mathrm{const}(\mathrm{pt}_{\mathcal{M}}) \in s\mathcal{M}$ (e.g. one coming from a *simplicio-spatial model structure* (see Definition 5.6.2)).[6]

Example 5.1.27. The Reedy trick generalizes from model categories to model ∞-categories without change. Recall that the walking span category $\mathrm{N}^{-1}(\Lambda^2_0) = (\bullet \leftarrow \bullet \rightarrow \bullet)$ admits a Reedy category structure determined by the degree function described by the picture $(0 \leftarrow 1 \rightarrow 2)$. Moreover, it is straightforward to verify that this Reedy category has fibrant constants (see e.g. the proof of [Hir03, Proposition 15.10.10]). Thus, for any model ∞-category $\mathcal{M}$, we obtain a Quillen adjunction

$$\mathrm{colim} : \mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_0), \mathcal{M})_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const},$$

in which the cofibrant objects of $\mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_1), \mathcal{M})_{\mathrm{Reedy}}$ are precisely the diagrams of the form $x \leftarrow y \rightarrowtail z$ for $x, y, z \in \mathcal{M}^c \subset \mathcal{M}$.

Example 5.1.28. Clearly, the poset $(\mathbb{N}, \leq)$ admits a Reedy structure (defined by the identity map, considered as a degree function) which has fibrant constants. Thus, for any model ∞-category $\mathcal{M}$, we obtain a Quillen adjunction

$$\mathrm{colim} : \mathrm{Fun}((\mathbb{N}, \leq), \mathcal{M})_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const},$$

in which the cofibrant objects of $\mathrm{Fun}((\mathbb{N}, \leq), \mathcal{M})_{\mathrm{Reedy}}$ are precisely those diagrams consisting of cofibrations between cofibrant objects.[7] Dually, we obtain a Quillen adjunction

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}((\mathbb{N}, \leq)^{op}, \mathcal{M})_{\mathrm{Reedy}} : \mathrm{lim},$$

in which the fibrant objects of $\mathrm{Fun}((\mathbb{N}, \leq)^{op}, \mathcal{M})_{\mathrm{Reedy}}$ are precisely those diagrams consisting of fibrations between fibrant objects.

We end this section by recording the following result.

Lemma 5.1.29. *Let $\mathcal{C}$ be a Reedy category, and let $c \in \mathcal{C}$.*

(1) (a) The latching category $\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)$ admits a Reedy structure with fibrant constants, in which the direct subcategory is the entire category and the inverse subcategory contains only the identity maps.

[6] If $\mathcal{C}$ is an ∞-category (which is finitely bicomplete and admits geometric realizations) and we equip $\mathcal{C}$ with the *trivial* model structure (see Example 1.2.2), then we do obtain a Quillen adjunction $|-| : s(\mathcal{C}_{\mathrm{triv}})_{\mathrm{Reedy}} \rightleftarrows \mathcal{C}_{\mathrm{triv}} : \mathrm{const}$. However, unwinding the definitions, we see that this is really just the Quillen adjunction $|-| : (s\mathcal{C})_{\mathrm{triv}} \rightleftarrows \mathcal{C}_{\mathrm{triv}} : \mathrm{const}$.

[7] In fact, this Reedy poset has cofibrant constants as well. However, the resulting Quillen adjunction will be trivial since this poset has an initial object.

(b) *With respect to the Reedy structure of part (a), the canonical functor* $\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right) \to \mathcal{C}$ *induces an isomorphism*

$$\partial\left(\overrightarrow{\partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)}_{/(d\to c)}\right) \xrightarrow{\cong} \partial\left(\overrightarrow{\mathcal{C}}_{/d}\right)$$

of latching categories (from that of the object $(d \to c) \in \partial\left(\overrightarrow{\mathcal{C}}_{/c}\right)$ *to that of the object* $d \in \mathcal{C}$*).*

(2) (a) *The matching category* $\partial\left(\overleftarrow{\mathcal{C}}_{c/}\right)$ *admits a Reedy structure with cofibrant constants, in which the direct subcategory contains only the identity maps and the inverse subcategory is the entire category.*

(b) *With respect to the Reedy structure of part (a), the canonical functor* $\partial\left(\overleftarrow{\mathcal{C}}_{c/}\right) \to \mathcal{C}$ *induces an isomorphism*

$$\partial\left(\overleftarrow{\partial\left(\overleftarrow{\mathcal{C}}_{c/}\right)}_{(c\to d)/}\right) \xrightarrow{\cong} \partial\left(\overleftarrow{\mathcal{C}}_{d/}\right)$$

of matching categories (from that of $(c \to d) \in \partial\left(\overleftarrow{\mathcal{C}}_{c/}\right)$ *to that of* $d \in \mathcal{C}$*).*

Proof. Parts (1)(a) and (2)(a) follow from the *proof* of [Hir03, Proposition 15.10.6], and parts (1)(b) and (2)(b) follow by inspection. □

5.2 Relative co/cartesian fibrations and bicartesian fibrations

In this brief section, we describe two enhancements of the theory of co/cartesian fibrations which we will need: in §5.2.1 we study *relative co/cartesian fibrations*, while in §5.2.2 we study *bicartesian fibrations*.

5.2.1 Relative co/cartesian fibrations

Suppose we are given a diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{F} & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty & \xrightarrow{\mathscr{L}} & \mathcal{C}\mathrm{at}_\infty \\ & & \Big\downarrow{\scriptstyle \mathrm{U}_{\mathcal{R}\mathrm{el}}} & & \\ & & \mathcal{C}\mathrm{at}_\infty. & & \end{array}$$

In our proof of Theorem 5.1.1, we will be interested in the relationship between the upper composite (of the componentwise localization of the diagram F of relative ∞-categories) and the cocartesian fibration

$$\mathrm{Gr}(\mathrm{U}_{\mathcal{R}\mathrm{el}} \circ F) \to \mathcal{C}.$$

In other words, we would like to take some sort of "fiberwise localization" of this cocartesian fibration. In order to do this, we must keep track of the morphisms which we would like to invert. This leads us to the following terminology.

Definition 5.2.1. Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, and suppose we are given a commutative diagram

$$\begin{array}{ccc} & & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \\ & \overset{F}{\nearrow} & \big\downarrow \mathrm{U}_{\mathcal{R}\mathrm{el}} \\ \mathcal{C} & \xrightarrow[\mathrm{U}_{\mathcal{R}\mathrm{el}} \circ F]{} & \mathcal{C}\mathrm{at}_\infty. \end{array}$$

Then, we write $\mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F)$ for the relative ∞-category obtained by equipping $\mathrm{Gr}(\mathrm{U}_{\mathcal{R}\mathrm{el}} \circ F)$ with the weak equivalences coming from the lift F of $\mathrm{U}_{\mathcal{R}\mathrm{el}} \circ F$. Note that its weak equivalences all map to equivalences in $\mathcal{C}$, so that we can consider the canonical projection as a map $\mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F) \to \mathrm{min}(\mathcal{C})$ of relative ∞-categories. We write $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}_{\mathcal{R}\mathrm{el}}(\mathcal{C})$ for the ∞-category of cocartesian fibrations over $\mathcal{C}$ equipped with such a relative ∞-category structure, and we call this the ∞-category of ***relative cocartesian fibrations*** over $\mathcal{C}$. The Grothendieck construction clearly lifts to an equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}_{\mathcal{R}\mathrm{el}}} \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}_{\mathcal{R}\mathrm{el}}(\mathcal{C}).$$

Of course, we have a dual notion of ***relative cartesian fibrations*** over $\mathcal{C}$; these assemble into an ∞-category $\mathcal{C}\mathcal{F}\mathrm{ib}_{\mathcal{R}\mathrm{el}}(\mathcal{C})$, which comes with an equivalence

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty) \xrightarrow[\sim]{\mathrm{Gr}^-_{\mathcal{R}\mathrm{el}}} \mathcal{C}\mathcal{F}\mathrm{ib}_{\mathcal{R}\mathrm{el}}(\mathcal{C}).$$

Remark 5.2.2. Note that an arbitrary cocartesian fibration over $\mathcal{C}$ equipped with a subcategory of weak equivalences which project to equivalences in $\mathcal{C}$ does not necessarily define a relative cocartesian fibration: it must be classified by a diagram of relative ∞-categories and relative functors between them (i.e. the cocartesian edges must intertwine the weak equivalences). A dual observation holds for cartesian fibrations.

We can now precisely state and prove our desired correspondence.

Proposition 5.2.3. *Let* $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$*, and let* $\mathcal{C} \xrightarrow{F} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *classify* $\mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F) \in \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}_{\mathcal{R}\mathrm{el}}(\mathcal{C})$*. Then the induced map*

$$\mathscr{L}(\mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F)) \to \mathcal{C}$$

is again a cocartesian fibration. Moreover, we have a canonical equivalence

$$\mathscr{L}(\mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F)) \simeq \mathrm{Gr}(\mathscr{L} \circ F)$$

in $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$*, i.e. this cocartesian fibration classifies the composite*

$$\mathcal{C} \xrightarrow{F} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty.$$

Proof. By [GHN, Theorem 7.4], we have a canonical equivalence

$$\mathrm{Gr}(\mathscr{L} \circ F) \simeq \mathrm{colim}\left(\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathcal{C}_{-/} \times (\mathscr{L} \circ F)} \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty \xrightarrow{- \times -} \mathcal{C}\mathrm{at}_\infty\right).$$

Since the composite $\mathcal{C}\mathrm{at}_\infty \xrightarrow{\min} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ is canonically equivalent to $\mathrm{id}_{\mathcal{C}\mathrm{at}_\infty}$ and the functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ commutes with finite products by Lemma 2.1.20, this can be rewritten as

$$\mathrm{Gr}(\mathscr{L} \circ F) \simeq \mathrm{colim}\left(\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{(\min \circ \mathcal{C}_{-/}) \times F} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{- \times -} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty\right).$$

Moreover, the functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ commutes with colimits (being a left adjoint), and so this can be rewritten further as

$$\mathrm{Gr}(\mathscr{L} \circ F) \simeq \mathscr{L}\left(\mathrm{colim}\left(\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{(\min \circ \mathcal{C}_{-/}) \times F} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{- \times -} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty\right)\right).$$

On the other hand, $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{U}_{\mathcal{R}\mathrm{el}}} \mathcal{C}\mathrm{at}_\infty$ also commutes with colimits (being a left adjoint as well) and is symmetric monoidal for the respective cartesian symmetric monoidal structures, and so we obtain that

$$\begin{aligned}
&\mathrm{U}_{\mathcal{R}\mathrm{el}}\left(\mathrm{colim}\left(\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{(\min \circ \mathcal{C}_{-/}) \times F} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{- \times -} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty\right)\right) \\
&\simeq \mathrm{colim}\left(\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{C_{-/} \times (\mathrm{U}_{\mathcal{R}\mathrm{el}} \circ F)} \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty \xrightarrow{- \times -} \mathcal{C}\mathrm{at}_\infty\right) \\
&\simeq \mathrm{Gr}(\mathrm{U}_{\mathcal{R}\mathrm{el}} \circ F),
\end{aligned}$$

again appealing to [GHN, Theorem 7.4]. In other words, the underlying ∞-category of the relative ∞-category

$$\operatorname{colim}\left(\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{(\min \circ \mathcal{C}_{-/}) \times F} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{-\times-} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty\right)$$

is indeed $\mathrm{Gr}(\mathrm{U}_{\mathcal{R}\mathrm{el}} \circ F)$; moreover, by definition its subcategory of weak equivalences is inherited from the functor F, and hence we have an equivalence

$$\mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F) \simeq \operatorname{colim}\left(\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{(\min \circ \mathcal{C}_{-/}) \times F} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{-\times-} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty\right)$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{/\min(\mathcal{C})}$.[8] Thus, we have obtained an equivalence

$$\mathrm{Gr}(\mathscr{L} \circ F) \simeq \mathscr{L}(\mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F))$$

in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$, which completes the proof of both claims. □

5.2.2 Bicartesian fibrations

Recall that an *adjunction* can be defined as a map to $[1] \in \mathcal{C}\mathrm{at}_\infty$ which is simultaneously a cocartesian fibration and a cartesian fibration. As we will be interested not just in adjunctions but in *families* of adjunctions (e.g. two-variable adjunctions), it will be convenient to introduce the following terminology.

Notation 5.2.4. Let $\mathcal{C}$ be an ∞-category. We denote by $\mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$ the ∞-category of *bicartesian fibrations* over $\mathcal{C}$. This is the underlying ∞-category of the bicartesian model structure of Theorem A.4.7.5.10; its objects are those functors to $\mathcal{C}$ which are simultaneously cocartesian fibrations and cartesian fibrations, and its morphisms are maps over $\mathcal{C}$ which are simultaneously morphisms of cocartesian fibrations and morphisms of cartesian fibrations (i.e. they preserve both cocartesian morphisms and cartesian morphisms). We thus have canonical forgetful functors

$$\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \hookleftarrow \mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \hookrightarrow \mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}),$$

[8] The structure map for the object on the right comes from its canonical projection to

$$\min(\mathrm{TwAr}(\mathcal{C})) \simeq \operatorname{colim}\left(\mathrm{TwAr}(\mathcal{C}) \xrightarrow{\mathrm{const}(\mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty})} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty\right)$$

followed by the composite projection $\min(\mathrm{TwAr}(\mathcal{C})) \to \min(\mathcal{C}^{op} \times \mathcal{C}) \to \min(\mathcal{C})$.

which are both inclusions of (non-full) subcategories, and which both admit left adjoints by Remark A.4.7.5.12. By Proposition A.4.7.5.17, the composite

$$\mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \hookrightarrow \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \xrightarrow[\sim]{\mathrm{Gr}} \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$$

identifies $\mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$ with a certain subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$,

- whose objects are those functors $\mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$ such that for every map $c_1 \to c_2$ in $\mathcal{C}$, the induced functor $F(c_1) \to F(c_2)$ is a left adjoint, and
- whose morphisms are those natural transformations satisfying a certain "right adjointableness" condition,

and dually for the composite

$$\mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \hookrightarrow \mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C}) \xrightarrow[\sim]{\mathrm{Gr}^-} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{C}\mathrm{at}_\infty).$$

Remark 5.2.5. Giving an adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ is equivalent to giving an object of $\mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}([1])$ equipped with certain identifications of its fibers, which data can be encoded succinctly as an object of the pullback

$$\lim \left(\begin{array}{ccc} & & \mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}([1]) \\ & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[(\mathcal{C},\mathcal{D})]{} & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty \end{array} \right)$$

in $\mathcal{C}\mathrm{at}_\infty$. In other words, the space of objects of this pullback is (canonically) equivalent to that of the ∞-category $\mathrm{Adjn}(\mathcal{C}; \mathcal{D})$. However, note that morphisms of bicartesian fibrations are quite different from morphisms in $\mathrm{Adjn}(\mathcal{C}; \mathcal{D})$: a map from an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ to an adjunction $F' : \mathcal{C} \rightleftarrows \mathcal{D} : G'$ is given

- in $\mathrm{Adjn}(\mathcal{C}; \mathcal{D})$, by either a natural transformation $F' \to F$ or a natural transformation $G \to G'$, but
- in $\mathrm{bi}\mathcal{C}\mathcal{F}\mathrm{ib}([1])$, a certain sort of *commutative square* in $\mathcal{C}\mathrm{at}_\infty$.

(So the latter is $(\infty, 1)$-categorical, while the former is inherently $(\infty, 2)$-categorical.) In fact, it is not hard to see that the above pullback in $\mathcal{C}\mathrm{at}_\infty$ actually defines an ∞-groupoid: really, this is just a more elaborate version of the difference between $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and

$$\lim \left(\begin{array}{ccc} & & \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1]) \\ & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[(\mathcal{C},\mathcal{D})]{} & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty \end{array} \right).$$

Despite Remark 5.2.5, we will have use for the following notation.

Notation 5.2.6. For $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$, we denote by $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1]; \mathcal{C}, \mathcal{D})$ the second pullback in Remark 5.2.5. We will use analogous notation for the various variants of this construction (namely cartesian, relative co/cartesesian, and bicartesian fibrations over $[1]$). For consistency, we will similarly write

$$\mathcal{C}\mathrm{at}_\infty([1]; \mathcal{C}, \mathcal{D}) = \lim \left(\begin{array}{ccc} & & (\mathcal{C}\mathrm{at}_\infty)_{/[1]} \\ & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[(\mathcal{C},\mathcal{D})]{} & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty \end{array} \right).$$

For any $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, we also set

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty([1]; \mathcal{R}_1, \mathcal{R}_2) = \lim \left(\begin{array}{ccc} & & (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{/\min([1])} \\ & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[(\mathcal{R}_1,\mathcal{R}_2)]{} & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \end{array} \right).$$

Remark 5.2.7. Using Notation 5.2.6, note that we can identify

$$\mathrm{N}_\infty(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))_\bullet \simeq \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1]; [\bullet] \times \mathcal{C}, \mathcal{D})^{\simeq}.$$

This identification (and related ones) will be useful in the proof of Lemma 5.4.5.

5.3 The proofs of Theorem 5.1.1 and Corollary 5.1.3

This section is devoted to proving the results stated in §5.1.1, namely

- Theorem 5.1.1 – that a Quillen adjunction has a canonical derived adjunction –, and
- Corollary 5.1.3 – that the derived adjunction of a Quillen equivalence is an adjoint equivalence.

We begin with the following key result, the proof of which is based on that of [BHH, Lemma 2.4.8].

Lemma 5.3.1. *Let $\mathcal{M}$ be a model ∞-category, and let $x \in \mathcal{M}$. Then*

$$\left(\mathbf{W}^f_{\mathcal{M}_{x/}}\right)^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}.$$

Proof. By [Cis10b, Lemme d'asphéricité], it suffices to show that for any finite directed set considered as a category $\mathcal{C} \in \mathbb{c}\mathrm{at}$, any functor $\mathcal{C} \to \mathbf{W}^f_{\mathcal{M}_{x/}}$ is connected to a constant functor by a zigzag of natural transformations in $\mathrm{Fun}(\mathcal{C}, \mathbf{W}^f_{\mathcal{M}_{x/}})$.[9] Note that such a functor is equivalent to the data of

- the composite functor $\mathcal{C} \to \mathbf{W}^f_{\mathcal{M}_{x/}} \to \mathbf{W}^f_{\mathcal{M}}$, which we will denote by $\mathcal{C} \xrightarrow{F} \mathbf{W}^f_{\mathcal{M}}$, along with
- a natural transformation $\mathrm{const}(x) \to F$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{W}_{\mathcal{M}})$.

We now appeal to Cisinski's theory of *left-derivable categories* introduced in [Cis10a, §1] (there called "catégories dérivables à gauche"), which immediately generalizes to a theory of *left-derivable ∞-categories*: one simply replaces sets with spaces and categories with ∞-categories.[10] Clearly the model ∞-category $\mathcal{M}$ is in particular a left-derivable ∞-category. Hence, considering $\mathcal{C} \xrightarrow{F} \mathbf{W}^f_{\mathcal{M}} \hookrightarrow \mathcal{M}$ as an object of $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, by [Cis10a, Proposition 1.29] there exists a factorization

$$F \xrightarrow{\approx} F' \twoheadrightarrow \mathrm{pt}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})} \simeq \mathrm{const}(\mathrm{pt}_{\mathcal{M}})$$

of the terminal map in $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, where $F \xrightarrow{\approx} F'$ is a componentwise weak equivalence and the map $F' \twoheadrightarrow \mathrm{pt}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}$ is a *boundary fibration* (there called "une fibration bordée"). In other words, F' is *fibrant on the boundaries* (there called "fibrant sur les bords"), and in particular by [Cis10a, Corollaire 1.24] it is objectwise fibrant. Thus, we can consider $F \to F'$ as a morphism in $\mathrm{Fun}(\mathcal{C}, \mathbf{W}^f_{\mathcal{M}})$, and hence for our main goal it suffices to assume that F itself is fibrant on the boundaries.

Now, our map $\mathrm{const}(x) \to F$ induces a canonical map $x \to \lim_{\mathcal{C}} F$ in $\mathcal{M}$ (where this limit exists because $\mathcal{M}$ is finitely complete), and this map in turn admits a factorization

$$x \xrightarrow{\approx} y \twoheadrightarrow \lim_{\mathcal{C}} F.$$

Moreover, [Cis10a, Proposition 1.18] implies that $\lim_{\mathcal{C}} F \in \mathcal{M}$ is fibrant, and hence

[9] [Cis10b, Lemme d'asphéricité] can also be proved invariantly (i.e. without reference to quasicategories) by using the theory of complete Segal spaces and replacing Cisinski's appeal to the Quillen equivalence $\mathrm{sd} : s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} : \mathrm{Ex}$ and the functor Ex^∞ to their ∞-categorical variants (see §1.6.3).

[10] However, the notion of *finite direct categories* (there called "catégories directes finies") need not be changed. Note that such categories are gaunt, so 1-categorical pushouts and pullbacks between them compute their respective ∞-categorical counterparts.

$y \in \mathcal{M}$ is fibrant as well. Further, in the commutative diagram

$$\begin{array}{ccc} \mathrm{const}(x) & \xrightarrow{\approx} & F \\ \wr\downarrow & \nearrow & \uparrow \\ \mathrm{const}(y) & \longrightarrow & \mathrm{const}(\lim_{\mathcal{C}} F) \end{array}$$

in $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, the dotted arrow is a componentwise weak equivalence by the two-out-of-three property (applied componentwise). This provides the desired zigzag connecting the object

$$(\mathcal{C} \xrightarrow{F} \mathbf{W}^f_{\mathcal{M}}, \mathrm{const}(x) \to F) \in \mathrm{Fun}(\mathcal{C}, \mathbf{W}^f_{\mathcal{M}_{x/}})$$

to a constant functor, namely the object

$$(\mathcal{C} \xrightarrow{\mathrm{const}(y)} \mathbf{W}^f_{\mathcal{M}}, \mathrm{const}(x) \to \mathrm{const}(y)) \in \mathrm{Fun}(\mathcal{C}, \mathbf{W}^f_{\mathcal{M}_{x/}}),$$

which proves the claim. □

This has the following convenient consequence.

Lemma 5.3.2. *For any model ∞-category $\mathcal{M}$, the inclusion $\mathbf{W}^f \hookrightarrow \mathbf{W}$ induces an equivalence under the functor $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$.*

Proof. This functor is final by Theorem A (3.4.10) and Lemma 5.3.1; note that for an object $x \in \mathbf{W}$, we have an identification

$$\mathbf{W}^f \times_{\mathbf{W}} \mathbf{W}_{x/} \simeq \mathbf{W}^f_{\mathcal{M}_{x/}}.$$

Hence, the assertion follows from Proposition 3.4.8. □

In turn, this allows us to prove the following pair of results, which we will need in the proof of Theorem 5.1.1.

Proposition 5.3.3. *For any model ∞-category $\mathcal{M}$, the inclusion $(\mathcal{M}^f, \mathbf{W}^f) \hookrightarrow (\mathcal{M}, \mathbf{W})$ induces an equivalence*

$$\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M}^f, \mathbf{W}^f) \to \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M}, \mathbf{W})$$

in $s\mathcal{S}$.

Proof. We must show that for every $n \geq 0$, the map

$$\mathrm{preN}^{\mathrm{R}}_{\infty}(\mathcal{M}^f, \mathbf{W}^f)_n \to \mathrm{preN}^{\mathrm{R}}_{\infty}(\mathcal{M}, \mathbf{W})_n$$

in $\mathcal{C}\mathrm{at}_\infty$ becomes an equivalence upon applying $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$. By definition, this is the map

$$\mathrm{Fun}([n], (\mathcal{M}^f, \mathbf{W}^f))^{\mathbf{W}} \to \mathrm{Fun}([n], (\mathcal{M}, \mathbf{W}))^{\mathbf{W}}.$$

But this is precisely the inclusion

$$\mathbf{W}^f_{\mathrm{Fun}([n],\mathcal{M})_{\mathrm{proj}}} \hookrightarrow \mathbf{W}_{\mathrm{Fun}([n],\mathcal{M})_{\mathrm{proj}}},$$

which becomes an equivalence upon groupoid completion by Lemma 5.3.2. □

Corollary 5.3.4. *For any model ∞-category $\mathcal{M}$, the inclusion $(\mathcal{M}^f, \mathbf{W}^f) \hookrightarrow (\mathcal{M}, \mathbf{W})$ is a weak equivalence in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$, i.e. it induces an equivalence*

$$\mathcal{M}^f[\![(\mathbf{W}^f)^{-1}]\!] \xrightarrow{\sim} \mathcal{M}[\![\mathbf{W}^{-1}]\!]$$

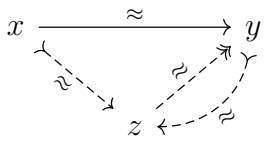
in $\mathcal{C}\mathrm{at}_\infty$.

Proof. This follows from Proposition 5.3.3 and the global universal property of the Rezk nerve (Proposition 2.3.9). □

We now give one more easy result which we will need in the proof of Theorem 5.1.1, which we refer to as ***Kenny Brown's lemma*** (for model ∞-categories).

Lemma 5.3.5. *Let $\mathcal{M}$ be a model ∞-category, and let $(\mathcal{R}, \mathbf{W}_{\mathcal{R}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ be a relative ∞-category such that $\mathbf{W}_{\mathcal{R}} \subset \mathcal{R}$ has the two-out-of-three property. If $\mathcal{M} \to \mathcal{R}$ is any functor of underlying ∞-categories which takes the subcategory $(\mathbf{W} \cap \mathbf{C})^c_{\mathcal{M}} \subset \mathcal{M}$ into $\mathbf{W}_{\mathcal{R}} \subset \mathcal{R}$, then it also takes the subcategory $\mathbf{W}^c_{\mathcal{M}} \subset \mathcal{M}$ into $\mathbf{W}_{\mathcal{R}} \subset \mathcal{C}$.*

Proof. Given any map $x \xrightarrow{\approx} y$ in $\mathbf{W}^c_{\mathcal{M}} \subset \mathcal{M}$, we can construct a diagram



in $\mathcal{M}$, i.e. a factorization of the chosen map and a section of the second map which are contained in the various subcategories defining the model structure on $\mathcal{M}$ as indicated, exactly as in [Hir03, Lemma 7.7.1] (only omitting the assertion of functoriality). Hence, our functor $\mathcal{M} \to \mathcal{R}$ must take our chosen map into $\mathbf{W}_{\mathcal{R}} \subset \mathcal{R}$ since this subcategory contains all the equivalences, has the two-out-of-three property, and is closed under composition. This proves the claim. □

We now turn to this section's primary goal.

Proof of Theorem 5.1.1. Let $(\mathcal{M} + \mathcal{N}) \to [1]$ denote the bicartesian fibration corresponding to the underlying adjunction $F \dashv G$ of the given Quillen adjunction. Let us equip this with the subcategory of weak equivalences inherited from $\mathbf{W}_{\mathcal{M}} \subset \mathcal{M}$ and $\mathbf{W}_{\mathcal{N}} \subset \mathcal{N}$; its structure map can then be considered as a map to $\min([1])$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.[11] Let us define full relative subcategories

$$(\mathcal{M}^c + \mathcal{N}^f), (\mathcal{M}^c + \mathcal{N}), (\mathcal{M} + \mathcal{N}^f) \subset (\mathcal{M} + \mathcal{N})$$

(which inherit maps to $\min([1])$) by restricting to the cofibrant objects of $\mathcal{M}$ and/or to the fibrant objects of $\mathcal{N}$, as indicated by the notation. Moreover, let us define the functors F^c and G^f to be the composites

$$\mathcal{M}^c \hookrightarrow \mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N} \hookleftarrow \mathcal{N}^f.$$

with F^c the composite $\mathcal{M}^c \hookrightarrow \mathcal{M} \xrightarrow{F} \mathcal{N}$ and G^f the composite $\mathcal{N}^f \hookrightarrow \mathcal{N} \xrightarrow{G} \mathcal{M}$.

Note that F^c and G^f both preserve weak equivalences by Kenny Brown's lemma (5.3.5). It follows that we have a canonical equivalence

$$(\mathcal{M}^c + \mathcal{N}) \simeq \mathrm{Gr}_{\mathcal{R}\mathrm{el}}(F^c)$$

in $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}_{\mathcal{R}\mathrm{el}}([1])$ and a canonical equivalence

$$(\mathcal{M} + \mathcal{N}^f) \simeq \mathrm{Gr}^-_{\mathcal{R}\mathrm{el}}(G^f)$$

in $\mathcal{C}\mathcal{F}\mathrm{ib}_{\mathcal{R}\mathrm{el}}([1])$. By Proposition 5.2.3 (and its dual), it follows that

$$\mathscr{L}(\mathcal{M}^c + \mathcal{N}) \simeq \mathrm{Gr}(\mathscr{L} \circ F^c)$$

in $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1])$ and that

$$\mathscr{L}(\mathcal{M} + \mathcal{N}^f) \simeq \mathrm{Gr}^-(\mathscr{L} \circ G^f)$$

in $\mathcal{C}\mathcal{F}\mathrm{ib}([1])$.

[11] Note that this will not generally make this map into a relative cocartesian fibration or a relative cartesian fibration: left and right Quillen functors are not generally functors of relative ∞-categories.

Now, by Lemma 5.3.6, the canonical inclusions induce weak equivalences

$$(\mathcal{M}^c + \mathcal{N}) \xleftarrow{\approx} (\mathcal{M}^c + \mathcal{N}^f) \xrightarrow{\approx} (\mathcal{M} + \mathcal{N}^f)$$

in $((\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{/\min([1])})_{\mathrm{BK}}$. Hence, applying $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ yields a diagram

$$\mathrm{Gr}(\mathscr{L} \circ F^c) \simeq \mathscr{L}(\mathcal{M}^c + \mathcal{N}) \xleftarrow{\sim} \mathscr{L}(\mathcal{M}^c + \mathcal{N}^f) \xrightarrow{\sim} \mathscr{L}(\mathcal{M} + \mathcal{N}^f) \simeq \mathrm{Gr}^-(\mathscr{L} \circ G^f)$$

in $(\mathcal{C}\mathrm{at}_\infty)_{/[1]}$, so that in particular the map $\mathscr{L}(\mathcal{M}^c + \mathcal{N}^f) \to [1]$ is a bicartesian fibration (which as a cocartesian fibration corresponds to F^c while as a cartesian fibration corresponds to G^f). Appealing to Corollary 5.3.4 (and its dual), we then obtain a diagram

$$\begin{array}{ccccccccc}
\mathcal{M}[\![\mathbf{W}_{\mathcal{M}}^{-1}]\!] & \xleftarrow{\sim} & \mathcal{M}^c[\![(\mathbf{W}_{\mathcal{M}}^c)^{-1}]\!] & \hookrightarrow & \mathscr{L}(\mathcal{M}^c + \mathcal{N}^f) & \hookleftarrow & \mathcal{N}^f[\![(\mathbf{W}_{\mathcal{N}}^f)^{-1}]\!] & \xrightarrow{\sim} & \mathcal{N}[\![\mathbf{W}_{\mathcal{N}}^{-1}]\!] \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \{0\} & \hookrightarrow & [1] & \hookleftarrow & \{1\} & &
\end{array}$$

in which the squares are fiber inclusions and which, upon making choices of inverses for the equivalences (the spaces of which are contractible), selects the desired adjunction. □

We now prove a key result which we needed in the proof of Theorem 5.1.1.

Lemma 5.3.6. *The inclusions*

$$(\mathcal{M}^c + \mathcal{N}) \hookleftarrow (\mathcal{M}^c + \mathcal{N}^f) \hookrightarrow (\mathcal{M} + \mathcal{N}^f)$$

are weak equivalences in $((\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{/\min([1])})_{\mathrm{BK}}$.

Proof. We will show that the inclusion

$$(\mathcal{M}^c + \mathcal{N}) \hookleftarrow (\mathcal{M}^c + \mathcal{N}^f)$$

is a weak equivalence in $((\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{/\min([1])})_{\mathrm{BK}}$; the other weak equivalence follows from a dual argument. By the global universal property of Rezk nerve (Proposition 2.3.9), it suffices to show that applying the functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S}$ to this map yields an equivalence. This is equivalent to showing that for every $n \geq 0$, the map

$$\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{M}^c + \mathcal{N}^f)_n \to \mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{M}^c + \mathcal{N})_n$$

in $\mathcal{C}\mathrm{at}_\infty$ becomes an equivalence upon groupoid completion. By definition, this is the postcomposition map

$$\mathrm{Fun}([n], (\mathcal{M}^c + \mathcal{N}^f))^{\mathbf{W}} \to \mathrm{Fun}([n], (\mathcal{M}^c + \mathcal{N}))^{\mathbf{W}}.$$

Now, observe that since neither $(\mathcal{M}^c + \mathcal{N}^f)$ nor $(\mathcal{M}^c + \mathcal{N})$ has any weak equivalences covering the unique non-identity map of $[1]$, then these ∞-categories decompose as coproducts (in $\mathcal{C}\mathrm{at}_\infty$) over the set of possible composite maps $[n] \to (\mathcal{M}^c + \mathcal{N}^{(f)}) \to [1]$, and moreover the map between them respects these decompositions. Thus, it suffices to show that for each choice of structure map $[n] \to [1]$, the resulting map

$$\mathrm{Fun}_{/[1]}([n], (\mathcal{M}^c + \mathcal{N}^f))^{\mathbf{W}} \to \mathrm{Fun}_{/[1]}([n], (\mathcal{M}^c + \mathcal{N}))^{\mathbf{W}}$$

in $\mathcal{C}\mathrm{at}_\infty$ becomes an equivalence upon applying $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$.

First of all, we obtain an equivalence of fibers over the constant map $[n] \xrightarrow{\mathrm{const}(0)} [1]$. Moreover, over the constant map $[n] \xrightarrow{\mathrm{const}(1)} [1]$, the above map reduces to

$$\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{N}^f, \mathbf{W}^f_{\mathcal{N}})_n \to \mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{N}, \mathbf{W}_{\mathcal{N}})_n,$$

in which situation the result follows from Proposition 5.3.3. Thus, let us restrict our attention to the intermediate cases, supposing that our structure map $[n] \to [1]$ is given by $0, \dots, i \mapsto 0$ and $i+1, \dots, n \mapsto 1$, where $0 \leq i < n$. Let us write $j = n - (i+1)$. Then, we can reidentify these ∞-categories as

$$\mathcal{C}^{c,f} = \mathrm{Fun}_{/[1]}([n], (\mathcal{M}^c + \mathcal{N}^f))^{\mathbf{W}} \simeq \lim \left(\begin{array}{ccc} & & \mathrm{Fun}([j], \mathcal{N}^f)^{\mathbf{W}} \\ & & \downarrow{\scriptstyle \{0\}} \\ & \mathrm{Fun}([1], \mathcal{N})^{\mathbf{W}} & \xrightarrow[\{1\}]{} \mathbf{W}_{\mathcal{N}} \\ & \downarrow{\scriptstyle \{0\}} & \\ \mathrm{Fun}([i], \mathcal{M}^c)^{\mathbf{W}} & \xrightarrow[F^c \circ \{i\}]{} \mathbf{W}_{\mathcal{N}} & \end{array} \right)$$

and

$$\mathcal{C}^{c} = \mathrm{Fun}_{/[1]}([n], (\mathcal{M}^c + \mathcal{N}))^{\mathbf{W}} \simeq \lim \left(\begin{array}{ccc} & & \mathrm{Fun}([j], \mathcal{N})^{\mathbf{W}} \\ & & \downarrow{\scriptstyle \{0\}} \\ & \mathrm{Fun}([1], \mathcal{N})^{\mathbf{W}} & \xrightarrow[\{1\}]{} \mathbf{W}_{\mathcal{N}} \\ & \downarrow{\scriptstyle \{0\}} & \\ \mathrm{Fun}([i], \mathcal{M}^c)^{\mathbf{W}} & \xrightarrow[F^c \circ \{i\}]{} \mathbf{W}_{\mathcal{N}} & \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{ccc} & & \mathrm{Fun}([j+1],\mathcal{N})^{\mathbf{W}} \\ & & \downarrow{\scriptstyle \{0\}} \\ \mathrm{Fun}([i],\mathcal{M}^c)^{\mathbf{W}} & \xrightarrow[F^c\circ\{i\}]{} & \mathbf{W}_{\mathcal{N}} \end{array} \right)$$

(with the evident map between them). By Theorem A (3.4.10) and Proposition 3.4.8, it suffices to show that for any object

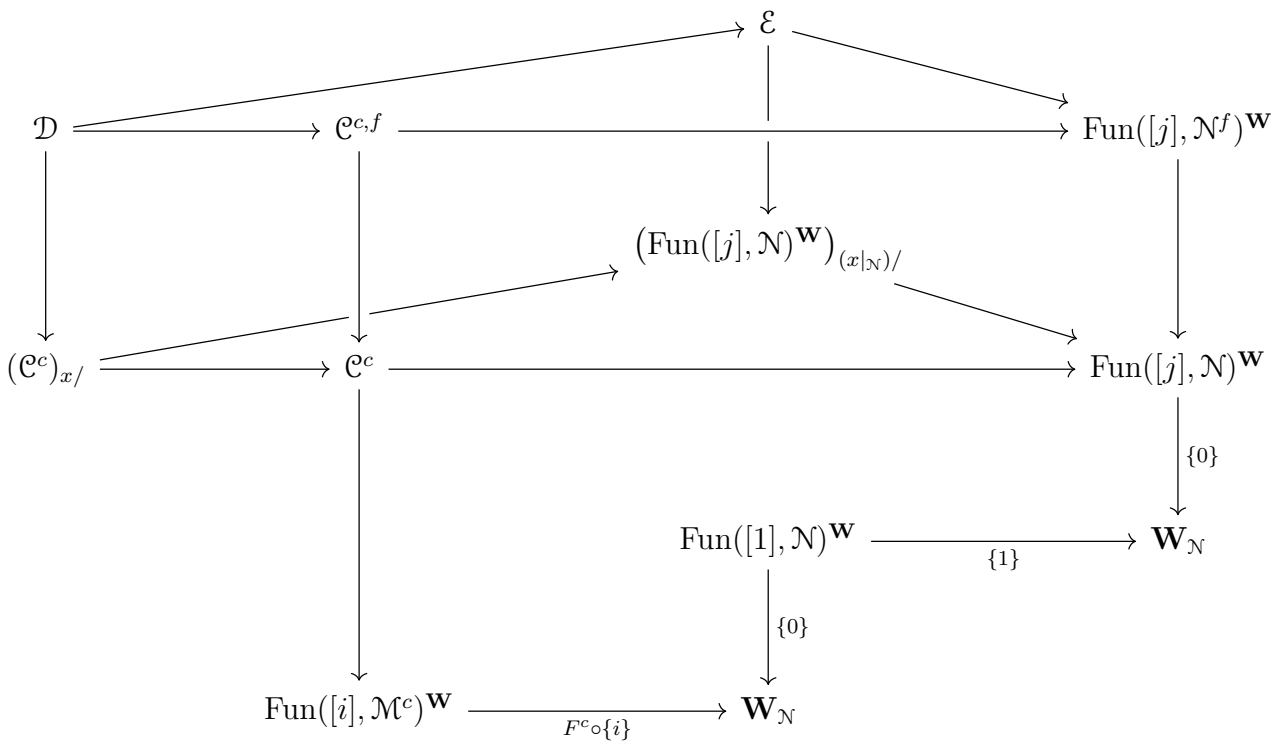
$$x = ((m_0 \to \cdots \to m_i), (F(m_i) \to n_0), (n_0 \to \cdots \to n_j)) \in \mathcal{C}^c,$$

the resulting comma ∞-category

$$\mathcal{D} = \mathcal{C}^{c,f} \underset{\mathcal{C}^c}{\times} (\mathcal{C}^c)_{x/}$$

has that $\mathcal{D}^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$.

Let us write $x|_{\mathcal{N}} = (n_0 \to \cdots \to n_j) \in \mathrm{Fun}([j],\mathcal{N})^{\mathbf{W}}$, and using this let us define the ∞-category $\mathcal{E}$ via the commutative diagram



in which all of $\mathcal{C}^{c,f}$, $\mathcal{D}$, and $\mathcal{E}$ are defined as pullbacks (which is what provides the functor $\mathcal{D} \to \mathcal{E}$). By applying Lemma 5.3.1 to the model ∞-category $\mathrm{Fun}([j],\mathcal{N})_{\mathrm{proj}}$

and the object $x|_{\mathcal{N}} \in \mathrm{Fun}([j], \mathcal{N})$, we obtain that $\mathcal{E}^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$. Moreover, unwinding the definitions, we see that the functor $\mathcal{D} \to \mathcal{E}$ is a right adjoint, with left adjoint given by taking the object

$$\begin{pmatrix} n_0 & \longrightarrow & \cdots & \longrightarrow & n_j \\ \wr\downarrow{\scriptstyle \nu_0} & & & & {\scriptstyle \nu_j}\downarrow\wr \\ n'_0 & \longrightarrow & \cdots & \longrightarrow & n'_j \end{pmatrix} \in \mathcal{E}$$

to the object

$$\left(\begin{pmatrix} m_0 & \longrightarrow & \cdots & \longrightarrow & m_i \\ \wr\downarrow{\scriptstyle \mathrm{id}_{m_0}} & & & & {\scriptstyle \mathrm{id}_{m_i}}\downarrow\wr \\ m_0 & \longrightarrow & \cdots & \longrightarrow & m_i \end{pmatrix}, \begin{pmatrix} F(m_i) & \longrightarrow & n_0 \\ \wr\downarrow{\scriptstyle \mathrm{id}_{F(m_i)}} & & {\scriptstyle \nu_0}\downarrow\wr \\ F(m_i) & \longrightarrow & n'_0 \end{pmatrix}, \begin{pmatrix} n_0 & \longrightarrow & \cdots & \longrightarrow & n_j \\ \wr\downarrow{\scriptstyle \nu_0} & & & & {\scriptstyle \nu_j}\downarrow\wr \\ n'_0 & \longrightarrow & \cdots & \longrightarrow & n'_j \end{pmatrix} \right) \in \mathcal{D}$$

and acting in the expected way on morphisms.[12] Hence, by Corollary 2.1.28, it follows that $\mathcal{D}^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$ as well. This proves the claim. □

Building on the proof of Theorem 5.1.1, we can now prove Corollary 5.1.3.

Proof of Corollary 5.1.3. We will prove that the unit of the derived adjunction

$$\mathbb{L}F : \mathcal{M}[\mathbf{W}_{\mathcal{M}}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}_{\mathcal{N}}^{-1}] : \mathbb{R}G$$

is a natural equivalence; that its counit is also a natural equivalence will follow from a dual argument. For this, choose any $x \in \mathcal{M}[\mathbf{W}_{\mathcal{M}}^{-1}]$, and choose any cofibrant representative $\tilde{x} \in \mathcal{M}^c$. Then by Theorem 5.1.1, $F(\tilde{x}) \in \mathcal{N}$ represents $(\mathbb{L}F)(x) \in \mathcal{N}[\mathbf{W}_{\mathcal{N}}^{-1}]$. Let us choose any fibrant replacement

$$F(\tilde{x}) \xrightarrow{\approx} \mathbb{R}(F(\tilde{x})) \twoheadrightarrow \mathrm{pt}_{\mathcal{N}}$$

in $\mathcal{N}$. Then, again by Theorem 5.1.1, $G(\mathbb{R}(F(\tilde{x}))) \in \mathcal{M}$ represents $(\mathbb{R}G)((\mathbb{L}F)(x)) \in \mathcal{M}[\mathbf{W}_{\mathcal{M}}^{-1}]$. Moreover, it follows from the proof of Theorem 5.1.1 that the unit map of $\mathbb{L}F \dashv \mathbb{R}G$ at $x \in \mathcal{M}[\mathbf{W}_{\mathcal{M}}^{-1}]$ is represented by the composite map

$$\tilde{x} \xrightarrow{\eta_{\tilde{x}}^{F \dashv G}} G(F(\tilde{x})) \to G(\mathbb{R}(F(\tilde{x})))$$

in $\mathcal{M}$. As this composite map is adjoint to the original weak equivalence $F(\tilde{x}) \xrightarrow{\approx} \mathbb{R}(F(\tilde{x}))$ in $\mathcal{N}$, it must itself be a weak equivalence in $\mathcal{M}$ since $F \dashv G$ is a Quillen equivalence. So the unit of the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ is indeed a natural equivalence. □

[12] Rather than exhibit all of the necessary coherences, this existence of this adjunction can be deduced via (the dual of) Proposition T.5.2.2.8 from the evident counit transformation.

5.4 Two-variable Quillen adjunctions

Recall that a model ∞-category $\mathcal{M}$ may be thought of as a *presentation* of its localization $\mathcal{M}[\mathbf{W}^{-1}]$. The foremost results of this chapter – Theorem 5.1.1 and Corollary 5.1.3 – assert that certain structures on model ∞-categories (namely, Quillen adjunctions and Quillen equivalences) descend to corresponding structures on their localizations (namely, derived adjunctions and derived adjoint equivalences). In this section, we elaborate further on this theme: we define *two-variable Quillen adjunctions* (see Definition 5.4.3), and prove that they induce canonical *derived two-variable adjunctions* (see Theorem 5.4.6). For a more leisurely discussion of two-variable Quillen adjunctions (between model 1-categories), we refer the reader to [Hov99, §4.2].

We begin with a few auxiliary definitions.

Definition 5.4.1. Suppose that we are given three ∞-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, along with a two-variable adjunction

$$\left(\mathcal{C} \times \mathcal{D} \xrightarrow{-\otimes-} \mathcal{E} \ , \ \mathcal{C}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_l(-,-)} \mathcal{D} \ , \ \mathcal{D}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_r(-,-)} \mathcal{C} \right)$$

between them.

- We define the corresponding ***pushout product*** bifunctor

$$\mathrm{Fun}([1], \mathcal{C}) \times \mathrm{Fun}([1], \mathcal{D}) \xrightarrow{-\square-} \mathrm{Fun}([1], \mathcal{E})$$

  to be given by

$$(c_1 \to c_2) \square (d_1 \to d_2) = \left((c_2 \otimes d_1) \coprod_{(c_1 \otimes d_1)} (c_1 \otimes d_2) \to d_1 \otimes d_2 \right).$$

- We define the corresponding ***left pullback product*** bifunctor

$$\mathrm{Fun}([1], \mathcal{C})^{op} \times \mathrm{Fun}([1], \mathcal{E}) \xrightarrow{\underline{\hom}_l^{\square}(-,-)} \mathrm{Fun}([1], \mathcal{D})$$

  to be given by

$$\underline{\hom}_l^{\square}((c_1 \to c_2)^{\circ}, e_1 \to e_2) = \left(\underline{\hom}_l(c_2, e_1) \to \underline{\hom}_l(c_2, e_1) \underset{\underline{\hom}_l(c_1, e_2)}{\times} \underline{\hom}_l(c_1, e_1) \right).$$

- We define the corresponding ***right pullback product*** bifunctor

$$\mathrm{Fun}([1], \mathcal{D})^{op} \times \mathrm{Fun}([1], \mathcal{E}) \xrightarrow{\underline{\hom}_r^{\square}(-,-)} \mathrm{Fun}([1], \mathcal{C})$$

  to be given by

$$\underline{\hom}_r^{\square}((d_1 \to d_2)^{\circ}, e_1 \to e_2) = \left(\underline{\hom}_r(d_2, e_1) \to \underline{\hom}_r(d_2, e_1) \underset{\underline{\hom}_r(d_1, e_2)}{\times} \underline{\hom}_r(d_1, e_1) \right).$$

Remark 5.4.2. In the situation of Definition 5.4.1, the bifunctor $\mathcal{C} \times \mathcal{D} \xrightarrow{-\otimes-} \mathcal{E}$ is a left adjoint and hence commutes with colimits. Thus, we obtain canonical equivalences $\varnothing_{\mathcal{C}} \otimes \mathcal{D} \simeq \varnothing_{\mathcal{E}} \simeq c \otimes \varnothing_{\mathcal{D}}$ for any $c \in \mathcal{C}$ and any $d \in \mathcal{D}$. It follows that we obtain identifications

$$(c_1 \to c_2) \square (\varnothing_{\mathcal{D}} \to d) \simeq (c_1 \to c_2) \otimes d.$$

and

$$(\varnothing_{\mathcal{C}} \to c) \square (d_1 \to d_2) \simeq c \otimes (d_1 \to d_2).$$

Similarly, we obtain an identification

$$(\varnothing_{\mathcal{C}} \to c) \square (\varnothing_{\mathcal{D}} \to d) \simeq (\varnothing_{\mathcal{E}} \to c \otimes d).$$

We can now given the main definition of this subsection.

Definition 5.4.3. Suppose that $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are model ∞-categories, and suppose we are given a two-variable adjunction

$$\left(\mathcal{C} \times \mathcal{D} \xrightarrow{-\otimes-} \mathcal{E} \ , \ \mathcal{C}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_l(-,-)} \mathcal{D} \ , \ \mathcal{D}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_r(-,-)} \mathcal{C} \right)$$

between their underlying ∞-categories. We say that these data define a ***Quillen adjunction of two variables*** (or simply a ***two-variable Quillen adjunction***) if any of the following equivalent conditions is satisfied:

- the pushout product bifunctor satisfies
  - $\mathbf{C}_{\mathcal{C}} \square \mathbf{C}_{\mathcal{D}} \subset \mathbf{C}_{\mathcal{E}}$,
  - $(\mathbf{W} \cap \mathbf{C})_{\mathcal{C}} \square \mathbf{C}_{\mathcal{D}} \subset (\mathbf{W} \cap \mathbf{C})_{\mathcal{E}}$, and
  - $\mathbf{C}_{\mathcal{C}} \square (\mathbf{W} \cap \mathbf{C})_{\mathcal{D}} \subset (\mathbf{W} \cap \mathbf{C})_{\mathcal{E}}$;
- the left pullback product bifunctor satisfies
  - $\underline{\hom}_l^{\square}(\mathbf{C}_{\mathcal{C}}, \mathbf{F}_{\mathcal{E}}) \subset \mathbf{F}_{\mathcal{D}}$,

  - $\underline{\hom}_l^{\square}((\mathbf{W}\cap\mathbf{C})_{\mathcal{C}},\mathbf{F}_{\mathcal{E}})\subset(\mathbf{W}\cap\mathbf{F})_{\mathcal{D}}$, and
  - $\underline{\hom}_l^{\square}(\mathbf{C}_{\mathcal{C}},(\mathbf{W}\cap\mathbf{F})_{\mathcal{E}})\subset(\mathbf{W}\cap\mathbf{F})_{\mathcal{D}}$;
- the right pullback product bifunctor satisfies
  - $\underline{\hom}_r^{\square}(\mathbf{C}_{\mathcal{D}},\mathbf{F}_{\mathcal{E}})\subset\mathbf{F}_{\mathcal{C}}$,
  - $\underline{\hom}_r^{\square}((\mathbf{W}\cap\mathbf{C})_{\mathcal{D}},\mathbf{F}_{\mathcal{E}})\subset(\mathbf{W}\cap\mathbf{F})_{\mathcal{C}}$, and
  - $\underline{\hom}_r^{\square}(\mathbf{C}_{\mathcal{D}},(\mathbf{W}\cap\mathbf{F})_{\mathcal{E}})\subset(\mathbf{W}\cap\mathbf{F})_{\mathcal{C}}$.

Before stating the main result of this subsection, we must introduce a parametrized version of Theorem 5.1.1.

Notation 5.4.4. Let $\mathcal{M}$ and $\mathcal{N}$ be model ∞-categories. We write $\mathrm{QAdjn}(\mathcal{M};\mathcal{N})\subset\mathrm{Adjn}(\mathcal{M};\mathcal{N})$ for the full subcategory on the Quillen adjunctions, and we write $\mathrm{LQAdjt}(\mathcal{M},\mathcal{N})\subset\mathrm{Fun}(\mathcal{M},\mathcal{N})$ (resp. $\mathrm{RQAdjt}(\mathcal{N},\mathcal{M})\subset\mathrm{Fun}(\mathcal{N},\mathcal{M})$) for the full subcategory of left (resp. right) Quillen functors.[13] Thus, there are evident equivalences

$$\mathrm{LQAdjt}(\mathcal{M},\mathcal{N})^{op}\xleftarrow{\sim}\mathrm{QAdjn}(\mathcal{M};\mathcal{N})\xrightarrow{\sim}\mathrm{RQAdjt}(\mathcal{N},\mathcal{M}).$$

Similarly, for model ∞-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, we write $\mathrm{QAdjn}(\mathcal{C},\mathcal{D};\mathcal{E})\subset\mathrm{Adjn}(\mathcal{C},\mathcal{D};\mathcal{E})$ for the full subcategory on the two-variable Quillen adjunctions.

Lemma 5.4.5. *For any model ∞-categories $\mathcal{M}$ and $\mathcal{N}$, the construction of Theorem 5.1.1 assembles canonically into a functor*

$$\mathrm{QAdjn}(\mathcal{M};\mathcal{N})\to\mathrm{Adjn}\left(\mathcal{M}[\![\mathbf{W}_{\mathcal{M}}^{-1}]\!];\mathcal{N}[\![\mathbf{W}_{\mathcal{N}}^{-1}]\!]\right).$$

We will prove Lemma 5.4.5 below. First, we state the main result of this section.

Theorem 5.4.6. *Suppose that $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are model ∞-categories. Then, a two-variable Quillen adjunction*

$$\left(\mathcal{C}\times\mathcal{D}\xrightarrow{-\otimes-}\mathcal{E}\ ,\ \mathcal{C}^{op}\times\mathcal{E}\xrightarrow{\underline{\hom}_l(-,-)}\mathcal{D}\ ,\ \mathcal{D}^{op}\times\mathcal{E}\xrightarrow{\underline{\hom}_r(-,-)}\mathcal{C}\right)$$

induces a canonical two-variable adjunction

$$\left(\begin{array}{c}\mathcal{C}[\![(\mathbf{W}^{\mathcal{C}})^{-1}]\!]\times\mathcal{D}[\![(\mathbf{W}^{\mathcal{D}})^{-1}]\!]\xrightarrow{\overset{\mathbb{L}}{-\otimes-}}\mathcal{E}[\![(\mathbf{W}^{\mathcal{E}})^{-1}]\!]\ ,\\ \mathcal{C}[\![(\mathbf{W}^{\mathcal{C}})^{-1}]\!]^{op}\times\mathcal{E}[\![(\mathbf{W}^{\mathcal{E}})^{-1}]\!]\xrightarrow{\mathbb{R}\underline{\hom}_l(-,-)}\mathcal{D}[\![(\mathbf{W}^{\mathcal{D}})^{-1}]\!]\ ,\\ \mathcal{D}[\![(\mathbf{W}^{\mathcal{D}})^{-1}]\!]^{op}\times\mathcal{E}[\![(\mathbf{W}^{\mathcal{E}})^{-1}]\!]\xrightarrow{\mathbb{R}\underline{\hom}_r(-,-)}\mathcal{C}[\![(\mathbf{W}^{\mathcal{C}})^{-1}]\!]\end{array}\right)$$

[13] More precisely, in the latter definitions we might refer only to those functors which *admit* right (resp. left) adjoints. The question of whether the resulting adjunction will be a Quillen adjunction is independent of that choice, however.

on localizations, whose constituent bifunctors are respectively obtained by applying the localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ *to the composites*

$$\left(\begin{array}{c} \mathcal{C}^c \times \mathcal{D}^c \hookrightarrow \mathcal{C} \times \mathcal{D} \xrightarrow{-\otimes-} \mathcal{E} \ , \\ (\mathcal{C}^c)^{op} \times \mathcal{E}^f \hookrightarrow \mathcal{C}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_l(-,-)} \mathcal{D} \ , \\ (\mathcal{D}^c)^{op} \times \mathcal{E}^f \hookrightarrow \mathcal{D}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_r(-,-)} \mathcal{C} \end{array}\right).$$

Moreover, this construction assembles canonically into a functor

$$\mathrm{QAdjn}(\mathcal{C}, \mathcal{D}; \mathcal{E}) \to \mathrm{Adjn}(\mathcal{C}[\![\mathbf{W}_{\mathcal{C}}^{-1}]\!], \mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!]; \mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!]).$$

We will prove Theorem 5.4.6 at the end of this section (after the proof of Lemma 5.4.5).

Definition 5.4.7. Given a two-variable Quillen adjunction, we refer to the resulting two-variable adjunction on localizations of Theorem 5.4.6 as its ***derived two-variable adjunction***, and we refer to its constituent bifunctors as the ***derived bifunctors*** of those of the original two-variable Quillen adjunction.

Remark 5.4.8. A two-variable adjunction can be thought of as a special sort of indexed family of adjunctions.[14] Thus, Lemma 5.4.5 provides a crucial ingredient for the proof of Theorem 5.4.6. As a result, it is essentially no more work to prove the parametrized version of Theorem 5.4.6 than it is to prove the unparametrized version.

Proof of Lemma 5.4.5. Our argument takes place in the diagram in $\mathcal{C}\mathrm{at}_\infty$ of Figure 5.1. Our asserted functor is the middle dotted vertical arrow. Moreover,

- the diagonal factorizations follow from Kenny Brown's lemma (5.3.5),
- the vertical maps out of the targets of these factorizations are those of Remark 2.1.23,
- the vertical equivalences follow from Corollary 5.3.4 (and its dual), and
- the vertical factorizations follow from Theorem 5.1.1.

Thus, it only remains to show that the diagram commutes, i.e. that the two shorter vertical dotted arrows – which by definition make the outer parts of the diagram commute – also make the part of the diagram between them commute.

The chief difficulty is in aligning the various sorts of fibrations over [1], which are the setting of the proof of Theorem 5.1.1, with our ∞-categories of adjunctions

[14] The "special" here refers to the fact that functor $\mathrm{Adjn}(\mathcal{C}, \mathcal{D}; \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Adjn}(\mathcal{D}; \mathcal{E}))$ will not generally be surjective.

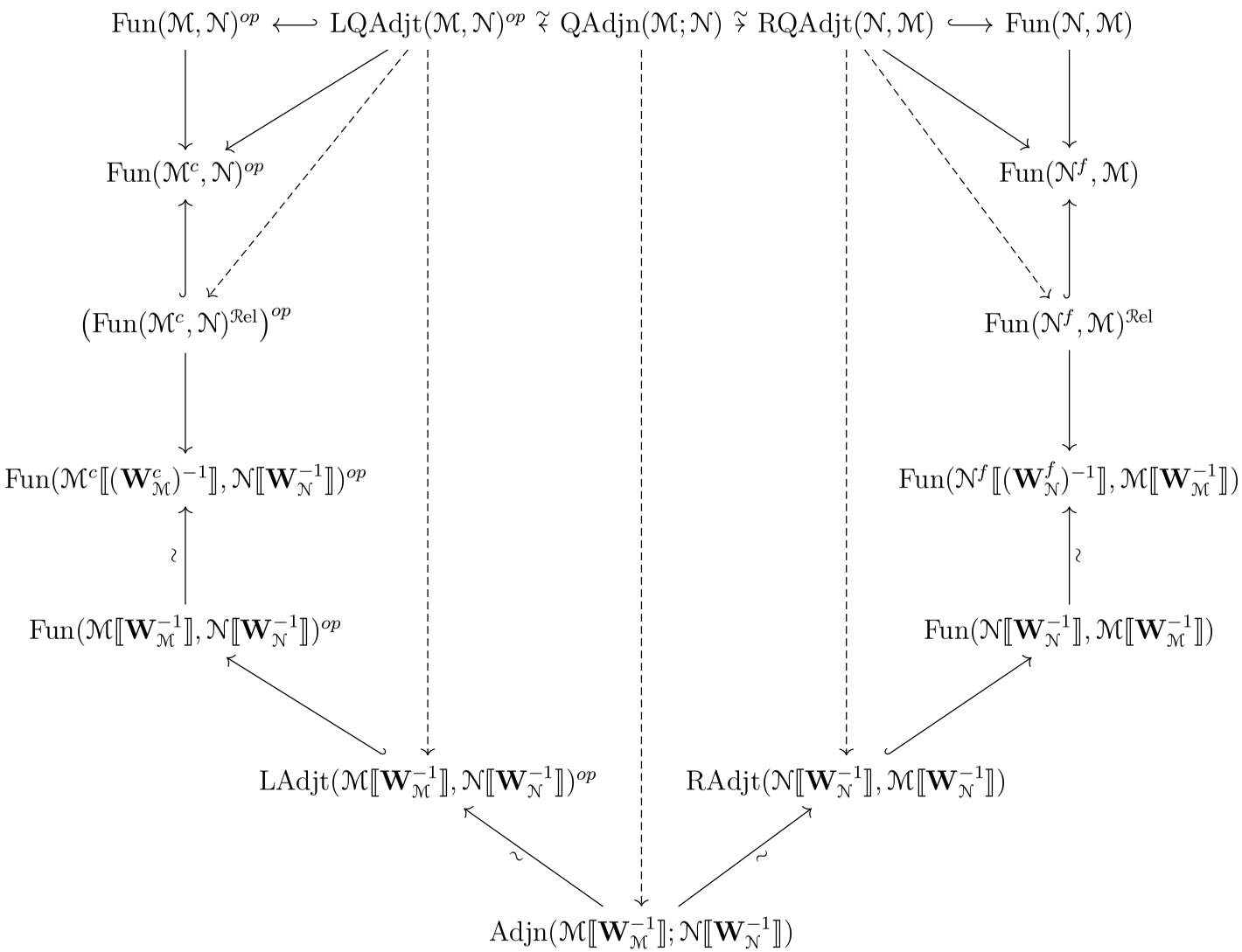


Figure 5.1: The main diagram in the proof of Lemma 5.4.5.

(recall Remark 5.2.5). We can solve this using Remark 5.2.7. For instance, via the equivalence $\mathrm{N}_\infty : \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{CSS}$, we can identify the right portion of the diagram of Figure 5.1 as in Figure 5.2.

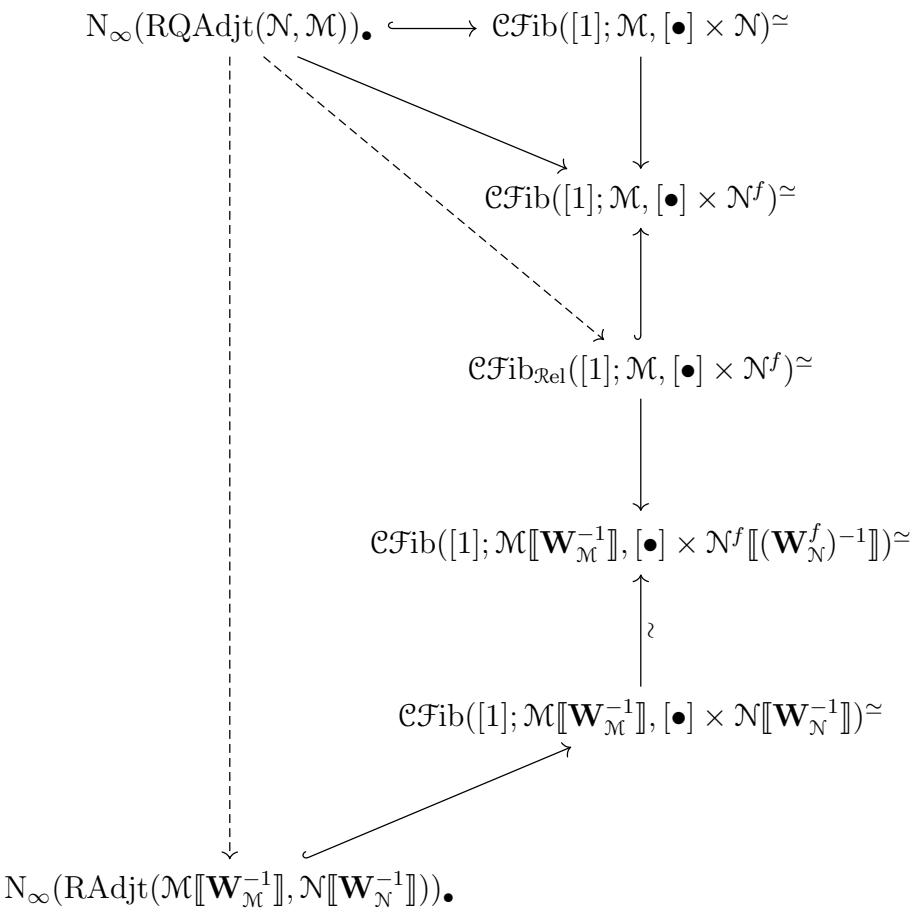


Figure 5.2: The nerve of the right portion of the diagram of Figure 5.1.

However, we have not quite reached a symmetric state of affairs: we would like to somehow relate this to the corresponding identifications of the nerves of the left side of the diagram of Figure 5.1, but for instance we have

$$\mathrm{N}_\infty(\mathrm{Fun}(\mathcal{M},\mathcal{N}))_\bullet \simeq \mathrm{co}\mathcal{CF}\mathrm{ib}([1];[\bullet]\times\mathcal{M},\mathcal{N})^{\simeq},$$

and the fibers here do not match up with those in Figure 5.2 (nor is this rectified by the fact that we're actually interested in $\mathrm{Fun}(\mathcal{M},\mathcal{N})^{op}$ (recall Remark 2.2.3)). To

rectify this, we observe that for any $n \geq 0$ and any $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$, we have a canonical map

$$\begin{aligned} \mathrm{N}_\infty(\mathrm{Fun}(\mathcal{C},\mathcal{D}))_n &\simeq \hom_{\mathcal{C}\mathrm{at}_\infty}([n] \times \mathcal{C}, \mathcal{D}) \\ &\to \hom_{\mathcal{C}\mathrm{at}_\infty}([n] \times \mathcal{C}, [n]) \times \hom_{\mathcal{C}\mathrm{at}_\infty}([n] \times \mathcal{C}, \mathcal{D}) \\ &\simeq \hom_{\mathcal{C}\mathrm{at}_\infty}([n] \times \mathcal{C}, [n] \times \mathcal{D}) \end{aligned}$$

selected by the point $\mathrm{pr}_{[n]} \in \hom_{\mathcal{C}\mathrm{at}_\infty}([n] \times \mathcal{C}, [n])$, and this target in turn admits a forgetful map

$$\hom_{\mathcal{C}\mathrm{at}_\infty}([n] \times \mathcal{C}, [n] \times \mathcal{D}) \simeq \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}([1]; [n] \times \mathcal{C}, [n] \times \mathcal{D}) \to \mathcal{C}\mathrm{at}_\infty([1]; [n] \times \mathcal{C}, [n] \times \mathcal{D}).$$

Bootstrapping this technique up to the relative case (and piecing the maps together for all objects $[n]^\circ \in \mathbf{\Delta}^{op}$), we obtain the diagram of Figure 5.3, which provides an inclusion of the right edge of the diagram of Figure 5.2 into various complete Segal spaces whose constituent spaces now consists of maps to $[1]$ whose fibers over *both* objects $0 \in [1]$ and $1 \in [1]$ are "fattened up".

From here, we only need mimic the proof of Theorem 5.1.1 and restrict further along the inclusion $\mathcal{M}^c \subset \mathcal{M}$: as displayed in the diagram of Figure 5.4, the lower part of the left edge of the diagram of Figure 5.3 admits an inclusion into a map which is now completely self-dual. This, finally, gives us a common home for the left and right sides of the diagram of Figure 5.1: its left side

- admits an identification of its nerve as in Figure 5.2, which in turn
- admits an inclusion into certain "fattened up" objects as in Figure 5.3, which finally
- connects, by restricting along the inclusion $\mathcal{N}^f \subset \mathcal{N}$, to the very same map

$$\begin{array}{c} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty([1]; [\bullet] \times \mathcal{M}^c, [\bullet] \times \mathcal{N}^f)^\simeq \\ \downarrow \\ \mathcal{C}\mathrm{at}_\infty([1]; [\bullet] \times \mathcal{M}[\![\mathbf{W}_{\mathcal{M}}^{-1}]\!], [\bullet] \times \mathcal{N}^f[\![(\mathbf{W}_{\mathcal{N}}^f)^{-1}]\!])^\simeq \end{array}$$

  as that on the left edge in Figure 5.4.

It is now simply a matter of unwinding the definitions to see that the middle part of the diagram in Figure 5.1 does indeed commute: all the localization functors admit full inclusions into the one indicated just above, and the ∞-category

$$\mathrm{Adjn}(\mathcal{M}[\![\mathbf{W}_{\mathcal{M}}^{-1}]\!]; \mathcal{N}[\![\mathbf{W}_{\mathcal{N}}^{-1}]\!])$$

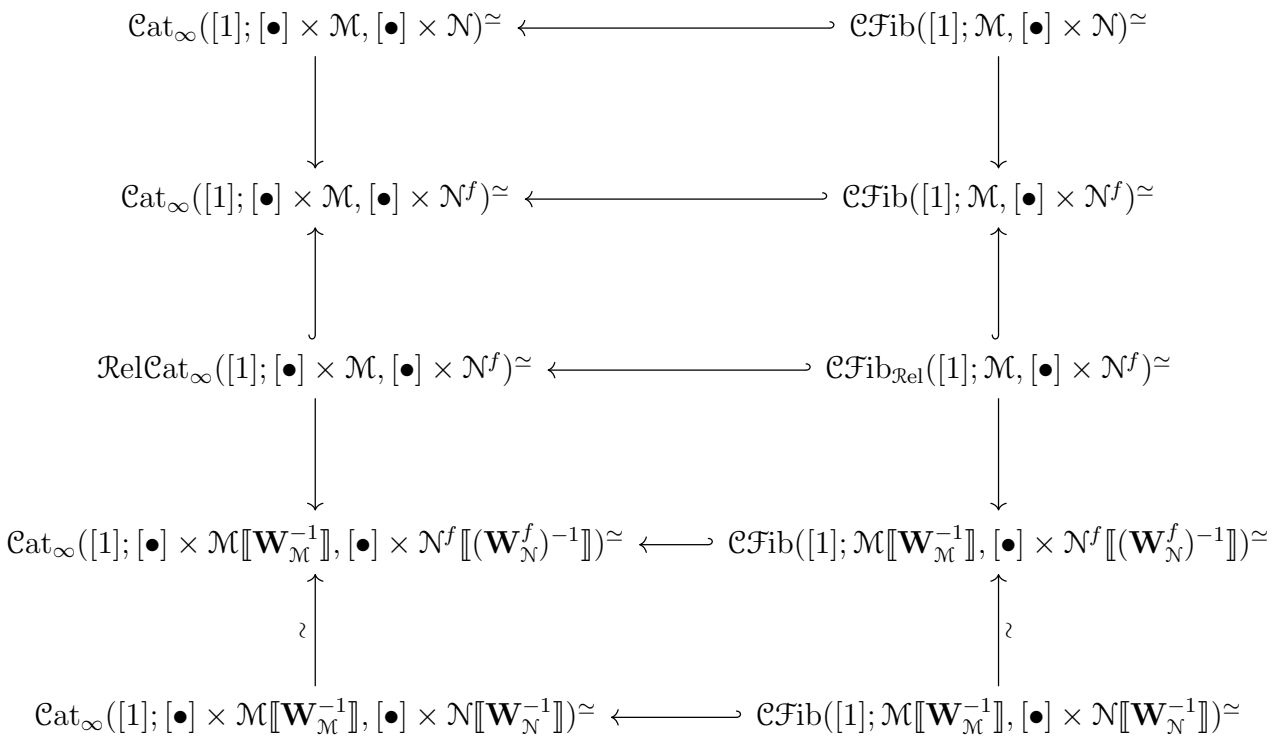


Figure 5.3: An inclusion of the right edge of the diagram of Figure 5.2.

includes as a full subcategory of its target by, after breaking symmetry, once again appealing to the trick of selecting a canonical projection map to $[n] \in \mathcal{C}\mathrm{at}_\infty$ (though the entire point is that the two different ways of obtaining this inclusion are canonically equivalent). This proves the claim. □

Proof of Theorem 5.4.6. By Remark 5.4.2, for any $c \in \mathcal{C}^c$ the induced adjunction

$$c \otimes - : \mathcal{D} \rightleftarrows \mathcal{E} : \underline{\hom}_l(c, -)$$

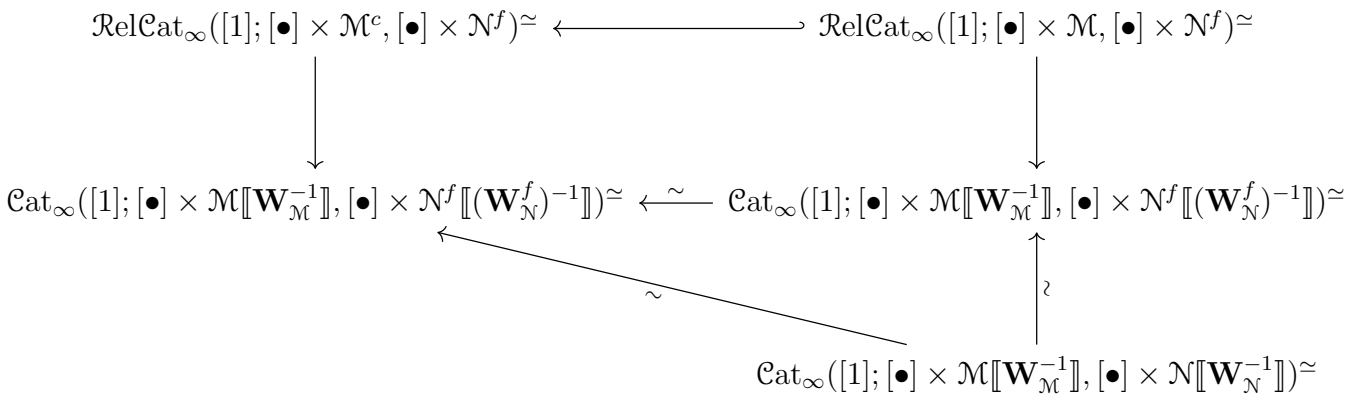


Figure 5.4: The restriction along $\mathcal{M}^c \subset \mathcal{M}$ of the lower part of the left edge of the diagram of Figure 5.3.

is a Quillen adjunction. Thus, we obtain a factorization

$$\begin{array}{ccc}
\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Fun}(\mathcal{D}^{op}\times\mathcal{E},\mathcal{S})) & \longhookrightarrow & \mathrm{Fun}(\mathcal{C}^{op},\mathrm{Adjn}(\mathcal{D};\mathcal{E})) \\
\uparrow & \nearrow & \downarrow \\
\mathrm{Adjn}(\mathcal{C},\mathcal{D};\mathcal{E}) & \longrightarrow & \mathrm{Fun}((\mathcal{C}^c)^{op},\mathrm{Adjn}(\mathcal{D};\mathcal{E})) \\
\uparrow & \nearrow & \uparrow \\
\mathrm{QAdjn}(\mathcal{C},\mathcal{D};\mathcal{E}) & \dashrightarrow & \mathrm{Fun}((\mathcal{C}^c)^{op},\mathrm{QAdjn}(\mathcal{D};\mathcal{E})),
\end{array}$$

which we compose the functor $(\mathcal{C}^c)^{op} \to \mathrm{QAdjn}(\mathcal{D},\mathcal{E})$ selected by our two-variable Quillen adjunction with the canonical functor of Lemma 5.4.5 to obtain a composite functor

$$(\mathcal{C}^c)^{op} \to \mathrm{QAdjn}(\mathcal{D};\mathcal{E}) \to \mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!];\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!]).$$

We claim that this composite functor takes weak equivalences to equivalences. To see this, suppose first that we are given an acyclic cofibration $c_1 \xrightarrow{\approx} c_2$ in $\mathcal{C}^c$. Again by Remark 5.4.2, for any $d \in \mathcal{D}^c$ the induced adjunction

$$- \otimes d : \mathcal{C} \rightleftarrows \mathcal{E} : \underline{\hom}_r(d,-)$$

is a Quillen adjunction, so that in particular we obtain an acyclic cofibration

$$c_1 \otimes d \xrightarrow{\approx} c_2 \otimes d$$

is an acyclic cofibration in $\mathcal{E}$. Since by Theorem 5.1.1 the derived left adjoints of these Quillen adjunctions $\mathcal{D} \rightleftarrows \mathcal{E}$ are computed by localizing the composite $\mathcal{D}^c \hookrightarrow \mathcal{D} \to \mathcal{E}$, it follows that the induced map $(c_1 \otimes -) \to (c_2 \otimes -)$ in

$$\mathrm{LQAdjt}(\mathcal{D}, \mathcal{E}) \simeq \mathrm{QAdjn}(\mathcal{D}; \mathcal{E})^{op}$$

does indeed descend to an equivalence in

$$\mathrm{LAdjt}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!], \mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!]) \simeq \mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!]; \mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])^{op}.$$

The claim now follows from Kenny Brown's lemma (5.3.5). We therefore obtain a factorization

$$\begin{array}{ccc} (\mathcal{C}^c)^{op} & \longrightarrow & \mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!]; \mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!]) \\ \downarrow & \nearrow & \\ (\mathcal{C}^c[\![(\mathbf{W}_{\mathcal{C}}^c)^{-1}]\!])^{op} & & \end{array}$$

which, appealing to Remark 2.1.23, in fact arises from the induced factorization in the diagram

$$\begin{array}{ccc} \mathrm{QAdjn}(\mathcal{C}, \mathcal{D}; \mathcal{E}) & \longrightarrow & \mathrm{Fun}((\mathcal{C}^c)^{op}, \mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!]; \mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])) \\ & \searrow & \uparrow \\ & & \mathrm{Fun}((\mathcal{C}^c)^{op}, \min(\mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!]; \mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])))^{\mathcal{R}\mathrm{el}} \\ & & \downarrow \\ & & \mathrm{Fun}(\mathcal{C}^c[\![(\mathbf{W}_{\mathcal{C}}^c)^{-1}]\!], \mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!]; \mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])). \end{array}$$

Thus, it only remains to show that we have a further factorization

$$\begin{array}{ccc} \mathrm{QAdjn}(\mathcal{C},\mathcal{D};\mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}^c[\![(\mathbf{W}_{\mathcal{C}}^c)^{-1}]\!],\mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!];\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])) \\ & \searrow & \uparrow \wr \\ & & \mathrm{Fun}(\mathcal{C}[\![\mathbf{W}_{\mathcal{C}}^{-1}]\!],\mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!];\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])) \\ & & \uparrow \\ & & \mathrm{Adjn}(\mathcal{C}[\![\mathbf{W}_{\mathcal{C}}^{-1}]\!],\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!];\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!]) \end{array}$$

which does not depend on our having privileged $\mathcal{C}$ among the model ∞-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ participating in our two-variable Quillen adjunction. We accomplish these tasks simultaneously by replacing $\mathcal{C}$ with $\mathcal{D}$ in the above arguments: by essentially the same argument as the one given in the proof of Theorem 5.4.6 for why the diagram of Figure 5.1 commutes, one sees that we have a commutative square

$$\begin{array}{ccc} & \mathrm{QAdjn}(\mathcal{C},\mathcal{D};\mathcal{E}) & \\ \swarrow & & \searrow \\ \mathrm{Fun}((\mathcal{C}[\![\mathbf{W}_{\mathcal{C}}^{-1}]\!])^{op},\mathrm{Adjn}(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!];\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])) & & \mathrm{Fun}((\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!])^{op},\mathrm{Adjn}(\mathcal{C}[\![\mathbf{W}_{\mathcal{C}}^{-1}]\!];\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])) \\ \searrow & & \swarrow \\ & \mathrm{Fun}((\mathcal{C}[\![\mathbf{W}_{\mathcal{C}}^{-1}]\!])^{op}\times(\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!])^{op}\times\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!],\mathcal{S}), & \end{array}$$

which shows

- that those trifunctors in the image of either of the two (equivalent) composites are indeed co/representable in all variables and hence define two-variable adjunctions, and
- that the resulting functor

$$\mathrm{QAdjn}(\mathcal{C},\mathcal{D};\mathcal{E})\to\mathrm{Adjn}(\mathcal{C}[\![\mathbf{W}_{\mathcal{C}}^{-1}]\!],\mathcal{D}[\![\mathbf{W}_{\mathcal{D}}^{-1}]\!];\mathcal{E}[\![\mathbf{W}_{\mathcal{E}}^{-1}]\!])$$

  is indeed completely independent of the choice of $\mathcal{C}$, since rotating the two-variable (Quillen) adjunctions involved – which really just amounts to reorder-

ing and passing to opposites as appropriate – clearly does not affect the induced functor either. □

5.5 Monoidal and symmetric monoidal model ∞-categories

In this section, we show that the localization of a (resp. *symmetric*) *monoidal model ∞-categories* is canonically *closed* (resp. *symmetric*) *monoidal*. For a more leisurely discussion of monoidal and symmetric monoidal model categories, we again refer the reader to [Hov99, §4.2].

Definition 5.5.1. Let $\mathcal{V} \in \mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)$ be a closed monoidal ∞-category, and suppose that $\mathcal{V}$ is equipped with a model structure. We say that these data make $\mathcal{V}$ into a ***monoidal model ∞-category*** if they satisfy the following evident ∞-categorical analogs of the usual axioms for a monoidal model category.

MM$_\infty$1 *(pushout product)* The underlying two-variable adjunction

$$\left(\mathcal{V} \times \mathcal{V} \xrightarrow{-\otimes-} \mathcal{V}\ ,\ \mathcal{V}^{op} \times \mathcal{V} \xrightarrow{\underline{\hom}_l(-,-)} \mathcal{V}\ ,\ \mathcal{V}^{op} \times \mathcal{V} \xrightarrow{\underline{\hom}_r(-,-)} \mathcal{V}\right)$$

is a two-variable Quillen adjunction.

MM$_\infty$2 *(unit)* There exists a cofibrant replacement $\varnothing_{\mathcal{V}} \rightarrowtail \mathbb{Q}\mathbf{1}_{\mathcal{V}} \xrightarrow{\approx} \mathbf{1}_{\mathcal{V}}$ such that the functors

$$\mathcal{V} \xrightarrow{(\mathbb{Q}\mathbf{1}_{\mathcal{V}} \to \mathbf{1}_{\mathcal{V}})\otimes-} \mathrm{Fun}([1], \mathcal{V})$$

and

$$\mathcal{V} \xrightarrow{-\otimes(\mathbb{Q}\mathbf{1}_{\mathcal{V}} \to \mathbf{1}_{\mathcal{V}})} \mathrm{Fun}([1], \mathcal{V})$$

take cofibrant objects to weak equivalences.

Remark 5.5.2. The unit axiom MM$_\infty$2 is automatically satisfied whenever the unit object $\mathbf{1}_{\mathcal{V}} \in \mathcal{V}$ is itself cofibrant.

We have the following key example.

Example 5.5.3. The model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$ of Theorem 1.4.4 is a monoidal model ∞-category with respect to its cartesian symmetric monoidal structure:

- that the underlying two-variable adjunction is a Quillen adjunction follows from (an identical argument to) the proof of [Hov99, Lemma 4.2.4] (see [Hov99, Corollary 4.2.5]), and

- the unit object $\mathrm{pt}_{s\mathcal{S}} \simeq \Delta^0 \in s\mathcal{S}_{\mathrm{KQ}}$ is cofibrant.

We then have the following result.

Proposition 5.5.4. *Suppose that $\mathcal{V}$ is a monoidal model ∞-category. Then the derived two-variable adjunction of its underlying two-variable Quillen adjunction itself underlies a canonical closed monoidal structure on its localization $\mathcal{V}[\![\mathbf{W}^{-1}]\!]$.*

Proof. Observe that the monoidal product preserves cofibrant objects. Hence, the underlying non-unital monoidal structure on $\mathcal{V}$ restricts to one on $\mathcal{V}^c$. Moreover, the structure maps for $\mathcal{V}^c \in \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}\mathrm{at}_\infty)$ preserve weak equivalences by Kenny Brown's lemma (5.3.5), so we obtain a natural lift to $\mathcal{V}^c \in \mathrm{Alg}^{\mathrm{nu}}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)$.

Now, the localization functor is symmetric monoidal by Lemma 2.1.20, so that we obtain $\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!] \in \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}\mathrm{at}_\infty)$. To see that this can in fact be canonically promoted to a unital monoidal structure, we use the guaranteed cofibrant replacement $\varnothing_{\mathcal{V}} \rightarrowtail \mathbb{Q}\mathbf{1}_{\mathcal{V}} \xrightarrow{\sim} \mathbf{1}_{\mathcal{V}}$. First of all, by assumption, the resulting natural transformations $(\mathbb{Q}\mathbf{1}_{\mathcal{V}} \otimes -) \to (\mathbf{1}_{\mathcal{V}} \otimes -)$ and $(- \otimes \mathbb{Q}\mathbf{1}_{\mathcal{V}}) \to (- \otimes \mathbf{1}_{\mathcal{V}})$ in $\mathrm{Fun}(\mathcal{V}, \mathcal{V})$ restrict to natural weak equivalences in $\mathrm{Fun}(\mathcal{V}^c, \mathcal{V})$. As the unit object comes equipped with equivalences

$$(\mathbf{1}_{\mathcal{V}} \otimes -) \simeq \mathrm{id}_{\mathcal{V}} \simeq (- \otimes \mathbf{1}_{\mathcal{V}}),$$

it follows that the restrictions along $\mathcal{V}^c \subset \mathcal{V}$ of these functors all lie in the full subcategory

$$\mathrm{Fun}(\mathcal{V}^c, \mathcal{V}^c)^{\mathcal{R}\mathrm{el}} \subset \mathrm{Fun}(\mathcal{V}^c, \mathcal{V}^c) \subset \mathrm{Fun}(\mathcal{V}^c, \mathcal{V}),$$

where they give rise to a diagram

$$(\mathbb{Q}\mathbf{1}_{\mathcal{V}} \otimes -) \xrightarrow{\sim} (\mathbf{1}_{\mathcal{V}} \otimes -) \simeq \mathrm{id}_{\mathcal{V}^c} \simeq (- \otimes \mathbf{1}_{\mathcal{V}}) \xleftarrow{\sim} (- \otimes \mathbb{Q}\mathbf{1}_{\mathcal{V}})$$

of natural weak equivalences. Applying the canonical functor

$$\mathrm{Fun}(\mathcal{V}^c, \mathcal{V}^c)^{\mathcal{R}\mathrm{el}} \to \mathrm{Fun}(\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!], \mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!])$$

of Remark 2.1.23 then yields a diagram

$$\left(\mathbb{Q}\mathbf{1}_{\mathcal{V}} \overset{\mathbb{L}}{\otimes} -\right) \xrightarrow{\sim} \mathrm{id}_{\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!]} \xleftarrow{\sim} \left(- \overset{\mathbb{L}}{\otimes} \mathbb{Q}\mathbf{1}_{\mathcal{V}}\right)$$

of natural equivalences. Thus, the map $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{\mathbb{Q}\mathbf{1}_{\mathcal{V}}} \mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!]$ is a quasi-unit (in the sense of Definition A.5.4.3.5) for the non-unital monoidal ∞-category $\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!] \in \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}\mathrm{at}_\infty)$. It then follows from Theorem A.5.4.3.8 (and Propositions A.4.1.2.15

and A.5.4.3.2) that there exists a unique refinement $\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!] \in \mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)$ to a monoidal ∞-category.[15]

The assertion is now clear: we have exhibited a canonical monoidal structure on $\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!] \simeq \mathcal{V}[\![\mathbf{W}^{-1}]\!]$ whose underlying monoidal product is precisely the left derived bifunctor of the original monoidal product on $\mathcal{V}$, and the derived bifunctors $\mathbb{R}\underline{\hom}_l(-,-)$ and $\mathbb{R}\underline{\hom}_r(-,-)$, being participants in the derived two-variable adjunction, have no choice but to define left and right internal hom-objects. □

We also have the following variant.

Definition 5.5.5. Let $\mathcal{V} \in \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ be a closed symmetric monoidal ∞-category, and suppose that $\mathcal{V}$ is equipped with a model structure. We say that these data make $\mathcal{V}$ into a ***symmetric monoidal model ∞-category*** if they make the underlying closed monoidal ∞-category $\mathcal{V} \in \mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)$ into a monoidal model ∞-category.

We then have the following corresponding result.

Proposition 5.5.6. *Suppose that $\mathcal{V}$ is a symmetric monoidal model ∞-category. Then the derived two-variable adjunction of its underlying two-variable Quillen adjunction itself underlies a canonical closed symmetric monoidal structure on its localization $\mathcal{V}[\![\mathbf{W}^{-1}]\!]$.*

Proof. In light of Proposition 5.5.4, it only remains to show that the symmetric monoidal structure on $\mathcal{V}$ descends canonically to one on $\mathcal{V}[\![\mathbf{W}^{-1}]\!]$ (extending its monoidal structure). Just as in the proof of that result, the underlying datum $\mathcal{V} \in \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}\mathrm{at}_\infty)$ restricts to give $\mathcal{V}^c \in \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}\mathrm{at}_\infty)$, which admits a natural lift $\mathcal{V}^c \in \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)$, and then the fact that the localization functor is symmetric monoidal yields $\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!] \in \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}\mathrm{at}_\infty)$. The existence of a canonical lift $\mathcal{V}^c[\![(\mathbf{W}^c)^{-1}]\!] \in \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ now follows from Corollary A.5.4.4.7. □

Remark 5.5.7. In the special case that our (resp. symmetric) monoidal model ∞-category $\mathcal{V}$ has that its unit object is cofibrant, then its localization $\mathcal{V}[\![\mathbf{W}^{-1}]\!]$ obtains a canonical (resp. symmetric) monoidal structure by Proposition A.4.1.3.4. However, this result does not alone guarantee a *closed* (resp. symmetric) monoidal structure, as does Proposition 5.5.4 (resp. Proposition 5.5.6).

Remark 5.5.8. Though they presumably exist, we do not pursue any notions of "$\mathcal{O}$-monoidal model ∞-category" for other ∞-operads $\mathcal{O}$ here.

[15] Note that Definition A.5.4.3.5 only requires the *existence* of a quasi-unit; the quasi-unit itself is not part of the data.

Remark 5.5.9. In Definitions 5.5.1 and 5.5.5, one could remove the requirement that there exist a suitable cofibrant replacement of the unit object (or even that there exist a unit object at all); then, Propositions 5.5.4 and 5.5.6 would admit non-unital variants.

5.6 Enriched model ∞-categories

In this final section, we show that the localization of a model ∞-category that is compatibly *enriched and bitensored* over a closed monoidal model ∞-category is itself enriched and bitensored over the localization of the enriching model ∞-category. For a more leisurely discussion of monoidal and symmetric monoidal model categories, we yet again refer the reader to [Hov99, §4.2] (beginning with [Hov99, Definition 4.2.18]).

Definition 5.6.1. Let $\mathcal{V} \in \mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)$ be a monoidal model ∞-category, let $\mathcal{M} \in \mathrm{RMod}_{\mathcal{V}}(\mathcal{C}\mathrm{at}_\infty)$ be a right $\mathcal{V}$-module (with respect to its underlying monoidal ∞-category structure) whose underlying action bifunctor extends to a two-variable adjunction

$$\left(\mathcal{M} \times \mathcal{V} \xrightarrow{-\odot-} \mathcal{M} \ , \ \mathcal{M}^{op} \times \mathcal{M} \xrightarrow{\underline{\hom}_{\mathcal{M}}(-,-)} \mathcal{V} \ , \ \mathcal{V}^{op} \times \mathcal{M} \xrightarrow{-\pitchfork-} \mathcal{M} \right),$$

and suppose that $\mathcal{M}$ is equipped with a model structure. We say that these these data make $\mathcal{M}$ into a ***$\mathcal{V}$-enriched model ∞-category*** (or simply a ***$\mathcal{V}$ model ∞-category***) if they satisfy the following evident ∞-categorical analogs of the usual axioms for an enriched model category.

$\mathrm{EM}_\infty 1$ *(pushout product)* The above two-variable adjunction is a two-variable Quillen adjunction.

$\mathrm{EM}_\infty 2$ *(unit)* There exists a cofibrant replacement $\varnothing_{\mathcal{V}} \xrightarrow{\approx} \mathbb{Q}\mathbf{1}_{\mathcal{V}} \xrightarrow{\approx} \mathbf{1}_{\mathcal{V}}$ such that the functor

$$\mathcal{M} \xrightarrow{-\odot(\mathbb{Q}\mathbf{1}_{\mathcal{V}} \to \mathbf{1}_{\mathcal{V}})} \mathrm{Fun}([1], \mathcal{M})$$

takes cofibrant objects to weak equivalences.

We use the same terminology in the case that $\mathcal{V} \in \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ is in fact a symmetric monoidal model ∞-category.

Definition 5.6.2. As a special case of Definition 5.6.1, we refer to a $s\mathcal{S}_{\mathrm{KQ}}$-enriched model ∞-category as a ***simplicial model ∞-category*** (recall Example 5.5.3).

Remark 5.6.3. If $\mathcal{M}$ is a simplicial model category (i.e. a $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$-enriched model category), then $\mathcal{M}$ can also be considered as a simplicial model ∞-category in which

- the co/tensoring over $s\mathcal{S}$ is obtained by precomposition with $\pi_0^{\mathrm{lw}} : s\mathcal{S} \to s\mathcal{S}\mathrm{et}$, and
- the internal hom is obtained by postcomposition with $\mathrm{disc}^{\mathrm{lw}} : s\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S}$.

Thus, the abuse of terminology is extremely slight.

Example 5.6.4. Given a model ∞-category $\mathcal{M}$, the resolution model ∞-category $s\mathcal{M}_{\mathrm{res}}$ (see Example 1.2.7) is simplicial, in direct analogy with the classical resolution model structure (see [DKS93, 3.1 and 5.3]).

Example 5.6.5. If $\mathcal{C}_{\mathrm{triv}}$ is an ∞-category equipped with the trivial model structure (see Example 1.2.2) and the underlying ∞-category $\mathcal{C}$ is bitensored, then $\mathcal{C}_{\mathrm{triv}}$ can be considered as a simplicial model ∞-category in which

- the co/tensoring over $s\mathcal{S}$ is obtained by precomposition with $|-| : s\mathcal{S} \to \mathcal{S}$, and
- the internal hom is obtained by postcomposition with $\mathrm{const} : \mathcal{S} \hookrightarrow s\mathcal{S}$.

Example 5.6.6. if $\mathcal{M}$ is a simplicial model ∞-category, then the levelwise action $s\mathcal{M} \odot^{\mathrm{lw}} s\mathcal{S} \to s\mathcal{M}$ given by $(x_\bullet \odot Y)_n = x_n \odot Y$ makes $s\mathcal{M}_{\mathrm{Reedy}}$ into a simplicial model ∞-category.

We now show that the structure of an enriched model ∞-category descends to localizations as claimed.

Proposition 5.6.7. *Suppose that $\mathcal{M}$ is a $\mathcal{V}$-enriched model ∞-category. Then the derived two-variable adjunction of its underlying two-variable Quillen adjunction itself underlies a canonical enrichment and bitensoring of $\mathcal{M}[\![\mathbf{W}_{\mathcal{M}}^{-1}]\!]$ over $\mathcal{V}[\![\mathbf{W}_{\mathcal{V}}^{-1}]\!]$.*

Proof. The proof is almost identical to that of Proposition 5.5.4, only now we replace the appeal to Theorem A.5.4.3.8 with an appeal to (the dual of) Proposition A.5.4.3.16. □

Remark 5.6.8. Let $\mathcal{M}$ be a simplicial model ∞-category. As being bitensored over $\mathcal{S}$ is actually a *condition* (rather than additional structure), it follows that the derived bitensoring over $s\mathcal{S}[\![\mathbf{W}_{\mathrm{KQ}}^{-1}]\!] \simeq \mathcal{S}$ of $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$ guaranteed by Proposition 5.6.7 must indeed be a bitensoring in the usual sense.

Chapter 6

Model ∞-categories III: the fundamental theorem

In this chapter, we prove that a model structure on a relative ∞-category $(\mathcal{M}, \mathbf{W})$ gives an efficient and computable way of accessing the hom-spaces $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x, y)$ in the localization. More precisely, we show that when the source $x \in \mathcal{M}$ is *cofibrant* and the target $y \in \mathcal{M}$ is *fibrant*, then this hom-space is a "quotient" of the hom-space $\hom_{\mathcal{M}}(x, y)$ by either of a *left homotopy relation* or a *right homotopy relation*.

6.0 Introduction

6.0.1 Model ∞-categories

A *relative ∞-category* is a pair $(\mathcal{M}, \mathbf{W})$ of an ∞-category $\mathcal{M}$ and a subcategory $\mathbf{W} \subset \mathcal{M}$ containing all the equivalences, called the subcategory of *weak equivalences*. Freely inverting the weak equivalences, we obtain the *localization* of this relative ∞-category, namely the initial functor

$$\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$$

from $\mathcal{M}$ which sends all maps in $\mathbf{W}$ to equivalences. In general, it is extremely difficult to access the localization. The purpose of this chapter is to show that the additional data of a *model structure* on $(\mathcal{M}, \mathbf{W})$ makes it far easier: we prove the following ***fundamental theorem of model ∞-categories***.[1]

[1] For the precise definition a model ∞-category, we refer the reader to §1.1. However, for the present discussion, it suffices to observe that it is simply a direct generalization of the standard definition of a model category.

Theorem (6.1.9)**.** *Suppose that $\mathcal{M}$ is a model ∞-category. Then, for any cofibrant object $x \in \mathcal{M}^c$ and any fibrant object $y \in \mathcal{M}^f$, the induced map*

$$\hom_{\mathcal{M}}(x, y) \to \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x, y)$$

on hom-spaces is a π_0-surjection. Moreover, this becomes an equivalence upon imposing either of a "left homotopy relation" or a "right homotopy relation" on the source (see Definition 6.1.7).

We view this result – and the framework of model ∞-categories more generally – as providing a theory of ***resolutions*** which is native to the ∞-categorical setting. To explain this perspective, let us recall Quillen's classical theory of model categories, in which for instance

- replacing a topological space by a CW complex constitutes a *cofibrant resolution* – that is, a choice of representative which is "good for mapping out of" – of its underlying object of $\mathcal{T}\mathrm{op}[\mathbf{W}_{\text{w.h.e.}}^{-1}]$ (i.e. its underlying weak homotopy type), while
- replacing an R-module by a complex of injectives constitutes a *fibrant resolution* – that is, a choice of representative which is "good for mapping into" – of its underlying object of $\mathrm{Ch}_R[\mathbf{W}_{\text{q.i.}}^{-1}]$.

Thus, a model structure on a relative (1- or ∞-)category $(\mathcal{M}, \mathbf{W})$ provides *simultaneously compatible* choices of objects of $\mathcal{M}$ which are "good for mapping out of" and "good for mapping into" with respect to the corresponding localization $\mathcal{M} \to \mathcal{M}[\mathbf{W}^{-1}]$.

A prototypical example of this phenomenon arises from the interplay of ***left and right derived functors*** (in the classical model-categorical sense), i.e. of left and right adjoint functors of ∞-categories. For instance,

- in a *left localization* adjunction $\mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C}$, we can think of the subcategory $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ as that of the "fibrant" objects, while *every* object is "cofibrant", while dually
- in a *right localization* adjunction $\mathrm{R}\mathcal{C} \rightleftarrows \mathcal{C}$, we can think of the subcategory $\mathrm{R}\mathcal{C} \subset \mathcal{C}$ as that of the "cofibrant" objects, while *every* object is "fibrant".[2]

As a model structure generally has neither all its objects cofibrant nor all its objects fibrant, it can therefore be seen as a *simultaneous generalization* of the notions of left localization and right localization.

[2]See Examples 1.2.12 and 1.2.17 for more details on such model structures.

Remark 6.0.1. Indeed, this observation encompasses one of the most important examples of a model ∞-category, which was in fact the original motivation for their theory.

Suppose we are given a presentable ∞-category $\mathcal{C}$ along with a set $\mathcal{G}$ of generators which we assume (without real loss of generality) to be closed under finite coproducts. Then, the corresponding ***nonabelian derived ∞-category*** is the ∞-category $\mathcal{P}_\Sigma(\mathcal{G}) = \mathrm{Fun}_\Sigma(\mathcal{G}^{op}, \mathcal{S})$ of those presheaves on $\mathcal{G}$ that take finite coproducts in $\mathcal{G}$ to finite products in $\mathcal{S}$. This admits a canonical projection

$$\begin{array}{ccc} & \overset{\hom^{\mathrm{lw}}_{\mathcal{C}}(=,-)}{\longleftarrow} & s\mathcal{C} \\ s(\mathcal{P}_\Sigma(\mathcal{G})) & & \\ & \underset{|-|}{\longrightarrow} & \mathcal{P}_\Sigma(\mathcal{G}), \end{array}$$

the composition of the (restricted) levelwise Yoneda embedding (a *right* adjoint) followed by (pointwise) geometric realization (a *left* adjoint): given a simplicial object $Y_\bullet \in s\mathcal{C}$ and a generator $S^\beta \in \mathcal{G}$, this composite is given by

$$\begin{array}{ccc} & \longmapsto & Y_\bullet \\ \hom^{\mathrm{lw}}_{\mathcal{C}}(S^\beta, Y_\bullet) & & \\ & \longmapsto & \left|\hom^{\mathrm{lw}}_{\mathcal{C}}(S^\beta, Y_\bullet)\right|, \end{array}$$

where we use the abbreviation "lw" to denote "levelwise". In fact, this composite is a *free* localization (but neither a left nor a right localization): denoting by $\mathbf{W}_{\mathrm{res}} \subset s\mathcal{C}$ the subcategory spanned by those maps which it inverts, it induces an equivalence

$$s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \xrightarrow{\sim} \mathcal{P}_\Sigma(\mathcal{G}).$$

In future work, we will provide a ***resolution model structure*** on the ∞-category $s\mathcal{C}$ in order to organize computations in the nonabelian derived ∞-category $\mathcal{P}_\Sigma(\mathcal{G})$. (The resolution model structure on the ∞-category $s\mathcal{C}$, which might also be called an "E^2 model structure", is based on work of Dwyer–Kan–Stover and Bousfield (see [DKS93] and [Bou03], resp.).)

Remark 6.0.2. In turn, the original motivation for the resolution model structure was provided by *Goerss–Hopkins obstruction theory* (see §1.0.3). However, the nonabelian derived ∞-category also features prominently for instance in Barwick's universal characterization of *algebraic K-theory* (see [Bara]), as well as in his theory of *spectral Mackey functors* (which provide an ∞-categorical model for genuine equivariant spectra) (see [Barc]).

6.0.2 Outline

We now provide a more detailed outline of the contents of this chapter.

- In §6.1, we give a precise statement of the ***fundamental theorem of model ∞-categories*** (6.1.9). This involves the notions of a *cylinder object* $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ and a *path object* $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ for our chosen source and target objects $x, y \in \mathcal{M}$, which generalize their corresponding model 1-categorical namesakes and play analogous roles thereto.

- In §6.2, we prove that the spaces of *left homotopy classes of maps* (defined in terms of a cylinder object $\mathrm{cyl}^\bullet(x)$) and of *right homotopy classes of maps* (defined in terms of a path object $\mathrm{path}_\bullet(y)$) are both equivalent to a more symmetric bisimplicial colimit (defined in terms of both $\mathrm{cyl}^\bullet(x)$ and $\mathrm{path}_\bullet(y)$).

- In §6.3, we prove that it suffices to consider the case that our cylinder and path objects are *special*.

- In §6.4, we digress to introduce *model diagrams*, which corepresent diagrams in a model ∞-category $\mathcal{M}$ of a specified type (i.e. whose constituent morphisms can be required to be contained in (one or more of) the various defining subcategories $\mathbf{W}, \mathbf{C}, \mathbf{F} \subset \mathcal{M}$).

- In §6.5, we prove that when our cylinder and path objects are both special, the bisimplicial colimit of §6.2 is equivalent to the groupoid completion of a certain ∞-category $\underline{\tilde{\mathbf{3}}}(x, y)$ of *special three-arrow zigzags* from x to y.

- In §6.6, we prove that the inclusion $\underline{\tilde{\mathbf{3}}}(x, y) \hookrightarrow \underline{\mathbf{3}}(x, y)$ into the ∞-category of (all) three-arrow zigzags from x to y induces an equivalence on groupoid completions.

- In §6.7, we prove that the inclusion $\underline{\mathbf{3}}(x, y) \hookrightarrow \underline{\mathbf{7}}(x, y)$ into a certain ∞-category of *seven-arrow zigzags* from x to y induces an equivalences on groupoid completions.

- In §6.8, in order to access the hom-spaces in the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$, we prove that the *Rezk nerve* $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M}, \mathbf{W})$ (see §2.3) of (the underlying relative ∞-category of) a model ∞-category is a Segal space. (By the local universal property of the Rezk nerve (Theorem 2.3.8), this Segal space necessarily presents the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$.)

- In §6.9, we prove that the groupoid completion $\underline{\mathbf{7}}(x,y)^{\mathrm{gpd}}$ of the ∞-category of seven-arrow zigzags from x to y is equivalent to the hom-space $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$.

- In §6.10, using the fundamental theorem of model ∞-categories (6.1.9), we prove that the Rezk nerve $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M},\mathbf{W})$ is in fact a *complete* Segal space.

6.0.3 Acknowledgments

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6.1 The fundamental theorem of model ∞-categories

Given an ∞-category $\mathcal{M}$ equipped with a subcategory $\mathbf{W} \subset \mathcal{M}$, the primary purpose of extending these data to a model structure is to obtain an efficient and computable

presentation of the hom-spaces in the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$. In this section, we work towards a precise statement of this presentation, which comprises the *fundamental theorem of model ∞-categories* (6.1.9).

A key feature of a model structure is that it allows one to say what it means for two maps in $\mathcal{M}$ to be "homotopic", that is, to become equivalent (in the ∞-categorical sense) upon application of the localization functor $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$. Classically, to pass to the homotopy category of a relative 1-category (i.e. to its 1-categorical localization), one simply *identifies* maps that are homotopic. In keeping with the core philosophy of higher category theory, we will instead want to *remember* these homotopies, and then of course we'll also want to keep track of the higher homotopies between them.

In the theory of model 1-categories, to abstractify the notion of a "homotopy" between maps from an object x to an object y, one introduces the dual notions of *cylinder objects* and *path objects*. In the ∞-categorical setting, at first glance it might seem that it will suffice to take cylinder and path objects to be as they were before (namely, as certain factorizations of the fold and diagonal maps, respectively): we'll recover a space of maps from a cylinder object for x to y, and we might hope that these spaces will keep track of higher homotopies for us. However, this is not necessarily the case: it might be that a particular homotopy *between* homotopies only exists after passing to a cylinder object on the cylinders themselves. Of course, it is not possible to guarantee that this process will terminate at some finite stage, and so we must allow for an infinite sequence of such maneuvers.

Although the geometric intuition here no longer corresponds to mere cylinders and paths, we nevertheless recycle the terminology.

Definition 6.1.1. Let $\mathcal{M}$ be a model ∞-category. A ***cylinder object*** for an object $x \in \mathcal{M}$ is a cosimplicial object $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ equipped with an equivalence $x \simeq \mathrm{cyl}^0(x)$, such that

- the codegeneracy maps $\mathrm{cyl}^n(x) \xrightarrow{\sigma^i} \mathrm{cyl}^{n-1}(x)$ are all in $\mathbf{W}$, and
- the latching maps $\mathrm{L}_n \mathrm{cyl}^\bullet(x) \to \mathrm{cyl}^n(x)$ are in $\mathbf{C}$ for all $n \geq 1$.

The cylinder object is called ***special*** if the codegeneracy maps are all also in $\mathbf{F}$ and the matching maps $\mathrm{cyl}^n(x) \to \mathrm{M}_n \mathrm{cyl}^\bullet(x)$ are in $\mathbf{W} \cap \mathbf{F}$ for all $n \geq 1$. We will use the notation ${}_\sigma\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ to denote a special cylinder object for $x \in \mathcal{M}$.

Dually, a ***path object*** for an object $y \in \mathcal{M}$ is a simplicial object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ equipped with an equivalence $y \simeq \mathrm{path}_0(y)$, such that

- the degeneracy maps $\mathrm{path}_n(y) \xrightarrow{\sigma_i} \mathrm{path}_{n+1}(y)$ are all in $\mathbf{W}$, and

- the matching maps $\mathrm{path}_n(y) \to \mathrm{M}_n\,\mathrm{path}_\bullet(y)$ are in $\mathbf{F}$ for all $n \geq 1$.

The path object is called ***special*** if the degeneracy maps are all also in $\mathbf{C}$ and the latching maps $\mathrm{L}_n\,\mathrm{path}_\bullet(y) \to \mathrm{path}_n(y)$ are in $\mathbf{W} \cap \mathbf{C}$ for all $n \geq 1$. We will use the notation ${}_\sigma\mathrm{path}_\bullet(y) \in s\mathcal{M}$ to denote a special path object for $y \in \mathcal{M}$.

Remark 6.1.2. Restricting a cylinder object $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ to the subcategory $\mathbf{\Delta}_{\leq 1} \subset \mathbf{\Delta}$ and employing the identification $x \simeq \mathrm{cyl}^0(x)$, we recover the classical notion of a cylinder object, i.e. a factorization

$$x \sqcup x \rightarrowtail \mathrm{cyl}^1(x) \xrightarrow{\approx} x$$

of the fold map; the specialness condition then restricts to the single requirement that the weak equivalence $\mathrm{cyl}^1(x) \xrightarrow{\approx} x$ also be a fibration. In particular, if $\mathrm{ho}(\mathcal{M})$ is a model category – recall from Example 1.2.11 that this will be the case as long as $\mathrm{ho}(\mathcal{M})$ satisfies limit axiom $\mathrm{M}_\infty 1$ (i.e. is finitely bicomplete), e.g. if $\mathcal{M}$ is itself a 1-category –, then a cylinder object $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ for $x \in \mathcal{M}$ gives rise to a cylinder object for $x \in \mathrm{ho}(\mathcal{M})$ in the classical sense. Of course, dual observations apply to path objects.

Remark 6.1.3. One might think of a cylinder object as a "cofibrant $\mathbf{W}$-cohypercover", and dually of a path object as a "fibrant $\mathbf{W}$-hypercover". Indeed, if $x \in \mathcal{M}^c$ then a cylinder object $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ defines a cofibrant replacement

$$\varnothing_{c\mathcal{M}} \rightarrowtail \mathrm{cyl}^\bullet(x) \xrightarrow{\approx} \mathrm{const}(x)$$

in $c\mathcal{M}_{\mathrm{Reedy}}$, and dually if $y \in \mathcal{M}^f$ then a path object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ defines a fibrant replacement

$$\mathrm{const}(y) \xrightarrow{\approx} \mathrm{path}_\bullet(y) \twoheadrightarrow \mathrm{pt}_{s\mathcal{M}}$$

in $s\mathcal{M}_{\mathrm{Reedy}}$.[3] Note, however, that under Definition 6.1.1, not every such co/fibrant replacement defines a cylinder/path object, simply because of our requirements that the 0^{th} objects remain unchanged. In turn, we have made this requirement so that Remark 6.1.2 is true, i.e. so that our definition recovers the classical one.

By contrast, in [DK80b, 4.3], Dwyer–Kan introduce the notions of "co/simplicial resolutions" of objects in a model category (with the "special" condition appearing in [DK80b, Remark 6.8]). These are functionally equivalent to our cylinder and path objects; the biggest difference is just that the 0^{th} object of one of their resolutions

[3] Since the object $[0] \in \mathbf{\Delta}$ is terminal we obtain an adjunction $(-)^0 : c\mathcal{M} \rightleftarrows \mathcal{M} : \mathrm{const}$, via which the equivalence $\mathrm{cyl}^0(x) \xrightarrow{\sim} x$ in $\mathcal{M}$ determines a map $\mathrm{cyl}^\bullet(x) \to \mathrm{const}(x)$ in $c\mathcal{M}$; the map $\mathrm{const}(y) \to \mathrm{path}_\bullet(y)$ arises dually.

is required to be a co/fibrant replacement of the original object. Of course, we'll ultimately only care about cylinder objects for cofibrant objects and path objects for fibrant objects, and on the other hand they eventually reduce their proofs to the case of co/simplicial resolutions in which this replacement map is the identity (so that in particular the original object is co/fibrant). Thus, in the end the difference is almost entirely aesthetic.

Remark 6.1.4. Since Definition 6.1.1 is somewhat involved, here we collect the intuition and/or justification behind each of the pieces of the definition, focusing on (special) path objects.

- A path object is supposed to be a sort of simplicial resolution. Thus, the first demand we should place on this simplicial object is that it be "homotopically constant", i.e. its structure maps should be weak equivalences. This is accomplished by the requirement that the degeneracy maps lie in $\mathbf{W} \subset \mathcal{M}$.

- On the other hand, a path object should also be "good for mapping into" (as discussed in Remark 6.1.3). This fibrancy-like property is encoded by the requirement that the matching maps lie in $\mathbf{F} \subset \mathcal{M}$. (By the dual of Lemma 6.2.2 (whose proof uses (the dual of) this condition), when $y \in \mathcal{M}$ is fibrant then so are all the objects $\mathrm{path}_n(y) \in \mathcal{M}$, for any path object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$.)

- The first condition for the specialness of $\mathrm{path}_\bullet(y)$ – that the degeneracy maps are (acyclic) cofibrations – guarantees that for each $n \geq 0$, the unique structure map $y \simeq \mathrm{path}_0(y) \to \mathrm{path}_n(y)$ is also a cofibration. This is necessary for Lemma 6.5.2 to even make sense, and also appears in the proof of the *factorization lemma* (6.4.24).

- The second condition for the specialness of $\mathrm{path}_\bullet(y)$ – that the latching maps be acyclic cofibrations – guarantees that special path objects are "weakly initial" among all path objects (in a sense made precise in Lemma 6.3.2(2)).

Of course, these notions are only useful because of the following existence result.

Proposition 6.1.5. *Let $\mathcal{M}$ be a model ∞-category.*

(1) Every object of $\mathcal{M}$ admits a special cylinder object.

(2) Every object of $\mathcal{M}$ admits a special path object.

Proof. We only prove part (2); part (1) will then follow by duality. So, suppose we are given any object $y \in \mathcal{M}$. First, set $\mathrm{path}_0(y) = y$. Then, we inductively define

$\mathrm{path}_n(y)$ by taking a factorization

$$\begin{array}{ccc} \mathrm{L}_n \,\mathrm{path}_\bullet(y) & \longrightarrow & \mathrm{M}_n \,\mathrm{path}_\bullet(y) \\ & \overset{\approx}{\searrow} \quad \nearrow & \\ & \mathrm{path}_n(y) & \end{array}$$

of the canonical map using factorization axiom $\mathrm{M}_\infty 5$.[4] As observed in Remark 5.1.15, this procedure suffices to define a simplicial object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$.

Now, by construction, above degree 0 the latching maps are all in $\mathbf{W} \cap \mathbf{C}$ while the matching maps are all in $\mathbf{F}$. Thus, it only remains to check that the degeneracy maps are all in $\mathbf{W} \cap \mathbf{C}$. For this, note that for any $n \geq 0$, every degeneracy map $\mathrm{path}_n(y) \xrightarrow{\sigma_i} \mathrm{path}_{n+1}(y)$ factors canonically as a composite

$$\mathrm{path}_n(y) \to \mathrm{L}_{n+1} \,\mathrm{path}_\bullet(y) \xrightarrow{\approx} \mathrm{path}_{n+1}(y)$$

in $\mathcal{M}$, where the first map is the inclusion into the colimit at the object

$$([n]^\circ \xrightarrow{\sigma_i} [n+1]^\circ) \in \partial \left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ} \right).$$

So, it suffices to show that this first map is also in $\mathbf{W} \cap \mathbf{C}$. This follows from applying Lemma 6.1.6 to the data of

- the model ∞-category $\mathcal{M}$,
- the Reedy category $\partial \left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ} \right)$,
- the maximal object $([n]^\circ \xrightarrow{\sigma_i} [n+1]^\circ) \in \partial \left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ} \right)$, and
- the composite functor

$$\partial \left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ} \right) \hookrightarrow \overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ} \to \overrightarrow{\mathbf{\Delta}^{op}} \hookrightarrow \mathbf{\Delta}^{op} \xrightarrow{\mathrm{path}_\bullet(y)} \mathcal{M}.$$

Indeed, $\partial \left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ} \right)$ is a Reedy category equal to its own direct subcategory by Lemma 5.1.29 (1) (a), and it is clearly a poset. Moreover, our composite functor

[4] At $n = 1$, the map $\mathrm{L}_1 \,\mathrm{path}_\bullet(y) \to \mathrm{M}_1 \,\mathrm{path}_\bullet(y)$ is just the diagonal map $y \to y \times y$.

satisfies the hypothesis of Lemma 6.1.6 by Lemma 5.1.29 (1) (b); in fact, all the latching maps are acyclic cofibrations except for possibly the one at the initial object

$$([0]^\circ \to [n+1]^\circ) \in \partial\left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ}\right).$$

Therefore, the degeneracy map $\mathrm{path}_n(y) \xrightarrow{\sigma_i} \mathrm{path}_{n+1}(y)$ is indeed an acyclic cofibration, and hence the object $\mathrm{path}_n(y) \in s\mathcal{M}$ defines a special path object for an arbitrary object $y \in \mathcal{M}$. □

The proof of Proposition 6.1.5 relies on the following result.

Lemma 6.1.6. *Let $\mathcal{M}$ be a model ∞-category, let $\mathcal{C}$ be a Reedy poset which is equal to its own direct subcategory, and let $m \in \mathcal{C}$ be a maximal element. Suppose that $\mathcal{C} \xrightarrow{F} \mathcal{M}$ is a functor such that for any $c \in \mathcal{C}$ which is incomparable to $m \in \mathcal{C}$ (i.e. such that $\hom_{\mathcal{C}}(c, m) = \varnothing_{\mathcal{S}\mathrm{et}}$), the latching map $\mathrm{L}_c F \to F(c)$ lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$. Then, the induced map $F(m) \to \mathrm{colim}_{\mathcal{C}}(F)$ also lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$.*

Proof. We begin by observing that for any object $c \in \mathcal{C}$, the forgetful map $\mathcal{C}_{/c} \to \mathcal{C}$ is actually the inclusion of a full subposet. Now, writing $\mathcal{C}' = (\mathcal{C} \backslash \{m\}) \subset \mathcal{C}$, it is easy to see that we have a pushout square

$$\begin{array}{ccc} \partial(\mathcal{C}_{/m}) & \longrightarrow & \mathcal{C}_{/m} \\ \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \mathcal{C} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ of inclusions of full subposets. By Proposition T.4.4.2.2, this induces a pushout square

$$\begin{array}{ccc} \mathrm{L}_m F & \longrightarrow & F(m) \\ \downarrow & & \downarrow \\ \mathrm{colim}_{\mathcal{C}'}(F) & \longrightarrow & \mathrm{colim}_{\mathcal{C}}(F) \end{array}$$

in $\mathcal{M}$ (where the colimits all exist by limit axiom $\mathrm{M}_\infty 1$, and where we simply write F again for its restriction to any subposet of $\mathcal{C}$).[5] Thus, it suffices to show that the map $\mathrm{L}_m F \to \mathrm{colim}_{\mathcal{C}'}(F)$ lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$, since this subcategory is closed under pushouts.

[5] In the statement of Proposition T.4.4.2.2, note that the requirement that one of the maps be a monomorphism (i.e. a cofibration in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$) guarantees that this pushout is indeed a homotopy pushout in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ (by the left properness of $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, or alternatively by the Reedy trick).

For this, let us choose an ordering

$$\mathcal{C}'\backslash\partial(\mathcal{C}_{/m}) = \{c_1, \dots, c_k\}$$

such that for every $1 \leq i \leq k$ the object c_i is minimal in the full subposet $\{c_i, \dots, c_k\} \subset \mathcal{C}$.[6] Let us write

$$\mathcal{C}_i = (\partial(\mathcal{C}_{/m}) \cup \{c_1, \dots, c_i\}) \subset \mathcal{C}'$$

for the full subposet, setting $\mathcal{C}_0 = \partial(\mathcal{C}_{/m})$ for notational convenience, so that we have the chain of inclusions

$$\partial(\mathcal{C}_{/m}) = \mathcal{C}_0 \subset \cdots \subset \mathcal{C}_k = \mathcal{C}'.$$

Our requirement on the ordering of the objects c_i guarantees that we have

$$\partial(\mathcal{C}_{/c_i}) \subset \mathcal{C}_{i-1},$$

and from here it is not hard to see that in fact we have a pushout square

$$\begin{array}{ccc} \partial(\mathcal{C}_{/c_i}) & \longrightarrow & \mathcal{C}_{i-1} \\ \downarrow & & \downarrow \\ \mathcal{C}_{/c_i} & \longrightarrow & \mathcal{C}_i \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ for all $1 \leq i \leq k$, from which by again applying Proposition T.4.4.2.2 we obtain a pushout square

$$\begin{array}{ccc} \mathrm{L}_{c_i} F & \longrightarrow & \mathrm{colim}_{\mathcal{C}_{i-1}}(F) \\ \downarrow & & \downarrow \\ F(c_i) & \longrightarrow & \mathrm{colim}_{\mathcal{C}_i}(F) \end{array}$$

in $\mathcal{M}$. But since $\hom_{\mathcal{C}}(c_i, m) = \varnothing_{\mathcal{S}\mathrm{et}}$ by assumption, our hypotheses imply that the map $\mathrm{L}_{c_i} F \to F(c_i)$ lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$; since this subcategory is closed under pushouts, it follows that it contains the map $\mathrm{colim}_{\mathcal{C}_{i-1}}(F) \to \mathrm{colim}_{\mathcal{C}_i}(F)$ as well. Thus, we have obtained the map $\mathrm{L}_m F \to \mathrm{colim}_{\mathcal{C}'}(F)$ as a composite

$$\mathrm{L}_m F = \mathrm{colim}_{\partial(\mathcal{C}_{/m})}(F) = \mathrm{colim}_{\mathcal{C}_0}(F) \xrightarrow{\approx} \cdots \xrightarrow{\approx} \mathrm{colim}_{\mathcal{C}_k}(F) = \mathrm{colim}_{\mathcal{C}'}(F)$$

of acyclic cofibrations in $\mathcal{M}$, so it is itself an acyclic cofibration. This proves the claim. □

[6] If the Reedy structure on $\mathcal{C}$ is induced by a degree function $\mathrm{N}(\mathcal{C})_0 \xrightarrow{\deg} \mathbb{N}$ (which must be possible by its finiteness), then this can be accomplished simply by requiring that $\deg(c_i) \leq \deg(c_{i+1})$ for all $1 \leq i < k$.

Now that we have shown that (special) cylinder and path objects always exist, we come to the following key definitions. These should be expected: taking the *quotient* by a relation in a 1-topos corresponds to taking the *geometric realization* of a simplicial object in an ∞-topos. (Among these, *equivalence* relations then correspond to ∞-*groupoid* objects (see Definition T.6.1.2.7).)

Definition 6.1.7. Let $\mathcal{M}$ be a model ∞, and let $x, y \in \mathcal{M}$. We define the space of ***left homotopy classes of maps*** from x to y with respect to a given cylinder object $\mathrm{cyl}^\bullet(x)$ for x to be

$$\hom^{l}_{\mathcal{M}}(x,y) = \left|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), y)\right|.$$

Dually, we define the space of ***right homotopy classes of maps*** from x to y with respect to a given path object $\mathrm{path}_\bullet(y)$ for y to be

$$\hom^{r}_{\mathcal{M}}(x,y) = \left|\hom^{\mathrm{lw}}_{\mathcal{M}}(x, \mathrm{path}_\bullet(y))\right|.$$

A priori these spaces depend on the choices of cylinder or path objects, but we nevertheless suppress them from the notation.

Remark 6.1.8. Note that $\hom^{\mathrm{lw}}_{\mathcal{M}}(x, \mathrm{path}_\bullet(y))$ is not itself an ∞-groupoid object in $\mathcal{S}$. To ask for this would be too strict: it would not allow for the "homotopies between homotopies" that we sought at the beginning of this section. (Correspondingly, by Yoneda's lemma this would also imply that $\mathrm{path}_\bullet(y)$ is itself an ∞-groupoid object in $\mathcal{M}$, which is clearly a far stronger condition than the "fibrant $\mathbf{W}$-hypercover" heuristic of Remark 6.1.3 would dictate.)

We can now state the ***fundamental theorem of model ∞-categories***, which says that under the expected co/fibrancy hypotheses, the spaces of left and right homotopy classes of maps both compute the hom-space in the localization.

Theorem 6.1.9. *Let $\mathcal{M}$ be a model ∞-category, suppose that $x \in \mathcal{M}^c$ is cofibrant and $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ is any cylinder object for x, and suppose that $y \in \mathcal{M}^f$ is fibrant and $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ is any path object for y. Then there is a diagram of equivalences*

$$\begin{array}{ccccc} \hom^{l}_{\mathcal{M}}(x,y) & \xrightarrow{\sim} & \left\|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y))\right\| & \xleftarrow{\sim} & \hom^{r}_{\mathcal{M}}(x,y) \\ & & \wr\downarrow & & \\ & & \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y) & & \end{array}$$

in $\mathcal{S}$.

Proof. The horizontal equivalences are proved as Proposition 6.2.1(3) and its dual. By Proposition 6.3.4, it suffices to assume that both $\mathrm{cyl}^\bullet(x)$ and $\mathrm{path}_\bullet(y)$ are special. The vertical equivalence is then obtained as the composite of the equivalences

$$\left\|\hom^{\mathrm{lw}}_{\mathcal{M}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y))\right\| \simeq \tilde{\mathbf{3}}(x,y)^{\mathrm{gpd}} \simeq \underline{\mathbf{3}}(x,y)^{\mathrm{gpd}} \simeq \underline{\mathbf{7}}(x,y)^{\mathrm{gpd}} \simeq \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$$

(where the as-yet-undefined objects of which will be explained in Notation 6.4.10 and Definition 6.4.15) which are respectively proved as Propositions 6.5.1 (and 6.3.4), 6.6.1, 6.7.1, and 6.9.1. □

Remark 6.1.10. The proof of the fundamental theorem of model ∞-categories (6.1.9) roughly follows that of [DK80b, Proposition 4.4] (and specifically the fix given in [Man99, §7] for [DK80b, 7.2(iii)]). Speaking ahistorically, the main difference is that we have replaced the ultimate appeal to the hammock localization as providing a model for the hom-space $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$ with an appeal to the (∞-categorical) Rezk nerve $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M},\mathbf{W})$, which we will prove (as Proposition 6.8.1) likewise provides a model for this hom-space (by the local universal property of the Rezk nerve (Theorem 2.3.8)).

An easy consequence of the fundamental theorem of model ∞-categories (6.1.9) is its "homotopy" version.

Corollary 6.1.11. *Let $\mathcal{M}$ be a model ∞-category, suppose that $x \in \mathcal{M}^c$ is cofibrant and $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ is any cylinder object for x, and suppose that $y \in \mathcal{M}^f$ is fibrant and $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ is any path object for y. Then there is a diagram of isomorphisms*

$$\left(\frac{[x,y]_{\mathcal{M}}}{[\mathrm{cyl}^1(x),y]_{\mathcal{M}}}\right) \xrightarrow{\sim} [x,y]_{\mathcal{M}[\mathbf{W}^{-1}]} \xleftarrow{\sim} \left(\frac{[x,y]_{\mathcal{M}}}{[x,\mathrm{path}_1(y)]_{\mathcal{M}}}\right)$$

in $\mathcal{S}\mathrm{et}$.

Proof. Observe that we have a commutative square

$$\begin{array}{ccc} s\mathcal{S} & \xrightarrow{\pi_0^{\mathrm{lw}}} & s\mathcal{S}\mathrm{et} \\ {\scriptstyle \mathrm{colim}^{\mathcal{S}}_{\mathbf{\Delta}^{op}}(-)}\downarrow & & \downarrow{\scriptstyle \mathrm{colim}^{\mathcal{S}\mathrm{et}}_{\mathbf{\Delta}^{op}}(-)} \\ \mathcal{S} & \xrightarrow[\pi_0]{} & \mathcal{S}\mathrm{et} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$, since all four functors are left adjoints and the resulting composite right adjoints coincide. The claim now follows immediately from Theorem 6.1.9. □

Remark 6.1.12. In the particular case that $\mathcal{M}$ is a model 1-category, we obtain equivalences $\mathrm{ho}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$ and $\mathrm{ho}(\mathcal{M}[\![\mathbf{W}^{-1}]\!]) \xrightarrow{\sim} \mathcal{M}[\mathbf{W}^{-1}]$. Hence, Corollary 6.1.11 specializes to recover the classical fundamental theorem of model categories (see e.g. [Hir03, Theorems 7.4.9 and 8.3.9]).

Remark 6.1.13. In contrast with Remark 6.1.8, the proof of [Hir03, Theorem 7.4.9] carries over without essential change to show that in the situation of Corollary 6.1.11, the diagram

$$\begin{array}{ccc} [\mathrm{cyl}^1(x), y]_{\mathcal{M}} & & [x, \mathrm{path}_1(y)]_{\mathcal{M}} \\ & \searrow \quad \swarrow & \\ & [x, y]_{\mathcal{M}} & \end{array}$$

does define a pair of equal equivalence relations (in $\mathcal{S}\mathrm{et}$).

6.2 The equivalence $\hom^{l}_{\mathcal{M}}(x,y) \simeq \left\|\hom^{\mathbf{lw}}_{\mathcal{M}}(\mathrm{cyl}^{\bullet}(x), \mathrm{path}_{\bullet}(y))\right\|$

Without first setting up any additional scaffolding, we can immediately prove the horizontal equivalences of Theorem 6.1.9. The following result is an analog of [DK80b, Proposition 6.2, Corollary 6.4, and Corollary 6.5].

Proposition 6.2.1. *Let $\mathcal{M}$ be a model ∞-category, suppose that $x \in \mathcal{M}^c$ is cofibrant, and let $\mathrm{cyl}^{\bullet}(x) \in c\mathcal{M}$ be any cylinder object for x.*

(1) The functor

$$\mathcal{M} \xrightarrow{\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^{\bullet}(x), -)} s\mathcal{S}$$

sends $(\mathbf{W} \cap \mathbf{F}) \subset \mathcal{M}$ into $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}} \subset s\mathcal{S}$.

(2) The same functor sends $(\mathcal{M}^f \cap \mathbf{W}) \subset \mathcal{M}$ into $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$.

(3) If $y \in \mathcal{M}^f$ is fibrant, then for any path object $\mathrm{path}_{\bullet}(y) \in s\mathcal{M}$ for y, the canonical map $\mathrm{const}(y) \to \mathrm{path}_{\bullet}(y)$ in $s\mathcal{M}$ induces an equivalence

$$\left|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^{\bullet}(x), y)\right| \xrightarrow{\sim} \left\|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^{\bullet}(x), \mathrm{path}_{\bullet}(y))\right\|.$$

Proof. To prove part (1), we use the criterion of Proposition 1.7.2 (that $s\mathcal{S}_{\mathrm{KQ}}$ has a set of generating cofibrations given by the boundary inclusions $I_{\mathrm{KQ}} = \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}$). First, note that to say that x is cofibrant is to say that the 0^{th} latching

map $\varnothing_{\mathcal{M}} \simeq \mathrm{L}_0\,\mathrm{cyl}^\bullet(x) \to \mathrm{cyl}^0(x) \simeq x$ of $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ is also a cofibration. Then, for any $n \geq 0$, suppose we are given an acyclic fibration $y \xrightarrow{\approx} z$ in $\mathcal{M}$ inducing the right map in any commutative square

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), y) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), z) \end{array}$$

in $s\mathcal{S}$. This commutative square is equivalent data to that of a commutative square

$$\begin{array}{ccc} \mathrm{L}_n\,\mathrm{cyl}^\bullet(x) & \longrightarrow & y \\ \downarrow & & \downarrow \wr \\ \mathrm{cyl}^n(x) & \longrightarrow & z, \end{array}$$

in $\mathcal{M}$, and moveover a lift in either one determines a lift in the other. But the latter admits a lift by lifting axiom $\mathrm{M}_\infty 4$. Hence, the induced map $\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), y) \to \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), z)$ is indeed in $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}}$.

Next, part (2) follows immediately from part (1) and the dual of Kenny Brown's lemma (5.3.5).

To prove part (3), note that all structure maps in any path object are weak equivalences, and note also that when y is fibrant, then any path object $\mathrm{path}_\bullet(y)$ consists of fibrant objects by the dual of Lemma 6.2.2. Hence, using

- Fubini's theorem for colimits,
- part (2), and
- the fact that simplicial objects whose structure maps are equivalences must be constant,

we obtain the string of equivalences

$$\begin{aligned} \left\| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y)) \right\| &= \mathrm{colim}_{([m]^\circ,[n]^\circ) \in \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}} \hom_{\mathcal{M}}(\mathrm{cyl}^m(x), \mathrm{path}_n(y)) \\ &\simeq \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} \left(\mathrm{colim}_{[m]^\circ \in \mathbf{\Delta}^{op}} \hom_{\mathcal{M}}(\mathrm{cyl}^m(x), \mathrm{path}_n(y)) \right) \\ &= \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} \left| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_n(y)) \right| \\ &\simeq \left| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_0(y)) \right| \\ &\simeq \left| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), y) \right|, \end{aligned}$$

proving the claim. □

We needed the following auxiliary result in the proof of Proposition 6.2.1.

Lemma 6.2.2. *If $x \in \mathcal{M}^c$ is cofibrant, then for any cylinder object $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ for x, for every $n \geq 0$ the object $\mathrm{cyl}^n(x) \in \mathcal{M}$ is cofibrant.*

Proof. Since $\mathrm{cyl}^0(x) \simeq x$ by definition, the claim holds at $n = 0$ by assumption. For $n \geq 1$, by definition we have a cofibration $\mathrm{L}_n \mathrm{cyl}^\bullet(x) \rightarrowtail \mathrm{cyl}^n(x)$, so it suffices to show that the object $\mathrm{L}_n \mathrm{cyl}^\bullet(x) \in \mathcal{M}$ is cofibrant. We prove this by induction: at $n = 0$, we have $\mathrm{L}_0 \mathrm{cyl}^\bullet(x) = \mathrm{cyl}^0(x) \sqcup \mathrm{cyl}^0(x) \simeq x \sqcup x$, which is cofibrant.

Now, recall that by definition,

$$\mathrm{L}_n \mathrm{cyl}^\bullet(x) = \mathrm{colim}_{\partial(\overrightarrow{\mathbf{\Delta}}_{/[n]})} \mathrm{cyl}^\bullet(x),$$

i.e. the latching object is given by the colimit of the composite

$$\partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right) \hookrightarrow \overrightarrow{\mathbf{\Delta}}_{/[n]} \to \overrightarrow{\mathbf{\Delta}} \hookrightarrow \mathbf{\Delta} \xrightarrow{\mathrm{cyl}^\bullet(x)} \mathcal{M}.$$

Now, by Lemma 5.1.29(1)(a), the latching category $\partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right)$ admits a Reedy category structure with fibrant constants, so that we obtain a Quillen adjunction

$$\mathrm{colim} : \mathrm{Fun}\left(\partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right), \mathcal{M}\right)_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const}$$

(since $\mathcal{M}$ is finitely cocomplete by limit axiom $\mathrm{M}_\infty 1$). Thus, it suffices to check that the above composite defines a cofibrant object of $\mathrm{Fun}\left(\partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right), \mathcal{M}\right)_{\mathrm{Reedy}}$. For this, given an object $([m] \hookrightarrow [n]) \in \partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right)$, by Lemma 5.1.29(1)(b), its latching category is given by

$$\partial\left(\overrightarrow{\partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right)}_{/([m]\hookrightarrow[n])}\right) \cong \partial\left(\overrightarrow{\mathbf{\Delta}}_{/[m]}\right).$$

Hence, the latching map of the above composite at this object simply reduces to the cofibration

$$\mathrm{L}_m \mathrm{cyl}^\bullet(x) \rightarrowtail \mathrm{cyl}^m(x).$$

Therefore, the above composite does indeed define a cofibrant object of $\mathrm{Fun}\left(\partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right), \mathcal{M}\right)_{\mathrm{Reedy}}$, which proves the claim. □

6.3 Reduction to the special case

In order to proceed with the string of equivalences in the proof of the fundamental theorem of model ∞-categories (6.1.9), we will need to be able to make the assumption that our cylinder and path objects are *special*. In this section, we therefore reduce to the special case.

Notation 6.3.1. Let $\mathcal{M}$ be a model ∞-category. For any $x \in \mathcal{M}$, we write

$$\{\mathrm{cyl}^\bullet(x)\} \subset \left(c\mathcal{M} \underset{(-)^0,\mathcal{M},x}{\times} \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \right)$$

for the full subcategory on the cylinder objects for x, and we write

$$\{\mathrm{path}_\bullet(x)\} \subset \left(s\mathcal{M} \underset{(-)_0,\mathcal{M},x}{\times} \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \right)$$

for the full subcategory on the path objects for x.

We now have the following analog of [DK80b, Propositions 6.9 and 6.10].

Lemma 6.3.2. *Suppose that $x \in \mathcal{M}$.*

(1) Every special cylinder object ${}_\sigma\mathrm{cyl}^\bullet(x) \in \{\mathrm{cyl}^\bullet(x)\}$ is weakly terminal: any $\mathrm{cyl}^\bullet(x) \in \{\mathrm{cyl}^\bullet(x)\}$ admits a map

$$\mathrm{cyl}^\bullet(x) \to {}_\sigma\mathrm{cyl}^\bullet(x)$$

in $\{\mathrm{cyl}^\bullet(x)\}$.

(2) Every special path object ${}_\sigma\mathrm{path}_\bullet(x) \in \{\mathrm{path}_\bullet(x)\}$ is weakly initial: any $\mathrm{path}_\bullet(x) \in \{\mathrm{path}_\bullet(x)\}$ admits a map

$${}_\sigma\mathrm{path}_\bullet(x) \to \mathrm{path}_\bullet(x)$$

in $\{\mathrm{path}_\bullet(x)\}$.

Proof. We only prove the first of two dual statements. We will construct the map by induction. The given equivalences

$$\mathrm{cyl}^0(x) \simeq x \simeq {}_\sigma\mathrm{cyl}^0(x)$$

imply that there is a unique way to begin in degree 0. Then, assuming the map has been constructed up through degree $(n-1)$, Definition 6.1.1 and lifting axiom $\mathrm{M}_\infty 4$ guarantee the existence of a lift in the commutative rectangle

$$\begin{array}{ccccc} \mathrm{L}_n\,\mathrm{cyl}^\bullet(x) & \longrightarrow & \mathrm{L}_n\,{}_\sigma\mathrm{cyl}^\bullet(x) & \longrightarrow & {}_\sigma\mathrm{cyl}^n(x) \\ \downarrow & \nearrow & & & \downarrow\wr \\ \mathrm{cyl}^n(x) & \longrightarrow & {}_\sigma\mathrm{cyl}^n(x) & \longrightarrow & \mathrm{M}_n\,{}_\sigma\mathrm{cyl}^\bullet(x) \end{array}$$

in $\mathcal{M}$, which provides an extension of the map up through degree n. □

Lemma 6.3.3. *Let $\mathcal{M}$ be a model ∞-category, let $x \in \mathcal{M}^c$ be cofibrant, let $y \in \mathcal{M}^f$ be fibrant, let $\mathrm{cyl}_1^\bullet(x) \to \mathrm{cyl}_2^\bullet(x)$ be a map in $\{\mathrm{cyl}^\bullet(x)\}$, and suppose that $\mathrm{path}_\bullet(y) \in \{\mathrm{path}_\bullet(y)\}$. Then the induced maps*

$$\left|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_2^\bullet(x), y)\right| \to \left|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_1^\bullet(x), y)\right|$$

and

$$\left\|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_2^\bullet(x), \mathrm{path}_\bullet(y))\right\| \to \left\|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_1^\bullet(x), \mathrm{path}_\bullet(y))\right\|$$

are equivalences in $\mathcal{S}$.

Proof. By Proposition 6.2.1(3) and its dual, these data induce a commutative diagram

$$\begin{array}{ccc} \left|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_2^\bullet(x), y)\right| & \longrightarrow & \left|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_1^\bullet(x), y)\right| \\ \downarrow\wr & & \downarrow\wr \\ \left\|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_2^\bullet(x), \mathrm{path}_\bullet(y))\right\| & \longrightarrow & \left\|\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}_1^\bullet(x), \mathrm{path}_\bullet(y))\right\| \\ & \nwarrow^{\sim} \quad \nearrow_{\sim} & \\ & \left|\hom^{\mathrm{lw}}_{\mathcal{M}}(x, \mathrm{path}_\bullet(y))\right| & \end{array}$$

of equivalences in $\mathcal{S}$. □

Proposition 6.3.4. *Let $\mathcal{M}$ be a model ∞-category, let $x, y \in \mathcal{M}$, let $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ be a cylinder object for x, and let $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ be a path object for y. Then there exist*

- *a map $\mathrm{cyl}^\bullet(x) \to {}_\sigma\mathrm{cyl}^\bullet(x)$ to a special cylinder object for x, and*
- *a map $\mathrm{path}_\bullet(y) \to {}_\sigma\mathrm{path}_\bullet(y)$ to a special path object for y,*

such that the induced square

$$\begin{array}{ccc} \hom^{\mathrm{lw}}_{\mathcal{M}}({}_{\sigma}\mathrm{cyl}^{\bullet}(x), {}_{\sigma}\mathrm{path}_{\bullet}(y)) & \longrightarrow & \hom^{\mathrm{lw}}_{\mathcal{M}}({}_{\sigma}\mathrm{cyl}^{\bullet}(x), \mathrm{path}_{\bullet}(y)) \\ \downarrow & & \downarrow \\ \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^{\bullet}(x), {}_{\sigma}\mathrm{path}_{\bullet}(y)) & \longrightarrow & \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^{\bullet}(x), \mathrm{path}_{\bullet}(y)) \end{array}$$

in $ss\mathcal{S}$ becomes an equivalence upon applying the colimit functor

$$ss\mathcal{S} \xrightarrow{\|-\|} \mathcal{S}.$$

Proof. The maps are obtained from Lemma 6.3.2; the claim then follows from Lemma 6.3.3. □

6.4 Model diagrams and left homotopies

In the remainder of the proof of the fundamental theorem of model ∞-categories (6.1.9), it will be convenient to have a framework for corepresenting diagrams of a specified type in our model ∞-category $\mathcal{M}$. This leads to the notion of a *model ∞-diagram*, which we introduce and study in §6.4.1. Then, in §6.4.2, we specialize this setup to describe the data that thusly corepresents a "left homotopy" in the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$. (In fact, in order to be completely concrete and explicit we will further specialize to deal only with *model diagrams* (as opposed to model ∞-diagrams), since in the end this is all that we will need.)

6.4.1 Model diagrams

We will be interested in ∞-categories of diagrams of a specified shape inside of a model ∞-category. These are corepresented, in the following sense.

Definition 6.4.1. A ***model ∞-diagram*** is an ∞-category $\mathcal{D}$ equipped with three wide subcategories $\mathbf{W}, \mathbf{C}, \mathbf{F} \subset \mathcal{D}$. These assemble into the evident ∞-category, which we denote by $\mathcal{M}\mathrm{odel}_{\infty}$. Of course, a model ∞-category can be considered as a model ∞-diagram. A ***model diagram*** is a model ∞-diagram whose underlying ∞-category is a 1-category. These assemble into a full subcategory $\mathcal{M}\mathrm{odel} \subset \mathcal{M}\mathrm{odel}_{\infty}$.

Remark 6.4.2. We introduced *model diagrams* in [MG16, Definition 3.1], where we required that the subcategory of weak equivalences satisfy the two-out-of-three property. As this requirement is superfluous for our purposes, we have omitted it from

Definition 6.4.1. (However, the wideness requirement is necessary: it guarantees that a map of model diagrams can take any map to an identity map, which in turn jibes with the requirement that the three defining subcategories of a model ∞-category be wide.)

Remark 6.4.3. A relative ∞-category $(\mathcal{R}, \mathbf{W})$ can be considered as a model ∞-diagram by taking $\mathbf{C} = \mathbf{F} = \mathcal{R}^{\simeq}$. In this way, we will identify $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \subset \mathcal{M}\mathrm{odel}_\infty$ and $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \subset \mathcal{M}\mathrm{odel}$ as full subcategories.[7]

Notation 6.4.4. In order to disambiguate our notation associated with various model ∞-diagrams, we will sometimes decorate them for clarity: for instance, we may write $(\mathcal{D}_1, \mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1)$ and $(\mathcal{D}_2, \mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2)$ to denote two arbitrary model ∞-diagrams. (This is consistent with both Notations 1.1.2 and 2.1.3.)

Remark 6.4.5. Among the axioms for a model ∞-category, all but limit axiom $\mathrm{M}_\infty 1$ (so two-out-of-three axiom $\mathrm{M}_\infty 2$, retract axiom $\mathrm{M}_\infty 3$, lifting axiom $\mathrm{M}_\infty 4$, and factorization axiom $\mathrm{M}_\infty 5$) can be encoded by requiring that the underlying model ∞-diagram has the extension property with respect to certain maps of model diagrams.

Since we will be working with a model ∞-category with chosen source and target objects of interest, we also introduce the following variant.

Definition 6.4.6. A ***doubly-pointed model ∞-diagram*** is a model ∞-diagram $\mathcal{D}$ equipped with a map $\mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \sqcup \mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \to \mathcal{D}$. The two inclusions $\mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \hookrightarrow \mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \sqcup \mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty}$ select objects $s, t \in \mathcal{D}$, which we call the ***source*** and ***target***; we will sometimes subscript these to remove ambiguity, e.g. as $s_{\mathcal{D}}$ and $t_{\mathcal{D}}$. These assemble into the evident ∞-category

$$(\mathcal{M}\mathrm{odel}_\infty)_{**} = (\mathcal{M}\mathrm{odel}_\infty)_{(\mathrm{pt}_{\mathcal{M}\mathrm{odel}_{**}} \sqcup \mathrm{pt}_{\mathcal{M}\mathrm{odel}_{**}})/}.$$

Of course, there is a forgetful functor $(\mathcal{M}\mathrm{odel}_\infty)_{**} \to \mathcal{M}\mathrm{odel}_\infty$. We will often implicitly consider a model ∞-diagram equipped with two chosen objects as a doubly-pointed model ∞-diagram. We write $\mathcal{M}\mathrm{odel}_{**} \subset (\mathcal{M}\mathrm{odel}_\infty)_{**}$ for the full subcategory of ***doubly-pointed model diagrams***, i.e. of those doubly-pointed model ∞-diagrams whose underlying ∞-category is a 1-category.

Remark 6.4.7. Similarly to Remark 6.4.3, we will consider $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{**} \subset (\mathcal{M}\mathrm{odel}_\infty)_{**}$ and $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{**} \subset \mathcal{M}\mathrm{odel}_{**}$ as full subcategories.

[7] This inclusion exhibits $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ as a right localization of $\mathcal{M}\mathrm{odel}_\infty$. In fact, $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ is also a *left* localization of $\mathcal{M}\mathrm{odel}_\infty$ via the inclusion which sets both $\mathbf{C}$ and $\mathbf{F}$ to be the entire underlying ∞-category, but this latter inclusion will not play any role here.

Notation 6.4.8. In order to simultaneously refer to the situations of unpointed and doubly-pointed model ∞-diagrams, we will use the notation $(\mathcal{M}\mathrm{odel}_\infty)_{(**)}$ (and similarly for other related notations). When we use this notation, we will mean for the entire statement to be interpreted either in the unpointed context or the doubly-pointed context. (This is consistent with Notation 4.2.3.)

It will be useful to expand on Definition 4.2.5 (in view of Remark 6.4.7) in the following way.

Definition 6.4.9. We define a ***model word*** to be a (possibly empty) word $\underline{\mathbf{m}}$ in any of the symbols $\mathbf{A}$, $\mathbf{W}$, $\mathbf{C}$, $\mathbf{F}$, $(\mathbf{W} \cap \mathbf{C})$, $(\mathbf{W} \cap \mathbf{F})$ or any of their inverses. Of course, these naturally define doubly-pointed model diagrams; we continue to employ the convention set in Definition 4.2.5 that we read our model words *forwards*, so that for instance the model word $\underline{\mathbf{m}} = [\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}]$ defines the doubly-pointed model diagram

$$s \rightarrowtail \bullet \xleftarrow{\approx} \bullet \longrightarrow t.$$

We denote this object by $\underline{\mathbf{m}} \in \mathcal{M}\mathrm{odel}_{**}$. Of course, via Remark 6.4.7, we can consider any relative word as a model word.

Notation 6.4.10. Since they will appear repeatedly, we make the abbreviation

$$\underline{\tilde{\mathbf{3}}} = [(\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}; (\mathbf{W} \cap \mathbf{C})^{-1}]$$

for the model word

$$s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} t$$

(which is a variant of Notation 4.3.2), and we make the abbreviation

$$\underline{\mathbf{7}} = [\mathbf{W}; \mathbf{W}^{-1}; \mathbf{W}; \mathbf{A}; \mathbf{W}; \mathbf{W}^{-1}; \mathbf{W}]$$

for the model word (in fact, relative word)

$$s \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} t.$$

We now make rigorous "the ∞-category of (either unpointed or doubly-pointed) $\mathcal{D}$-shaped diagrams in $\mathcal{M}$ (and either natural transformations or natural weak equivalences between them)".

Notation 6.4.11. Recall from Notation 2.1.6 that $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ is a cartesian closed symmetric monoidal ∞-category, with internal hom-object given by

$$\left(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}}\right) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$$

for $(\mathcal{R}_1, \mathbf{W}_1), (\mathcal{R}_2, \mathbf{W}_2) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. It is not hard to see that $\mathcal{M}\mathrm{odel}_\infty$ is enriched and tensored over $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \times)$. Namely, for any

$$(\mathcal{D}_1, \mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1), (\mathcal{D}_2, \mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2) \in \mathcal{M}\mathrm{odel}_\infty,$$

we define

$$\left(\mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathcal{M}\mathrm{odel}}, \mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}}\right) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$$

by setting

$$\mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathcal{M}\mathrm{odel}} \subset \mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)$$

to be the full subcategory on those functors which send the subcategories $\mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1 \subset \mathcal{D}_1$ into $\mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2 \subset \mathcal{D}_2$ respectively, and setting

$$\mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathcal{M}\mathrm{odel}}$$

to be the (generally non-full) subcategory on the natural weak equivalences; moreover, the tensoring is simply the cartesian product in $\mathcal{M}\mathrm{odel}_\infty$ (composed with the inclusion $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \subset \mathcal{M}\mathrm{odel}_\infty$ of Remark 6.4.3).

Notation 6.4.12. Similarly to Notations 6.4.11 and 4.2.2, $(\mathcal{M}\mathrm{odel}_\infty)_{**}$ is enriched and tensored over $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \times)$. As for the enrichment, for any

$$(\mathcal{D}_1, \mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1), (\mathcal{D}_2, \mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2) \in (\mathcal{M}\mathrm{odel}_\infty)_{**},$$

in analogy with Notation 4.2.2 we define the object

$$\left(\mathrm{Fun}_{**}(\mathcal{D}_1, \mathcal{D}_2)^{\mathcal{M}\mathrm{odel}}, \mathrm{Fun}_{**}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}}\right) = \lim \left(\begin{array}{ccc} & & \left(\mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathcal{M}\mathrm{odel}}, \mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}}\right) \\ & & \downarrow{\scriptstyle (\mathrm{ev}_{s_1}, \mathrm{ev}_{t_1})} \\ \mathrm{pt}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} & \xrightarrow[(s_2,t_2)]{} & (\mathcal{D}_2, \mathbf{W}_2) \times (\mathcal{D}_2, \mathbf{W}_2) \end{array} \right)$$

of $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ (where we write $s_1, t_1 \in \mathcal{D}_1$ and $s_2, t_2 \in \mathcal{D}_2$ to distinguish between the source and target objects). Then, the tensoring is obtained by taking $(\mathcal{R}, \mathbf{W}_\mathcal{R}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ and $(\mathcal{D}, \mathbf{W}_\mathcal{D}, \mathbf{C}_\mathcal{D}, \mathbf{F}_\mathcal{D}) \in (\mathcal{M}\mathrm{odel}_\infty)_{**}$ to the pushout

$$\mathrm{colim} \left(\begin{array}{ccc} \mathcal{R} \times \{s, t\} & \longrightarrow & \mathcal{R} \times \mathcal{D} \\ \downarrow & & \\ \mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \times \{s, t\} & & \end{array} \right)$$

in $\mathcal{M}\mathrm{odel}_\infty$, with its double-pointing given by the natural map from $\mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \sqcup \mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \simeq \mathrm{pt}_{\mathcal{M}\mathrm{odel}_\infty} \times \{s, t\}$.

Remark 6.4.13. While we are using the notation $\mathrm{Fun}(-,-)^{\mathbf{W}}$ both in the context of relative ∞-categories and model ∞-diagrams, due to the identification $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \subset \mathcal{M}\mathrm{odel}_\infty$ of Remark 6.4.3 this is actually *not* an abuse of notation. The notation $\mathrm{Fun}_{**}(-,-)^{\mathbf{W}}$ is similarly unambiguous.

Notation 6.4.14. Similarly to Notation 4.2.4, we will write

$$(\mathcal{M}\mathrm{odel}_\infty)_{(**)} \times \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{-\odot-} (\mathcal{M}\mathrm{odel}_\infty)_{(**)}$$

to denote either tensoring of Notation 6.4.11 or of Notation 6.4.12 (using the convention of Notation 6.4.8).

Corresponding to Definition 6.4.9, we expand on Definition 4.2.9 as follows.

Definition 6.4.15. Given a model ∞-diagram $\mathcal{M} \in \mathcal{M}\mathrm{odel}_\infty$ (e.g. a model ∞-category) equipped with two chosen objects $x, y \in \mathcal{M}$, and given a model word $\underline{\mathbf{m}} \in \mathcal{M}\mathrm{odel}_{**}$, we define the ∞-category of ***zigzags*** in $\mathcal{M}$ from x to y of type $\underline{\mathbf{m}}$ to be

$$\underline{\mathbf{m}}_{\mathcal{M}}(x,y) = \mathrm{Fun}_{**}(\underline{\mathbf{m}}, \mathcal{M})^{\mathbf{W}}.$$

If the model ∞-diagram $\mathcal{M}$ is clear from context, we will simply write $\underline{\mathbf{m}}(x,y)$.

Definition 6.4.16. For any model ∞-diagram $\mathcal{M}$ and any objects $x, y \in \mathcal{M}$, we will refer to

$$\tilde{\underline{\mathbf{3}}}(x,y) = \mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, \mathcal{M})^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$$

as the ∞-category of ***special three-arrow zigzags*** in $\mathcal{M}$ from x to y (which is a variant of Definition 4.3.3), and we will refer to

$$\underline{\mathbf{7}}(x,y) = \mathrm{Fun}_{**}(\underline{\mathbf{7}}, \mathcal{M})^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$$

as the ∞-category of ***seven-arrow zigzags*** in $\mathcal{M}$ from x to y.

Now, the reason we are interested in the tensorings of Notation 6.4.14 is the following construction.

Notation 6.4.17. We define a functor

$$(\mathcal{M}\mathrm{odel}_\infty)_{(**)} \xrightarrow{c^\bullet_{(**)}} c(\mathcal{M}\mathrm{odel}_\infty)_{(**)}$$

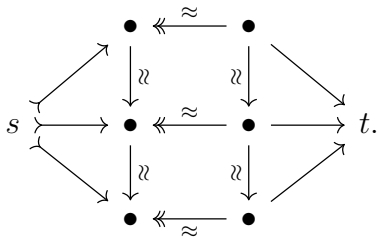
by setting

$$c^\bullet_{(**)}\mathcal{D} = \mathcal{D} \odot [\bullet]_{\mathbf{W}}$$

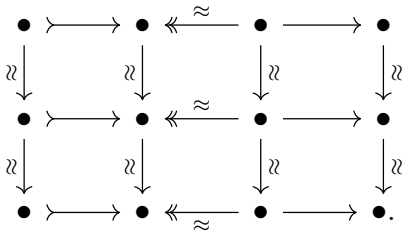
for any $\mathcal{D} \in (\mathcal{M}\mathrm{odel}_\infty)_{(**)}$ (where $[\bullet]_{\mathbf{W}}$ denotes the composite $\mathbf{\Delta} \hookrightarrow \mathcal{C}\mathrm{at} \xrightarrow{\max} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$). Of course, this restricts to a functor

$$\mathcal{M}\mathrm{odel}_{(**)} \xrightarrow{c^\bullet_{(**)}} c\mathcal{M}\mathrm{odel}_{(**)}.$$

Example 6.4.18. If we consider $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \in \mathcal{M}\mathrm{odel}_{**}$, then $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \odot [2]_{\mathbf{W}} \in \mathcal{M}\mathrm{odel}_{**}$ is given by



On the other hand, if we consider $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \in \mathcal{M}\mathrm{odel}$, then $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \odot [2]_{\mathbf{W}} \in \mathcal{M}\mathrm{odel}$ is given by



In turn, Notation 6.4.17 is itself useful for the following reason.

Lemma 6.4.19. *For any $\mathcal{D}, \mathcal{M} \in (\mathcal{M}\mathrm{odel}_\infty)_{(**)}$, we have an equivalence*

$$\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(c^\bullet_{(**)}\mathcal{D}, \mathcal{M}) \simeq \mathrm{N}_\infty \left(\mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}}\right)$$

in $s\mathcal{S}$ which is natural in both variables.

Proof. For any $n \geq 0$ we have a composite equivalence

$$\begin{aligned}
\mathrm{N}_\infty \left(\mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}}\right)_n &= \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}}\right) \\
&\simeq \hom_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty}\left([n]_{\mathbf{W}}, \left(\mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathcal{M}\mathrm{odel}}, \mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}}\right)\right) \\
&\simeq \hom_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D} \odot [n]_{\mathbf{W}}, \mathcal{M}) \\
&= \hom_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(c^n_{(**)}\mathcal{D}, \mathcal{M})
\end{aligned}$$

which clearly commutes with the simplicial structure maps on both sides. □

We now introduce slightly more elaborate versions of the concepts we have been exploring – an ∞-categorical version of [MG16, Variant 3.3] – which will be used in the proofs of Proposition 6.6.1, Proposition 6.7.1, and Lemma 6.8.2.

Definition 6.4.20. A ***decorated model ∞-diagram*** is a model ∞-diagram with some subdiagrams decorated as colimit or limit diagrams. For instance, if we define $\mathcal{D}$ to be the "walking pullback square", then for any other model ∞-diagram $\mathcal{M}$, we let $\hom^{*}_{\mathcal{M}\mathrm{odel}_\infty}(\mathcal{D},\mathcal{M}) \subset \hom_{\mathcal{M}\mathrm{odel}_\infty}(\mathcal{D},\mathcal{M})$, $\mathrm{Fun}^{*}(\mathcal{D},\mathcal{M})^{\mathcal{M}\mathrm{odel}} \subset \mathrm{Fun}(\mathcal{D},\mathcal{M})^{\mathcal{M}\mathrm{odel}}$, and $\mathrm{Fun}^{*}(\mathcal{D},\mathcal{M})^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{D},\mathcal{M})^{\mathbf{W}}$ denote the subobjects spanned by those morphisms $\mathcal{D} \to \mathcal{M}$ of model ∞-diagrams which select a pullback square in $\mathcal{M}$. Of course, we define a ***doubly-pointed decorated model ∞-diagram*** similarly.

In fact, we will only use this variant in the doubly-pointed case, and then only for pushout and pullback squares. So, in the interest of easing our Ti*k*Zographical burden, we will simply superscript these model diagrams with "p.o." and/or "p.b." as appropriate; the question of which square we are referring to is fully disambiguated by the fact that our pushouts will only be of acyclic cofibrations while our pullbacks will only be of acyclic fibrations.

Note that the constructions $\hom^{*}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D},\mathcal{M}) \in \mathcal{S}$ and $\mathrm{Fun}^{*}_{(**)}(\mathcal{D},\mathcal{M})^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$ are not generally functorial in the target $\mathcal{M}$. On the other hand, they are functorial for *some* maps in the source $\mathcal{D}$. We will refer to such maps as ***decoration-respecting***. These define an ∞-category $(\mathcal{M}\mathrm{odel}_\infty)^{*}_{(**)}$. (Note the distinction between $\hom_{(\mathcal{M}\mathrm{odel}_\infty)^{*}_{(**)}}(-,-)$ and $\hom^{*}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(-,-)$.) We consider $(\mathcal{M}\mathrm{odel}_\infty)_{(**)} \subset (\mathcal{M}\mathrm{odel}_\infty)^{*}_{(**)}$ simply by considering undecorated model ∞-diagrams as being trivially decorated. We will not need a general theory for understanding which maps of decorated model diagrams are decoration-respecting; rather, it will suffice to observe once and for all that given a square which is decorated as a pushout or pullback square, it is decoration-respecting to either

- take it to another similarly decorated square, or
- collapse it onto a single edge (since a commutative square in which two parallel edges are equivalences is both a pushout and a pullback).

Note that if the source of a map of decorated model ∞-diagrams is actually undecorated, then the map is automatically decoration-respecting; in other words, we must only check that maps in which the *source* is decorated are decoration-respecting.

Remark 6.4.21. Of course, adding in Definition 6.4.20 allows us to also demand finite bicompleteness of a model ∞-diagram via lifting conditions, and hence all of the axioms for a model ∞-diagram to be a model ∞-category can now be encoded in this language (recall Remark 6.4.5).

We will need the following analog of Lemma 4.3.5 for model ∞-diagrams.

Lemma 6.4.22. *Given a pair of maps $\mathcal{D}_1 \rightrightarrows \mathcal{D}_2$ in $(\mathcal{M}\mathrm{odel}_\infty)^{*}_{(**)}$, a morphism between them in $\mathrm{Fun}^{*}_{(**)}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}}$ induces, for any $\mathcal{M} \in (\mathcal{M}\mathrm{odel}_\infty)_{(**)}$, a natural transformation between the two induced functors*

$$\mathrm{Fun}^{*}_{(**)}(\mathcal{D}_2, \mathcal{M})^{\mathbf{W}} \rightrightarrows \mathrm{Fun}^{*}_{(**)}(\mathcal{D}_1, \mathcal{M})^{\mathbf{W}}.$$

Proof. It is not hard to see that the proof of Lemma 4.3.5 carries over without essential change (this time using the enrichment of $(\mathcal{M}\mathrm{odel}_\infty)_{(**)}$ over $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$). □

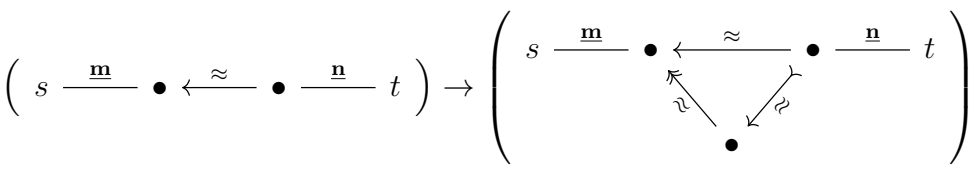
In order to state the final result of this subsection, we need to introduce a bit of notation.

Notation 6.4.23. For any objects $x, y \in \mathcal{M}$, we denote

- by
$$\mathbf{W}_{x\wr} \subset \mathbf{W}_{x/}$$
the full subcategory on those objects $(x \xrightarrow{\approx} z) \in \mathbf{W}_{x/}$ whose structure map is a cofibration,
- by
$$\mathbf{W}_{\downarrow y} \subset \mathbf{W}_{/y}$$
the full subcategory on those objects $(z \xrightarrow{\approx} y) \in \mathbf{W}_{/y}$ whose structure map is a fibration, and
- by
$$\mathbf{W}_{x\wr\downarrow y} = \mathbf{W}_{x\wr} \times_{\mathbf{W}} \mathbf{W}_{\downarrow y} \subset \mathbf{W}_{x//y}$$
the full subcategory on those objects $(x \xrightarrow{\approx} z \xrightarrow{\approx} y) \in \mathbf{W}_{x//y}$ whose structure maps are respectively a cofibration and fibration (as indicated).

We now give an extremely useful result, an analog of [DK80b, 8.1], which will appear in the proofs of Proposition 6.6.1, Proposition 6.7.1, and Lemma 6.8.2. We refer to it as the ***factorization lemma***.

Lemma 6.4.24. *Let $\mathcal{M}$ be a model ∞-category, and let $x, y \in \mathcal{M}$. For any model words $\underline{\mathbf{m}}$ and $\underline{\mathbf{n}}$, applying $\mathrm{Fun}_{**}(-, \mathcal{M})^{\mathbf{W}}$ to the evident inclusion*



in $\mathcal{M}\mathrm{odel}_{**}$ induces a map in $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$.

Proof. We first observe that the target of this inclusion in $\mathcal{M}\mathrm{odel}_{**}$ is isomorphic to the model word

$$[\underline{\mathbf{m}}; (\mathbf{W} \cap \mathbf{F})^{-1}; (\mathbf{W} \cap \mathbf{C})^{-1}; \underline{\mathbf{n}}],$$

it is just drawn so that the "evident inclusion" is truly evident. So, the induced map can be expressed as

$$[\underline{\mathbf{m}}; (\mathbf{W} \cap \mathbf{F})^{-1}; (\mathbf{W} \cap \mathbf{C})^{-1}; \underline{\mathbf{n}}](x, y) \to [\underline{\mathbf{m}}; \mathbf{W}^{-1}; \underline{\mathbf{n}}](x, y).$$

To abbreviate notation, we will write this map in $\mathcal{C}\mathrm{at}_\infty$ simply as $\mathcal{C}_1 \to \mathcal{C}_2$.

Now, showing that the induced map $\mathcal{C}_1^{\mathrm{gpd}} \to \mathcal{C}_2^{\mathrm{gpd}}$ is an equivalence in $\mathcal{S}$ is equivalent to showing that the induced map $(\mathcal{C}_1^{op})^{\mathrm{gpd}} \to (\mathcal{C}_2^{op})^{\mathrm{gpd}}$ is an equivalence in $\mathcal{S}$, and for this by Proposition 3.4.8 it suffices to show that the functor $\mathcal{C}_1^{op} \to \mathcal{C}_2^{op}$ is final. According to the characterization of Theorem A (3.4.10), this is equivalent to showing that for any object

$$f = \left(x \xrightarrow{\underline{\mathbf{m}}} x_1 \xleftarrow{\approx} y_1 \xrightarrow{\underline{\mathbf{n}}} y \right) \in \mathcal{C}_2,$$

the groupoid completion of the comma ∞-category

$$(\mathcal{C}_1)^{op} \underset{(\mathcal{C}_2)^{op}}{\times} ((\mathcal{C}_2)^{op})_{f^\circ/} \simeq \left(\mathcal{C}_1 \underset{\mathcal{C}_2}{\times} (\mathcal{C}_2)_{/f} \right)^{op}$$

is contractible, which is in turn equivalent to showing that the groupoid completion of the comma ∞-category

$$\mathcal{C}_3 = \mathcal{C}_1 \underset{\mathcal{C}_2}{\times} (\mathcal{C}_2)_{/f}$$

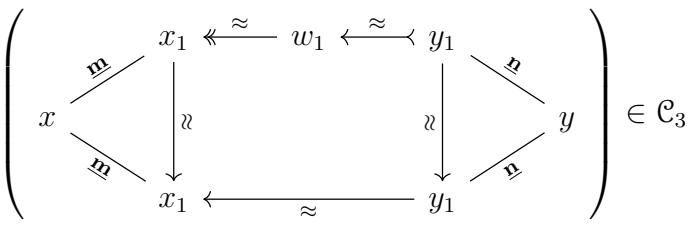
is contractible.

For this, let us first choose a factorization $y_1 \overset{\approx}{\rightarrowtail} z_1 \overset{\approx}{\twoheadrightarrow} x_1$ in $\mathcal{M}$ using factorization axiom $\mathrm{M}_\infty 5$; we can consider this as defining an object $Z_1 = (y_1 \overset{\approx}{\rightarrowtail} z_1 \overset{\approx}{\twoheadrightarrow} x_1) \in \mathcal{M}_{y_1 // x_1}$. Then, working in the model ∞-category $\mathcal{M}_{y_1 // x_1}$ (see Example 1.2.3), we apply Proposition 6.1.5(2) to obtain a special path object $\mathrm{path}_\bullet(Z_1) \in s(\mathcal{M}_{y_1 // x_1})$. Note that every constituent object $\mathrm{path}_n(Z_1) \in \mathcal{M}_{y_1 // x_1}$ is in fact bifibrant: it is cofibrant since specialness implies that the unique structure map $Z_1 \simeq \mathrm{path}_0(Z_1) \to \mathrm{path}_n(Z_1)$ (a composite of degeneracy maps) is an acyclic cofibration and Z_1 itself is cofibrant, and it is fibrant by the dual of Lemma 6.2.2 since Z_1 itself is fibrant. Moreover, since $\mathbf{W}$ has the two-out-of-three property, it follows that in fact $\mathrm{path}_\bullet(Z_1) \in s(\mathbf{W}_{y_1 \downarrow \downarrow x_1})$.

Now, observe that there is a natural functor

$$\mathbf{W}_{y_1 \downarrow \downarrow x_1} \to \mathcal{C}_3$$

which takes an object $(y_1 \xrightarrow{\approx} w_1 \xrightarrow{\approx} x_1) \in \mathbf{W}_{y_1 \downarrow\downarrow x_1}$ to the object



(in which diagram the bottom zigzag is the chosen object $f \in \mathcal{C}_2$ and the top zigzag (an object of $\mathcal{C}_1$) is obtained by simply splicing the zigzag $x_1 \xleftarrow{\approx} w_1 \xleftarrow{\approx} y_1$ into it, and all vertical weak equivalences (including those not pictured) are identity maps). Thus, we obtain a composite

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{path}_\bullet(Z_1)} \mathbf{W}_{y_1 \downarrow\downarrow x_1} \to \mathcal{C}_3,$$

which we will again denote simply by $\mathrm{path}_\bullet(Z_1) \in s(\mathcal{C}_3)$. Since $(\mathbf{\Delta}^{op})^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$ (as $\mathbf{\Delta}^{op}$ is sifted), again referring to Proposition 3.4.8 we see that it suffices to show that this functor is final. Then, again referring to Theorem A (3.4.10), we see that this is equivalent to showing that for any object

$$g = \left(\begin{array}{ccccc} & x_2 & \xleftarrow{\approx} & z_2 & \xleftarrow{\approx} y_2 \\ x & \downarrow & & \downarrow & \downarrow \\ & x_1 & \xleftarrow[\approx]{} & y_1 & \end{array} \right) \in \mathcal{C}_3$$

(in which diagram the bottom zigzag is again the chosen object $f \in \mathcal{C}_2$ but now the top zigzag is an arbitrary object of $\mathcal{C}_1$), the groupoid completion of the comma ∞-category

$$\mathcal{C}_4 = \mathbf{\Delta}^{op} \underset{\mathcal{C}_3}{\times} (\mathcal{C}_3)_{g/}$$

is contractible.

For this, let us define a simplicial space $Y \in s\mathcal{S}$ by setting

$$Y_\bullet = \hom^{\mathrm{lw}}_{\mathcal{C}_3}(g, \mathrm{path}_\bullet(Z_1)).$$

On the one hand, considering $Y \in s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$, we have an equivalence

$$\mathrm{srep}(Y) \simeq \mathrm{N}_\infty(\mathcal{C}_4)$$

in $s\mathcal{S}$: for any $n \geq 0$ we have an equivalence

$$\begin{aligned} \mathrm{srep}(Y)_n &\simeq \coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} Y_{\alpha(0)} \\ &= \coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \hom_{\mathcal{C}_3}(g, \mathrm{path}_{\alpha(0)}(Z_1)) \\ &\simeq \mathrm{N}_\infty(\mathcal{C}_4)_n, \end{aligned}$$

and it is not hard to see that these respect the structure maps of the two simplicial spaces. But on the other hand, unwinding the definitions we obtain an identification

$$Y_\bullet \simeq \lim \left(\begin{array}{ccc} & & \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(z_2, \mathrm{path}_\bullet(Z_1)) \\ & & \big\downarrow \\ \mathrm{pt}_{s\mathcal{S}} & \longrightarrow & \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(y_2, \mathrm{path}_\bullet(Z_1)) \end{array} \right),$$

in which pullback

- we implicitly consider $\mathrm{path}_\bullet(Z_1) \in s(\mathbf{W}_{/x_1})$ via the evident forgetful functor $\mathbf{W}_{y_1 \downarrow\downarrow x_1} \to \mathbf{W}_{/x_1}$,
- the vertical map is given by levelwise precomposition with $y_2 \xrightarrow{\approx} z_2$, and
- the horizontal map is given by the composite

  $$\mathrm{pt}_{s\mathcal{S}} \to \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(z_1, \mathrm{path}_\bullet(Z_1)) \to \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(y_1, \mathrm{path}_\bullet(Z_1)) \to \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(y_2, \mathrm{path}_\bullet(Z_1))$$

  of the canonical point of $\hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(z_1, \mathrm{path}_\bullet(Z_1))$ followed by the maps induced by precomposition with the composite $y_2 \xrightarrow{\approx} y_1 \xrightarrow{\approx} z_1$.

Considering $\mathcal{M}_{/x_1}$ as a model ∞-category (again see Example 1.2.3), the simplicial object $\mathrm{path}_\bullet(Z_1) \in s(\mathcal{M}_{/x_1})$ defines a path object for the fibrant object $z_1 \in (\mathcal{M}_{/x_1})^f$. Thus, by the dual of Proposition 6.2.1(1), the vertical map in this pullback lies in $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}} \subset s\mathcal{S}$. Hence, by Proposition 1.6.5 (and Proposition 1.7.2) it follows that $|Y_\bullet| \simeq \mathrm{pt}_{\mathcal{S}}$. Finally, combining the two equivalences we have just obtained with the Bousfield–Kan colimit formula (Theorem 3.5.8) and Proposition 2.2.4, we obtain the string of equivalences

$$\mathrm{pt}_{\mathcal{S}} \simeq |Y_\bullet| \simeq |\mathrm{srep}(Y)_\bullet| \simeq |\mathrm{N}_\infty(\mathcal{C}_4)_\bullet| \simeq (\mathcal{C}_4)^{\mathrm{gpd}},$$

which completes the proof. □

6.4.2 Left homotopies

Given two parallel maps $\mathcal{D}_1^\bullet \rightrightarrows \mathcal{D}_2^\bullet$ in $c\mathcal{M}\mathrm{odel}_{(**)}$, and any $\mathcal{M} \in \mathcal{M}\mathrm{odel}_{(**)}$, applying the functor

$$c\mathcal{M}\mathrm{odel}_{(**)} \xrightarrow{\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(-,\mathcal{M})} s\mathcal{S}$$

yields two parallel maps

$$\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \rightrightarrows \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$

in $s\mathcal{S}$. We will be interested explicitly describing additional data which causes these maps become equivalent upon geometric realization. This motivates the following definition.

Definition 6.4.25. Given two parallel maps $f, g \in \hom_{s\mathcal{S}}(Y, Z)$, a ***left homotopy*** from f to g (in the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$) is a map $h \in \hom_{s\mathcal{S}}(Y \times \Delta^1, Z)$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} Y & \xrightarrow{\sim} & Y \times \Delta^{\{0\}} & \longrightarrow & Y \times \Delta^1 & \longleftarrow & Y \times \Delta^{\{1\}} & \xleftarrow{\sim} & Y \\ & \searrow^{f} & & & \downarrow{\scriptstyle h} & & & \swarrow_{g} & \\ & & & & Z & & & & \end{array}$$

in $s\mathcal{S}$.

Of course, this comes with the following expected result.

Lemma 6.4.26. *A left homotopy $Y \times \Delta^1 \to Z$ in $s\mathcal{S}_{\mathrm{KQ}}$ between two parallel maps $Y \rightrightarrows Z$ in $s\mathcal{S}$ induces an equivalence between the two induced parallel maps $|Y| \rightrightarrows |Z|$ in $\mathcal{S}$.*

Proof. The maps $Y \simeq Y \times \Delta^{\{i\}} \to Y \times \Delta^1$ are in $\mathbf{W}_{\mathrm{KQ}}$ since geometric realization (as a sifted colimit) commutes with finite products. Hence, the diagram

$$\begin{array}{ccccccccc} Y & \xrightarrow{\sim} & Y \times \Delta^{\{0\}} & \xrightarrow{\approx} & Y \times \Delta^1 & \xleftarrow{\approx} & Y \times \Delta^{\{1\}} & \xleftarrow{\sim} & Y \\ & \searrow & & & \downarrow & & & \swarrow & \\ & & & & Z & & & & \end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$ induces, upon geometric realization, the diagram

$$\begin{array}{ccccccccc} |Y| & \xrightarrow{\sim} & |Y \times \Delta^{\{0\}}| & \xrightarrow{\sim} & |Y \times \Delta^1| & \xleftarrow{\sim} & |Y \times \Delta^{\{1\}}| & \xleftarrow{\sim} & |Y| \\ & \searrow & & & \downarrow & & & \swarrow & \\ & & & & |Z| & & & & \end{array}$$

in $\mathcal{S}$, which selects the desired equivalence between the two induced maps $|Y| \rightrightarrows |Z|$. □

In our cases of interest, the left homotopy between two parallel maps

$$\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \rightrightarrows \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$

will be natural in the variable $\mathcal{M} \in (\mathcal{M}\mathrm{odel}_\infty)_{(**)}$. By Yoneda's lemma, the data of such a left homotopy itself will be corepresentable by some additional data relating $\mathcal{D}_1^\bullet$ and $\mathcal{D}_2^\bullet$. This leads us to the following definition.

Definition 6.4.27. Given $\varphi^\bullet, \psi^\bullet \in \hom_{c\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{D}_2^\bullet)$, a ***left homotopy corepresentation*** from $\varphi^\bullet$ to $\psi^\bullet$ is a family of maps

$$\{h_n^i \in \hom_{\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_1^{n+1}, \mathcal{D}_2^n)\}_{0 \le i \le n \ge 0}$$

satisfying the identities

$$\begin{aligned}
h_n^0 \delta^0 &= \varphi^n \\
h_n^n \delta^{n+1} &= \psi^n \\
h_n^j \delta^i &= \begin{cases} \delta^i h_{n-1}^{j-1}, & i < j \\ h_n^{j-1} \delta^i, & i = j \neq 0 \\ \delta^{i-1} h_{n-1}^j, & i > j+1 \end{cases} \\
h_n^j \sigma^i &= \begin{cases} \sigma^j h_{n+1}^{j+1}, & i \le j \\ \sigma^{i-1} h_{n+1}^j, & i > j. \end{cases}
\end{aligned}$$

Remark 6.4.28. These identities are nothing but the duals of those defining a "simplicial homotopy" in the classical sense (see e.g. [May92, Definitions 5.1]).

Then, we have the following expected result.

Lemma 6.4.29. *Fix some $\varphi^\bullet, \psi^\bullet \in \hom_{c\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{D}_2^\bullet)$. Then, giving a left homotopy corepresentation*

$$\{h_n^i \in \hom_{\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_1^{n+1}, \mathcal{D}_2^n)\}_{0 \le i \le n \ge 0}$$

from $\varphi^\bullet$ to $\psi^\bullet$ is equivalent to giving a left homotopy

$$\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \times \Delta^1 \to \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$

from $\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\varphi^\bullet, \mathcal{M})$ to $\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\psi^\bullet, \mathcal{M})$ which is natural in the variable $\mathcal{M} \in (\mathcal{M}\mathrm{odel}_\infty)_{(**)}$.

Proof. Suppose we have such a natural left homotopy. If we apply it to $\mathcal{D}_2^n$, the natural map

$$\Delta^n \to \hom^{\mathrm{lw}}_{\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{D}_2^n)$$

in $s\mathcal{S}$ corresponding to $\mathrm{id}_{\mathcal{D}_2^n}$ gives rise to the composite map

$$\Delta^n \times \Delta^1 \to \hom^{\mathrm{lw}}_{\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{D}_2^n) \times \Delta^1 \to \hom^{\mathrm{lw}}_{\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{D}_2^n).$$

Evaluating this at the $n+1$ nondegenerate $(n+1)$-simplices of $\Delta^n \times \Delta^1$ and ranging over all $n \geq 0$ yields the maps defining the left homotopy corepresentation; that these satisfy the identities follows from applying the natural left homotopy to the cosimplicial structure maps of $\mathcal{D}_2^\bullet \in c\mathcal{M}\mathrm{odel}_{(**)}$.

Conversely, given a left homotopy representation, we define a natural left homotopy given in level n by the map

$$\hom_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^n, \mathcal{M}) \times (\Delta^1)_n \simeq \coprod_{(\Delta^1)_n} \hom_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^n, \mathcal{M}) \to \hom_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_1^n, \mathcal{M})$$

which, on the summand corresponding to the element of $(\Delta^1)_n \cong \hom_{\boldsymbol{\Delta}}([n],[1])$ associated to the decomposition

$$[n] = \{0, \ldots, n-i\} \sqcup \{(n+1)-i, \ldots, n\}$$

(for $i \in \{0, \ldots, n+1\}$), is corepresented by the map

$$\begin{cases} \varphi^n = h_n^0 \delta^0, & i = 0 \\ h_n^{i-1}\delta^i = h_n^i \delta^n, & 0 < i < n+1 \\ \psi^n = h_n^n \delta^{n+1}, & i = n+1 \end{cases}$$

in $\hom_{\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_1^n, \mathcal{D}_2^n)$; that these do indeed define a left homotopy follows from the fact that our choices here are induced by the simplicial structure maps of $\Delta^1 \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$. □

Definition 6.4.30. In the situation of Lemma 6.4.29, we refer to an induced map

$$\hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \times \Delta^1 \to \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$

as a ***corepresented left homotopy*** (in the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$) associated to the left homotopy corepresentation.

6.5 The equivalence $\left\|\hom^{\mathbf{lw}}_{\mathcal{M}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y))\right\| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathbf{gpd}}$

We now proceed with an analog of [Man99, Proposition 7.3].

Proposition 6.5.1. *Suppose we have $x, y \in \mathcal{M}$ with x cofibrant and y fibrant, and let ${}_\sigma\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ and ${}_\sigma\mathrm{path}_\bullet(y) \in s\mathcal{M}$ be a special cylinder object for x and a special path object for y, respectively. Then*

$$\left\|\hom^{\mathrm{lw}}_{\mathcal{M}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y))\right\| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}}.$$

Proof. To prove the claim, we construct a commutative diagram

$$\begin{array}{ccccc} M_\bullet & \longrightarrow & Q_\bullet & \longleftarrow & P_\bullet \\ \downarrow & \nearrow & \uparrow & \nwarrow & \downarrow \\ N_\bullet & \longleftarrow & P_\bullet & \longrightarrow & P_\bullet \times \Delta^1 \end{array}$$

in $s\mathcal{S}$ whose maps are all in $\mathbf{W}_{\mathrm{KQ}}$, such that

$$|M_\bullet| \simeq \left\|\hom^{\mathrm{lw}}_{\mathcal{M}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y))\right\|$$

and

$$|Q_\bullet| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}}.$$

We first define the simplicial spaces of the diagram. Certain auxiliary definitions will appear superfluous, but they will be used later in the proof.

- We begin by defining the object $M_\bullet \in s\mathcal{S}$ by

$$M_\bullet = \mathrm{srep}\left(\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{\hom_{\mathcal{M}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y))} \mathcal{S}\right)_\bullet.$$

  By the Bousfield–Kan colimit formula (Theorem 3.5.8), we have that

$$|M_\bullet| \simeq \left\|\hom^{\mathrm{lw}}_{\mathcal{M}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y))\right\|,$$

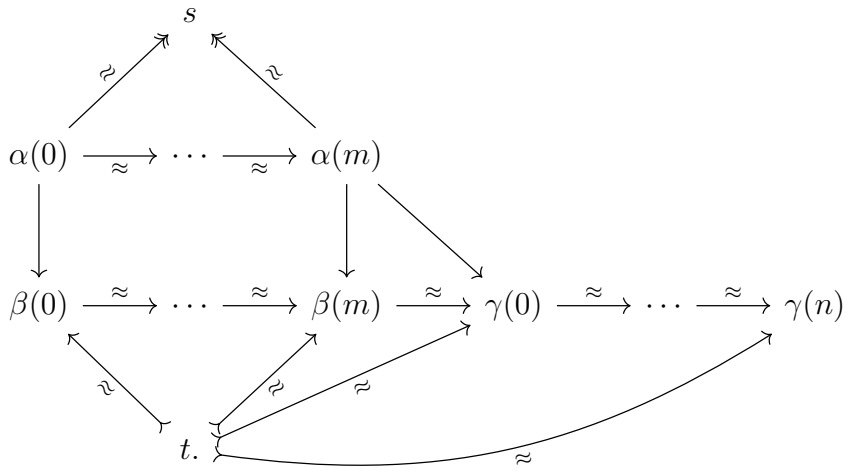
  as desired. Note that, since $[n] \in \mathcal{C}\mathrm{at}$ and $\mathbf{\Delta} \times \mathbf{\Delta}^{op} \in \mathcal{C}\mathrm{at}$ are gaunt, up to making the identification

$$\hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathbf{\Delta}^{op}) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}([n]^{op}, \mathbf{\Delta}^{op}) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathbf{\Delta}),$$

  we have that

$$\begin{aligned} M_n &\simeq \mathrm{colim}_{(\alpha,\beta)\in \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathbf{\Delta}\times\mathbf{\Delta}^{op})} \hom_{\mathcal{M}}({}_\sigma\mathrm{cyl}^{\alpha(n)}(x), {}_\sigma\mathrm{path}_{\beta(0)}(y)) \\ &\simeq \coprod_{(\alpha,\beta)\in \mathrm{N}(\mathbf{\Delta})_n \times \mathrm{N}(\mathbf{\Delta}^{op})_n} \hom_{\mathcal{M}}({}_\sigma\mathrm{cyl}^{\alpha(n)}(x), {}_\sigma\mathrm{path}_{\beta(0)}(y)). \end{aligned}$$

- We define the objects $N_\bullet, Q_\bullet, P_\bullet \in s\mathcal{S}$ simultaneously, as follows. For any $m, n \geq 0$, let $\underline{\mathbf{p}}^{m,n}$ denote the doubly-pointed model diagram



Moreover, let $\underline{\mathbf{n}}^{m,n} \subset \underline{\mathbf{p}}^{m,n}$ denote the full subcategory on the objects

$$\{s, t, \alpha(i), \gamma(j)\}_{0 \leq i \leq m, 0 \leq j \leq n}$$

and let $\underline{\mathbf{q}}^{m,n} \subset \underline{\mathbf{p}}^{m,n}$ denote the full subcategory on the objects

$$\{s, t, \alpha(i), \beta(j)\}_{0 \leq i,j \leq m},$$

both considered as doubly-pointed model diagrams in the evident way. Let us use the placeholders $Y \in \{N, Q, P\}$ and $\underline{\mathbf{y}} \in \{\underline{\mathbf{n}}, \underline{\mathbf{q}}, \underline{\mathbf{p}}\}$. Then, the various objects $\underline{\mathbf{y}}^{m,n} \in \mathcal{M}\mathrm{odel}_{**}$ assemble into the evident bicosimplicial object $\underline{\mathbf{y}}^{\bullet\bullet} \in c\mathcal{M}\mathrm{odel}_{**}$, and we auxiliarily define

$$Y_{\bullet\bullet} = \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{**}}(\underline{\mathbf{y}}^{\bullet\bullet}, \mathcal{M}) \in ss\mathcal{S}.$$

Then, we define $\underline{\mathbf{y}}^{\bullet} = \mathrm{diag}^*(\underline{\mathbf{y}}^{\bullet\bullet}) \in c\mathcal{M}\mathrm{odel}_{**}$, and we set

$$Y_\bullet = \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{**}}(\underline{\mathbf{y}}^{\bullet}, \mathcal{M}) \in s\mathcal{S},$$

so that $Y_\bullet \simeq \mathrm{diag}^*(Y_{\bullet\bullet})$.

We now provide alternative identifications of the simplicial spaces $N_\bullet$ and $Q_\bullet$.

– As for $N_\bullet$, we clearly have

$$N_n \simeq \mathrm{colim}_{(\alpha,\gamma) \in \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathbf{W}_{\downarrow x} \times \mathbf{W}_{y \downarrow}\right)} \hom_{\mathcal{M}}(\alpha(n), \gamma(0)).$$

Moreover, examining the structure maps of $N_\bullet \in s\mathcal{S}$, we see that up to making the identification

$$\hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \left(\mathbf{W}_{\downarrow x}\right)^{op}\right) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n]^{op}, \left(\mathbf{W}_{\downarrow x}\right)^{op}\right) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathbf{W}_{\downarrow x}\right),$$

we have that

$$N_\bullet \simeq \mathrm{srep}\left(\left(\mathbf{W}_{\downarrow x}\right)^{op} \times \mathbf{W}_{y\mathbb{I}} \xrightarrow{\left((x' \xrightarrow{\approx} x)^\circ, (y \xrightarrow{\approx} y')\right) \mapsto \hom_{\mathcal{M}}(x', y')} \mathcal{S}\right)_\bullet.$$

– As for $Q_\bullet$, note first of all that $\underline{\mathbf{q}}^{m,n} \in \mathcal{M}\mathrm{odel}_{**}$ (and hence $Q_{m,n} \in \mathcal{S}$) is independent of n. Moreover, since we have an evident isomorphism $\underline{\mathbf{q}}^\bullet \cong c^\bullet_{**}\underline{\tilde{\mathbf{3}}}$ in $c\mathcal{M}\mathrm{odel}_{**}$ – indeed, the only difference is that we have named the intermediate objects of the constituent model diagrams of $\underline{\mathbf{q}}^\bullet \in c\mathcal{M}\mathrm{odel}_{**}$ – it follows from Lemma 6.4.19 that

$$Q_\bullet \simeq \mathrm{N}_\infty(\underline{\tilde{\mathbf{3}}}(x,y))_\bullet.$$

Hence, Proposition 2.2.4 this implies that

$$|Q_\bullet| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}},$$

as desired.

Finally, we observe that since $\mathbf{\Delta}^{op} \xrightarrow{\mathrm{diag}} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ is final (as $\mathbf{\Delta}^{op}$ is sifted), then by Fubini's theorem for colimits, continuing to use the placeholder $Y \in \{N, Q, P\}$ we have an identification

$$\begin{aligned} |Y_\bullet| &\simeq \|Y_{\bullet\bullet}\| \\ &= \mathrm{colim}_{([m]^\circ,[n]^\circ)\in \mathbf{\Delta}^{op}\times\mathbf{\Delta}^{op}} Y_{m,n} \\ &\simeq \mathrm{colim}_{[n]^\circ\in\mathbf{\Delta}^{op}}\left(\mathrm{colim}_{[m]^\circ\in\mathbf{\Delta}^{op}} Y_{m,n}\right) \\ &= \mathrm{colim}_{[n]^\circ\in\mathbf{\Delta}^{op}} |Y_{\bullet,n}|, \end{aligned}$$

and similarly we have an identification

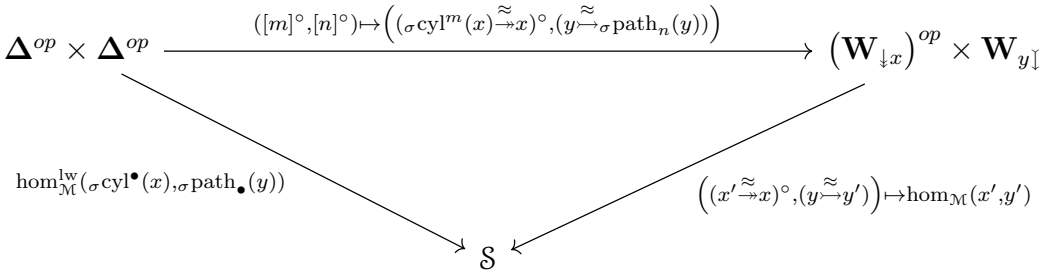
$$|Y_\bullet| \simeq \mathrm{colim}_{[m]^\circ\in\mathbf{\Delta}^{op}} |Y_{m,\bullet}|.$$

We now define the maps in the diagram, and along the way we show that the subdiagram

$$\begin{array}{ccccc} M_\bullet & & Q_\bullet & \longleftarrow & P_\bullet \\ \downarrow & & & & \downarrow \\ N_\bullet & \longleftarrow & P_\bullet & \longrightarrow & P_\bullet \times \Delta^1 \end{array}$$

lies in $\mathbf{W}_{\mathrm{KQ}}$, which suffices to prove that the entire diagram is in $\mathbf{W}_{\mathrm{KQ}}$ by the two-out-of-three property.[8]

- We have a commutative diagram



  in $\mathcal{C}\mathrm{at}_\infty$; considering this as a map in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{S}}$, we obtain the map $M_\bullet \to N_\bullet$ from Proposition 3.5.13(2). The upper map in this diagram is the product of two functors which are each final, the second by Lemma 6.5.2 and the first by the opposite of its dual. Hence, this functor is itself final by Proposition 3.4.9. Thus, the map $M_\bullet \to N_\bullet$ is in $\mathbf{W}_{\mathrm{KQ}}$ by the Bousfield–Kan colimit formula (Theorem 3.5.8).

- The map $N_\bullet \to Q_\bullet$ is corepresented by the morphism in $\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^\bullet, \underline{\mathbf{n}}^\bullet)$ given in level n by the unique functor satisfying $\alpha(i) \mapsto \alpha(i)$ and $\beta(i) \mapsto \gamma(i)$. (Note that there are composite morphisms $\alpha(i) \to \beta(i)$ implicit in the diagram defining $\underline{\mathbf{n}}^n$.)

- The map $M_\bullet \to Q_\bullet$ is the composition $M_\bullet \to N_\bullet \to Q_\bullet$.

- The map $P_\bullet \to N_\bullet$ is corepresented by the morphism in $\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{n}}^\bullet, \underline{\mathbf{p}}^\bullet)$ which is simply the defining inclusion in each level. Note that this is obtained by applying $cc\mathcal{M}\mathrm{odel}_{**} \xrightarrow{\mathrm{diag}^*} c\mathcal{M}\mathrm{odel}_{**}$ to the morphism in $\hom_{cc\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{n}}^{\bullet\bullet}, \underline{\mathbf{p}}^{\bullet\bullet})$ which is again simply the defining inclusion in each bidegree. This latter map corepresents a map $P_{\bullet\bullet} \to N_{\bullet\bullet}$ in $ss\mathcal{S}$, from which the map $P_\bullet \to N_\bullet$ in $s\mathcal{S}$ is therefore obtained by applying $ss\mathcal{S} \xrightarrow{\mathrm{diag}^*} s\mathcal{S}$.

  Now, since $|P_\bullet| \simeq \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} |P_{\bullet,n}|$ and $|N_\bullet| \simeq \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} |N_{\bullet,n}|$, to prove that the map $P_\bullet \to N_\bullet$ is in $\mathbf{W}_{\mathrm{KQ}}$, it suffices to prove that for each $[n]^\circ \in \mathbf{\Delta}^{op}$,

[8] Of course, really it would already have sufficed to obtain the zigzag $M_\bullet \to N_\bullet \leftarrow P_\bullet \to Q_\bullet$ of maps in $\mathbf{W}_{\mathrm{KQ}}$, but this proof is almost no more work and has the added benefit of showing that the map inducing the equivalence is the expected one.

the map $|P_{\bullet,n}| \to |N_{\bullet,n}|$ is an equivalence in $\mathcal{S}$, i.e. that the map $P_{\bullet,n} \to N_{\bullet,n}$ is in $\mathbf{W}_{\mathrm{KQ}}$.

To see this, we construct an inverse up to left homotopy in $s\mathcal{S}_{\mathrm{KQ}}$ for this map. This is corepresented by the map in $\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{p}}^{\bullet,n}, \underline{\mathbf{n}}^{\bullet,n})$ given in level m by the unique functor satisfying $\alpha(i) \mapsto \alpha(i)$, $\beta(i) \mapsto \gamma(0)$, and $\gamma(i) \mapsto \gamma(i)$. As the resulting composite map $\underline{\mathbf{n}}^{\bullet,n} \to \underline{\mathbf{p}}^{\bullet,n} \to \underline{\mathbf{n}}^{\bullet,n}$ in $c\mathcal{M}\mathrm{odel}_{**}$ is the identity, it follows that the corepresented composite map $N_{\bullet,n} \to P_{\bullet,n} \to N_{\bullet,n}$ is also the identity.

On the other hand, the composite map $\underline{\mathbf{p}}^{\bullet,n} \to \underline{\mathbf{n}}^{\bullet,n} \to \underline{\mathbf{p}}^{\bullet,n}$ is not equal to the identity. However, it suffices to give a left homotopy corepresentation

$$\{{}_{\underline{\mathbf{p}}}h^i_m \in \hom_{\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{p}}^{m+1,n}, \underline{\mathbf{p}}^{m,n})\}_{0\leq i\leq m\geq 0}$$

from this composite to $\mathrm{id}_{\underline{\mathbf{p}}^{\bullet,n}}$, which we define by taking ${}_{\underline{\mathbf{p}}}h^i_m$ to be the unique functor satisfying

$$\alpha(j) \mapsto \begin{cases} \alpha(j), & j \leq i \\ \alpha(j-1), & j > i \end{cases}$$
$$\beta(j) \mapsto \begin{cases} \beta(j), & j \leq i \\ \gamma(0), & j > i \end{cases}$$
$$\gamma(j) \mapsto \gamma(j).$$

(It is tedious but straightforward to verify that these formulas do indeed define such a left homotopy corepresentation.) By Lemma 6.4.29 this gives us a left homotopy in $s\mathcal{S}_{\mathrm{KQ}}$ from the corepresented composite map $P_{\bullet,n} \to N_{\bullet,n} \to P_{\bullet,n}$ to $\mathrm{id}_{P_{\bullet,n}}$, and so by Lemma 6.4.26 this corepresented composite map becomes equivalent upon geometric realization to $\mathrm{id}_{|P_{\bullet,n}|}$. Thus, the map $P_{\bullet,n} \to N_{\bullet,n}$ does indeed lie in $\mathbf{W}_{\mathrm{KQ}}$ for all $[n]^\circ \in \mathbf{\Delta}^{op}$, so that the map $P_\bullet \to N_\bullet$ lies in $\mathbf{W}_{\mathrm{KQ}}$ as well.

- The vertical map $P_\bullet \to Q_\bullet$ is of course given by the composition $P_\bullet \to N_\bullet \to Q_\bullet$. More explicitly, it is corepresented by the morphism in $\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^\bullet, \underline{\mathbf{p}}^\bullet)$ given in level n by the unique functor satisfying $\alpha(i) \mapsto \alpha(i)$ and $\beta(i) \mapsto \gamma(i)$.

- The horizontal map $P_\bullet \to Q_\bullet$ is corepresented by the morphism in $\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^\bullet, \underline{\mathbf{p}}^\bullet)$ which is simply the the defining inclusion in each level. Note that this is obtained by applying $cc\mathcal{M}\mathrm{odel}_{**} \xrightarrow{\mathrm{diag}^*} c\mathcal{M}\mathrm{odel}_{**}$ to the morphism in $\hom_{cc\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^{\bullet\bullet}, \underline{\mathbf{p}}^{\bullet\bullet})$ which is again simply the defining inclusion in each bidegree. This latter map

corepresents a map $P_{\bullet\bullet} \to Q_{\bullet\bullet}$ in $ss\mathcal{S}$, from which the horizontal map $P_\bullet \to Q_\bullet$ in $s\mathcal{S}$ is therefore obtained by applying $ss\mathcal{S} \xrightarrow{\mathrm{diag}^*} s\mathcal{S}$.

Now, since $|P_\bullet| \simeq \mathrm{colim}_{[m]^\circ \in \mathbf{\Delta}^{op}} |P_{m,\bullet}|$ and $|Q_\bullet| \simeq \mathrm{colim}_{[m]^\circ \in \mathbf{\Delta}^{op}} |Q_{m,\bullet}|$, to prove that the horizontal map $P_\bullet \to Q_\bullet$ is in $\mathbf{W}_{\mathrm{KQ}}$, it suffices to prove that for each $[m]^\circ \in \mathbf{\Delta}^{op}$, the map $|P_{m,\bullet}| \to |Q_{m,\bullet}| \simeq Q_m$ is an equivalence in $\mathcal{S}$ (where the given equivalence comes from the fact that $Q_{m,\bullet} \simeq \mathrm{const}(Q_m)$).

Via the map $P_{m,\bullet} \to Q_{m,\bullet} \simeq \mathrm{const}(Q_m)$, we can consider $P_{m,\bullet}$ as a simplicial object

$$\mathbf{\Delta}^{op} \xrightarrow{P_{m,\bullet}} \mathcal{S}_{/Q_m};$$

moreover, $|P_{m,\bullet}|$ is still its colimit in this ∞-category since colimits in $\mathcal{S}_{/Q_m}$ are created in $\mathcal{S}$. Now, we have a composite equivalence

$$\mathrm{Fun}(Q_m, \mathcal{S}) \underset{\sim}{\xrightarrow{\mathrm{Gr}}} \mathcal{LF}\mathrm{ib}(Q_m) \simeq \mathcal{S}_{/Q_m}$$

(recall Remark 3.1.5), under which the above simplicial object corresponds to a simplicial object

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{Gr}^{-1}(P_{m,\bullet})} \mathrm{Fun}(Q_m, \mathcal{S}).$$

Hence, to show that $|P_{m,\bullet}| \in \mathcal{S}_{/Q_m}$ is a terminal object (i.e. to show that $|P_{m,\bullet}| \xrightarrow{\sim} Q_m$), it suffices to obtain an equivalence

$$|\mathrm{Gr}^{-1}(P_{m,\bullet})| \simeq \mathrm{pt}_{\mathrm{Fun}(Q_m,\mathcal{S})}.$$

As colimits in $\mathrm{Fun}(Q_m, \mathcal{S})$ are computed pointwise, for this it suffices to show that for any point $q \in Q_m$, we have

$$|\mathrm{Gr}^{-1}(P_{m,\bullet})(q)| \simeq \mathrm{pt}_{\mathcal{S}}.$$

Moreover, the naturality of the Grothendieck construction implies that we can identify the constituent simplicial spaces of this geometric realization as

$$\mathrm{Gr}^{-1}(P_{m,n})(q) \simeq \lim \left(\begin{array}{ccc} & & P_{m,n} \\ & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[q]{} & Q_m \end{array} \right)$$

for all $n \geq 0$ in a way compatible with the simplicial structure maps; in other words, we have an equivalence

$$\mathrm{Gr}^{-1}(P_{m,\bullet})(q) \simeq \lim \left(\begin{array}{ccc} & & P_{m,\bullet} \\ & & \downarrow \\ \mathrm{pt}_{s\mathcal{S}} & \xrightarrow[\mathrm{const}(q)]{} & \mathrm{const}(Q_m) \end{array} \right)$$

in $s\mathcal{S}$.

Now, by definition $Q_m = \hom_{(\mathcal{M}\mathrm{odel}_\infty)_{**}}(\underline{\mathbf{q}}^m, \mathcal{M})$, and so our point $q \in Q_m$ corresponds to some map $\underline{\mathbf{q}}^m \xrightarrow{q'} \mathcal{M}$ in $(\mathcal{M}\mathrm{odel}_\infty)_{**}$. Via this map we can consider $\mathcal{M} \in ((\mathcal{M}\mathrm{odel}_\infty)_{**})_{\underline{\mathbf{q}}^m/}$, and it is not hard to see that we have equivalences

$$\begin{aligned} \lim \left(\begin{array}{ccc} & & P_{m,\bullet} \\ & & \downarrow \\ \mathrm{pt}_{s\mathcal{S}} & \xrightarrow[\mathrm{const}(q)]{} & \mathrm{const}(Q_m) \end{array} \right) &\simeq \hom^{\mathrm{lw}}_{((\mathcal{M}\mathrm{odel}_\infty)_{**})_{\underline{\mathbf{q}}^m/}}(\underline{\mathbf{p}}^{m,\bullet}, \mathcal{M}) \\ &\simeq \mathrm{N}_\infty \left(\left(\mathbf{W}_{y\wr} \right)_{(y \xrightarrow{\approx} q'(\beta(i)))/} \right). \end{aligned}$$

But this last simplicial space is the nerve of an ∞-category with an initial object, so it has contractible geometric realization by Proposition 2.2.4 and the opposite of Corollary 3.4.11. Thus, we have shown that $|P_{m,\bullet}| \xrightarrow{\sim} Q_m$, which as we have seen implies that $|P_\bullet| \xrightarrow{\sim} |Q_\bullet|$, i.e. that $P_\bullet \to Q_\bullet$ lies in $\mathbf{W}_{\mathrm{KQ}}$.

- The maps $P_\bullet \to P_\bullet \times \Delta^1$ are given by

$$P_\bullet \simeq P_\bullet \times \Delta^{\{i\}} \to P_\bullet \times \Delta^1,$$

where we take $i = 0$ for the horizontal map and $i = 1$ for the vertical map. These lie in $\mathbf{W}_{\mathrm{KQ}}$ since the geometric realization functor $|-| : s\mathcal{S} \to \mathcal{S}$ (as a sifted colimit) commutes with finite products.

- The map $P_\bullet \times \Delta^1 \to Q_\bullet$ is the corepresented left homotopy associated to the left homotopy corepresentation

$$\{\{{}_{\underline{\mathbf{q}}}h^i_n \in \hom_{\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^{n+1}, \underline{\mathbf{p}}^n)\}_{0 \leq i \leq n}\}_{n \geq 0}$$

given by defining ${}_{\underline{\mathbf{q}}}h^i_n$ to be the unique functor satisfying

$$\alpha(j) \mapsto \begin{cases} \alpha(j), & j \leq i \\ \alpha(j-1), & j > i \end{cases}$$
$$\beta(j) \mapsto \begin{cases} \beta(j), & j \leq i \\ \gamma(j), & j > i. \end{cases}$$

(It is tedious but straightforward to verify that these formulas do indeed define a suitable left homotopy corepresentation.)

Thus, we have exhibited the above original commutative diagram in $s\mathcal{S}$ and shown that it lies entirely in $\mathbf{W}_{\mathrm{KQ}}$. In particular, it follows that $|M_\bullet| \xrightarrow{\sim} |Q_\bullet|$, i.e. that

$$\left\| \hom^{\mathrm{lw}}_{\mathcal{M}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y)) \right\| \xrightarrow{\sim} \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}},$$

as desired. □

We now prove an auxiliary result which was needed in the proof of Proposition 6.5.1, an analog of [DK80b, Proposition 6.11].[9]

Lemma 6.5.2. *If $y \in \mathcal{M}^f$ is fibrant and ${}_\sigma\mathrm{path}_\bullet(y) \in s\mathcal{M}$ is any special path object for y, then the functor*

$$\begin{aligned} \mathbf{\Delta}^{op} &\to \mathbf{W}_{y\downarrow} \\ [n]^\circ &\mapsto (y \xrightarrow{\approx} {}_\sigma\mathrm{path}_n(y)) \end{aligned}$$

is final.

Proof. According to the characterization of Theorem A (3.4.10), it suffices to show that for any object $(y \xrightarrow{\approx} z) \in \mathbf{W}_{y\downarrow}$, the groupoid completion of the comma ∞-category

$$\mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\downarrow}}{\times} \left(\mathbf{W}_{y\downarrow}\right)_{(y \xrightarrow{\approx} z)/}$$

[9] The proof of [DK80b, Proposition 6.11] contains a mild but rather confusing typo. There, it is claimed that a certain category is isomorphic to the homotopy colimit of a simplicial set, which is then claimed to have the same homotopy type as another simplicial set. In fact, it is the *nerve* of the category which is *isomorphic* to the first simplicial set itself (without saying "homotopy colimit"), and then this simplicial set is equivalent to the other simplicial set because the latter is the nerve of the category of simplices of the former. This last statement can be seen as coming from the fact that there are two ways to take the homotopy colimit of a simplicial set: either by taking its usual geometric realization, or by taking the geometric realization of its simplicial replacement.

is contractible.

First of all, note that the chosen equivalence $y \simeq {}_\sigma\mathrm{path}_0(y)$ endows the object $\hom^{\mathrm{lw}}_{\mathcal{M}}(y, {}_\sigma\mathrm{path}_\bullet(y)) \in s\mathcal{S}$ with a canonical basepoint $\mathrm{pt}_{s\mathcal{S}} \to \hom^{\mathrm{lw}}_{\mathcal{M}}(y, {}_\sigma\mathrm{path}_\bullet(y))$. The dual of Proposition 6.2.1(1) implies that the map

$$\hom^{\mathrm{lw}}_{\mathcal{M}}(z, {}_\sigma\mathrm{path}_\bullet(y)) \to \hom^{\mathrm{lw}}_{\mathcal{M}}(y, {}_\sigma\mathrm{path}_\bullet(y))$$

is in $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}}$, which implies (by Proposition 1.6.5) that its fiber over that basepoint has contractible geometric realization. As fibers (being limits) in $s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$ are computed objectwise, this fiber is given in level n by

$$\hom_{\left(\mathbf{W}_{y\downarrow}\right)}\left(y \xrightarrow{\approx} z, y \xrightarrow{\approx} {}_\sigma\mathrm{path}_n(y)\right).$$

(Note that the inclusions $\mathbf{W}_{y\downarrow} \subset \mathbf{W}_{y/} \subset \mathcal{M}_{y/}$ are both inclusions of full subcategories (the latter by the two-out-of-three property).) By the Bousfield–Kan colimit formula (Theorem 3.5.8), the geometric realization of this simplicial space is equivalent to the geometric realization of its simplicial replacement when considered in $s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$. In level n, this simplicial replacement is given by

$$\coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \hom_{\left(\mathbf{W}_{y\downarrow}\right)}\left(y \xrightarrow{\approx} z, y \xrightarrow{\approx} {}_\sigma\mathrm{path}_{\alpha(0)}(y)\right).$$

We claim that this latter simplicial space is precisely the nerve of the comma ∞-category

$$\mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\downarrow}}{\times} \left(\mathbf{W}_{y\downarrow}\right)_{(y \xrightarrow{\approx} z)/}.$$

To see this, observe that

$$\begin{aligned}
\mathrm{N}_\infty\left(\mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\downarrow}}{\times} \left(\mathbf{W}_{y\downarrow}\right)_{(y \xrightarrow{\approx} z)/}\right)_n &= \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\downarrow}}{\times} \left(\mathbf{W}_{y\downarrow}\right)_{(y \xrightarrow{\approx} z)/}\right) \\
&\simeq \lim \left(\begin{array}{ccc} & & \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \left(\mathbf{W}_{y\downarrow}\right)_{(y \xrightarrow{\approx} z)/}\right) \\ & & \downarrow \\ \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathbf{\Delta}^{op}) & \longrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathbf{W}_{y\downarrow}\right) \end{array} \right).
\end{aligned}$$

Since $\hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathbf{\Delta}^{op}) \simeq \mathrm{N}(\mathbf{\Delta}^{op})_n$ is discrete, this pullback is equivalent to a coproduct over its elements of the corresponding fibers. Over the element $\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n$,

this fiber is

$$\lim \left(\begin{array}{ccc} & & \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \left(\mathbf{W}_{y\downarrow}\right)_{(y\xrightarrow{\approx} z)/}\right) \\ & & \downarrow \\ \left\{[n] \xrightarrow{\alpha} \mathbf{\Delta}^{op} \to \mathbf{W}_{y\downarrow}\right\} & \hookrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathbf{W}_{y\downarrow}\right) \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{ccccc} & & & & \{(y \xrightarrow{\approx} z)\} \\ & & & & \downarrow \\ & & \hom_{\mathcal{C}\mathrm{at}_\infty}\left(\{(-1) \to \cdots \to n\}, \mathbf{W}_{y\downarrow}\right) & \longrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty}\left(\{(-1)\}, \mathbf{W}_{y\downarrow}\right) \\ & & \downarrow & & \\ \left\{[n] \xrightarrow{\alpha} \mathbf{\Delta}^{op} \to \mathbf{W}_{y\downarrow}\right\} & \hookrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathbf{W}_{y\downarrow}) & & \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{ccccc} & & & & \{(y \xrightarrow{\approx} z)\} \\ & & & & \downarrow \\ & & \hom_{\mathcal{C}\mathrm{at}_\infty}\left(\{(-1) \to 0\}, \mathbf{W}_{y\downarrow}\right) & \longrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty}\left(\{(-1)\}, \mathbf{W}_{y\downarrow}\right) \\ & & \downarrow & & \\ \left\{(y \xrightarrow{\approx} {}_\sigma\mathrm{path}_{\alpha(0)}(y))\right\} & \hookrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty}\left([0], \mathbf{W}_{y\downarrow}\right) & & \end{array} \right)$$

$$\simeq \hom_{\left(\mathbf{W}_{y\downarrow}\right)}\left(y \xrightarrow{\approx} z, y \xrightarrow{\approx} {}_\sigma\mathrm{path}_{\alpha(0)}(y)\right).$$

Moreover, it is clear that the structure maps of this simplicial space agree with those of the above simplicial replacement: both are ultimately induced by the structure maps of ${}_\sigma\mathrm{path}_\bullet(y) \in s\mathcal{M}$. So, these are indeed equivalent simplicial spaces.

We have just shown that the geometric realization of the complete Segal space

$$\mathrm{N}_\infty\left(\mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\downarrow}}{\times} \left(\mathbf{W}_{y\downarrow}\right)_{(y\xrightarrow{\approx} z)/}\right)$$

is contractible. Thus, by Proposition 2.2.4, the groupoid completion

$$\left(\mathbf{\Delta}^{op} \underset{\mathbf{W}_{d\downarrow}}{\times} (\mathbf{W}_{d\downarrow})_{(d \xrightarrow{\approx} d')/} \right)^{\mathrm{gpd}}$$

is indeed contractible. □

6.6 The equivalence $\underline{\tilde{\mathbf{3}}}(x,y)^{\mathbf{gpd}} \simeq \underline{\mathbf{3}}(x,y)^{\mathbf{gpd}}$

We now prove that the ∞-category of three-arrow zigzags from x to y has equivalent groupoid completion to that of its subcategory of special three-arrow zigzags.

Proposition 6.6.1. *For any model ∞-category $\mathcal{M}$ and any $x, y \in \mathcal{M}$, the unique map $\underline{\mathbf{3}} \to \underline{\tilde{\mathbf{3}}}$ in $\mathcal{M}\mathrm{odel}_{**}$ induces an equivalence*

$$\underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}} \xrightarrow{\sim} \underline{\mathbf{3}}(x,y)^{\mathrm{gpd}}$$

on groupoid completions of ∞-categories of zigzags in $\mathcal{M}$ from x to y.

Proof. We apply the functor $\left(\mathrm{Fun}^{\sharp}_{**}(-,\mathcal{M})^{\mathbf{W}}\right)^{\mathrm{gpd}}$ to the sequence of maps in $\mathcal{M}\mathrm{odel}^{\sharp}_{**}$ given in the proof of [MG16, Proposition 3.11(1)] (which factors the unique map $\underline{\mathbf{3}} \to \underline{\tilde{\mathbf{3}}}$ in $\mathcal{M}\mathrm{odel}_{**}$). To show that the induced maps in $\mathcal{S}$ are all equivalences, the arguments given there generalize as follows.

- To show that the maps φ_1 and φ_4 defined there induce equivalences in $\mathcal{S}$, we replace the appeal to [MG16, Lemma 3.9(1)] with an appeal to the factorization lemma (6.4.24).

- The maps φ_2 and φ_5 defined there even induce equivalences in $\mathcal{C}\mathrm{at}_\infty$ upon application of $\mathrm{Fun}^{\sharp}_{**}(-,\mathcal{M})^{\mathbf{W}}$; to see this, we use the argument given in the proof of Proposition 6.7.1 for why the maps φ_2, φ_4, φ_9, and φ_{11} (of that proof) have this same property.

- To show that the maps φ_3 and φ_6 defined there induce equivalences in $\mathcal{S}$, we use the argument given in the proof of Proposition 6.7.1 for why the maps φ_7 and φ_{14} (of that proof) have this same property.

Thus, we obtain the desired equivalence $\underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}} \simeq \underline{\mathbf{3}}(x,y)^{\mathrm{gpd}}$ in $\mathcal{S}$. □

6.7 The equivalence $\underline{\mathbf{3}}(x,y)^{\mathbf{gpd}} \simeq \underline{\mathbf{7}}(x,y)^{\mathbf{gpd}}$

We now prove that the ∞-categories of three-arrow zigzags and seven-arrow zigzags from x to y have equivalent groupoid completions.

Proposition 6.7.1. *If $\mathcal{M}$ is a model ∞-category, then for any $x, y \in \mathcal{M}$, the map $\underline{\mathbf{7}} \to \underline{\mathbf{3}}$ in Model_{**} given by collapsing the middle four instances of $\mathbf{W}^{\pm}$ induces an equivalence*

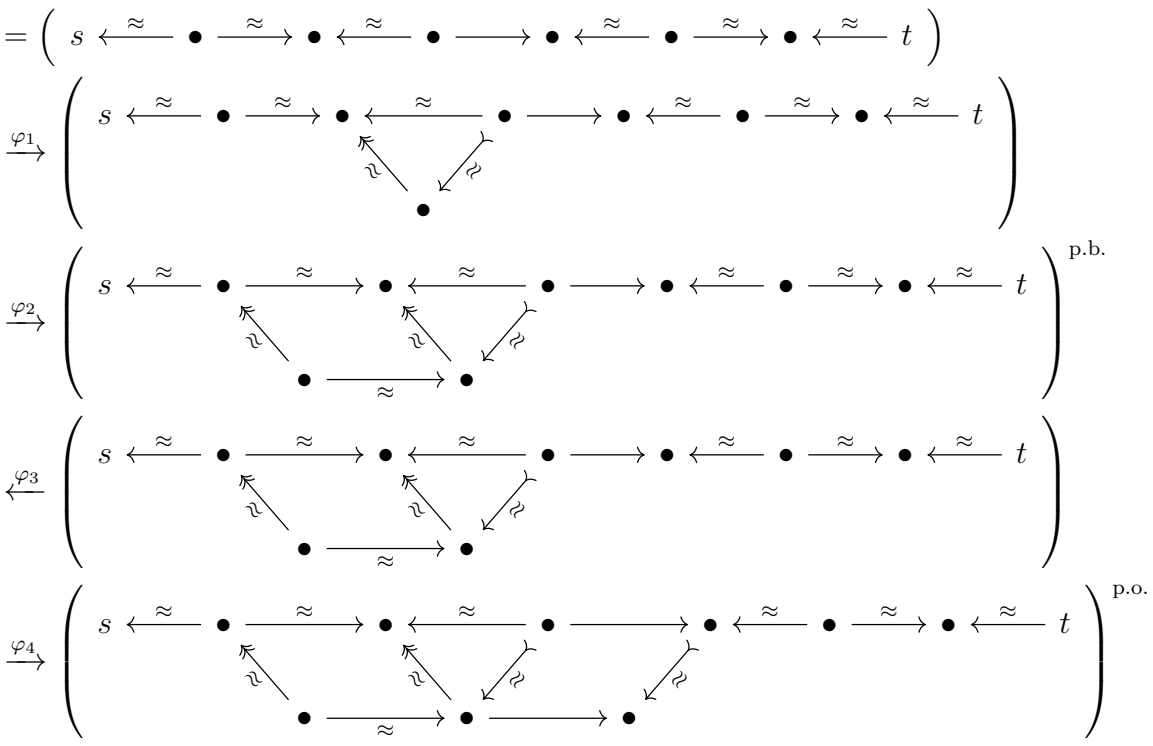
$$\underline{\mathbf{3}}(x,y)^{\mathrm{gpd}} \xrightarrow{\sim} \underline{\mathbf{7}}(x,y)^{\mathrm{gpd}}$$

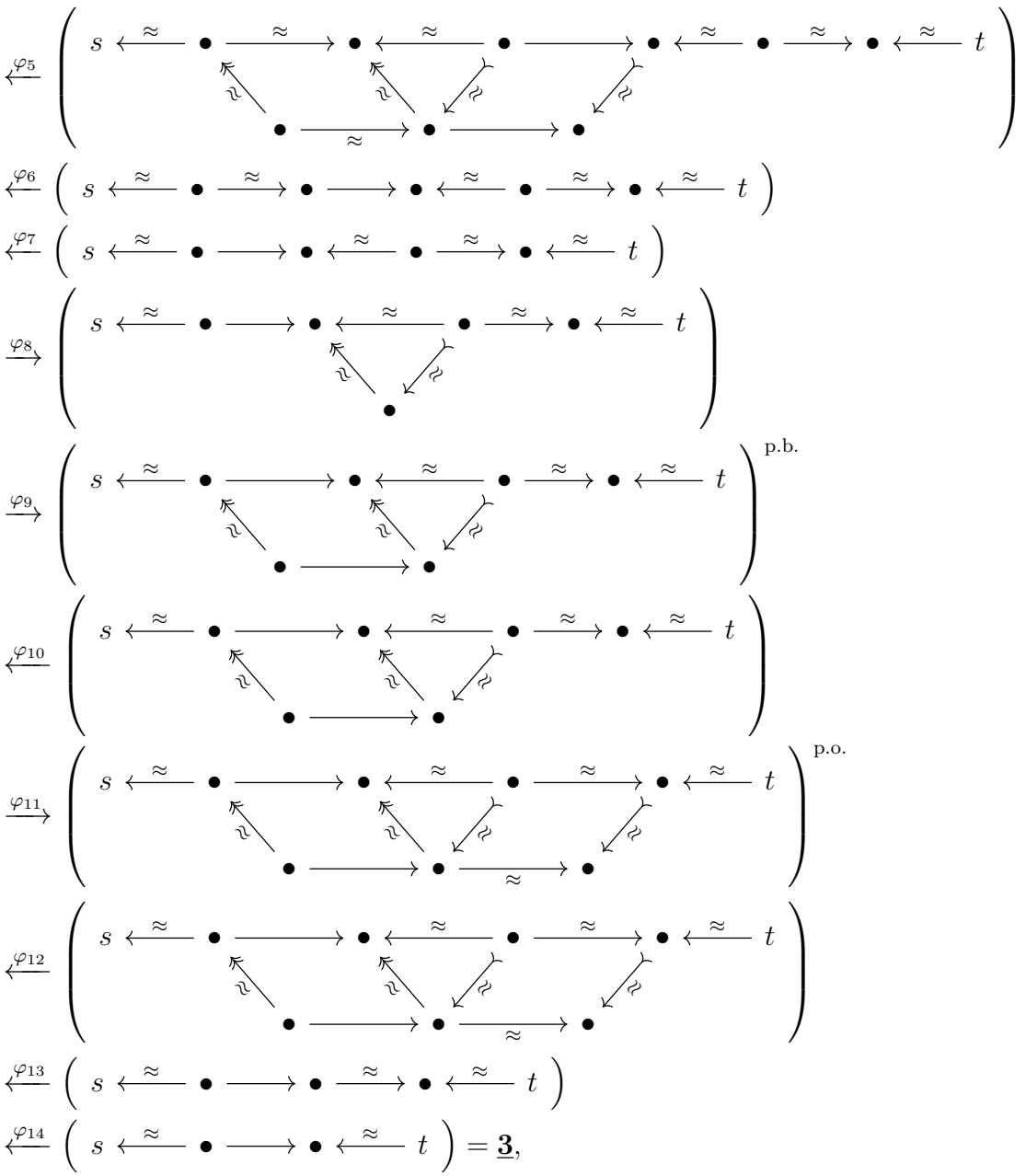
on groupoid completions of ∞-categories of zigzags in $\mathcal{M}$ from x to y.

Proof. In essence, we use the factorization lemma (6.4.24) to remove each instance of $\mathbf{W}^{-1}$ in $\underline{\mathbf{7}}$ which is adjacent to the unique instance of $\mathbf{A}$, and then we "compose out" the remaining instances of $\mathbf{W}$. To be precise, we define a diagram

$$\underline{\mathbf{7}} \xrightarrow{\varphi_1} \mathcal{I}_1 \xrightarrow{\varphi_2} \mathcal{I}_2 \xleftarrow{\varphi_3} \mathcal{I}_3 \xrightarrow{\varphi_4} \mathcal{I}_4 \xleftarrow{\varphi_5} \mathcal{I}_5 \xleftarrow{\varphi_6} \mathcal{I}_6 \xleftarrow{\varphi_7} \mathcal{I}_7 \xrightarrow{\varphi_8} \mathcal{I}_8 \xrightarrow{\varphi_9} \mathcal{I}_9 \xleftarrow{\varphi_{10}} \mathcal{I}_{10} \xrightarrow{\varphi_{11}} \mathcal{I}_{11} \xrightarrow{\varphi_{12}} \mathcal{I}_{12} \xleftarrow{\varphi_{13}} \mathcal{I}_{13} \xleftarrow{\varphi_{14}} \underline{\mathbf{3}}$$

in Model_{**}^{*}, given by





where all maps are the completely evident inclusions, except that

- φ_6 and φ_{13} are the "lower inclusions" (whose images omit any objects in the upper rows that are the source or target of a drawn-in diagonal arrow – note that there are certain "hidden" diagonal maps in $\mathcal{I}_5$ and $\mathcal{I}_{12}$, which are only composites of drawn-in arrows), and

- φ_7 and φ_{14} are obtained by taking the unique copy of $\mathbf{A}$ onto the composite $[\mathbf{W};\mathbf{A}]$ or $[\mathbf{A};\mathbf{W}]$, respectively.

We claim that this induces a diagram of equivalences in $\mathcal{S}$ upon application of $\big(\mathrm{Fun}^{\ast}_{\ast\ast}(-,\mathcal{M})^{\mathbf{W}}\big)^{\mathrm{gpd}}$. The arguments can be grouped as follows.

- The maps φ_1 and φ_8 induce equivalences in $\mathcal{S}$ by the factorization lemma (6.4.24).

- The maps φ_2, φ_4, φ_9, and φ_{11} actually even induce equivalences in $\mathcal{C}\mathrm{at}_\infty$ upon application of $\mathrm{Fun}^{\ast}_{\ast\ast}(-,\mathcal{M})^{\mathbf{W}}$; this follows from the facts that

  - $\mathcal{M}$ is finitely bicomplete,
  - the subcategories $(\mathbf{W}\cap\mathbf{F}), (\mathbf{W}\cap\mathbf{C}) \subset \mathcal{M}$ are respectively closed under pullbacks and pushouts, and
  - the subcategory $\mathbf{W}\subset\mathcal{M}$ has the two-out-of-three property

  (see e.g. Proposition T.4.3.2.15).

- Upon application of $\mathrm{Fun}^{\ast}_{\ast\ast}(-,\mathcal{M})^{\mathbf{W}}$, the maps φ_3 and φ_{10} induce functors which admit left adjoints, and so they induce equivalences in $\mathcal{S}$ upon application of $\big(\mathrm{Fun}^{\ast}_{\ast\ast}(-,\mathcal{M})^{\mathbf{W}}\big)^{\mathrm{gpd}}$ by Corollary 2.1.28. Dually, the maps φ_5 and φ_{12} also induce equivalences in $\mathcal{S}$.

- The maps φ_6, φ_7, φ_{13}, and φ_{14} admit evident retractions ψ_6, ψ_7, ψ_{13}, and ψ_{14}, respectively. Moreover,

  - there are evident cospans of doubly-pointed natural weak equivalences connecting $\mathrm{id}_{\mathbb{J}_5}$ with $\varphi_6\circ\psi_6$ and connecting $\mathrm{id}_{\mathbb{J}_{12}}$ with $\varphi_{13}\circ\psi_{13}$, and
  - there are evident doubly-pointed natural weak equivalences $\varphi_7\circ\psi_7 \xrightarrow{\approx} \mathrm{id}_{\mathbb{J}_6}$ and $\mathrm{id}_{\mathbb{J}_{13}} \xrightarrow{\approx} \varphi_{14}\circ\psi_{14}$.

  Hence, by Lemmas 6.4.22 and 2.1.26, these maps all induce equivalences in $\mathcal{S}$.

Thus, we obtain the desired equivalence $\underline{\mathbf{3}}(x,y)^{\mathrm{gpd}} \simeq \underline{\mathbf{7}}(x,y)^{\mathrm{gpd}}$ in $\mathcal{S}$ which, tracing back through the above zigzag in $\mathcal{M}\mathrm{odel}^{\ast}_{\ast\ast}$, it is clear is indeed induced by the asserted map $\underline{\mathbf{7}} \to \underline{\mathbf{3}}$ in $\mathcal{M}\mathrm{odel}_{\ast\ast}$. $\square$

6.8 Localization of model ∞-categories

So far, given a model ∞-category $\mathcal{M}$ and suitably co/fibrant objects $x, y \in \mathcal{M}$, we have related the spaces of left/right homotopy classes of maps from x to y to the groupoid completions of various ∞-categories of zigzags from x to y. However, in order to show that these are all actually equivalent to the space $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x, y)$ of maps from x to y in the localization $\mathcal{M}[\mathbf{W}^{-1}]$, we must access this latter hom-space. This aim is one of the primary purposes of the local universal property of the Rezk nerve (Theorem 2.3.8) and the calculus theorem (4.5.1), which we now bring to fruition. The following result will be strictly generalized by Theorem 6.10.1, but the latter actually requires the full force of the fundamental theorem of ∞-categories (Theorem 6.1.9). Thus, to avoid circularity, we prove only this weaker version first.

Proposition 6.8.1. *If $\mathcal{M}$ is a model ∞-category with underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$, then $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M}, \mathbf{W}) \in \mathcal{SS}$, and moreover the morphism $\mathrm{N}_{\infty}(\mathcal{M}) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M}, \mathbf{W}))$ in $\mathcal{CSS}$ corresponds to the morphism $\mathcal{M} \to \mathcal{M}[\mathbf{W}^{-1}]$ in $\mathcal{C}\mathrm{at}_{\infty}$.*

Proof. The first claim is obtained by combining Lemma 6.8.2 and the calculus theorem (4.5.1(1)), while the second claim follows from the local universal property of the Rezk nerve (Theorem 2.3.8). □

We now give an auxiliary result on which the proof of Proposition 6.8.1 relies.

Lemma 6.8.2. *If $\mathcal{M}$ is a model ∞-category, then its underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$ admits a homotopical three-arrow calculus.*

Proof. After choosing any pair of objects $x, y \in \mathcal{M}$, we apply the functor $\big(\mathrm{Fun}^{*}_{**}(-, \mathcal{M})^{\mathbf{W}}\big)^{\mathrm{gpd}}$ to the diagram in $\mathcal{M}\mathrm{odel}^{*}_{**}$ given in the proof of [MG16, Proposition 3.16(1)]. To show that the induced maps in $\mathcal{S}$ are all equivalences, the arguments given there generalize as follows.

- To show that the map ρ_1 defined there induces an equivalence in $\mathcal{S}$, we replace the appeal to [MG16, Lemma 3.9(1)] with an appeal to the factorization lemma (6.4.24).

- The map ρ_2 defined there even induces an equivalence in $\mathcal{C}\mathrm{at}_{\infty}$ upon application of $\mathrm{Fun}^{*}_{**}(-, \mathcal{M})^{\mathbf{W}}$; to see this, we repeatedly apply the argument given in the proof of Proposition 6.7.1 for why the maps φ_2, φ_4, φ_9, and φ_{11} (of that proof) have this same property.

- The map ρ_3 defined there induces an equivalence in $\mathcal{S}$ in exactly the same manner; we replace the appeal to [MG16, Lemma 3.10] with an appeal to Lemmas 6.4.22 and 2.1.26.

Thus, the underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$ of the model ∞-category $\mathcal{M}$ does indeed admit a homotopical three-arrow calculus. □

6.9 The equivalence $\underline{\mathbf{7}}(x,y)^{\mathbf{gpd}} \simeq \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$

In this section, we show that the groupoid completion of the ∞-category of seven-arrow zigzags from x to y is equivalent to the hom-space $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$, thus completing the string of equivalences in the proof of the fundamental theorem of model ∞-categories (6.1.9).

Proposition 6.9.1. *For any model ∞-category $\mathcal{M}$ and any $x, y \in \mathcal{M}$, we have a canonical equivalence*

$$\underline{\mathbf{7}}(x,y)^{\mathrm{gpd}} \xrightarrow{\sim} \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y).$$

Proof. First of all, by Proposition 6.8.1 (and Remark 4.1.5), we have

$$\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y) \simeq \lim \left(\begin{array}{ccc} & & \mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M},\mathbf{W})_1 \\ & & \downarrow{\scriptstyle (s,t)} \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[(x,y)]{} & \mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M},\mathbf{W})_0 \times \mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M},\mathbf{W})_0 \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{ccc} & & \mathrm{pt}_{\mathcal{S}} \\ & & \downarrow{\scriptstyle y} \\ & \mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M},\mathbf{W})_1 & \xrightarrow[t]{} \mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M},\mathbf{W})_0 \\ & \downarrow{\scriptstyle s} & \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[x]{} \mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M},\mathbf{W})_0 & \end{array} \right)$$

$$= \lim \left(\begin{array}{ccc} & & \mathrm{pt}_{\mathcal{S}} \\ & & \downarrow{\scriptstyle y} \\ & (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \xrightarrow[t^{\mathrm{gpd}}]{} (\mathrm{Fun}([0],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} \\ & \downarrow{\scriptstyle s^{\mathrm{gpd}}} & \\ \mathrm{pt}_{\mathcal{S}} \xrightarrow[x]{} & (\mathrm{Fun}([0],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{ccc} & & (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} \\ & & \downarrow{\scriptstyle y^{\mathrm{gpd}}} \\ & (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \xrightarrow[t^{\mathrm{gpd}}]{} \mathbf{W}^{\mathrm{gpd}} \\ & \downarrow{\scriptstyle s^{\mathrm{gpd}}} & \\ (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} & \xrightarrow[x^{\mathrm{gpd}}]{} \mathbf{W}^{\mathrm{gpd}} & \end{array} \right).$$

Note that this final limit is that of a diagram in $\mathcal{S}$ coming from a diagram in $\mathcal{C}\mathrm{at}_\infty$ via postcomposition with $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$. We will compute this limit by first computing the pullback of the lower left cospan (defined by the maps x and s) and then computing the pullback of the resulting cospan; for both pullbacks we will appeal to Theorems B_n and C_n (3.4.23 and 3.4.26), noting once and for all that $\mathbf{W}^{op}$ has property C_3 by Lemmas 6.9.2 and 6.8.2.

First of all, by Theorem C_n (3.4.26), the functor

$$(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op} \xrightarrow{x^\circ} \mathbf{W}^{op}$$

has property B_3. Hence, by Theorem B_n (3.4.23), we have a homotopy pullback square

$$\begin{array}{ccc} (x^\circ((\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op}) \downarrow_3 s^{op}((\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{op})) & \xrightarrow{t} & (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{op} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s^{op}} \\ (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op} & \xrightarrow[x^\circ]{} & \mathbf{W}^{op} \end{array}$$

in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$; unwinding the definitions, we can identify the homotopy pullback as

$$(\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{op},$$

where the object $x \in \mathcal{M}$ determines the pointing. As homotopy pullback squares in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ are preserved under the involution $(-)^{op} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$, it follows that

we have a pullback square

$$\begin{array}{ccc} (\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \longrightarrow & (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} \\ \downarrow & & \downarrow{\scriptstyle s^{\mathrm{gpd}}} \\ (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} & \xrightarrow[x^{\mathrm{gpd}}]{} & \mathbf{W}^{\mathrm{gpd}} \end{array}$$

in $\mathcal{S}$, and hence we can simplify the above limit computing $\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$ to give the identification

$$\hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y) \simeq \lim \left(\begin{array}{ccc} & & (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} \\ & & \downarrow{\scriptstyle y} \\ (\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \xrightarrow[t^{\mathrm{gpd}}]{} & \mathbf{W}^{\mathrm{gpd}} \end{array} \right).$$

Then, again by Theorem C_n (3.4.26), the functor

$$(\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{op} \xrightarrow{t^{op}} \mathbf{W}^{op}$$

has property B_3, so that by Theorem B_n (3.4.23) we have a homotopy pullback square

$$\begin{array}{ccc} (t^{op}((\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{op}) \downarrow_3 y^\circ((\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op})) & \xrightarrow{t} & (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle y^\circ} \\ (\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{op} & \xrightarrow[t^{op}]{} & \mathbf{W}^{op} \end{array}$$

in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$; this time, unwinding the definitions we can identify the homotopy pullback as

$$(\mathrm{Fun}_{**}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A};\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1}],\mathcal{M})^{\mathbf{W}})^{op},$$

where the objects $x, y \in \mathcal{M}$ determine the double-pointing. Hence we obtain an equivalence

$$\underline{\mathbf{7}}(x,y)^{\mathrm{gpd}} = (\mathrm{Fun}_{**}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A};\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1}],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} \xrightarrow{\sim} \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y),$$

as desired. □

We now provide a result which was needed in the proof of Proposition 6.9.1.

Lemma 6.9.2. *If $(\mathcal{R},\mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ admits a homotopical three-arrow calculus and $\mathbf{W} \subset \mathcal{R}$ has the two-out-of-three property, then $\mathbf{W}^{op}$ has property C_3.*

Proof. To show that $\mathbf{W}^{op}$ has property C_3, we must show that any functor $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{r^\circ} \mathbf{W}^{op}$ (selecting an object $r^\circ \in \mathbf{W}^{op}$) has property B_3, i.e. that the induced functor

$$\mathbf{W}^{op} \xrightarrow{(r^\circ(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})\downarrow_3 -)} \mathcal{C}\mathrm{at}_\infty$$

has property Q, i.e. that for any map $z^\circ \xrightarrow{\varphi^\circ} y^\circ$ in $\mathbf{W}^{op}$ (opposite to a map $z \xleftarrow{\varphi} y$ in $\mathbf{W}$), the induced map

$$(r^\circ(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}) \downarrow_3 z^\circ) \to (r^\circ(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}) \downarrow_3 y^\circ)$$

is in $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$. Unwinding the definitions, we can identify this map simply as the functor

$$\underline{\mathbf{3}}_{(\mathbf{W},\mathbf{W})}(r, z) \to \underline{\mathbf{3}}_{(\mathbf{W},\mathbf{W})}(r, y)$$

that postconcatenates a zigzag $r \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} z$ with the map φ (considered as a $[\mathbf{W}^{-1}]$-shaped zigzag) and then composes the last two maps.[10] Thus, the nerve of the above map in $\mathcal{C}\mathrm{at}_\infty$ sits as the upper composite in a commutative square

$$\begin{array}{ccccc}
\mathrm{N}_\infty(\underline{\mathbf{3}}(r,z)) & \longrightarrow & \mathrm{N}_\infty([\mathbf{W}^{-1}; \mathbf{A}; (\mathbf{W}^{-1})^{\circ 2}](r,y)) & \longrightarrow & \mathrm{N}_\infty(\underline{\mathbf{3}}(r,y)) \\
\wr\downarrow & & & & \wr\downarrow \\
\underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(r,z) & & \xrightarrow[\chi^{\mathscr{L}^H(\mathbf{W},\mathbf{W})}_{r,z,y}(-,\varphi^{-1})]{\approx} & & \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(r,y)
\end{array}$$

in $s\mathcal{S}_{\mathrm{KQ}}$, in which

- the lower map
  - is the evaluation of the composition map

    $$\underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(y,z) \times \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(z,r) \xrightarrow{\chi^{\mathscr{L}^H(\mathbf{W},\mathbf{W})}_{z,y,r}} \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(y,r)$$

    in $\mathscr{L}^H(\mathbf{W},\mathbf{W}) \in \mathcal{C}\mathrm{at}_{s\mathcal{S}}$ (recall Definition 4.1.8) at the point chosen by the composite

    $$\mathrm{pt}_{s\mathcal{S}} \to \mathrm{N}_\infty([\mathbf{W}^{-1}](z,y)) \to \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(z,y)$$

    in which the first map is selected by φ and the second map is the defining inclusion into the colimit, and

[10] Recall that $\underline{\mathbf{z}}_3 = (s \to \bullet \leftarrow \bullet \to t)$ (see Notation 3.4.14) while $\underline{\mathbf{3}} = (s \xleftarrow{\approx} \bullet \to \bullet \xleftarrow{\approx} t)$, so there are *two* orientation-reversals going on here (counting the passage between $\mathbf{W}^{op}$ and $\mathbf{W}$), which cancel each other out.

– lies in $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ by Proposition 4.4.8,

- the triangle commutes by the definition of the hammock simplicial space as a colimit over $\mathcal{Z}^{op}$ (see Definition 4.2.17),
- the trapezoid commutes by the definition of composition in the hammock localization (see §4.4), and
- the vertical maps are in $\mathbf{W}_{\mathrm{KQ}}$ by the fundamental theorem of homotopical three-arrow calculi (4.3.4) since the relative ∞-category $(\mathbf{W}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ admits a homotopical three-arrow calculus by Lemma 6.9.3.

The upper map is therefore also in $\mathbf{W}_{\mathrm{KQ}}$ since $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ has the two-out-of-three property, and hence the result follows from Proposition 2.2.4. □

In the proof of Lemma 6.9.2, we needed the following stability property of homotopical three-arrow calculi.

Lemma 6.9.3. *If $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ admits a homotopical three-arrow calculus and $\mathbf{W} \subset \mathcal{R}$ has the two-out-of-three property, then $(\mathbf{W}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ also admits a homotopical three-arrow calculus.*

Proof. This follows directly from Definition 4.3.1: if $\mathbf{W} \subset \mathcal{R}$ has the two-out-of-three property, then the vertical maps in the commutative square

$$\begin{array}{ccc} \mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathbf{W})^{\mathbf{W}} & \longrightarrow & \mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathbf{W})^{\mathbf{W}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}} & \longrightarrow & \mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}} \end{array}$$

induced by the map $(\mathbf{W}, \mathbf{W}) \to (\mathcal{R}, \mathbf{W})$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ induce monomorphisms in $\mathcal{S}$ upon groupoid completion. □

6.10 Localization of model ∞-categories, redux

For completeness, we include the following improvement of Proposition 6.8.1, whose proof relies on the fundamental theorem of model ∞-categories (6.1.9).

Theorem 6.10.1. *If $\mathcal{M}$ is a model ∞-category with underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$, then $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M}, \mathbf{W}) \in \mathcal{CSS}$, and moreover the morphism $\mathrm{N}_\infty(\mathcal{M}) \to \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M}, \mathbf{W})$ in $\mathcal{CSS}$ corresponds to the morphism $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$.*

Proof. In light of Proposition 6.8.1, it only remains to show that $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{M}, \mathbf{W})$ is not just a Segal space, but is in fact complete. By the calculus theorem (4.5.1(2)), this follows from Lemma 6.10.2 and the fact that $\mathbf{W} \subset \mathcal{M}$ satisfies the two-out-of-three property. □

We needed the following result in the proof of Theorem 6.10.1.

Lemma 6.10.2. *If $\mathcal{M}$ is a model ∞-category, then its underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$ is saturated.*

Proof. We would like to show that the localization functor $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$ creates the subcategory $\mathbf{W} \subset \mathcal{M}$. This is equivalent to showing that the functor $\mathrm{ho}(\mathcal{M}) \to \mathrm{ho}(\mathcal{M}[\![\mathbf{W}^{-1}]\!])$ creates the subcategory $\mathrm{ho}(\mathbf{W}) \subset \mathrm{ho}(\mathcal{M})$. For this, we must show that if a map $x \to y$ in $\mathrm{ho}(\mathcal{M})$ is taken to an isomorphism in $\mathrm{ho}(\mathcal{M}[\![\mathbf{W}^{-1}]\!])$, then it lies in the subcategory $\mathrm{ho}(\mathbf{W})$. By two-out-of-three axiom $\mathrm{M}_{\infty}2$, it suffices to show this in the case that both objects $x, y \in \mathcal{M}^{cf} \subset \mathcal{M}$ are bifibrant. From here, with Corollary 6.1.11 in hand, the proof runs identically to that of [Hir03, Theorem 7.8.5]. □

Chapter 7

Goerss–Hopkins obstruction theory for ∞-categories

In this chapter, we construct Goerss–Hopkins obstruction theory for an arbitrary presentably symmetric monoidal stable ∞-category.

7.0 Introduction

For now, we refer the reader to to §0.3 (particularly §§0.3.3 and 0.3.4) for an overview; a future version of this thesis (available on the author's webpage) will contain a proper introduction to this chapter.

7.0.1 Acknowledgments

This chapter has provided the underlying motivation for much of the material in this thesis, and as a result, anyone who has contributed to another chapter has effectively contributed to this chapter as well.

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7.1 The resolution model structure

We lift results from [GJ09, Chapter II] in order to provide sufficient conditions for the existence of certain simplicial model ∞-category structures.

Remark 7.1.1. In this section, we will be constructing certain *resolution* model structures. These are closely related to the model structures of [DKS93] and [Bou03]; indeed, it is straightforward (but tedious) to verify that the proof of [Bou03, Theorem 3.3] immediately generalizes to an arbitrary right proper model ∞-category $\mathcal{M}$ equipped with a set of h-cogroup objects (in the model ∞-categorical sense). However, those model structures are in a sense more difficult: they're built by modifying $(s\mathcal{M})_{\mathrm{Reedy}}$, and in the end the fibrant objects are exactly the Reedy fibrant objects. By contrast, using model ∞-categories effectively allows us to obtain a model structure presenting the desired ∞-category by starting with a *trivial* model ∞-category (so that the Reedy model structure on simplicial objects therein will also be trivial).

7.1.1 Enrichments and bitensorings in the presence of presentability

We begin by providing sufficient conditions for constructing enrichments and bitensorings among presentable ∞-categories, and for lifting adjunctions between ∞-

categories equipped with these to enriched adjunctions.

Proposition 7.1.2. *Let $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^L)$ be a presentably monoidal ∞-category, and let $\mathcal{D} \in \mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)$ be a presentable ∞-category equipped with a left action of $\mathcal{V}$. Then this action $- \odot - : \mathcal{V} \times \mathcal{D} \to \mathcal{D}$ extends to an enrichment and bitensoring of $\mathcal{D}$ over $\mathcal{V}$, encoded by a two-variable adjunction*

$$\left(\mathcal{V} \times \mathcal{D} \xrightarrow{-\odot-} \mathcal{D} \ , \ \mathcal{V}^{op} \times \mathcal{D} \xrightarrow{-\pitchfork-} \mathcal{D} \ , \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{\underline{\hom}_{\mathcal{D}}(-,-)} \mathcal{V} \right).$$

Proof. The fact that the action takes place in the symmetric monoidal ∞-category Pr^L guarantees that it commutes with colimits separately in each variable. From here, presentability guarantees the co/representability required by the definition of a two-variable adjunction. □

Lemma 7.1.3. *Let $\mathcal{D}$ be a bicomplete ∞-category, and let $\mathcal{I} \in \mathcal{C}\mathrm{at}_\infty$ be a diagram ∞-category. Then the levelwise tensoring of $\mathrm{Fun}(\mathcal{I}, \mathcal{D})$ over $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ commutes with colimits separately in each variable and extends to an action $\mathrm{Fun}(\mathcal{I}, \mathcal{D}) \in \mathrm{LMod}_{\mathrm{Fun}(\mathcal{I},\mathcal{S})}(\mathrm{Pr}^L)$.*

Proof. The levelwise tensoring is given by the composite

$$\mathrm{Fun}(\mathcal{I}, \mathcal{S}) \times \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{I}, \mathcal{S} \times \mathcal{D}) \xrightarrow{\mathrm{Fun}(\mathcal{I}, -\odot-)} \mathrm{Fun}(\mathcal{I}, \mathcal{D});$$

indeed, we obtain $\mathrm{Fun}(\mathcal{I}, \mathcal{D}) \in \mathrm{LMod}_{\mathrm{Fun}(\mathcal{I},\mathcal{S})}(\mathcal{C}\mathrm{at}_\infty)$ by applying $\mathrm{Fun}(\mathcal{I}, -)$ to the data of $\mathcal{D} \in \mathrm{LMod}_{\mathcal{S}}(\mathcal{C}\mathrm{at}_\infty)$. Moreover, by definition the tensoring $- \odot - : \mathcal{S} \times \mathcal{D} \to \mathcal{D}$ commutes with colimits separately in each variable; as colimits in a functor ∞-category are computed pointwise, the above composite commutes with colimits separately in each variable as well. □

Corollary 7.1.4. *For any $\mathcal{D} \in \mathrm{Pr}^L$, the levelwise tensoring of $s\mathcal{D}$ over $s\mathcal{S}$ extends to an enrichment and bitensoring.*

Proof. By Lemma 7.1.3, the levelwise tensoring defines an action $s\mathcal{D} \in \mathrm{Mod}_{s\mathcal{S}}(\mathrm{Pr}^L)$, and so the claim follows from Proposition 7.1.2. □

Observation 7.1.5. Given two ∞-categories $\mathcal{D}$ and $\mathcal{E}$, one can define an adjunction $\mathcal{D} \rightleftarrows \mathcal{E}$ to be a functor $A : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{S}$ satisfying certain co/representability conditions (see item A(25)). If for some closed monoidal ∞-category $\mathcal{V}$ these ∞-categories are equipped with lifts $\underline{\mathcal{D}}$ and $\underline{\mathcal{E}}$ to $\mathcal{V}$-enriched ∞-categories, then an enriched adjunction $\underline{\mathcal{D}} \rightleftarrows \underline{\mathcal{E}}$ can be defined as a functor $\underline{A} : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{V}$ satisfying analogous co/representability conditions. (This recovers an ordinary adjunction between the underlying unenriched ∞-categories by postcomposition with the functor $\underline{\hom}_{\mathcal{V}}(\mathbf{1}_{\mathcal{V}}, -) : \mathcal{V} \to \mathcal{S}$.)

Lemma 7.1.6. *Let $\mathcal{V} \in \mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)$ be a presentable monoidal ∞-category, suppose that two ∞-categories $\mathcal{D}$ and $\mathcal{E}$ are enriched and bitensored over $\mathcal{V}$, and suppose we are given an adjunction $F : \mathcal{D} \rightleftarrows \mathcal{E} : G$ between their underlying ∞-categories. Suppose further that we have a natural equivalence $F(- \odot_{\mathcal{D}} -) \simeq (-) \odot_{\mathcal{E}} F(-)$ in $\mathrm{Fun}(\mathcal{V} \times \mathcal{D}, \mathcal{E})$. Then the adjunction $F \dashv G$ lifts to a $\mathcal{V}$-enriched adjunction $\underline{F} : \underline{\mathcal{D}} :\rightleftarrows \underline{\mathcal{E}} : \underline{G}$, and moreover we have a natural equivalence $G(- \pitchfork_{\mathcal{E}} -) \simeq (-) \pitchfork_{\mathcal{D}} G(-)$ in $\mathrm{Fun}(\mathcal{V}^{op} \times \mathcal{E}, \mathcal{D})$.*

Proof. First of all, the final claim follows from our assumption (and the Yoneda lemma) by the string of natural equivalences

$$\begin{aligned}\hom_{\mathcal{D}}(d, G(v \pitchfork_{\mathcal{E}} e)) &\simeq \hom_{\mathcal{E}}(F(d), v \pitchfork_{\mathcal{E}} e) \simeq \hom_{\mathcal{E}}(v \odot_{\mathcal{E}} F(d), e) \\ &\simeq \hom_{\mathcal{E}}(F(v \odot_{\mathcal{D}} d), e) \simeq \hom_{\mathcal{D}}(v \odot_{\mathcal{D}} d, G(e)) \\ &\simeq \hom_{\mathcal{D}}(d, v \pitchfork_{\mathcal{D}} G(e)).\end{aligned}$$

Now, consider the functor $\mathcal{D}^{op} \times \mathcal{E} \to \mathcal{P}(\mathcal{V})$ which takes a pair of objects $(d^\circ, e) \in \mathcal{D}^{op} \times \mathcal{E}$ to the presheaf taking $v^\circ \in \mathcal{V}^{op}$ to the space

$$\hom_{\mathcal{D}}(v \odot_{\mathcal{D}} d, Ge) \simeq \hom_{\mathcal{E}}(F(v \odot_{\mathcal{D}} d), e) \simeq \hom_{\mathcal{E}}(v \odot_{\mathcal{E}} F(d), e) \simeq \hom_{\mathcal{E}}(F(d), v \pitchfork_{\mathcal{E}} e).$$

Since $\mathcal{V}$ is presentable, this factors through the Yoneda embedding $\mathcal{V} \overset{\text{よ}\mathcal{V}}{\hookrightarrow} \mathcal{P}(\mathcal{V})$. By construction, this defines an enriched adjunction $\underline{F} : \underline{\mathcal{D}} \rightleftarrows \underline{\mathcal{E}} : \underline{G}$ lifting the original adjunction $F \dashv G$. □

Corollary 7.1.7. *For any $\mathcal{D} \in \mathrm{Pr}^L$ and any monad $t \in \mathrm{Alg}(\mathrm{End}(s\mathcal{D}))$, we obtain a canonical enrichment and bitensoring of $\mathrm{Alg}_t(s\mathcal{D})$ over $s\mathcal{S}$, and moreover the adjunction $\mathrm{F}_t : s\mathcal{D} \rightleftarrows \mathrm{Alg}_t(s\mathcal{D}) : \mathrm{U}_t$ is canonically enriched over $s\mathcal{S}$.*

Proof. As any object of $\mathrm{Alg}_t(s\mathcal{D})$ is a colimit of free objects, for any $K \in s\mathcal{S}$ and any $Y \in \mathrm{Alg}_t(s\mathcal{D})$ we define

$$K \odot Y = \mathrm{colim}_{(X \to \mathrm{U}_t(Y)) \in s\mathcal{N}_{/\mathrm{U}_t(Y)}} \mathrm{F}_t(K \odot X)$$

(using the action $s\mathcal{D} \in \mathrm{LMod}_{s\mathcal{S}}(\mathrm{Pr}^L)$ of Corollary 7.1.4). This defines a bifunctor $- \odot - : s\mathcal{S} \times \mathrm{Alg}_t(s\mathcal{D}) \to \mathrm{Alg}_t(s\mathcal{D})$ which by construction commutes with colimits separately in each variable. Thus it defines an action $\mathrm{Alg}_t(s\mathcal{D}) \in \mathrm{LMod}_{s\mathcal{S}}(\mathrm{Pr}^L)$, and so by Proposition 7.1.2 extends to an enrichment and bitensoring of $\mathrm{Alg}_t(s\mathcal{D})$ over $s\mathcal{S}$. Then, the enrichment of the adjunction $\mathrm{F}_t \dashv \mathrm{U}_t$ follows from Lemma 7.1.6. □

7.1.2 Simplicial model structures

We now provide a lifting theorem for constructing simplicial model ∞-category structures. This requires two auxiliary pieces of terminology.

Definition 7.1.8. Given a set I of homotopy classes of maps in $\mathcal{C}$, the subcategory I-proj of ***I-projectives*** is the subcategory of maps with $\mathrm{llp}(I)$.

Definition 7.1.9. Let $\mathcal{V}$ be a monoidal model ∞-category, and suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{V}$-enriched model ∞-categories. Then a ***$\mathcal{V}$-enriched Quillen adjunction*** between $\mathcal{M}$ and $\mathcal{N}$ is a $\mathcal{V}$-enriched adjunction $\underline{F} : \underline{\mathcal{M}} \rightleftarrows \underline{\mathcal{N}} : \underline{G}$ such that the underlying adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen adjunction.

Theorem 7.1.10. *Let $\mathcal{M}$ be a bicomplete ∞-category, and let $F : s\mathcal{S} \rightleftarrows \mathcal{M} : G$ be an adjunction such that G commutes with filtered colimits. Write $\mathbf{W}^{\mathcal{M}} = G^{-1}(\mathbf{W}^{s\mathcal{S}}_{\mathrm{KQ}})$, $\mathbf{F}^{\mathcal{M}} = G^{-1}(\mathbf{F}^{s\mathcal{S}}_{\mathrm{KQ}})$, and $\mathbf{C}^{\mathcal{M}} = (\mathbf{W} \cap \mathbf{F})^{\mathcal{M}}$-proj. Suppose that the following condition holds:*

$$\big(\mathbf{C}^{\mathcal{M}} \cap (\mathbf{F}^{\mathcal{M}}\text{-proj})\big) \subset \mathbf{W}^{\mathcal{M}}. \tag{$*$}$$

*Then $\mathcal{M}$ admits a **resolution model structure**, denoted $\mathcal{M}_{\mathrm{res}}$, with $\mathbf{W}^{\mathcal{M}}_{\mathrm{res}} = \mathbf{W}^{\mathcal{M}}$, $\mathbf{C}^{\mathcal{M}}_{\mathrm{res}} = \mathbf{C}^{\mathcal{M}}$, and $\mathbf{F}^{\mathcal{M}}_{\mathrm{res}} = \mathbf{F}^{\mathcal{M}}$, and the above adjunction becomes a Quillen adjunction $F : s\mathcal{S}_{\mathrm{KQ}} \rightleftarrows \mathcal{M}_{\mathrm{res}} : G$.*

Proof. The proof is almost identical to that of [GJ09, Theorem II.4.1] (despite the fact that there they only work in the special case of a category of simplicial objects); the only modification which must be made is that in the proofs of [GJ09, Lemmas II.4.2 and II.4.3] (which construct required factorizations) one must take a coproduct over *homotopy classes* of commutative squares. □

Remark 7.1.11. In practice, there seems to more-or-less always be (at least) one thing that's difficult to check in constructing a model structure. In this case, condition $(*)$ of Theorem 7.1.10 effectively requires that those would-be cofibrations that moreover have the left lifting property for all would-be fibrations are also would-be weak equivalences. We will give sufficient conditions for this condition to hold in §7.1.3.

Remark 7.1.12. It follows from the proof of Theorem 7.1.10 that one can replace the condition $(*)$ with the following pair of conditions:

- $(*')$ for every map $\Lambda^n_i \to \Delta^n$ in $J^{s\mathcal{S}}_{\mathrm{KQ}}$, the induced map $F(\Lambda^n_i) \to F(\Delta^n)$ lies in $\mathbf{W}^{\mathcal{M}} \subset \mathcal{M}$;
- $(*'')$ the maps in $(\mathbf{W} \cap \mathbf{C})^{\mathcal{M}}$ are closed under coproducts, pushouts, and sequential colimits.

This is explained in [GJ09, Remark II.4.5].

Theorem 7.1.13. *In the setting of Theorem 7.1.10, suppose that we have an action $\mathcal{M} \in \mathrm{LMod}_{s\mathcal{S}}(\mathcal{C}\mathrm{at}_\infty)$, denoted $- \odot - : s\mathcal{S} \times \mathcal{M} \to \mathcal{M}$, such that this bifunctor commutes with colimits separately in each variable, and suppose that we have a natural equivalence $F(- \times -) \simeq (-) \odot F(-)$ in $\mathrm{Fun}(s\mathcal{S} \times s\mathcal{S}, \mathcal{M})$. Then the resolution model structure canonically enhances to a simplicial model ∞-category $\underline{\mathcal{M}}_{\mathrm{res}}$, and the Quillen adjunction canonically enhances to an $s\mathcal{S}_{\mathrm{KQ}}$-enriched Quillen adjunction $\underline{F} : \underline{s\mathcal{S}}_{\mathrm{KQ}} \rightleftarrows \underline{\mathcal{M}}_{\mathrm{res}} : \underline{G}$.*

Proof. Using Lemma 7.1.6, the proof is identical to that of [GJ09, Theorem II.4.4]. □

7.1.3 Sufficient criteria for the satisfaction of condition $(*)$ of Theorem 7.1.10

We now provide various conditions guaranteeing that condition $(*)$ of Theorem 7.1.10 is satisfied.

The key result is the following.

Proposition 7.1.14. *In the setting of Theorem 7.1.10, suppose that there exists an endofunctor $\mathbb{R} : \mathcal{M} \to \mathcal{M}$ which factors through the subcategory $\mathbf{F}^{\mathcal{M}} \subset \mathcal{M}$ and which admits a map $\mathrm{id}_{\mathcal{M}} \to \mathbb{R}$ whose components lie in $\mathbf{W}^{\mathcal{M}}$. Then condition $(*)$ holds.*

Proof. The proof is identical to that of [GJ09, Lemma II.5.1]. □

Corollary 7.1.15. *In the setting of Theorem 7.1.10, suppose that for every object $X \in \mathcal{M}$ the terminal map $X \to \mathrm{pt}_{\mathcal{M}}$ lies in $\mathbf{F}^{\mathcal{M}}$. Then condition $(*)$ holds.*

Proof. This follows from Proposition 7.1.14, taking $\mathbb{R} = \mathrm{id}_{\mathcal{M}}$ (equipped with the identity coaugmentation). □

Corollary 7.1.16. *Let $\mathcal{N}$ be a bicomplete ∞-category, and for any object $Z \in \mathcal{N}$ consider the adjunction*

$$- \odot \mathrm{const}(Z) : s\mathcal{S} \rightleftarrows s\mathcal{N} : \mathrm{hom}^{\mathrm{lw}}_{\mathcal{N}}(Z, -).$$

If the object $Z \in \mathcal{N}$ is small, then this adjunction satisfies condition $(*)$ of Theorem 7.1.10.

Proof. With the theory of the Ex^∞ functor for $s\mathcal{S}_{\mathrm{KQ}}$ of §1.6 in hand (specifically Proposition 1.6.22 and Remark 1.6.23), this follows from Proposition 7.1.14 by an identical argument to that of [GJ09, Proposition II.5.5]. □

Remark 7.1.17. The technique of Corollary 7.1.16 cannot work for a general (bicomplete) ∞-category equipped with a right adjoint functor to $s\mathcal{S}$: it must be an ∞-category of simplicial objects. In effect, this is because the endofunctor Ex is a right adjoint, but it is not an *enriched* right adjoint. Indeed, the functor $\underline{\hom}_{s\mathcal{S}}(\Delta^1, -) : s\mathcal{S} \to s\mathcal{S}$ is an example of an enriched limit and so commutes with any enriched right adjoint, but the canonical map $\mathrm{Ex}(\underline{\hom}_{s\mathcal{S}}(\Delta^1, -)) \to \underline{\hom}_{s\mathcal{S}}(\Delta^1, \mathrm{Ex}(-))$ is not an equivalence; this can be seen by evaluating on Δ^1, since the source has three 0-simplices but the target has five.

Corollary 7.1.18. *Let $\mathcal{N} \in \mathrm{Pr}^L$, and let $Z \in \mathcal{N}$ be a small object. Then with the enrichment and bitensoring of $s\mathcal{N}$ over $s\mathcal{S}$ of Corollary 7.1.4, there exists a simplicial model structure on $s\mathcal{N}$ created by the $s\mathcal{S}$-enriched Quillen adjunction*

$$- \odot \mathrm{const}(Z) : \underline{s\mathcal{S}}_{\mathrm{KQ}} \rightleftarrows \underline{s\mathcal{N}}_{\mathrm{res}} : \hom^{\mathrm{lw}}_{\mathcal{N}}(Z, -).$$

Proof. By Corollary 7.1.16, this adjunction satisfies condition $(*)$ of Theorem 7.1.10 and hence creates a model structure on $s\mathcal{N}$. By Lemma 7.1.3, this adjunction furthermore satisfies the hypotheses of Theorem 7.1.13, so that $s\mathcal{N}_{\mathrm{res}}$ and the Quillen adjunction becomes compatibly $s\mathcal{S}_{\mathrm{KQ}}$-enriched. □

We will also be interested in the following "many-object" version of Corollary 7.1.18.

Theorem 7.1.19. *Let $\mathcal{N} \in \mathrm{Pr}^L$, and suppose we are given a set of small objects $Z_\alpha \in \mathcal{N}$. Then with the enrichment and bitensoring of $s\mathcal{N}$ over $s\mathcal{S}$ of Corollary 7.1.4, there exists a simplicial model structure on $s\mathcal{N}$ created by the $s\mathcal{S}$-enriched Quillen adjunction*

$$\coprod_\alpha \mathrm{pr}_\alpha(-) \odot \mathrm{const}(Z_\alpha) : \prod_\alpha \underline{s\mathcal{S}}_{\mathrm{KQ}} \rightleftarrows \underline{s\mathcal{N}}_{\mathrm{res}} : \left(\hom^{\mathrm{lw}}_{\mathcal{N}}(Z_\alpha, -)\right).$$

Proof. Given the above results, the proof is essentially identical to that of [GJ09, Proposition II.5.9]. □

Remark 7.1.20. In Theorem 7.1.19, if the objects Z_α form a set of compact projective generators (in the sense of Definition T.5.5.8.23) and the ∞-category $\mathcal{N}$ has enough projectives, then weak equivalences and fibrations in $s\mathcal{N}_{\mathrm{res}}$ will be detected by *all* projective objects (see [GJ09, Example II.5.10]).

We now identify the underlying ∞-category of the resolution model structure of Theorem 7.1.19.

Definition 7.1.21. For an ∞-category $\mathcal{D}$ admitting finite coproducts, we write $\mathcal{P}_\Sigma(\mathcal{D}) = \mathrm{Fun}^\times(\mathcal{D}^{op}, \mathcal{S})$ for its ***nonabelian derived ∞-category*** of product-preserving presheaves (i.e. of functors taking finite coproducts in $\mathcal{D}$ to finite products in $\mathcal{S}$). We write $\mathcal{P}^\delta_\Sigma(\mathcal{D}) \subset \mathcal{P}_\Sigma(\mathcal{D})$ for its subcategory of discrete objects; thus $\mathcal{P}^\delta_\Sigma(\mathcal{D}) \simeq \mathrm{Fun}^\times(\mathcal{D}^{op}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Fun}^\times(\mathrm{ho}(\mathcal{D})^{op}, \mathcal{S}\mathrm{et})$.

Theorem 7.1.22. *In the situation of Theorem 7.1.19, writing $\mathcal{G} \subset \mathcal{N}$ for the full subcategory generated by the objects Z_α under finite coproducts, we have a canonical Quillen adjunction*

$$\mathrm{Fun}(\mathcal{G}^{op}, s\mathcal{S}_{\mathrm{KQ}})_{\mathrm{proj}} \rightleftarrows s\mathcal{N}_{\mathrm{res}}$$

with derived adjunction given by the canonical adjunction

$$\mathcal{P}(\mathcal{G}) \rightleftarrows \mathcal{P}_\Sigma(\mathcal{G})$$

whose right adjoint is the defining inclusion.

Proof. The projective model structure can also be seen as lifted via Theorem 7.1.19 from the same product of copies of the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$, which implies that this is indeed a Quillen adjunction. As the functor $|-| : s\mathcal{S} \to \mathcal{S}$ commutes with finite products, it follows that the derived right adjoint factors through the subcategory $\mathcal{P}_\Sigma(\mathcal{G}) \subset \mathcal{P}(\mathcal{G})$. Moreover, as $\mathcal{N}$ is presentable, the restricted Yoneda embedding participates in an adjunction $\mathcal{P}_\Sigma(\mathcal{G}) \rightleftarrows \mathcal{N}$, from which it follows that this derived right adjoint surjects onto $\mathcal{P}_\Sigma(\mathcal{G})$ (by taking the constant simplicial object on a given object of $\mathcal{N}$, seen as a product-preserving presheaf on $\mathcal{G}$). So, it will suffice to show that the functor $s\mathcal{N}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \to \mathcal{P}(\mathcal{G})$ is fully faithful. First of all, taking any $X \in s\mathcal{N}^f_{\mathrm{res}}$, since $s\mathcal{N}_{\mathrm{res}}$ is simplicial, for any $K \in s\mathcal{S}\mathrm{et} = s\mathcal{S}^c_{\mathrm{KQ}}$ we have that

$$\begin{aligned}
\hom_{s\mathcal{N}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]}(K \odot \mathrm{const}(Z_\alpha), X) &\simeq \underline{\hom}_{s\mathcal{N}}(K \odot \mathrm{const}(Z_\alpha), X) \\
&\simeq \underline{\hom}_{s\mathcal{S}}(K, \underline{\hom}_{s\mathcal{N}}(\mathrm{const}(Z_\alpha), X)) \\
&\simeq \underline{\hom}_{s\mathcal{S}}(K, \hom^{\mathrm{lw}}_{\mathcal{N}}(Z_\alpha, X)) \\
&\simeq \hom_{\mathcal{S}}(|K|, |\hom^{\mathrm{lw}}_{\mathcal{N}}(Z_\alpha, X)|)
\end{aligned}$$

(where the last equivalence uses the fact that $s\mathcal{S}_{\mathrm{KQ}}$ is a simplicial model ∞-category). The claim now follows from the fact that $I^{s\mathcal{N}}_{\mathrm{res}} = \{I^{s\mathcal{S}}_{\mathrm{KQ}} \odot \mathrm{const}(Z_\alpha)\}$ forms a set of generating cofibrations of $s\mathcal{N}_{\mathrm{res}}$, so that we can construct a cofibrant replacement of any object as a transfinite composition of pushouts of these maps. □

We end this subsection with the following result, which gives a convenient class of examples for which the condition of Corollary 7.1.15 holds (i.e. that all objects are ("would-be") fibrant). It is an ∞-categorical analog of the classical fact that every simplicial group is in particular a Kan complex.

Lemma 7.1.23. *In the adjunction* $\mathrm{F}_{s\mathcal{G}\mathrm{rp}(\mathcal{S})} : s\mathcal{S} \rightleftarrows s\mathcal{G}\mathrm{rp}(\mathcal{S}) : \mathrm{U}_{s\mathcal{G}\mathrm{rp}(\mathcal{S})}$*, the right adjoint factors through the subcategory* $s\mathcal{S}^f_{\mathrm{KQ}} \subset s\mathcal{S}$ *of fibrant objects with respect to the Kan–Quillen model structure.*

Proof. Observe that the adjunction $\mathrm{F}_{\mathcal{G}\mathrm{rp}(\mathcal{S})} : \mathcal{S} \rightleftarrows \mathcal{G}\mathrm{rp}(\mathcal{S}) : \mathrm{U}_{\mathcal{G}\mathrm{rp}(\mathcal{S})}$ factors as the composite adjunction

$$\mathcal{S} \underset{\mathrm{U}_{\mathrm{Mon}(\mathcal{S})}}{\overset{\mathrm{F}_{\mathrm{Mon}(\mathcal{S})}}{\rightleftarrows}} \mathrm{Mon}(\mathcal{S}) \underset{}{\overset{(-)^{\mathrm{gp}}}{\rightleftarrows}} \mathcal{G}\mathrm{rp}(\mathcal{S}).$$

We claim that the diagram

$$\begin{array}{ccccc} \mathcal{S}\mathrm{et} & \xrightarrow{\mathrm{F}_{\mathrm{Mon}}} & \mathrm{Mon} & \xrightarrow{(-)^{\mathrm{gp}}} & \mathcal{G}\mathrm{rp} \\ \downarrow & & & & \downarrow \\ \mathcal{S} & \xrightarrow[\mathrm{F}_{\mathrm{Mon}(\mathcal{S})}]{} & \mathrm{Mon}(\mathcal{S}) & \xrightarrow[(-)^{\mathrm{gp}}]{} & \mathcal{G}\mathrm{rp}(\mathcal{S}) \end{array}$$

commutes.[1] Indeed, recall the factorization

$$\begin{array}{ccc} \mathrm{Mon}(\mathcal{S}) & \xrightarrow{(-)^{\mathrm{gp}}} & \mathcal{G}\mathrm{rp}(\mathcal{S}), \\ & {\scriptstyle B}\searrow \quad \nearrow{\scriptstyle \Omega} & \\ & \mathcal{S}_* & \end{array}$$

and recall that the functor $\mathrm{Mon}(\mathcal{S}) \xrightarrow{B} \mathcal{S}_*$ can itself be obtained as the composite

$$\mathrm{Mon}(\mathcal{S}) \xrightarrow{\mathfrak{B}} (\mathcal{C}\mathrm{at}_\infty)_* \xrightarrow{(-)^{\mathrm{gpd}}} (\mathcal{G}\mathrm{pd}_\infty)_* \simeq \mathcal{S}_*$$

(where $\mathfrak{B}$ denotes the "categorical delooping" functor). The claim now follows from the commutativity of the diagram

$$\begin{array}{ccccccc} \mathcal{S}\mathrm{et} & \xrightarrow{\mathrm{F}_{\mathrm{Mon}}} & \mathrm{Mon} & \xrightarrow{\mathfrak{B}} & \mathcal{C}\mathrm{at}_* & \xrightarrow{(-)^{\mathrm{gpd}}} & \mathcal{G}\mathrm{pd}_* \\ \downarrow & & & & & & \downarrow \\ \mathcal{S} & \xrightarrow[\mathrm{F}_{\mathrm{Mon}(\mathcal{S})}]{} & \mathrm{Mon}(\mathcal{S}) & \xrightarrow[\mathfrak{B}]{} & (\mathcal{C}\mathrm{at}_\infty)_* & \xrightarrow[(-)^{\mathrm{gpd}}]{} & (\mathcal{G}\mathrm{pd}_\infty)_*, \end{array}$$

[1] If we were to add in the middle vertical inclusion $\mathrm{Mon} \hookrightarrow \mathrm{Mon}(\mathcal{S})$, the left square would commute (simply by inspection of the functor $\mathrm{F}_{\mathrm{Mon}(\mathcal{S})}$), but the right square would not: its extreme failure to do so is encoded by [McD79, Theorem 1].

which itself follows from [DK80c, 5.4].

Now, applying $\mathrm{Fun}(\mathbf{\Delta}^{op}, -)$ to the original commutative rectangle, we obtain a commutative square

$$\begin{array}{ccc} s\mathcal{S}\mathrm{et} & \xrightarrow{\mathrm{F}_{s\mathcal{G}\mathrm{rp}}} & s\mathcal{G}\mathrm{rp} \\ \downarrow & & \downarrow \\ s\mathcal{S} & \xrightarrow[\mathrm{F}_{s\mathcal{G}\mathrm{rp}(\mathcal{S})}]{} & s\mathcal{G}\mathrm{rp}(\mathcal{S}). \end{array}$$

In particular, the image of any element $\Lambda^n_i \to \Delta^n$ of $J^{s\mathcal{S}\mathrm{et}}_{\mathrm{KQ}} = J^{s\mathcal{S}}_{\mathrm{KQ}}$ under the composite

$$s\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S} \xrightarrow{\mathrm{F}_{s\mathcal{G}\mathrm{rp}(\mathcal{S})}} s\mathcal{G}\mathrm{rp}(\mathcal{S})$$

admits a retraction (see e.g. [GJ09, Lemma I.3.4]). This proves the claim. □

7.2 Topology

In this section, we lay out the basic topological framework (absent any operadic structure).

7.2.1 Foundations of topology

Assumption 7.2.1. We begin with a presentably symmetric monoidal stable ∞-category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1})$. By presentability, this will automatically be closed (i.e. admit an internal hom bifunctor).

Remark 7.2.2. When it is convenient, we will consider $\mathcal{C}$ as being enriched over the symmetric monoidal ∞-category $(\mathcal{S}_*, \wedge, S^0)$ of pointed spaces equipped with the smash product: the basepoint $0 \in \hom_{\mathcal{C}}(X, Y)$ is given by the unique "zero map" $X \to 0_{\mathcal{C}} \to Y$, and the fact that the composition maps factor through the smash products amounts to the observation that any sequence of composable maps in which at least one of the maps is a zero map composes canonically to another zero map. Moreover, $\mathcal{C}$ admits a canonical bitensoring over $\mathcal{S}_*$ which is compatible with this enrichment. (It is not hard to make these assertions precise using the formalism of [GHa].)

Notation 7.2.3. We write $\mathrm{D} = \underline{\hom}_{\mathcal{C}}(-, \mathbf{1}) : \mathcal{C}^{op} \to \mathcal{C}$ for the "linear dual" functor, and we write $\mathcal{C}^{\mathrm{inv}} \subset \mathcal{C}^{\mathrm{d}} \subset \mathcal{C}$ for the full subcategories of invertible objects and of dualizable objects.

Assumption 7.2.4. We assume that the unit object $\mathbf{1} \in \mathcal{C}$ is compact, i.e. that the functor $\hom_{\mathcal{C}}(\mathbf{1}, -) : \mathcal{C} \to \mathcal{S}$ commutes with filtered colimits.

Observation 7.2.5. It follows immediately from Assumption 7.2.4 that any invertible object of $\mathcal{C}$ is necessarily compact. In fact, because of the assumption that the symmetric monoidal structure commutes with colimits separately in each variable, it follows that any dualizable object is compact as well: this is a consequence of the natural equivalence $\hom_{\mathcal{C}}(X, -) \simeq \hom_{\mathcal{C}}(\mathbf{1}, \mathrm{D}X \otimes -)$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$.

Assumption 7.2.6. We assume the existence of a small subcategory $\mathcal{G} \subset \mathcal{C}$ of (strong) generators, which we generally denote by $S^\beta \in \mathcal{G}$ (with the "S" and "β" chosen to evoke the notion of a "bigraded sphere" (from motivic stable homotopy theory)); that is, we assume that the functors

$$\hom_{\mathcal{C}}(S^\beta, -) : \mathcal{C} \to \mathcal{S}$$

are jointly conservative. We moreover assume that the subcategory $\mathcal{G} \subset \mathcal{C}$

- contains the unit object $\mathbf{1} \in \mathcal{C}$,
- is closed under de/suspensions,
- consists of invertible objects, and
- is closed under the monoidal product of $\mathcal{C}$.

We write $S^{n+\beta} = \Sigma^n S^\beta$ for any $n \in \mathbb{Z}$.

Notation 7.2.7. We write $\mathcal{G}^\delta = \pi_0(\mathcal{G}^{\simeq}) \in \mathrm{Ab}\mathcal{G}\mathrm{rp}$ for the abelian group of equivalence classes of objects of $\mathcal{G}$, with addition given by the monoidal product of $\mathcal{C}$. We denote the element corresponding to $S^\beta \in \mathcal{G}$ simply by $\beta \in \mathcal{G}^\delta$.

Definition 7.2.8. For any $\beta \in \mathcal{G}^\delta$, we refer to the equivalence $S^\beta \otimes - : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ as the ***β-fold suspension***. The ordinary notion of suspension is recovered as $(\Sigma^n \mathbf{1}) \otimes - : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$. We will henceforth refer to *any* β-fold suspension as a "suspension", and refer to this latter more restrictive notion as a ***categorical suspension***. We denote β-fold suspension by Σ^β, and categorical suspension simply by Σ^n. (Note that these conventions jibe with those of Assumption 7.2.6.) While through this definition the term "desuspension" technically becomes superfluous, we will nevertheless continue to employ it for aesthetic reasons.

Notation 7.2.9. We write $\underline{\mathcal{A}} = \mathrm{Fun}(\mathcal{G}^\delta, \mathrm{Ab})$ for the category of $\mathcal{G}^\delta$-graded abelian groups, equipped with the Day convolution monoidal structure relative to $(\mathcal{G}^\delta, +) = (\mathcal{G}^\delta, \otimes_{\mathcal{C}})$ and $(\mathrm{Ab}, \otimes_{\mathbb{Z}})$. This receives a "homotopy" functor $\pi_* : \mathcal{C} \to \underline{\mathcal{A}}$, given by

$\pi_\beta X = (\pi_* X)(S^\beta) = [S^\beta, X]_{\mathcal{C}}$.[2] This functor is is itself lax monoidal, and in fact descends along the monoidal functor $\mathcal{C} \to \mathrm{ho}(\mathcal{C})$ to another lax monoidal functor $\pi_* : \mathrm{ho}(\mathcal{C}) \to \underline{\mathcal{A}}$.

Remark 7.2.10. As a result of Assumption 7.2.6, to say that $\mathcal{G} \subset \mathcal{C}$ is a subcategory of strong generators is precisely to say that the functor $\pi_* : \mathcal{C} \to \underline{\mathcal{A}}$ creates the equivalences in $\mathcal{C}$.

Remark 7.2.11. One could alternatively consider the "homotopy" functor as taking values in $\mathcal{P}^\delta_\Sigma(\mathcal{G}^\vee) = \mathrm{Fun}((\mathcal{G}^\vee)^{op}, \mathcal{S}\mathrm{et})$, the category of product-preserving presheaves of sets on the closure of $\mathcal{G} \subset \mathcal{C}$ under finite coproducts (which remain coproducts in $\mathrm{ho}(\mathcal{C})$ since $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$ preserves products). This is analogous to the "Π-algebra" perspective taken by Dwyer–Kan–Stover in [DKS95] and by Blanc–Dwyer–Goerss in [BDG04]. However, in order to obtain a computable obstruction theory, Goerss–Hopkins take an alternative route, considering the homotopy groups of a spectrum simply as a $\mathbb{Z}$-graded abelian group (rather than as a module over the stable homotopy groups of spheres).[3]

We conclude this subsection with a few remarks concerning the choice of ambient ∞-category.

Remark 7.2.12. If we remove the requirement that $\mathcal{C}$ be stable, it becomes necessary to assume that the generators admit desuspensions in order for Lemma 7.2.45 to hold. It also becomes necessary to assume that the generators are h-cogroup objects (with respect to the wedge sum) in order to construct the relevant spectral sequence, but of course this is a strictly weaker assumption. More broadly, a great many of the arguments would become substantially more delicate.

Remark 7.2.13. If we only require $\mathcal{C}$ to be monoidal (instead of symmetric monoidal), then by the so-called "microcosm principle" it will only make sense to discuss associative algebras in $\mathcal{C}$, instead of commutative algebras. In the setting of ordinary spectra, associative algebras can be constructed via Hopkins–Miller obstruction theory (see [Rez98]), which is far simpler than Goerss–Hopkins obstruction theory since it is not necessary to resolve the associative operad (see §7.3.3.2). On the other hand, if we set our sights lower and remove the operad from the picture entirely, we simply recover an abstract version of Blanc–Dwyer–Goerss obstruction theory

[2] This is the composite of the canonical projection $\mathcal{C} \to \mathrm{ho}(\mathcal{C})$ followed by the restricted Yoneda embedding along the functor $\mathcal{G}^\delta \to \mathrm{ho}(\mathcal{C})$; note that we have a canonical equivalence $\mathcal{G}^\delta \simeq (\mathcal{G}^\delta)^{op}$ since this category has no nonidentity morphisms.

[3] Nevertheless, product-preserving presheaves pervade this story. We will mostly suppress them, but we will need to discuss them explicitly in §7.4.4.

(see [BDG04]). In any case, we expect that practical examples of interest will carry symmetric monoidal structures anyways.

7.2.2 The resolution model structure

Notation 7.2.14. Let $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{C}))$ be a homotopy commutative algebra object in $\mathcal{C}$. This induces $E_* = \pi_* E \in \mathrm{CAlg}(\underline{\mathcal{A}})$, and we write $\mathcal{A} = \mathrm{Mod}_{E_*}(\underline{\mathcal{A}})$ for its category of modules. Then we obtain a "homology" functor $E_* : \mathcal{C} \to \mathcal{A}$ by $E_* X = \pi_*(E \otimes X)$.

Definition 7.2.15. An ***E_*-equivalence*** in $\mathcal{C}$ is a morphism which becomes an isomorphism under the functor $E_* : \mathcal{C} \to \mathcal{A}$.

Notation 7.2.16. By definition, the E_*-equivalences are created by the composite $\mathcal{C} \xrightarrow{E\otimes -} \mathcal{C} \xrightarrow{\pi_*} \underline{\mathcal{A}}$ (as isomorphisms in $\mathcal{A}$ are created in $\underline{\mathcal{A}}$). However, Remark 7.2.10 implies that they are also created by the functor $\mathcal{C} \xrightarrow{E\otimes -} \mathcal{C}$. Our assumption that $\mathcal{C}$ is presentably symmetric monoidal immediately implies that the E_*-equivalences are strongly saturated (in the sense of Definition T.5.5.4.5), and so by Proposition T.5.5.4.15 there exists a left localization adjunction $\mathrm{L}_{E_*} : \mathcal{C} \rightleftarrows \mathrm{L}_E(\mathcal{C}) : \mathrm{U}_{E_*}$.

Definition 7.2.17. We define the subcategory $\mathcal{A}_{\mathrm{proj}} \subset \mathcal{A}$ of ***projective*** objects just as in classical algebra.

Assumption 7.2.18. We assume henceforth that E satisfies ***Adams's condition***, and fix a witnessing datum: this consists of a filtered diagram $E_\bullet : \mathcal{J} \to \mathcal{C}^{\mathrm{d}}_{/E} = \mathcal{C}^{\mathrm{d}} \times_{\mathcal{C}} \mathcal{C}_{/E}$ with $\mathrm{colim}(\mathcal{J} \xrightarrow{E_\bullet} \mathcal{C}) \xrightarrow{\sim} E$, such that for every $\alpha \in \mathcal{J}$,

- $E_* \mathrm{D}E_\alpha \in \mathcal{A}_{\mathrm{proj}}$, and
- for every $M \in \mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))$, the canonical map

$$\begin{aligned} [\mathrm{D}E_\alpha, M]_{\mathcal{C}} &\to \hom_{\mathcal{A}}(E_* \mathrm{D}E_\alpha, \pi_* M) \\ \left(\mathrm{D}E_\alpha \xrightarrow{f} M\right) &\mapsto \left(E_* \mathrm{D}E_\alpha \xrightarrow{E_*(f)} E_* M = \pi_*(E \otimes M) \to \pi_* M\right) \end{aligned}$$

  is an isomorphism.

Remark 7.2.19. The canonical map of Assumption 7.2.18 can be equivalently seen as the composite

$$[\mathrm{D}E_\alpha, M]_{\mathcal{C}} \cong [E \otimes \mathrm{D}E_\alpha, M]_{\mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))} \xrightarrow{\pi_*} \hom_{\mathcal{A}}(E_* \mathrm{D}E_\alpha, \pi_* M).$$

Observation 7.2.20. For any $X \in \mathcal{C}$ and any $\beta \in \mathcal{G}^\delta$, we have the string of isomorphisms

$$\begin{aligned}\mathrm{colim}_{\alpha\in\mathcal{J}}[\Sigma^\beta \mathrm{D}E_\alpha, X]_{\mathcal{C}} &\cong \mathrm{colim}_{\alpha\in\mathcal{J}}[S^\beta, E_\alpha \otimes X]_{\mathcal{C}} \cong [S^\beta, \mathrm{colim}_{\alpha\in\mathcal{J}}(E_\alpha \otimes X)]_{\mathcal{C}} \\ &\cong [S^\beta, \mathrm{colim}_{\alpha\in\mathcal{J}}(E_\alpha) \otimes X]_{\mathcal{C}} \cong [S^\beta, E \otimes X]_{\mathcal{C}} = E_\beta X\end{aligned}$$

in Ab.

Notation 7.2.21. Strings of adjunction isomorphisms having the same flavor as that of Observation 7.2.20 will frequently be useful to us. Rather than spell out the isomorphisms each time, we simply refer to this line of reasoning as a ***colimit argument***.

Notation 7.2.22. We write $\mathcal{G}^E_{\mathcal{C}} \subset \mathcal{C}$ for the smallest full subcategory containing $\mathcal{G}$ and $\{\mathrm{D}E_\alpha\}_{\alpha\in\mathcal{J}}$ that is closed under de/suspension and finite coproducts. We generally write $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$ for an arbitrary object (the letter "ε" being suggestive of the letter "E"), although we continue to write $S^\beta \in \mathcal{G} \subset \mathcal{G}^E_{\mathcal{C}}$ for an arbitrary object of $\mathcal{G}$ when considered as an object of $\mathcal{G}^E_{\mathcal{C}}$. We write $\mathcal{G}^{E\,\delta}_{\mathcal{C}} = \pi_0((\mathcal{G}^E_{\mathcal{C}})^{\simeq})$, and so (just as we write $\beta \in \mathcal{G}$) we also simply write $\varepsilon \in \mathcal{G}^{E\,\delta}_{\mathcal{C}}$ to denote an arbitrary element.

Observation 7.2.23. For any $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$ and any $M \in \mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))$, we have an isomorphism

$$[S^\varepsilon, M] \xrightarrow{\cong} \hom_{\mathcal{A}}(E_* S^\varepsilon, \pi_* M).$$

This can be seen as follows.

- For $S^\varepsilon = \mathrm{D}E_\alpha$, this follows from Assumption 7.2.18.
- For $S^\varepsilon = S^\beta \in \mathcal{G}$, note that $E_* S^\beta \cong E_* \otimes_{\mathbf{1}_*} \pi_* S^\beta$, and so we are interested in the composite

  $$[S^\beta, M]_{\mathcal{C}} \cong [E \otimes S^\beta, M]_{\mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))} \xrightarrow{\pi_*} \hom_{\mathcal{A}}(E_* S^\beta, \pi_* M) \cong \hom_{\mathrm{Mod}_{\mathbf{1}_*}(\underline{A})}(\pi_* S^\beta, \pi_* M),$$

  which is an isomorphism with inverse given by evaluation at the universal element of $\pi_\beta S^\beta$.
- In general, this property is preserved both by de/suspension and by the formation of finite coproducts.

Notation 7.2.24. Recall that $s\mathcal{C}$ is canonically enriched and bitensored over $s\mathcal{S}$ (see Corollary 7.1.4); these data assemble into a two-variable adjunction, which we denote by

$$\left(s\mathcal{S} \times s\mathcal{C} \xrightarrow{-\odot-} s\mathcal{C} \ , \ s\mathcal{S}^{op} \times s\mathcal{C} \xrightarrow{-\pitchfork-} s\mathcal{C} \ , \ s\mathcal{C}^{op} \times s\mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} s\mathcal{S} \right).$$

Definition 7.2.25. We fix the following terminology.

(1) A morphism in $\mathrm{ho}(\mathcal{C})$ is called a $\mathcal{G}_{\mathcal{C}}^{E}$***-epimorphism*** if the restricted Yoneda functor $\mathrm{ho}(\mathcal{C}) \to \mathcal{P}_{\Sigma}^{\delta}(\mathcal{G}_{\mathcal{C}}^{E})$ takes it to a componentwise surjection.

(2) An object of $\mathrm{ho}(\mathcal{C})$ is called $\mathcal{G}_{\mathcal{C}}^{E}$***-projective*** if it has the extension property for all $\mathcal{G}_{\mathcal{C}}^{E}$-epimorphisms.

(3) A morphism in $\mathrm{ho}(\mathcal{C})$ is called a $\mathcal{G}_{\mathcal{C}}^{E}$***-projective cofibration*** if it has the left lifting property for all $\mathcal{G}_{\mathcal{C}}^{E}$-epimorphisms.

Theorem 7.2.26. *There is a* ***resolution model structure*** *on $s\mathcal{C}$, denoted $s\mathcal{C}_{\mathrm{res}}$, which enjoys the following properties.*

(1) Its weak equivalences and fibrations are created by the functor

$$s\mathcal{C} \xrightarrow{X \mapsto (S^{\varepsilon} \mapsto \hom_{\mathcal{C}}^{\mathrm{lw}}(S^{\varepsilon}, X))} \prod_{\mathcal{G}_{\mathcal{C}}^{E\,\delta}} s\mathcal{S}_{\mathrm{KQ}}.$$

(2) It is simplicial.

(3) Its cofibrations are precisely those morphisms whose relative latching maps are $\mathcal{G}_{\mathcal{C}}^{E}$-projective cofibrations.

(4) All objects are fibrant in it.

(5) It is cofibrantly generated by the sets

$$I_{\mathrm{res}}^{s\mathcal{C}} = \{I_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^{\varepsilon})\}_{S^{\varepsilon} \in \mathcal{G}_{\mathcal{C}}^{E}} = \{\partial\Delta^{n} \odot \mathrm{const}(S^{\varepsilon}) \to \Delta^{n} \odot \mathrm{const}(S^{\varepsilon})\}_{n \geq 0, S^{\varepsilon} \in \mathcal{G}_{\mathcal{C}}^{E}}$$

and

$$J_{\mathrm{res}}^{s\mathcal{C}} = \{J_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^{\varepsilon})\}_{S^{\varepsilon} \in \mathcal{G}_{\mathcal{C}}^{E}} = \{\Lambda_{i}^{n} \odot \mathrm{const}(S^{\varepsilon}) \to \Delta^{n} \odot \mathrm{const}(S^{\varepsilon})\}_{0 \leq i \leq n \geq 1, S^{\varepsilon} \in \mathcal{G}_{\mathcal{C}}^{E}}.$$

Proof. This follows from Theorem 7.1.19 and Lemma 7.1.23. □

Remark 7.2.27. It will follow from the localized spiral exact sequence of Construction 7.2.52 that the weak equivalences of $s\mathcal{C}_{\mathrm{res}}$ are created by the functor

$$s\mathcal{C} \xrightarrow{[=,-]_{\mathcal{C}}^{\mathrm{lw}}} s\mathrm{Fun}(\mathcal{G}_{\mathcal{C}}^{E}, \mathrm{Ab}) \simeq \mathrm{Fun}(\mathcal{G}_{\mathcal{C}}^{E}, s\mathrm{Ab}_{\mathrm{KQ}})_{\mathrm{proj}}.$$

(In fact, the fibrations are as well.)

Definition 7.2.28. We define the subcategory of ***E_*-equivalences***, denoted $\mathbf{W}_{E^{\mathrm{lw}}_*} = \mathbf{W}^{s\mathcal{C}}_{E^{\mathrm{lw}}_*} \subset s\mathcal{C}$, to be created by pulling back the subcategory $\mathbf{W}^{s\mathcal{A}}_{\mathrm{KQ}} \subset s\mathcal{A}_{\mathrm{KQ}}$ under the functor $E^{\mathrm{lw}}_* : s\mathcal{C} \to s\mathcal{A}_{\mathrm{KQ}}$.

Notation 7.2.29. Rather than overburden notation, we simply write $\pi_n : s\mathrm{Ab} \to \mathrm{Ab}$ for the composite

$$s\mathrm{Ab} \xrightarrow{|-|} \mathrm{Ab}\mathcal{G}\mathrm{rp}(\mathcal{S}_*) \xrightarrow{\mathrm{Ab}\mathcal{G}\mathrm{rp}(\pi_n)} \mathrm{Ab}\mathcal{G}\mathrm{rp}(\mathcal{S}\mathrm{et}_*) = \mathrm{Ab}.$$

This can be obtained more abstractly as a "homotopy" functor from a derived ∞-category to its heart, and indeed we use this same notation π_n to denote all corresponding functors $s\mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}_*$, $s\underline{\mathcal{A}} \to \underline{\mathcal{A}}$, $s\mathcal{A} \to \mathcal{A}$, etc.

Observation 7.2.30. Suppose that $X \xrightarrow{\approx} Y$ is a weak equivalence in $s\mathcal{C}_{\mathrm{res}}$. By Remark 7.2.27, this means that for every $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$ we obtain a weak equivalence $[S^\varepsilon, X]^{\mathrm{lw}}_{\mathcal{C}} \xrightarrow{\approx} [S^\varepsilon, Y]^{\mathrm{lw}}_{\mathcal{C}}$ in $s\mathrm{Ab}_{\mathrm{KQ}}$, i.e. that we obtain isomorphisms $\pi_n([S^\varepsilon, X]^{\mathrm{lw}}_{\mathcal{C}}) \xrightarrow{\cong} \pi_n([S^\varepsilon, Y]^{\mathrm{lw}}_{\mathcal{C}})$ in Ab for all $n \geq 0$. In particular, letting S^ε range over the set $\{\Sigma^\beta \mathrm{D}E_\alpha\}_{\beta\in\mathcal{G}^\delta, \alpha\in\mathcal{J}}$, by Observation 7.2.20 and since homotopy groups in $s\mathcal{S}\mathrm{et}_*$ commute with filtered colimits, we obtain a weak equivalence $E^{\mathrm{lw}}_* X \xrightarrow{\approx} E^{\mathrm{lw}}_* Y$ in $s\mathcal{A}_{\mathrm{KQ}}$. In other words, we have an inclusion $\mathbf{W}_{\mathrm{res}} \subset \mathbf{W}_{E^{\mathrm{lw}}_*}$ of subcategories of $s\mathcal{C}$.

Observation 7.2.31. In our setting, after a colimit argument the standard filtration spectral sequence for an object $X \in s\mathcal{C}$ runs $\pi_n E^{\mathrm{lw}}_\beta X \Rightarrow E_{\beta+n}|X|$. (This agrees with the spectral sequence associated to the localized spiral exact sequence of Construction 7.2.52 (see [GHb, Lemma 3.1.5 and Remark 3.1.6]).) Thus, an E_*-equivalence in $s\mathcal{C}$ (for instance a weak equivalence in $s\mathcal{C}_{\mathrm{res}}$, by Observation 7.2.30) induces an isomorphism on E^2 pages of this spectral sequence. In other words, there exists a factorization

$$\begin{array}{ccccc} s\mathcal{C} & \xrightarrow{|-|} & \mathcal{C} & \xrightarrow{E_*} & \mathcal{A} \\ \downarrow & & & \nearrow & \\ s\mathcal{C}[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] & & & & \end{array}$$

through the localization functor.

Definition 7.2.32. We refer to this spectral sequence $\mathrm{E}^2 = \pi_n E^{\mathrm{lw}}_\beta X \Rightarrow \mathrm{E}^\infty = E_{\beta+n}|X|$ as the ***spiral spectral sequence***.

Remark 7.2.33. By Theorem 7.1.22, the resolution model structure presents the nonabelian derived ∞-category $\mathcal{P}_\Sigma(\mathcal{G}^E_{\mathcal{C}})$. Moreover, the composite $\mathcal{C} \xrightarrow{\mathrm{const}} s\mathcal{C} \to s\mathcal{C}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \simeq \mathcal{P}_\Sigma(\mathcal{G}^E_{\mathcal{C}})$ clearly coincides with the restricted Yoneda embedding. We will generally omit this from the notation.

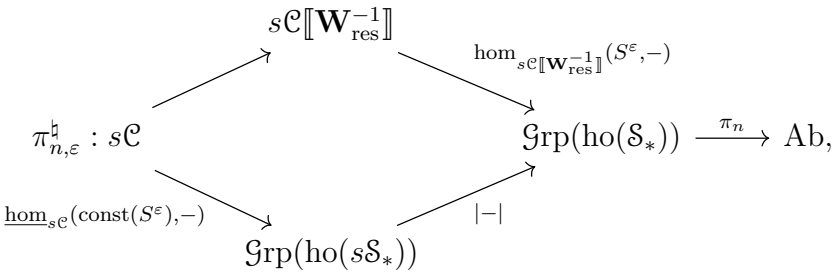
7.2.3 The spiral exact sequence

Definition 7.2.34. Choose any $n \geq 0$ and any $\varepsilon \in \mathcal{G}_{\mathcal{C}}^{E^\delta}$.

(1) We define the corresponding ***classical homotopy group*** functor to be the composite

$$\pi_n \pi_\varepsilon : s\mathcal{C} \xrightarrow{[S^\varepsilon,-]^{\mathrm{lw}}_{\mathcal{C}}} s\mathrm{Ab} \xrightarrow{\pi_n} \mathrm{Ab}.$$

(2) We define the corresponding ***natural homotopy group*** functor to be either equivalent composite



where

- the commutativity of the square follows from the fact that $s\mathcal{C}_{\mathrm{res}}$
  - is simplicial,
  - has $\mathrm{const}(S^\varepsilon) \in s\mathcal{C}^c_{\mathrm{res}}$ cofibrant, and
  - has all objects fibrant,

  and
- the fact that the down-and-right functors land in h-group objects follows from the fact that $S^\varepsilon \in \mathcal{C}$ is an h-cogroup object (so that $\mathrm{const}(S^\varepsilon) \in s\mathcal{C}$ is as well).

Definition 7.2.35. Let $K \in s\mathcal{S}_*$, and let $X \in s\mathcal{C}$. We define the ***reduced tensoring*** of X over K to be the pushout

$$\begin{array}{ccc} \mathrm{pt}_{s\mathcal{S}} \odot X & \longrightarrow & K \odot X \\ \downarrow & & \downarrow \\ \mathrm{pt}_{s\mathcal{S}} \odot 0_{s\mathcal{C}} & \longrightarrow & K \overline{\odot} X \end{array}$$

in $s\mathcal{C}$. This assembles into an action $s\mathcal{S}_* \times s\mathcal{C} \to s\mathcal{C}$.

Notation 7.2.36. We write $D^n_{\mathbf{\Delta}} = \Delta^n/\Lambda^n_0 \in s\mathcal{S}\mathrm{et}_* \subset s\mathcal{S}_*$ for the "reduced pointed simplicial n-disk" and $S^n_{\mathbf{\Delta}} = \Delta^n/\partial\Delta^n \in s\mathcal{S}\mathrm{et}_* \subset s\mathcal{S}_*$ for the "reduced pointed simplicial n-sphere".

Observation 7.2.37. The canonical composite

$$S^{n-1}_{\mathbf{\Delta}} \to D^n_{\mathbf{\Delta}} \to S^n_{\mathbf{\Delta}}$$

(where the first map is obtained by considering $\Delta^{n-1} \cong \Delta^{\{0,\dots,n-1\}} \subset \Delta^n$) is a cofiber sequence not just in $s\mathcal{S}\mathrm{et}_*$ but also in $s\mathcal{S}_*$.

Lemma 7.2.38. *For any $n \geq 0$ and any $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$, there is a natural isomorphism*

$$\pi^\natural_{n,\varepsilon}(-) \cong [S^n_{\mathbf{\Delta}} \overline{\odot} \mathrm{const}(S^\varepsilon), -]_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}$$

in $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. In light of the facts

- that $s\mathcal{C}_{\mathrm{res}}$ is simplicial,
- that $S^n_{\mathbf{\Delta}} \overline{\odot} \mathrm{const}(S^\varepsilon) \in s\mathcal{C}^c_{\mathrm{res}}$ is cofibrant, and
- that all objects of $s\mathcal{C}_{\mathrm{res}}$ are fibrant,

we have the string of natural isomorphisms

$$\begin{aligned}
[S^n_{\mathbf{\Delta}} \overline{\odot} \mathrm{const}(S^\varepsilon), -]_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]} &\cong \pi_0 |\underline{\hom}_{s\mathcal{C}}(S^n_{\mathbf{\Delta}} \overline{\odot} \mathrm{const}(S^\varepsilon), -)| \\
&\cong \pi_0 \left| \lim \left(\begin{array}{ccc} & & \underline{\hom}_{s\mathcal{C}}(S^n_{\mathbf{\Delta}} \odot \mathrm{const}(S^\varepsilon), -) \\ & & \downarrow \\ \underline{\hom}_{s\mathcal{C}}(\mathrm{pt}_{s\mathcal{S}} \odot 0_{\mathcal{C}}, -) & \longrightarrow & \underline{\hom}_{s\mathcal{C}}(\mathrm{pt}_{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon), -) \end{array} \right) \right| \\
&\cong \pi_0 \left| \lim \left(\begin{array}{ccc} & & \underline{\hom}_{s\mathcal{S}}(S^n_{\mathbf{\Delta}}, \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)) \\ & & \downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathrm{pt}_{s\mathcal{S}} & \xrightarrow[\underline{0}]{} & \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -) \end{array} \right) \right|.
\end{aligned}$$

In order to continue the string of isomorphisms, we make the following observations.

- The compatibility of $s\mathcal{C}_{\mathrm{res}}$ with $s\mathcal{S}_{\mathrm{KQ}}$ implies that the vertical map in this last expression is a fibration, so that we can commute the limit with the geometric realization.

- As $\mathrm{const}(S^\varepsilon) \in s\mathcal{C}^c_{\mathrm{res}}$ is cofibrant and all objects of $s\mathcal{C}_{\mathrm{res}}$ are fibrant, then $\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -) : s\mathcal{C} \to s\mathcal{S}^f_{\mathrm{KQ}}$ takes values in fibrant objects of $s\mathcal{S}_{\mathrm{KQ}}$.

- The object $S^n_{\mathbf{\Delta}} \in s\mathcal{S}^c_{\mathrm{KQ}}$ is cofibrant.

Using these, we continue as

$$
\begin{aligned}
&\cong \pi_0 \lim \left(\begin{array}{ccc} & & |\underline{\hom}_{s\mathcal{S}}(S^n_{\mathbf{\Delta}}, \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -))| \\ & & \big\downarrow{\scriptstyle |\mathrm{ev}_*|} \\ |\mathrm{pt}_{s\mathcal{S}}| & \xrightarrow[|\underline{0}|]{} & |\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)| \end{array} \right) \\
&\cong \pi_0 \lim \left(\begin{array}{ccc} & & \hom_{\mathcal{S}}(|S^n_{\mathbf{\Delta}}|, |\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)|) \\ & & \big\downarrow{\scriptstyle |\mathrm{ev}_*|} \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[|\underline{0}|]{} & |\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)| \end{array} \right) \\
&\cong \pi_0 \lim \left(\begin{array}{ccc} & & \hom_{\mathcal{S}}(S^n, \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, -)) \\ & & \big\downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[0]{} & \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, -) \end{array} \right) \\
&\cong \pi_0 \hom_{\mathcal{S}_*}(S^n, \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, -)) \\
&\cong \pi_n \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, -),
\end{aligned}
$$

proving the claim. □

Definition 7.2.39. Let $K \in s\mathcal{S}_*$, and let $X \in s\mathcal{C}$. We define the ***reduced cotensoring*** of K into X to be the pullback

$$
\begin{array}{ccc}
K \overline{\pitchfork} X & \longrightarrow & K \pitchfork X \\
\big\downarrow & & \big\downarrow \\
\mathrm{pt}_{s\mathcal{S}} \pitchfork 0_{s\mathcal{C}} & \longrightarrow & \mathrm{pt}_{s\mathcal{S}} \pitchfork X
\end{array}
$$

in $s\mathcal{C}$. This assembles into an action $(s\mathcal{S}_*)^{op} \times s\mathcal{C} \to s\mathcal{C}$.

Observation 7.2.40. The reduced co/tensoring bifunctors participate into an evident two-variable adjunction

$$\left(s\mathcal{S}_* \times s\mathcal{C} \xrightarrow{-\overline{\odot}-} s\mathcal{C} \ , \ (s\mathcal{S}_*)^{op} \times s\mathcal{C} \xrightarrow{-\overline{\pitchfork}-} s\mathcal{C} \ , \ s\mathcal{C}^{op} \times s\mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} s\mathcal{S}_* \right),$$

obtained by recognizing that the (enriched) hom-objects of $s\mathcal{C}$ are naturally pointed since $s\mathcal{C}$ has a zero object.

Observation 7.2.41. If

- on the one hand we restrict the reduced tensoring bifunctor to the constant simplicial objects of $\mathcal{C}$ via the composite

  $$s\mathcal{S}_* \times \mathcal{C} \xrightarrow{\mathrm{id}_{s\mathcal{S}_*} \times \mathrm{const}} s\mathcal{S}_* \times s\mathcal{C} \xrightarrow{-\overline{\odot}-} s\mathcal{C},$$

  while

- on the other hand we postcompose the reduced cotensoring bifunctor with the limit functor to obtain the composite

  $$(s\mathcal{S}_*)^{op} \times s\mathcal{C} \xrightarrow{-\overline{\pitchfork}-} s\mathcal{C} \xrightarrow{(-)_0} \mathcal{C},$$

then we similarly obtain a two-variable adjunction

$$\left(s\mathcal{S}_* \times \mathcal{C} \xrightarrow{-\overline{\odot}\mathrm{const}(-)} s\mathcal{C} \ , \ (s\mathcal{S}_*)^{op} \times s\mathcal{C} \xrightarrow{(-\overline{\pitchfork}-)_0} \mathcal{C} \ , \ \mathcal{C}^{op} \times s\mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} s\mathcal{S}_* \right).$$

Notation 7.2.42. In analogy with the "generalized matching object" bifunctor

$$\mathrm{M}_{(-)}(-) : s\mathcal{S}^{op} \times s\mathcal{C} \xrightarrow{(-\pitchfork-)_0} \mathcal{C},$$

we write

$$\overline{\mathrm{M}}_{(-)}(-) : (s\mathcal{S}_*)^{op} \times s\mathcal{C} \xrightarrow{(-\overline{\pitchfork}-)_0} \mathcal{C}$$

for the "reduced generalized matching object" bifunctor.

Definition 7.2.43. We define the ***(nonabelian) normalized*** n***-chains*** functor to be

$$N_n : s\mathcal{C} \xrightarrow{\overline{\mathrm{M}}_{D^n_{\Delta}}(-)} \mathcal{C},$$

and we define the ***(nonabelian)*** n***-cycles*** functor to be

$$Z_n : s\mathcal{C} \xrightarrow{\overline{\mathrm{M}}_{S^n_{\Delta}}(-)} \mathcal{C}.$$

Note that these would reduce to the usual notions if $\mathcal{C}$ were an abelian category.

Observation 7.2.44. The cofiber sequence $S^{n-1}_{\mathbf{\Delta}} \to D^n_{\mathbf{\Delta}} \to S^n_{\mathbf{\Delta}}$ in $s\mathcal{S}_*$ of Observation 7.2.37 induces a fiber sequence

$$Z_n \to N_n \to Z_{n-1}$$

in $\mathrm{Fun}(s\mathcal{C}, \mathcal{C})$.

Lemma 7.2.45. *For any $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$, there is a natural isomorphism*

$$[S^\varepsilon, N_n(-)]_{\mathcal{C}} \cong N_n[S^\varepsilon, -]^{\mathrm{lw}}_{\mathcal{C}}$$

in $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. Fix a test object $X \in s\mathcal{C}$. As by definition $N_n(X) = \overline{\mathrm{M}}_{D^n_{\mathbf{\Delta}}}(X)$, we have a pullback square

$$\begin{array}{ccc} N_n(X) & \longrightarrow & \mathrm{M}_{D^n_{\mathbf{\Delta}}}(X) \\ \downarrow & & \downarrow \\ \mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(0_{s\mathcal{C}}) & \longrightarrow & \mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(X) \end{array}$$

in $\mathcal{C}$. In light of the pushout square

$$\begin{array}{ccc} \Lambda^n_0 & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & D^n_{\mathbf{\Delta}} \end{array}$$

both in $s\mathcal{S}\mathrm{et}$ and in $s\mathcal{S}$, we also have a pullback square

$$\begin{array}{ccc} \mathrm{M}_{D^n_{\mathbf{\Delta}}}(X) & \longrightarrow & \mathrm{M}_{\Delta^n}(X) \\ \downarrow & & \downarrow \\ \mathrm{M}_{\Delta^0}(X) & \longrightarrow & \mathrm{M}_{\Lambda^n_0}(X) \end{array}$$

in $\mathcal{C}$, which simplifies to a pullback square

$$\begin{array}{ccc} \mathrm{M}_{D^n_{\mathbf{\Delta}}}(X) & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & \mathrm{M}_{\Lambda^n_0}(X) \end{array}$$

As the relevant corepresenting maps $\mathrm{pt}_{s\mathcal{S}} \to D^n_{\mathbf{\Delta}}$ and $\Delta^0 \to D^n_{\mathbf{\Delta}}$ in $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ coincide, we obtain the composite pullback square

$$\begin{array}{ccccc} N_n(X) & \longrightarrow & \mathrm{M}_{D^n_{\mathbf{\Delta}}}(X) & \longrightarrow & \mathrm{M}_{\Delta^n}(X) \\ \downarrow & & & & \downarrow \\ \mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(0_{s\mathcal{C}}) & \longrightarrow & \mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(X) \simeq \mathrm{M}_{\Delta^0}(X) & \longrightarrow & \mathrm{M}_{\Lambda^n_0}(X) \end{array}$$

in $\mathcal{C}$, which simplifies to a pullback square

$$\begin{array}{ccc} N_n(X) & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ 0_{\mathcal{C}} & \longrightarrow & \mathrm{M}_{\Lambda^n_0}(X) \end{array}$$

in $\mathcal{C}$. Moreover, replacing $0 \in [n]$ with any $i \in [n]$, we obtain analogous pullback squares

$$\begin{array}{ccc} \overline{\mathrm{M}}_{(\Delta^n/\Lambda^n_i)}(X) & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ 0_{\mathcal{C}} & \longrightarrow & \mathrm{M}_{\Lambda^n_i}(X) \end{array}$$

in $\mathcal{C}$. From here, the (dual of the corresponding cosimplicial) double induction argument of [GJ09, Chapter VIII, Lemma 1.8] yields the claim. □

Lemma 7.2.46. *For any $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$, there is a natural exact sequence*

$$[S^\varepsilon, N_{n+1}(-)]_{\mathcal{C}} \to [S^\varepsilon, Z_n(-)]_{\mathcal{C}} \to \pi^\natural_{n,\varepsilon}(-) \to 0$$

in $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. For any test object $X \in s\mathcal{C}$, we have

$$\pi^\natural_{n,\varepsilon} X = \pi_n \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, X) \cong \pi_0 \hom_{\mathcal{S}_*}(S^n, \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, X)).$$

Now, since $\mathrm{const}(S^\varepsilon) \in s\mathcal{C}^c_{\mathrm{res}}$ and $X \in s\mathcal{C}^f_{\mathrm{res}}$, we have that $\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X) \in s\mathcal{S}^f_{\mathrm{KQ}}$ and moreover $|\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X)| \simeq \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, X)$. On the other hand, $S^n_{\mathbf{\Delta}} \in s\mathcal{S}^c_{\mathrm{KQ}}$. Since co/fibrancy in $(s\mathcal{S}_*)_{\mathrm{KQ}}$ is created in $s\mathcal{S}_{\mathrm{KQ}}$, the fundamental theorem of model ∞-categories applied to $(s\mathcal{S}_*)_{\mathrm{KQ}}$ implies that we have a surjection

$$\hom_{s\mathcal{S}_*}(S^n_{\mathbf{\Delta}}, \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X)) \to \hom_{\mathcal{S}_*}(S^n, \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, X))$$

in $\mathcal{S}$. Applying π_0, by adjunction this yields a surjection

$$[S^\varepsilon, Z_n(X)]_{\mathcal{C}} \to \pi^\natural_{n,\varepsilon} X$$

in $\mathcal{S}\mathrm{et}$. As epimorphisms are Ab are created in $\mathcal{S}\mathrm{et}$, this proves exactness at $\pi^\natural_{n,\varepsilon}(-)$.

Now, suppose we are given an element of $\ker([S^\varepsilon, Z_n(X)]_{\mathcal{C}} \to \pi^\natural_{n,\varepsilon} X)$: this is witnessed by an extension

$$\begin{array}{ccc} S^n & \longrightarrow & \hom_{s\mathcal{C}[\mathbf{W}^{-1}_{\mathrm{res}}]}(S^\varepsilon, X) \\ \downarrow & \nearrow & \\ \mathrm{pt}_{\mathcal{S}_*} & & \end{array}$$

in $\mathcal{S}_*$. Since $D^{n+1}_{\mathbf{\Delta}} \in (s\mathcal{S}_{S^n_{\mathbf{\Delta}}/})^c_{\mathrm{KQ}}$ and $\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X) \in (s\mathcal{S}_{S^n_{\mathbf{\Delta}}/})^f_{\mathrm{KQ}}$, the fundamental theorem of model ∞-categories applied to $(s\mathcal{S}_{S^n_{\mathbf{\Delta}}/})_{\mathrm{KQ}}$ implies that the above extension in $\mathcal{S}_*$ is presented by an extension

$$\begin{array}{ccc} S^n_{\mathbf{\Delta}} & \longrightarrow & \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X) \\ \downarrow & \nearrow & \\ D^{n+1}_{\mathbf{\Delta}} & & \end{array}$$

in $s\mathcal{S}_*$. This proves exactness at $[S^\varepsilon, Z_n(-)]_{\mathcal{C}}$. □

Corollary 7.2.47. *There is a natural isomorphism* $\pi_0\pi_\varepsilon(-) \cong \pi^\natural_{0,\varepsilon}(-)$ *in* $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. Fix a test object $X \in s\mathcal{C}$. Applying Lemma 7.2.46 in the case that $n = 0$, we obtain an isomorphism

$$\mathrm{coker}([S^\varepsilon, N_1(X)]_{\mathcal{C}} \to [S^\varepsilon, Z_0(X)]_{\mathcal{C}}) \xrightarrow{\cong} \pi^\natural_{0,\varepsilon} X$$

in Ab. Unwinding the definition of $Z_0(X)$, we see that $Z_0(X) \simeq X_0 \in \mathcal{C}$, so that

$$[S^\varepsilon, Z_0(X)]_{\mathcal{C}} \cong [S^\varepsilon, X_0]_{\mathcal{C}} = ([S^\varepsilon, X]^{\mathrm{lw}}_{\mathcal{C}})_0.$$

Under this identification, unwinding the definition of $N_1 X$, we see that the image of the map

$$[S^\varepsilon, N_1 X]_{\mathcal{C}} \to [S^\varepsilon, Z_0 X]_{\mathcal{C}} \cong ([S^\varepsilon, X]^{\mathrm{lw}}_{\mathcal{C}})_0$$

is the set of those 0-simplices in $[S^\varepsilon, X]^{\mathrm{lw}}_{\mathcal{C}} \in s\mathrm{Ab}$ that are the "source" of a 1-simplex with "target" the basepoint 0-simplex $0 \in ([S^\varepsilon, X]^{\mathrm{lw}}_{\mathcal{C}})_0 \in \mathrm{Ab}$. So we obtain an isomorphism

$$\mathrm{coker}([S^\varepsilon, N_1(X)]_{\mathcal{C}} \to [S^\varepsilon, Z_0(X)]_{\mathcal{C}}) \cong \pi_0\pi_\varepsilon X,$$

from which the claim follows. □

Construction 7.2.48. For any object $X \in s\mathcal{C}$ and any $S^\varepsilon \in \mathcal{G}^E_{\mathcal{C}}$, by Observation 7.2.44 we have long exact sequences

$$\cdots \to [S^{\varepsilon+1}, Z_{n-1}(X)]_{\mathcal{C}} \to [S^\varepsilon, Z_n(X)]_{\mathcal{C}} \to [S^\varepsilon, N_n(X)]_{\mathcal{C}} \to [S^\varepsilon, Z_{n-1}(X)]_{\mathcal{C}}$$

in Ab (which actually continue indefinitely to the right as well since $\mathcal{C}$ is stable). These splice together into an exact couple

$$\begin{array}{ccc} [S^{\varepsilon+i+1}, Z_{n-1}(X)]_{\mathcal{C}} & \xrightarrow{(\varepsilon+i+1)\rightsquigarrow(\varepsilon+i)} & [S^{\varepsilon+i+1}, Z_n(X)]_{\mathcal{C}} \\ & \nwarrow \quad \swarrow & \\ & [S^{\varepsilon+i+1}, N_n(X)]_{\mathcal{C}}. & \end{array}$$

Using Lemmas 7.2.45 and 7.2.46, we can identify its derived long exact sequence as

$$\begin{aligned} \cdots \longrightarrow \pi_{i+1}\pi_\varepsilon(X) \xrightarrow{\delta} \pi^\natural_{i-1,\varepsilon+1}(X) \longrightarrow \pi^\natural_{i,\varepsilon}(X) \longrightarrow \pi_i\pi_\varepsilon(X) \xrightarrow{\delta} \cdots \\ \cdots \xrightarrow{\delta} \pi^\natural_{0,\varepsilon+1}(X) \longrightarrow \pi^\natural_{1,\varepsilon}(X) \longrightarrow \pi_1\pi_\varepsilon(X) \longrightarrow 0. \end{aligned}$$

We refer to this as the ***spiral exact sequence***.

7.2.4 The localized spiral exact sequence

In the end, we will not be interested in the natural and classical homotopy groups, but rather in their corresponding E-homology groups.

Notation 7.2.49. We simply write $E : s\mathcal{C} \xrightarrow{(E\otimes-)^{\mathrm{lw}}} s\mathcal{C}$ for the "tensor levelwise with E" functor.

Definition 7.2.50. Choose any $n \geq 0$ and any $\beta \in \mathcal{G}^\delta$.

(1) We define the corresponding ***classical E-homology group*** functor to be the composite

$$\pi_n E_\beta : s\mathcal{C} \xrightarrow{E} s\mathcal{C} \xrightarrow{\pi_n\pi_\beta} \mathrm{Ab}.$$

(2) We define the corresponding ***natural E-homology group*** functor to be the composite

$$E^\natural_{n,\beta} : s\mathcal{C} \xrightarrow{E} s\mathcal{C} \xrightarrow{\pi^\natural_{n,\beta}} \mathrm{Ab}.$$

When considered as indexed over all $\beta \in \mathcal{G}$ simultaneously, we write these functors simply as $\pi_n E_*$ and $E^\natural_{n,*}$, respectively.

Lemma 7.2.51. *There is a natural isomorphism* $\pi_0 E_\beta(-) \cong E^\natural_{0,\beta}(-)$ *in* $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. This follows from Corollary 7.2.47 and a colimit argument. □

Construction 7.2.52. For any $X \in s\mathcal{C}$, the spiral exact sequence for $EX \in s\mathcal{C}$ with respect to any $\beta \in \mathcal{G}^\delta$ becomes

$$\cdots \longrightarrow \pi_{i+1} E_\beta X \xrightarrow{\delta} E^\natural_{i-1,\beta+1} X \longrightarrow E^\natural_{i,\beta} X \longrightarrow \pi_i E_\beta X \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} E^\natural_{0,\beta+1} X \longrightarrow E^\natural_{1,\beta} X \longrightarrow \pi_1 E_\beta X \longrightarrow 0.$$

We refer to this as the ***localized spiral exact sequence***.

7.3 Algebraic topology

In this section, we add operadic structures to the mix.

7.3.1 Foundations of algebraic topology

Definition 7.3.1. By ***operad*** we mean what might otherwise be called a "single-colored ∞-operad". These are presented by monoids for the composition product in symmetric sequences in topological spaces or in simplicial sets (via the "operadic nerve" of Definition A.2.1.1.23). We write Op for the ∞-category of operads. For any $\mathcal{O} \in \mathrm{Op}$, we write $\mathcal{O}(n) \in \mathrm{Fun}(B\mathfrak{S}_n, \mathcal{S})$ for the space of n-ary operations, equipped with its canonical action of the symmetric group $\mathfrak{S}_n$.

Notation 7.3.2. For any $\mathcal{O} \in \mathrm{Op}$, we write $\mathrm{Alg}_\mathcal{O}(\mathcal{C})$ for the ∞-category of $\mathcal{O}$-algebras in $\mathcal{C}$, and we write

$$\mathrm{F}_\mathcal{O} : \mathcal{C} \rightleftarrows \mathrm{Alg}_\mathcal{O}(\mathcal{C}) : \mathrm{U}_\mathcal{O}$$

for the corresponding free/forget monadic adjunction.

Observation 7.3.3. The monad corresponding to the monadic adjunction $\mathrm{F}_\mathcal{O} \dashv \mathrm{U}_\mathcal{O}$ can be computed as

$$\mathrm{U}_\mathcal{O}(\mathrm{F}_\mathcal{O}(X)) \simeq \coprod_{n \geq 0} (\mathcal{O}(n) \odot X^{\otimes n})_{\mathfrak{S}_n}$$

(where we use the diagonal action to form the quotient).

Observation 7.3.4. Any map $\mathcal{O} \xrightarrow{\varphi} \mathcal{O}'$ in Op determines an adjunction

$$\varphi_* : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) : \varphi^*$$

between ∞-categories of algebras in $\mathcal{C}$, whose right adjoint is given by restriction of structure. The assignment $\varphi \mapsto \varphi_*$ assembles into a functor

$$\mathrm{Alg}_{(-)}(\mathcal{C}) : \mathrm{Op} \to \mathrm{Pr}^L.$$

Remark 7.3.5. We restrict to single-colored operads for simplicity, and because most operads of interest are single-colored. However, note that if one were interested in obtaining e.g. a commutative algebra $A \in \mathrm{CAlg}(\mathcal{C})$ as well as a module $M \in \mathrm{Mod}_A(\mathcal{C})$, one might proceed in steps, first using a single-colored obstruction theory in $\mathcal{C}$ to produce A, and then using a single-colored obstruction theory in $\mathrm{Mod}_A(\mathcal{C})$ to produce M.

7.3.2 Simplicial algebraic topology

Definition 7.3.6. Let $T \in s\mathrm{Op}$ be a simplicial object in operads. We define the ∞-category $\mathrm{Alg}_T(s\mathcal{C})$ of ***simplicial T-algebras in*** $\mathcal{C}$ to be the lax limit of the composite

$$\mathbf{\Delta}^{op} \xrightarrow{T} \mathrm{Op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{C})} \mathrm{Pr}^L.$$

Remark 7.3.7. The composite

$$\mathbf{\Delta}^{op} \xrightarrow{T} \mathrm{Op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{C})} \mathrm{Pr}^L \xrightarrow{\mathrm{U}_{\mathrm{Pr}^L}} \mathcal{C}\mathrm{at}_\infty$$

classifies a cocartesian fibration, which is in fact a bicartesian fibration; by (the dual of) [GHN, Proposition 7.1] (combined with Proposition T.5.5.3.13), its ∞-category of sections is precisely $\mathrm{Alg}_T(s\mathcal{C})$. Thus, we can think of a simplicial T-algebra $X = X_\bullet \in \mathrm{Alg}_T(s\mathcal{C})$ as being specified by the following data:

- for each object $[n]^\circ \in \mathbf{\Delta}^{op}$, an object $X_n \in \mathrm{Alg}_{T_n}(\mathcal{C})$;
- for each morphism $[n]^\circ \xrightarrow{\varphi} [m]^\circ$ in $\mathbf{\Delta}^{op}$, a morphism from $X_n \in \mathrm{Alg}_{T_n}(\mathcal{C})$ to $X_m \in \mathrm{Alg}_{T_m}(\mathcal{C})$ in (the bicartesian fibration over $[1]$ corresponding to) the adjunction
$$(T_\varphi)_* : \mathrm{Alg}_{T_n}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{T_m}(\mathcal{C}) : (T_\varphi)^*$$
arising from the induced map $T_n \xrightarrow{T_\varphi} T_m$ in Op, i.e. a point in the space
$$\hom_{\mathrm{Alg}_{T_n}(\mathcal{C})}(X_n, (T_\varphi)^* X_m) \simeq \hom_{\mathrm{Alg}_{T_m}(\mathcal{C})}((T_\varphi)_* X_n, X_m);$$

- higher coherence data for these structure maps corresponding to strings of composable morphisms in $\mathbf{\Delta}^{op}$.

Observation 7.3.8. Any map $T \xrightarrow{\varphi} T'$ in $s\mathrm{Op}$ determines an adjunction

$$\varphi_* : \mathrm{Alg}_T(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_{T'}(s\mathcal{C}) : \varphi^*$$

between ∞-categories of simplicial algebras in $\mathcal{C}$, whose right adjoint is given by restriction of structure. In particular, taking T to be trivial yields a monadic adjunction

$$\mathrm{F}_{T'} : s\mathcal{C} \rightleftarrows \mathrm{Alg}_{T'}(s\mathcal{C}) : \mathrm{U}_{T'},$$

whose underlying monad is computed levelwise.

Observation 7.3.9. Let $\mathcal{O} \in \mathrm{Op}$ be an operad, and consider the the corresponding constant simplicial operad $\mathrm{const}(\mathcal{O}) \in s\mathrm{Op}$. Since the resulting composite

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{const}(\mathcal{O})} \mathrm{Op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{C})} \mathrm{Pr}^L$$

is constant at $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, it follows that we have a canonical equivalence

$$\mathrm{Alg}_{\mathrm{const}(\mathcal{O})}(s\mathcal{C}) \simeq s(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})).$$

Observation 7.3.10. For any $T \in s\mathrm{Op}$, we have a canonical composite adjunction

$$\mathrm{Alg}_T(s\mathcal{C}) \underset{(\eta_T)^*}{\overset{(\eta_T)_*}{\rightleftarrows}} \mathrm{Alg}_{\mathrm{const}(|T|)}(s\mathcal{C}) \simeq s(\mathrm{Alg}_{|T|}(\mathcal{C})) \underset{\mathrm{const}}{\overset{|-|}{\rightleftarrows}} \mathrm{Alg}_{|T|}(\mathcal{C}),$$

where

- the first adjunction follows by applying Observation 7.3.8 to the component $T \xrightarrow{\eta_T} \mathrm{const}(|T|)$ of the unit of the adjunction $|-| : s\mathrm{Op} \rightleftarrows \mathrm{Op} : \mathrm{const}(-)$;
- the equivalence is that of Observation 7.3.9; and
- the second adjunction is the colimit/constant adjunction in $\mathrm{Alg}_{|T|}(\mathcal{C})$.

Notation 7.3.11. For simplicity, we simply write

$$|-| : \mathrm{Alg}_T(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_{|T|}(\mathcal{C}) : \mathrm{const}$$

for the composite adjunction of Observation 7.3.10. When convenient and unambiguous, we will omit the right adjoint from the notation.

Lemma 7.3.12. *The diagram*

$$\begin{array}{ccc} \mathrm{Alg}_T(s\mathcal{C}) & \xrightarrow{|-|} & \mathrm{Alg}_{|T|}(\mathcal{C}) \\ {\scriptstyle \mathrm{U}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{|T|}} \\ s\mathcal{C} & \xrightarrow[|-|]{} & \mathcal{C} \end{array}$$

commutes.

Proof. Both vertical functors are right adjoints which commute with sifted colimits. □

Theorem 7.3.13. *There is a* ***resolution model structure*** *on* $\mathrm{Alg}_T(s\mathcal{C})$*, denoted* $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$*; it is obtained by lifting the resolution model structure* $s\mathcal{C}_{\mathrm{res}}$ *along the adjunction*

$$\mathrm{F}_T : s\mathcal{C} \rightleftarrows \mathrm{Alg}_T(s\mathcal{C}) : \mathrm{U}_T,$$

which therefore becomes a Quillen adjunction. It enjoys the following properties.

(1) Its weak equivalences and fibrations are created by pullback along the right adjoint U_T*.*

(2) It is simplicial.

(3) All objects are fibrant in it.

(4) It is cofibrantly generated by the sets

$$\begin{aligned} I_{\mathrm{res}}^{\mathrm{Alg}_T(s\mathcal{C})} &= \mathrm{F}_T(I_{\mathrm{res}}^{s\mathcal{C}}) = \{\mathrm{F}_T(I_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon))\}_{S^\varepsilon \in \mathcal{G}_{\mathcal{C}}^E} \\ &= \{\mathrm{F}_T(\partial\Delta^n \odot \mathrm{const}(S^\varepsilon)) \to \mathrm{F}_T(\Delta^n \odot \mathrm{const}(S^\varepsilon))\}_{n\geq 0, S^\varepsilon \in \mathcal{G}_{\mathcal{C}}^E} \end{aligned}$$

and

$$\begin{aligned} J_{\mathrm{res}}^{\mathrm{Alg}_T(s\mathcal{C})} &= \mathrm{F}_T(J_{\mathrm{res}}^{s\mathcal{C}}) = \{\mathrm{F}_T(J_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon))\}_{S^\varepsilon \in \mathcal{G}_{\mathcal{C}}^E} \\ &= \{\mathrm{F}_T(\Lambda_i^n \odot \mathrm{const}(S^\varepsilon)) \to \mathrm{F}_T(\Delta^n \odot \mathrm{const}(S^\varepsilon))\}_{0\leq i\leq n\geq 1, S^\varepsilon \in \mathcal{G}_{\mathcal{C}}^E}. \end{aligned}$$

Proof. The model structure follows from Theorem 7.1.10, the enrichment and bitensoring over $s\mathcal{S}$ follows from Corollary 7.1.7, and their compatibility follows from Theorem 7.1.13. □

Notation 7.3.14. Extending Definition 7.2.28, we write $\mathbf{W}_{E^{\mathrm{lw}}_*} = \mathbf{W}^{\mathrm{Alg}_T(s\mathcal{C})}_{E^{\mathrm{lw}}_*} \subset \mathrm{Alg}_T(s\mathcal{C})$ for the preimage of $\mathbf{W}^{s\mathcal{C}}_{E^{\mathrm{lw}}_*} \subset s\mathcal{C}$ under the forgetful functor $\mathrm{U}_T : \mathrm{Alg}_T(s\mathcal{C}) \to s\mathcal{C}$. Since $\mathbf{W}^{s\mathcal{C}}_{\mathrm{res}} \subset \mathbf{W}^{s\mathcal{C}}_{E^{\mathrm{lw}}_*}$ by Observation 7.2.30, then also $\mathbf{W}^{\mathrm{Alg}_T(s\mathcal{C})}_{\mathrm{res}} \subset \mathbf{W}^{\mathrm{Alg}_T(s\mathcal{C})}_{E^{\mathrm{lw}}_*}$.

Observation 7.3.15. In the end, our moduli spaces of interest will not be subgroupoids of the localization $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$, but rather of the further localization $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$. However, in order to compute hom-spaces in this latter localization, it suffices to observe that the induced functor $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ is actually a left localization: then, we can simply work in $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$ but require that our target objects present local objects in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$ (with respect to this left localization). It follows from Theorem 7.1.22 (and the monadic derived adjunction underlying the monadic Quillen adjunction $\mathrm{F}_T \dashv \mathrm{U}_T$) that $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$ is presentable, so we can apply the recognition result Proposition T.5.5.4.15: it suffices to show that the image in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$ of $\mathbf{W}_{E^{\mathrm{lw}}_*} \subset \mathrm{Alg}_T(s\mathcal{C})$ is strongly saturated (in the sense of Definition T.5.5.4.5). The first two conditions follow from [GHb, Lemma 1.5.2], while the two-out-of-three property follows from the fact that it is ultimately pulled back from a subcategory $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{A}$ which has the two-out-of-three property.

Notation 7.3.16. We will write $\mathrm{L}_{E^{\mathrm{lw}}_*} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] : \mathrm{U}_{E^{\mathrm{lw}}_*}$ for the left localization adjunction of Observation 7.3.15.

Remark 7.3.17. The existence of a fully faithful right adjoint to the canonical functor $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ should not be surprising: in [GHb], this is constructed as a left Bousfield localization (cf. [GHb, Theorems 1.4.9 and 1.5.1]).

Remark 7.3.18. Taking T to be trivial, we obtain a left localization adjunction $\mathrm{L}_{E^{\mathrm{lw}}_*} : s\mathcal{C}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \rightleftarrows s\mathcal{C}[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] : \mathrm{U}_{E^{\mathrm{lw}}_*}$.

Remark 7.3.19. Whereas we have identified $s\mathcal{C}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$ as a nonabelian derived ∞-category, it appears that $s\mathcal{C}[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ does not generally take this form. It will become clear over the course of the construction that we really do need to be working in a nonabelian derived ∞-category.

7.3.3 Operads, revisited

We give a brief unified treatment of all of the sorts of operads, their homotopy, and their related structures that we will be considering. The material in this subsection is undergirded by the foundational work [CH].

7.3.3.1 Operads and their algebras

Notation 7.3.20. For an ∞-category $\mathcal{V}$, we write $\mathcal{V}^{\mathfrak{S}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}^{\simeq}, \mathcal{V})$ for the ∞-category of symmetric sequences in $\mathcal{V}$. Given $\mathcal{O} \in \mathcal{V}^{\mathfrak{S}}$, we write $\mathcal{O}(n) = \mathcal{O}(\{1, \dots, n\})$ for simplicity. Assuming $\mathcal{V}$ has an initial object, we consider $\mathcal{V} \subset \mathcal{V}^{\mathfrak{S}}$ via left Kan extension along $\{\mathrm{pt}_{\mathcal{S}\mathrm{et}}\} \hookrightarrow \mathcal{S}\mathrm{et}^{\simeq}$. When $\mathcal{V}$ additionally admits a symmetric monoidal structure that commutes with colimits separately in each variable (e.g. if the symmetric monoidal structure is closed), the ∞-category $\mathcal{V}^{\mathfrak{S}}$ acquires a composition product monoidal structure $(\mathcal{V}^{\mathfrak{S}}, \circ, \mathbf{1}_{\mathcal{V}})$, algebras for which are precisely ("single colored") $\mathcal{V}$-operads (a/k/a "operads internal to $\mathcal{V}$"). We denote the ∞-category of these by $\mathrm{Op}(\mathcal{V})$, and write

$$\mathrm{F}_{\mathrm{Op}(\mathcal{V})} : \mathcal{V}^{\mathfrak{S}} \rightleftarrows \mathrm{Op}(\mathcal{V}) : \mathrm{U}_{\mathrm{Op}(\mathcal{V})}$$

for the resulting monadic adjunction. For brevity, we will simply say that $\mathcal{V}$ "admits operads" in this case.

When $\mathcal{V}$ is the ∞-category $\mathcal{S}$ of spaces (equipped with the cartesian symmetric monoidal structure), we (continue to) omit it from all our notation and terminology; in particular, we (continue to) refer to the objects of Op simply as "operads". For emphasis, we may refer to objects of $\mathrm{Op}(\mathcal{V})$ for some possibly unspecified $\mathcal{V}$ as "internal operads".

Notation 7.3.21. Let $\mathcal{D} \in \mathrm{LMod}_{\mathcal{V}}(\mathcal{C}\mathrm{at}_{\infty})$ be an ∞-category admitting an action of $\mathcal{V}$, and assume that $\mathcal{D}$ is cocomplete and finitely complete. Then for any $\mathcal{O} \in \mathrm{Op}(\mathcal{V})$ we denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$ the ∞-category of $\mathcal{O}$-algebras in $\mathcal{D}$. This is monadic over $\mathcal{D}$, and we write

$$\mathrm{F}_{\mathcal{O}} : \mathcal{D} \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) : \mathrm{U}_{\mathcal{O}}$$

for the monadic adjunction.

Observation 7.3.22. Let $\mathcal{V}$ be an ∞-category that admits operads, and let $\mathcal{I}$ be any diagram ∞-category. Then $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ also admits operads: it inherits a componentwise symmetric monoidal structure from $\mathcal{V}$, and colimits (including the empty colimit) are computed componentwise. In fact, it is not hard to see that we have an equivalence

$$\mathrm{Op}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})) \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Op}(\mathcal{V})).$$

Proposition 7.3.23. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category that admits operads and admits finite limits, and suppose that the unit object $\mathbf{1}_{\mathcal{V}} \in \mathcal{V}$ is compact. Then there exists a* ***Boardman–Vogt model structure*** *on the ∞-category of $s\mathcal{V}$-operads, denoted $\mathrm{Op}(s\mathcal{V})_{\mathrm{BV}}$, which is simplicial and participates in a Quillen adjunction*

$$\coprod_{n \geq 0} s\mathcal{S}_{\mathrm{KQ}} \rightleftarrows \mathrm{Op}(s\mathcal{V})_{\mathrm{BV}} : \left(\hom^{\mathrm{lw}}_{\mathcal{V}}(\mathbf{1}_{\mathcal{V}}, \mathrm{U}_{\mathfrak{S}_n}((-)(n)))\right)_{n \geq 0}$$

of simplicial model ∞-categories, where $\mathrm{F}_{\mathfrak{S}_n} : \mathcal{V} \rightleftarrows \mathrm{Fun}(B\mathfrak{S}_n, \mathcal{V}) : \mathrm{U}_{\mathfrak{S}_n}$ *denotes the left Kan extension adjunction for the canonical functor* $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \to B\mathfrak{S}_n$.

Proof. This follows from Theorems 7.1.10 and 7.1.13. □

Remark 7.3.24. In the end, we will only use Proposition 7.3.23 in situations when $\mathcal{V}$ is a 1-category. In this case, the result is ultimately more-or-less just a consequence of [Qui67, Chapter II, §4, Theorem 4]. The name of the model structure pays homage to the foundational work [BV73], which introduced the study of homotopy-coherent algebraic structures. The Boardman–Vogt model structure of Proposition 7.3.23 is also closely related to those of [BM03, Theorems 3.1 and 3.2], as explained in [BM03, Example 3.3.1].

Observation 7.3.25. Let $\mathcal{V}$ and $\mathcal{V}'$ be two ∞-categories equipped with symmetric monoidal structures that commute with colimits separately in each variable. Then any lax symmetric monoidal functor $\mathcal{V} \to \mathcal{V}'$ induces a functor $\mathrm{Op}(\mathcal{V}) \to \mathrm{Op}(\mathcal{V}')$.

We single out two particular cases of interest.

- The functor $- \odot \mathbf{1} : \mathcal{S} \to \mathcal{C}$ is symmetric monoidal (with respect to $(\mathcal{S}, \times, \mathrm{pt}_{\mathcal{S}})$ and $(\mathcal{C}, \otimes, \mathbf{1})$).

- The homology functor $E_* : \mathcal{C} \to \mathcal{A}$ is lax symmetric monoidal: for any $X, Y \in \mathcal{C}$, we have a canonical map $E_*X \otimes_{E_*} E_*Y \to E_*(X \otimes Y)$ in $\mathcal{A}$, which takes the element
$$\left(S^\beta \xrightarrow{\varphi} E \otimes X\right) \otimes \left(S^{\beta'} \xrightarrow{\varphi'} E \otimes Y\right)$$
to the element
$$\left(S^{\beta+\beta'} \simeq S^\beta \otimes S^{\beta'} \xrightarrow{\varphi\otimes\varphi'} E \otimes X \otimes E \otimes Y \simeq E^{\otimes 2} \otimes X \otimes Y \xrightarrow{\mu_E \otimes \mathrm{id}_X \otimes \mathrm{id}_Y} E \otimes X \otimes Y\right).$$

It follows that the composite functor
$$\mathcal{S} \xrightarrow{-\odot\mathbf{1}} \mathcal{C} \xrightarrow{E_*} \mathcal{A}$$
is lax symmetric monoidal, and hence induces a composite functor on internal operads, which for brevity we denote simply as
$$E_* : \mathrm{Op} = \mathrm{Op}(\mathcal{S}) \xrightarrow{\mathrm{Op}(-\odot\mathbf{1})} \mathrm{Op}(\mathcal{C}) \xrightarrow{\mathrm{Op}(E_*)} \mathrm{Op}(\mathcal{A}).$$

7.3.3.2 Resolutions of operads

Definition 7.3.26. We say that an operad $\mathcal{O} \in \mathrm{Op}$ is π_0-$\mathfrak{S}$-***free*** if for each $n \geq 0$ the induced action of $\mathfrak{S}_n$ on $\pi_0(\mathcal{O}(n))$ is free.

Remark 7.3.27. As early in the literature as [May72, Definition 1.1], the term "$\mathfrak{S}$-free" is used to describe a point-set operad (e.g. in topological spaces) whose symmetric group actions are free at the point-set level. Of course, such an operad need not present a π_0-$\mathfrak{S}$-free operad in the sense of Definition 7.3.26.

Lemma 7.3.28. *The functor* $\mathrm{F}_{\mathrm{Op}} : \mathcal{S}^{\mathfrak{S}} \to \mathrm{Op}$ *takes values in* π_0-$\mathfrak{S}$-*free operads.*

Proof. This is immediate from the explicit description of F_{Op} that follows from [Rez96, Proposition A.0.2 and Remark A.0.1]. □

Notation 7.3.29. We simply write

$$\mathrm{Bar}(-)_\bullet : \mathrm{Op} \xrightarrow{\mathrm{Bar}(\mathrm{pt}_{\mathcal{S}}, \mathrm{U}_{\mathrm{Op}}\mathrm{F}_{\mathrm{Op}}, -)_\bullet} s\mathrm{Op}$$

for the bar construction on the monad $\mathrm{U}_{\mathrm{Op}}\mathrm{F}_{\mathrm{Op}} \in \mathrm{Alg}(\mathrm{End}(\mathcal{S}^{\mathfrak{S}}))$ with respect to the left module given by the unit $\mathrm{pt}_{\mathcal{S}} \in \mathcal{S}^{\mathfrak{S}}$ and an unspecified operad considered as a right module.

Corollary 7.3.30. *The functor* $\mathrm{Bar} : \mathrm{Op} \to s\mathrm{Op}$ *takes values in levelwise* π_0-$\mathfrak{S}$-*free simplicial operads, and admits a natural equivalence* $|\mathrm{Bar}(-)_\bullet| \simeq \mathrm{id}_{\mathrm{Op}}$ *in* $\mathrm{Fun}(\mathrm{Op}, \mathrm{Op})$.

Proof. This follows from Lemma 7.3.28. □

Corollary 7.3.31. *Given an operad* $\mathcal{O}$*, suppose that* $E_*(\mathcal{O}(n)) \in \mathcal{A}_{\mathrm{proj}}$ *for all* $n \geq 0$*. Then* $E_*^{\mathrm{lw}}\mathrm{Bar}(\mathcal{O})_\bullet \in s\mathrm{Op}(\mathcal{A}) \simeq \mathrm{Op}(s\mathcal{A})_{\mathrm{BV}}$ *is cofibrant, and the augmentation* $\mathrm{Bar}(\mathcal{O})_\bullet \to \mathrm{const}(\mathcal{O})$ *induces a weak equivalence* $E_*^{\mathrm{lw}}\mathrm{Bar}(\mathcal{O})_\bullet \xrightarrow{\approx} \mathrm{const}(E_*\mathcal{O})$ *in* $\mathrm{Op}(s\mathcal{A})_{\mathrm{BV}}$.

Proof. This is immediate from the explicit description of F_{Op} that follows from [Rez96, Proposition A.0.2 and Remark A.0.1]. □

Remark 7.3.32. While we will ultimately be interested in a simplicial operad resolving our operad of primary interest, much of the theory goes through equally well for any simplicial operad.

7.4 Algebra

7.4.1 Foundations of algebra

Recall that we write $\mathcal{G}^\delta = \pi_0(\mathcal{G})$ for our chosen group of Picard elements, $\underline{\mathcal{A}} = \mathrm{Fun}(\mathcal{G}^\delta, \mathrm{Ab})$ for the category of $\mathcal{G}^\delta$-graded abelian groups, and $\mathcal{A} = \mathrm{Mod}_{E_*}(\underline{\mathcal{A}})$ for the category of E_*-modules in $\underline{\mathcal{A}}$.

Assumption 7.4.1. We assume that $E_*E \in \mathcal{A}$ is flat.

Notation 7.4.2. It follows from Assumption 7.4.1 that (E_*, E_*E) is a Hopf algebroid in $\underline{\mathcal{A}}$. We write $\tilde{\mathcal{A}} = \mathrm{Comod}_{(E_*, E_*E)}$ for its category of left comodules (which in light of Assumption 7.4.1 is abelian by [Rav86, Theorem A1.1.3]), and we consider our homology theory as a functor $E_* : \mathcal{C} \to \tilde{\mathcal{A}}$ taking values in (E_*, E_*E)-comodules.

Remark 7.4.3. In general, the forgetful functor $\tilde{\mathcal{A}} \xrightarrow{\mathrm{U}_{\tilde{\mathcal{A}}}} \mathcal{A}$ does not admit a left adjoint (e.g. it does not preserve products (see [Hov04, §1.2])).

Observation 7.4.4. For any $\beta \in \mathcal{G}^\delta$ we obtain an evident endofuctor $\Sigma^\beta : \tilde{\mathcal{A}} \xrightarrow{\sim} \tilde{\mathcal{A}}$. This allows us to consider $\tilde{\mathcal{A}}$ as enriched over $\mathcal{A}$, where for $M, N \in \tilde{\mathcal{A}}$ we set

$$\underline{\hom}_{\tilde{\mathcal{A}}}(M, N) = \{\hom_{\tilde{\mathcal{A}}}(\Sigma^\beta M, N)\}_{\beta \in \mathcal{G}^\delta} \in \mathcal{A}.$$

7.4.2 Compatibility

The resolutions of operads considered in §7.3.3.2 are necessary but not alone sufficient to render the obstruction theory to be tractable: we have introduced a new simplicial direction on the topology side, but we have not yet exerted any control on the simplicial direction that results on the algebra side. Indeed, this will bring our E-homology computations into the realm of homotopical algebra, with its own attendant notions of "cofibrant resolution", and we must ensure that our homology functor E_* preserves resolutions.

We introduce three increasingly general notions of compatibility; the first is merely to fix ideas, the second is auxiliary, and the last is our real goal.

Definition 7.4.5. We say that an operad $\mathcal{O} \in \mathrm{Op}$ is ***adapted*** to E if it comes with a corresponding monad $\mathcal{O}_E \in \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$ admitting a lift

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) & \overset{E_*}{\dashrightarrow} & \mathrm{Alg}_{\mathcal{O}_E}(\mathcal{A}) \\ {\scriptstyle \mathrm{U}_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\mathcal{O}_E}} \\ \mathcal{C} & \xrightarrow[E_*]{} & \mathcal{A} \end{array}$$

such that the following condition holds:

- for any $Z \in \mathcal{C}$ with $E_* Z \in \mathcal{A}_{\mathrm{proj}}$, the natural map $\mathrm{F}_{\mathcal{O}_E}(E_* Z) \to E_*(\mathrm{F}_{\mathcal{O}}(Z))$ is an isomorphism in $\mathrm{Alg}_{\mathcal{O}_E}(\mathcal{A})$.

Definition 7.4.6. We say that a simplicial operad $T \in s\mathrm{Op}$ is ***adapted*** to E if it comes with a corresponding monad $T_E \in \mathrm{Alg}(\mathrm{End}(s\mathcal{A}))$ admitting a lift

$$\begin{array}{ccc} \mathrm{Alg}_T(s\mathcal{C}) & \overset{E_*^{\mathrm{lw}}}{\dashrightarrow} & \mathrm{Alg}_{T_E}(s\mathcal{A}) \\ {\scriptstyle \mathrm{U}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{T_E}} \\ s\mathcal{C} & \xrightarrow[E_*]{} & s\mathcal{A} \end{array}$$

such that the following condition holds:

- for any $Z \in s\mathcal{C}$ with $E_*^{\mathrm{lw}} Z \in s\mathcal{A}^c_{\mathrm{KQ}}$, the natural map $\mathrm{F}_{T_E}(E_*^{\mathrm{lw}} Z) \to E_*^{\mathrm{lw}}(\mathrm{F}_T(Z))$ is an isomorphism in $\mathrm{Alg}_{T_E}(s\mathcal{A})$.

This has the following consequence.

Lemma 7.4.7 ([GHb, Lemma 1.4.15])**.** *If $T \in s\mathrm{Op}$ is adapted to E, then any cofibration between cofibrant objects in $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$ is a retract of a map $X \xrightarrow{\varphi} Y$ such that the underlying map of degeneracy diagrams of $E_*^{\mathrm{lw}}(\varphi)$ is isomorphic to one of the form $E_*^{\mathrm{lw}}(X) \to E_*^{\mathrm{lw}}(X) \coprod T_E(M)$, where M is s-free on an object of $\mathcal{A}_{\mathrm{proj}}$.* □

Definition 7.4.8. Suppose that the simplicial operad $T \in s\mathrm{Op}$ is adapted to $E_*^{\mathrm{lw}} : s\mathcal{C} \to s\mathcal{A}$. We then say that T is ***homotopically adapted*** to E if there exists a monad $\tilde{T}_E \in \mathrm{Alg}(\mathrm{End}(s\tilde{\mathcal{A}}))$ which lifts the monad $T_E \in \mathrm{Alg}(\mathrm{End}(s\mathcal{A}))$ (i.e. they're intertwined by $s(\mathrm{U}_{\tilde{\mathcal{A}}})$) and which admits a lift

$$\begin{array}{ccc} \mathrm{Alg}_T(s\mathcal{C}) & \overset{E_*^{\mathrm{lw}}}{\dashrightarrow} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \\ {\scriptstyle \mathrm{U}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\tilde{T}_E}} \\ s\mathcal{C} & \xrightarrow[E_*]{} & s\tilde{\mathcal{A}} \end{array}$$

such that the following conditions hold:

- the adjunction $\mathrm{F}_{T_E} : s\mathcal{A} \rightleftarrows \mathrm{Alg}_{T_E}(s\mathcal{A}) : \mathrm{U}_{T_E}$ creates a simplicial model structure on $\mathrm{Alg}_{T_E}(s\mathcal{A})$; and

- there exists a simplicial model structure on $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$ such that the forgetful functor $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \to \mathrm{Alg}_{T_E}(s\mathcal{A})$ creates weak equivalences and preserves fibrations.

Building on Lemma 7.4.7, this has the following key consequence.

Lemma 7.4.9 ([GHb, Corollary 1.4.18])**.** *If $T \in s\mathrm{Op}$ is homotopically adapted to E, then the induced functor $E^{\mathrm{lw}}_{*} : \mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}} \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_*}$ preserves both weak equivalences as well as cofibrations between cofibrant objects.* □

This result, in turn, has the following ∞-categorical significance.

Corollary 7.4.10. *If $T \in s\mathrm{Op}$ is homotopically adapted to E, then the functor $E^{\mathrm{lw}}_{*} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\pi_*}]\!]$ preserves colimits.*

Proof. This follows by combining Lemma 7.4.9 with the theory of homotopy colimits in model ∞-categories of §5.1.2; more specifically, the model ∞-categories $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$ and $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_*}$ are both cofibrantly generated and hence admit projective model structures, and the functor of model ∞-categories preserves projective cofibrancy by Lemma 7.4.9. □

Remark 7.4.11. Given two ∞-categories that admit finite coproducts and a functor between them that preserves these, applying the functor $\mathcal{P}_\Sigma$ automatically gives a cocontinuous functor: up to further left localizations (which commute with colimits), this is precisely the situation that Corollary 7.4.10 addresses. However, it is only through Theorem 7.1.22 that we can identify it as such.

Assumption 7.4.12. We henceforth assume that T is homotopically adapted to E, and fix the corresponding monad $\tilde{T}_E \in \mathrm{Alg}(\mathrm{End}(s\tilde{\mathcal{A}}))$.

Example 7.4.13. For any $\mathcal{O} \in \mathrm{Op}$, we can take T to be a cofibrant object of $\mathrm{Op}(s\mathcal{S}\mathrm{et})_{\mathrm{BV}}$ which presents it: each $T(n)$ will have a free $\mathfrak{S}_n$-action (as a simplicial set), and we can take $\tilde{T}_E$ to be the monad corresponding to the operad $E_*T \in \mathrm{Op}(s\tilde{\mathcal{A}})$.

7.4.3 The module structure on the localized spiral exact sequence

Definition 7.4.14. An ***augmentation*** of the monad $\tilde{T}_E \in \mathrm{Alg}(\mathrm{End}(s\tilde{\mathcal{A}}))$ is the data of a monad $\Phi \in \mathrm{Alg}(\mathrm{End}(\tilde{\mathcal{A}}))$ and a natural isomorphism making the diagram

$$\begin{array}{ccc} s\tilde{\mathcal{A}} & \xrightarrow{\tilde{T}_E} & s\tilde{\mathcal{A}} \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ \tilde{\mathcal{A}} & \xrightarrow[\Phi]{} & \tilde{\mathcal{A}} \end{array}$$

commute, satisfying the diagrammatic coherence conditions of [GHb, Definition 2.5.7]. We write this as $\tilde{T}_E \downarrow \Phi$, though note that this does not depict a morphism in any category.

Assumption 7.4.15. We henceforth assume the existence of an augmentation $\tilde{T}_E \downarrow \Phi$.

In order to describe the key consequence of Assumption 7.4.15, we must introduce some terminology.

Definition 7.4.16. For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, we define the category of ***A-modules (relative to Φ)*** as the category $\mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}}) = \mathrm{Ab}(\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{/A})$ of abelian group objects in its overcategory. To align our notation with standard intuition, we write

$$\begin{array}{ccccc} \tilde{\mathcal{A}} & \xleftarrow{\mathrm{U}_A} & \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}}) & \xrightarrow{-\ltimes A} & \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}) \\ \ker^{\tilde{\mathcal{A}}}(\varphi) & \longleftarrow\!\shortmid & (B \xrightarrow{\varphi} A) & \shortmid\!\longrightarrow & B \end{array}$$

for the two forgetful functors.

Lemma 7.4.17 ([GHb, Propositions 2.5.9 and 2.5.10])**.** *There exists a canonical lift*

$$\begin{array}{ccccc} & & & & \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}) \\ & & \overset{\pi_0}{\dashrightarrow} & & \downarrow{\scriptstyle \mathrm{U}_\Phi} \\ \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) & \xrightarrow[\mathrm{U}_{\tilde{T}_E}]{} & s\tilde{\mathcal{A}} & \xrightarrow[\pi_0]{} & \tilde{\mathcal{A}}, \end{array}$$

and this lift is the left adjoint in an adjunction

$$\pi_0 : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \rightleftarrows \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}) : \mathrm{const}.$$

Moreover, for any $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$ and any $n \geq 1$, the object $\pi_n X \in \tilde{\mathcal{A}}$ admits a canonical lift through the functor

$$\mathrm{Mod}^{\Phi}_{\pi_0 X}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_{\pi_0 X}} \tilde{\mathcal{A}}. \qquad \square$$

Corollary 7.4.18. *There exists a canonical lift*

$$\begin{array}{ccccccc} & & & & & & \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \\ & & & \overset{\pi_0 E^{\mathrm{lw}}_{*}}{\nearrow} & & & \downarrow{\scriptstyle \mathrm{U}_{\Phi}} \\ \mathrm{Alg}_{T}(s\mathcal{C}) & \xrightarrow[E^{\mathrm{lw}}_{*}]{} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) & \xrightarrow[\mathrm{U}_{\tilde{T}_E}]{} & s\tilde{\mathcal{A}} & \xrightarrow[\pi_0]{} & \tilde{\mathcal{A}}. \end{array}$$

Moreover, for any $X \in \mathrm{Alg}_T(s\mathcal{C})$ and any $n \geq 1$, the object $\pi_n E^{\mathrm{lw}}_{*} X \in \tilde{\mathcal{A}}$ admits a canonical lift through the functor

$$\mathrm{Mod}^{\Phi}_{\pi_0 E^{\mathrm{lw}}_{*} X}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_{\pi_0 E^{\mathrm{lw}}_{*} X}} \tilde{\mathcal{A}}. \qquad \square$$

We record a useful fact about the adjunction of Lemma 7.4.17.

Lemma 7.4.19. *The adjunction of Lemma 7.4.17 lifts to a Quillen adjunction*

$$\pi_0 : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_*} \rightleftarrows \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{\mathrm{triv}} : \mathrm{const},$$

whose derived adjunction is a left localization adjunction.

Proof. To see that this is a Quillen adjunction, we observe that the left adjoint

- trivially preserves cofibrations, and
- preserves acyclic cofibrations by definition of the subcategory $\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$.

Then, to see that the derived adjunction is a left localization adjunction, we check that its counit is a componentwise equivalence. Since every object of $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{\mathrm{triv}}$ is fibrant, the composite

$$\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{const}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\pi_*}]\!]$$

computes the derived right adjoint $\mathbb{R}\mathrm{const}$. Now, let

$$\varnothing_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})} \rightarrowtail \mathbb{Q}\mathrm{const}(A) \xrightarrow{\approx} \mathrm{const}(A)$$

be a cofibrant replacement in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_*}$. Then by definition the induced map

$$\pi_0(\mathbb{Q}\mathrm{const}(A)) \to \pi_0(\mathrm{const}(A)) \cong A$$

is an isomorphism in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$. So the counit is indeed an equivalence. $\square$

Notation 7.4.20. As both functors in the Quillen adjunction of Lemma 7.4.19 preserve all weak equivalences, we will simply write

$$\pi_0 : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \rightleftarrows \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) : \mathrm{const}$$

for its derived adjunction (as opposed to $\mathbb{L}\pi_0 \dashv \mathbb{R}\mathrm{const}$). Moreover, we will often leave implicit both the right Quillen functor as well as its derived right adjoint.

We have just seen that classical homology groups admit certain algebraic structure. In fact, natural homology groups do too.

Lemma 7.4.21 ([GHb, Examples 3.1.14 and 3.1.17])**.** *There exists a canonical lift*

$$\begin{array}{ccc} & & \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \\ & \overset{E^{\natural}_{0,*}}{\nearrow} & \big\downarrow {\scriptstyle \mathrm{U}_{\mathcal{A}}\circ\mathrm{U}_{\tilde{\mathcal{A}}}\circ\mathrm{U}_{\Phi}} \\ \mathrm{Alg}_T(s\mathcal{C}) & \xrightarrow[E^{\natural}_{0,*}]{} & \underline{\mathcal{A}}. \end{array}$$

Moreover, for any $X \in \mathrm{Alg}_T(s\mathcal{C})$ and any $n \geq 1$, the object $E^{\natural}_{n,*}X \in \underline{\mathcal{A}}$ admits a canonical lift through the functor

$$\mathrm{Mod}^{\Phi}_{E^{\natural}_{0,*}X}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_{\mathcal{A}}\circ\mathrm{U}_{\tilde{\mathcal{A}}}\circ\mathrm{U}_{E^{\natural}_{0,*}X}} \underline{\mathcal{A}}.$$ □

Moreover, these algebraic structures are compatible in the following way.

Lemma 7.4.22 ([GHb, Corollary 3.1.18])**.** *The isomorphism $\pi_0 E^{\mathrm{lw}}_*(-) \cong E^{\natural}_{0,*}(-)$ in $\mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \underline{\mathcal{A}})$ of Lemma 7.2.51 is compatible with the lifts to $\mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}))$ of Corollary 7.4.18 and Lemma 7.4.21.* □

Notation 7.4.23. For simplicity, we may write $E_0 : \mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ for the functor $\pi_0 E^{\mathrm{lw}}_* \cong E^{\natural}_{0,*}$.

Lemma 7.4.24 ([GHb, Example 3.1.13])**.** *For any $A \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ and any $n \geq 1$, the endofunctor $\Omega^n : \tilde{\mathcal{A}} \xrightarrow{\sim} \tilde{\mathcal{A}}$ lifts to an endofunctor $\Omega^n : \mathrm{Mod}^{\Phi}_A(\tilde{\mathcal{A}}) \xrightarrow{\sim} \mathrm{Mod}^{\Phi}_A(\tilde{\mathcal{A}})$.* □

Remark 7.4.25. In fact, if we define $\Sigma^{\beta}_{+}S^{\varepsilon} = (\mathbf{1} \oplus S^{\beta}) \otimes S^{\varepsilon}$, then the construction of [GHb, Example 3.1.13] generalizes to define lifted endofunctors $\Omega^{\beta} : \mathrm{Mod}^{\Phi}_A(\tilde{\mathcal{A}}) \xrightarrow{\sim} \mathrm{Mod}^{\Phi}_A(\tilde{\mathcal{A}})$ for any $\beta \in \mathcal{G}^{\delta}$.

We can now give the module structure on the localized spiral exact sequence.

Proposition 7.4.26 ([GHb, Corollary 3.1.18])**.** *For any $X \in \mathrm{Alg}_T(s\mathcal{C})$, assembling the localized spiral exact sequence in* Ab *over all $\beta \in \mathcal{G}^\delta$, we obtain an exact sequence*

$$\cdots \longrightarrow \pi_{i+1}E_*X \xrightarrow{\delta} \Omega(E^\natural_{i-1,*}X) \longrightarrow E^\natural_{i,*}X \longrightarrow \pi_i E_*X \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} \Omega(E^\natural_{0,*}X) \longrightarrow E^\natural_{1,*}X \longrightarrow \pi_1 E_*X \longrightarrow 0$$

in $\mathrm{Mod}^\Phi_{E_0X}(\tilde{\mathcal{A}})$. □

7.4.4 The module structure on the spiral exact sequence

We will make certain computations *before* appealing to a colimit argument, and for these we will need to obtain analogous structure on the unlocalized spiral exact sequence. In fact, this is an input to the module structure on the localized spiral exact sequence (via a colimit argument, as always), but the algebraic objects at play are slightly less familiar so we have reversed their order here. However, the story is nearly identical to that of §7.4.3, and so we only highlight the key points.

Notation 7.4.27. We write $T(\mathcal{G}^E_{\mathcal{C}}) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$ for the full subcategory spanned by the image of the composite

$$\mathcal{G}^E_{\mathcal{C}} \hookrightarrow \mathcal{C} \xrightarrow{\mathrm{const}} s\mathcal{C} \xrightarrow{\mathrm{F}_T} \mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!].$$

Observation 7.4.28. The functor $\mathcal{G}^E_{\mathcal{C}} \xrightarrow{\mathrm{F}_T} T(\mathcal{G}^E_{\mathcal{C}})$ preserves coproducts, and so induces a forgetful functor $\mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_{\mathcal{C}})) \xrightarrow{\mathrm{U}_{T(\mathcal{G}^E_{\mathcal{C}})}} \mathcal{P}^\delta_\Sigma(\mathcal{G}^E_{\mathcal{C}})$.

Definition 7.4.29. For any $A \in \mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_{\mathcal{C}}))$, we define the category of ***A*-modules (relative to** $T(\mathcal{G}^E_{\mathcal{C}})$) as the category $\mathrm{Mod}_A^{T(\mathcal{G}^E_{\mathcal{C}})}(\mathcal{P}^\delta_\Sigma(\mathcal{G}^E_{\mathcal{C}})) = \mathrm{Ab}(\mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_{\mathcal{C}}))_{/A})$ of abelian group objects in its overcategory. This admits two forgetful functors, which we denote by

$$\mathcal{P}^\delta_\Sigma(\mathcal{G}^E_{\mathcal{C}}) \xleftarrow{\mathrm{U}_A} \mathrm{Mod}_A^{T(\mathcal{G}^E_{\mathcal{C}})}(\mathcal{P}^\delta_\Sigma(\mathcal{G}^E_{\mathcal{C}})) \xrightarrow{-\ltimes A} \mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_{\mathcal{C}})).$$
$$\ker^{\mathcal{P}^\delta_\Sigma(\mathcal{G}^E_{\mathcal{C}})}(\varphi) \longleftarrow\!\shortmid\ (B \xrightarrow{\varphi} A) \longmapsto B$$

The following example will be of use later.

Notation 7.4.30. Let $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$. Then we obtain an object $\text{ɬ}^E(A) \in \mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_\mathcal{C}))$ by declaring that

$$\text{ɬ}^E(A)(\mathrm{F}_T(S^\varepsilon)) = \hom_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})}(\pi_0 E^{\mathrm{lw}}_* \mathrm{F}_T(S^\varepsilon), A).$$

Similarly, if $M \in \mathrm{Mod}^\Phi_A(\tilde{\mathcal{A}})$, we obtain an object $\text{ɬ}^E(M) \in \mathrm{Mod}^{T(\mathcal{G}^E_\mathcal{C})}_{\text{ɬ}^E_*(A)}(\mathcal{P}^\delta_\Sigma(\mathcal{G}^E_\mathcal{C}))$ by declaring that

$$\text{ɬ}^E(M)(S^\varepsilon) = \hom_{\tilde{\mathcal{A}}}(\pi_0 E^{\mathrm{lw}}_* S^\varepsilon, M);$$

technically, the A-action arises through Definitions 7.4.16 and 7.4.29 (in terms of abelian objects in overcategories), but morally it just comes from postcomposition.

Observation 7.4.31. As the functor $\mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{ho}(\mathrm{Alg}_T(s\mathcal{C}))$ preserves finite coproducts, by adjunction both composite functors

$$\mathrm{Alg}_T(s\mathcal{C}) \xrightarrow{\mathrm{U}_T} s\mathcal{C} \underset{\pi_0\pi^{\mathrm{lw}}_*}{\overset{\pi^\natural_{0,*}}{\rightrightarrows}} \mathcal{P}^\delta_\Sigma(\mathcal{G}^E_\mathcal{C})$$

admit lifts through $\mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_\mathcal{C})) \xrightarrow{\mathrm{U}_{T(\mathcal{G}^E_\mathcal{C})}} \mathcal{P}^\delta_\Sigma(\mathcal{G}^E_\mathcal{C})$ for any $n \geq 0$.

Lemma 7.4.32. *The isomorphisms $\pi_0\pi^{\mathrm{lw}}_\varepsilon(-) \cong \pi^\natural_{0,\varepsilon}(-)$ in $\mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \mathrm{Fun}(\mathcal{G}^{E^\delta}_\mathcal{C}, \mathrm{Ab}))$ of Corollary 7.2.47 are compatible with the lifts to $\mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_\mathcal{C})))$ of Observation 7.4.31.*

Notation 7.4.33. For simplicity, we may write $\pi_0 : \mathrm{Alg}_T(s\mathcal{C}) \to \mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_\mathcal{C}))$ for the functor $\pi_0\pi^{\mathrm{lw}}_* \cong \pi^\natural_{0,*}$.

Proposition 7.4.34 ([GHb, Theorem 3.1.15])**.** *For any $X \in \mathrm{Alg}_T(s\mathcal{C})$, assembling the spiral exact sequence in* Ab *over all $\varepsilon \in \mathcal{G}^{E^\delta}_\mathcal{C}$, we obtain an exact sequence*

$$\cdots \longrightarrow \pi_{i+1}\pi_* X \xrightarrow{\delta} \Omega(\pi^\natural_{i-1,*}X) \longrightarrow \pi^\natural_{i,*}X \longrightarrow \pi_i\pi_* X \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} \Omega(\pi^\natural_{0,*}X) \longrightarrow \pi^\natural_{1,*}X \longrightarrow \pi_1\pi_* X \longrightarrow 0$$

in $\mathrm{Mod}^{T(\mathcal{G}^E_\mathcal{C})}_{\pi_0 X}(\mathcal{P}^\delta_\Sigma(\mathcal{G}^E_\mathcal{C}))$. □

7.5 Homotopical algebra

7.5.1 Postnikov towers in algebra

Definition 7.5.1. For any $n \geq 0$, an object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ is called n-***truncated*** if $\pi_{>n}X = 0$. Such objects form a full subcategory $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq n} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, and as n varies these subcategories are evidently nested as

$$\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \hookleftarrow \cdots \hookleftarrow \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq 1} \hookleftarrow \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq 0}.$$

By presentability, these inclusions admit left adjoints, and we denote the corresponding left localization adjunctions by

$$P_n^{\mathrm{alg}} : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \rightleftarrows \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq n} : \mathrm{U}_n^{\mathrm{alg}}.$$

We therefore obtain a tower of functors

$$\mathrm{id}_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]} \to \cdots \to P_1^{\mathrm{alg}} \to P_0^{\mathrm{alg}}.$$

We refer to its value on an object of $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ as its ***Postnikov tower***. We write

$$\mathrm{id}_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]} \xrightarrow{\tau_n^{\mathrm{alg}}} P_n^{\mathrm{alg}}$$

for the natural transformation (or for its composite with $\mathrm{U}_m^{\mathrm{alg}}$ for any $m \geq 0$), which we refer to as the n-***truncation map***.

7.5.2 Cohomology

Our obstructions will take place in (André–Quillen) cohomology groups in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. We will only need to consider them with respect to a base object lying in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, so we restrict to this special case.

We begin by defining the representing objects for cohomology.

Definition 7.5.2. Let $A \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, let $M \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$, and let $n \geq 1$.

(1) We say that an object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ is of ***type*** K_A if there exists an equivalence $X \simeq A$, i.e. if

- there exists an isomorphism $\pi_0 X \cong A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, and
- $\pi_i X = 0$ for $i > 0$.

(2) We say that an object $Y \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ is of ***type*** $K_A(M,n)$ if

- there exists an isomorphism $\pi_0 Y \cong A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$,
- there exists an isomorphism $\pi_n Y \cong M$ via the resulting equivalence of categories $\mathrm{Mod}^{\Phi}_{\pi_0 Y}(\tilde{\mathcal{A}}) \simeq \mathrm{Mod}^{\Phi}_{A}(\tilde{\mathcal{A}})$, and
- $\pi_i Y = 0$ for $i \notin \{0, n\}$.

(3) We say that a morphism $X \to Y$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ is of ***type*** $\vec{K}_A(M,n)$ if

- X is of type K_A,
- Y is of type $K_A(M,n)$, and
- the map $\pi_0 X \to \pi_0 Y$ is an isomorphism in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$.

(4) We say that an object is of ***type*** $K_A(M,0)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ if it is of type $K_{M \ltimes A}$, and we say that a morphism in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ is of ***type*** $\vec{K}_A(M,0)$ if it admits an equivalence to the map $\mathrm{const}(A \to M \ltimes A)$.

We refer to objects of type K_A and $K_A(M,n)$ collectively as ***algebraic Eilenberg–Mac Lane objects***, and to morphisms of type $\vec{K}_A(M,n)$ collectively as ***algebraic Eilenberg–Mac Lane morphisms***. We will see that these all exist and are unique in Propositions 7.5.25 and 7.5.26; justified by this, we may simply write K_A or $K_A(M,n)$ for convenience when referring to an algebraic Eilenberg–Mac Lane object of the indicated type.

Observation 7.5.3. Suppose that $X \to Y$ is morphism in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ of type $\vec{K}_A(M,n)$ for some $n \geq 1$. Then $P_0^{\mathrm{alg}}(Y)$ is of type K_A, and the composite

$$X \to Y \xrightarrow{\tau_0^{\mathrm{alg}}} P_0^{\mathrm{alg}}(Y)$$

with the canonical 0-truncation map is an equivalence. Fixing an equivalence $X \simeq A$ then allows us to consider

$$K_A(M,n) \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]_{A//A}.$$

Of course, such consideration is immediate for $n = 0$.

Observation 7.5.4. For any $n \geq 0$, taking the pullback of a map of type $\vec{K}_A(M, n+1)$ with itself yields a fiber square

$$\begin{array}{ccc} K_A(M,n) & \xrightarrow{\tau_0^{\mathrm{alg}}} & A \\ {\scriptstyle \tau_0^{\mathrm{alg}}}\downarrow & & \downarrow \\ A & \longrightarrow & K_A(M,n+1) \end{array}$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. Hence, the objects

$$\left\{ K_A(M,n) \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{A//A} \right\}_{n \geq 0}$$

assemble into an Ω-spectrum object

$$K_A M \in \mathrm{Stab}\left(\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{A//A} \right).$$

Definition 7.5.5. Let $A \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, let $M \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$, and let $n \geq 0$. Suppose that $k \to A = \mathrm{const}(A)$ is a morphism in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, and use this to consider $K_A(M,n) \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k//A}$. Then, choose any object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k//A}$.

(1) We define the $n^{\mathbf{th}}$ ***(André–Quillen) cohomology group*** of X with coefficients in M to be the abelian group

$$H^n_{\tilde{T}_E}(X/k; M) = [X, K_A(M,n)]_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k//A}} \in \mathrm{Ab}.$$

(2) We define the $n^{\mathbf{th}}$ ***(André–Quillen) cohomology space*** of X with coefficients in M to be the based space

$$\mathscr{H}^n_{\tilde{T}_E}(X/k; M) = \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k//A}}(X, K_A(M,n)) \in \mathcal{S}_*.$$

Thus, we have that

$$H^n_{\tilde{T}_E}(X/k; M) = \pi_0(\mathscr{H}^n_{\tilde{T}_E}(X/k; M)),$$

and moreover it follows from Observation 7.5.4 that

$$H^{n-i}_{\tilde{T}_E}(X/k; M) = \pi_i(\mathscr{H}^n_{\tilde{T}_E}(X/k; M))$$

for $0 \leq i \leq n$. (In particular, cohomology groups are indeed abelian groups, and cohomology spaces are infinite loopspaces.)

Observation 7.5.6. In the setting of Definition 7.5.5, there is an evident pullback square

$$\begin{array}{ccc} \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]_{X//A}}(A, K_A(M,n)) & \longrightarrow & \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]_{k//A}}(A, K_A(M,n)) \\ \downarrow & & \downarrow \\ \{X \to A \to K_A(M,n)\} & \hookrightarrow & \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]_{k//A}}(X, K_A(M,n)) \end{array}$$

in $\mathcal{S}_*$, which is by definition a pullback square

$$\begin{array}{ccc} \mathscr{H}^n_{\tilde{T}_E}(A/X; M) & \longrightarrow & \mathscr{H}^n_{\tilde{T}_E}(A/k; M) \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathscr{H}^n_{\tilde{T}_E}(X/k; M). \end{array}$$

This gives rise to a long exact sequence

$$0 \longrightarrow H^0_{\tilde{T}_E}(A/X; M) \longrightarrow H^0_{\tilde{T}_E}(A/k; M) \longrightarrow H^0_{\tilde{T}_E}(X/k; M) \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} H^n_{\tilde{T}_E}(A/X; M) \longrightarrow H^n_{\tilde{T}_E}(A/k; M) \longrightarrow H^n_{\tilde{T}_E}(X/k; M) \xrightarrow{\delta} H^{n+1}_{\tilde{T}_E}(A/X; M) \longrightarrow \cdots$$

in Ab; exactness at $H^0_{\tilde{T}_E}(A/X; M)$ follows from the fact that the space

$$\hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]_{k//A}}(X, K_A(M,0)) \simeq \hom_{\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{\pi_0 k//A}}(\pi_0 X, M \ltimes A)$$

is discrete (and so in particular has vanishing π_1). We refer to this as the ***transitivity sequence***.

Remark 7.5.7. When $M \in \tilde{\mathcal{A}}$ is an *extended* comodule, these cohomology computations reduce to analogous ones in $\mathrm{Alg}_{T_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ (see [GHb, Proposition 2.4.7]).

7.5.3 Moduli spaces in algebra

We will be interested in various moduli spaces of algebraic objects: ultimately, our obstruction theory will be based on homotopy groups in the ∞-category $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$.

In order to be able to effectively control these homotopy groups, we need to make the following assumption.

Assumption 7.5.8. We assume that $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ has ***Blakers–Massey excision***: for any pushout square

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Z \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\rho} \\ Y & \longrightarrow & W \end{array}$$

such that $\pi_{<m}(\mathrm{fib}(\varphi)) = \pi_{<n}(\mathrm{fib}(\psi)) = 0$, the map $\pi_k(\mathrm{fib}(\varphi)) \to \pi_k(\mathrm{fib}(\rho))$ is an isomorphism for $k < m+n$ and is surjective for $k = m+n$.

Corollary 7.5.9 ([GHb, Corollary 2.3.15])**.** *Suppose that*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Z \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

is a pushout square in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ *such that* $\pi_{<m}(\mathrm{fib}(\varphi)) = \pi_{<n}(\mathrm{fib}(\psi)) = 0$. *Then there is an induced partial long exact sequence*

$$\pi_{m+n}(Y) \oplus \pi_{m+n}(Z) \longrightarrow \pi_{m+n}(W) \xrightarrow{\delta} \pi_{m+n-1}(X) \longrightarrow \cdots$$
$$\cdots \xrightarrow{\delta} \pi_0(X) \longrightarrow \pi_0(Y) \oplus \pi_0(Z) \longrightarrow \pi_0(W) \longrightarrow 0$$

in $\tilde{\mathcal{A}}$*, which we refer to as the* ***Blakers–Massey long exact sequence****.*

Remark 7.5.10. Assumption 7.5.8 holds in examples of interest, e.g. when $\tilde{T}_E$ is the monad corresponding to an operad $E_*(T) \in \mathrm{Op}(s\tilde{\mathcal{A}})$ for any $T \in \mathrm{Op}(s\mathcal{S}\mathrm{et})$ (see [GHb, Theorem 2.3.13 and Remark 2.3.14]).

Our moduli spaces will be related by the following natural construction.

Construction 7.5.11. Let $X \xrightarrow{\varphi} Y$ be a map in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$, and write

$$\mathrm{p}_0^{\mathrm{alg}}(\varphi) = Y \coprod_X P_0^{\mathrm{alg}} X = \mathrm{colim} \left(\begin{array}{ccc} X & \xrightarrow{\tau_0^{\mathrm{alg}}} & P_0^{\mathrm{alg}}(X) \\ {\scriptstyle\varphi}\downarrow & & \\ Y & & \end{array} \right)$$

for the indicated pushout. For any $n \geq 0$ we obtain a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\tau_0^{\mathrm{alg}}} & P_0^{\mathrm{alg}}(X) & & \\ {\scriptstyle\varphi}\downarrow & & \downarrow & \searrow^{\delta_n(\varphi)} & \\ Y & \longrightarrow & \mathrm{p}_0^{\mathrm{alg}}(\varphi) & \xrightarrow[\tau_{n+1}^{\mathrm{alg}}]{} & P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}(\varphi)) \end{array}$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$, and we refer to the map $\delta_n(\varphi)$ as the $n^{\mathbf{th}}$ ***difference construction*** on the map φ. This defines an augmented endofunctor on $\mathrm{Fun}([1], \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}])$. We will generally only apply this in the case that $n \geq 1$, and in the case that $\pi_{<n}(\varphi)$ is an isomorphism.

Lemma 7.5.12. *Suppose that the map $X \xrightarrow{\varphi} Y$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$ is an isomorphism on $\pi_{<n}$ for some $n \geq 1$. Write $A = \pi_0 X \cong \pi_0 Y \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ and $M = \pi_n \mathrm{fib}(\varphi) \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$. Then, the map*

$$P_0^{\mathrm{alg}}(X) \xrightarrow{\delta_n(\varphi)} P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}(\varphi))$$

is of type $\vec{K}_A(M, n+1)$.

Proof. This follows from Assumption 7.5.8. □

Corollary 7.5.13 ([GHb, Proposition 2.5.13])**.** *Let $X \xrightarrow{\varphi} Y$ be a map in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$. Suppose that $\pi_* \mathrm{fib}(\varphi)$ is concentrated in degree n. The the square*

$$\begin{array}{ccc} X & \xrightarrow{\tau_0^{\mathrm{alg}}} & P_0^{\mathrm{alg}}(X) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\delta_n(\varphi)} \\ Y & \longrightarrow & P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}(\varphi)) \end{array}$$

is a pullback in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]$. □

Observation 7.5.14. In the setting of Corollary 7.5.13, if additionally X (and hence Y) is n-truncated, then we can identify the map $X \to Y$ as $\tau_{\leq n}^{\mathrm{alg}} X \to \tau_{\leq (n-1)}^{\mathrm{alg}} X$, and from here Lemma 7.5.12 allows us to identify the pullback square of Corollary 7.5.13 as

$$\begin{array}{ccc} P_n^{\mathrm{alg}} X & \longrightarrow & K_A \\ {\scriptstyle\tau_{n-1}^{\mathrm{alg}}}\downarrow & & \downarrow \\ P_{n-1}^{\mathrm{alg}} X & \longrightarrow & K_A(M, n+1) \end{array}$$

(in which the right vertical map is of type $\vec{K}_A(M, n+1)$). This is a functorial construction of k-invariants in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$.

Notation 7.5.15. We fix an object $k \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. We will generally work in its undercategory $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$; in particular, we will generally have fixed a map $k \to A = \mathrm{const}(A)$. Everything will take place in this undercategory, so that e.g. a morphism in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$ of type $\vec{K}_A(M, n)$ will be understood to mean a commutative triangle

$$\begin{array}{ccc} & k & \\ \swarrow & & \searrow \\ K_A & \longrightarrow & K_A(M,n) \end{array}$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ in which the left vertical arrow identifies with the fixed map.

Notation 7.5.16. Suppose that $Y \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$ is $(n-1)$-truncated for some $n \geq 1$, write $A = \pi_0 Y \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{k/}$, and suppose $M \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$. We write

$$\mathscr{M}_k(Y \oplus (M, n)) \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$$

for the moduli space of those objects X such that

- X is n-truncated,
- there exists an equivalence $P_{n-1}^{\mathrm{alg}} X \xrightarrow{\sim} Y$, and
- there exists an isomorphism $\pi_n X \cong M$ via the resulting equivalence $\mathrm{Mod}_{\pi_0 X}^{\Phi}(\tilde{\mathcal{A}}) \simeq \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$.

Notation 7.5.17. In our moduli spaces, we will use the symbol $\twoheadrightarrow$ to denote a restriction to morphisms which are isomorphisms on homotopy groups in those dimensions for which *both* the source and the target have nonvanishing homotopy.

Proposition 7.5.18 ([GHb, Theorem 2.5.16])**.** *Suppose that* $Y \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$ *is* $(n-1)$*-truncated for some* $n \geq 1$*, write* $A = \pi_0 Y \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{k/}$*, and suppose* $M \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$*. Then the functor*

$$X \mapsto \left(P_{n-1}^{\mathrm{alg}}(X) \to P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}((\tau_{n-1}^{\mathrm{alg}})_X)) \xleftarrow{\delta_n((\tau_{n-1}^{\mathrm{alg}})_X)} P_0^{\mathrm{alg}}(X) \right)$$

determines an equivalence

$$\mathscr{M}_k(Y \oplus (M,n)) \xrightarrow{\sim} \mathscr{M}_k(Y \looparrowright K_A(M,n+1) \hookleftarrow K_A)$$

in $\mathcal{S}$.

Proof. An inverse is provided by the pullback functor. □

Notation 7.5.19. For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, we write

$$\mathscr{M}_{A/k} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$$

for the moduli space of objects of type $K_{A/k}$. For any $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$ and any $n \geq 1$, we write

$$\mathscr{M}_{A/k}(M,n) \subset \mathrm{Fun}([1], \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/})$$

for the moduli space of morphisms of type $\vec{K}_{A/k}(M,n)$.

Notation 7.5.20. It will be of auxiliary use to write

$$\mathscr{M}_{A/k}(M,0)$$

for the moduli space of pairs of an object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ and an abelian (∞-)group object $Y \in \mathrm{Ab}(\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{/X})$ in its overcategory which are in the image of (A,M) under the derived right adjoint

$$\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{/A} \xrightarrow{\mathrm{const}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{/A}$$

of the Quillen adjunction of Lemma 7.4.19.

Proposition 7.5.21. *Let* $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, *let* $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, *and let* $n \geq 0$. *Then the functor*

$$(X \to Y) \mapsto \lim^{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{X/}} \begin{pmatrix} & & X \\ & & \downarrow \\ X & \longrightarrow & Y \end{pmatrix}$$

defines an equivalence

$$\mathscr{M}_{A/k}(M,n+1) \xrightarrow{\sim} \mathscr{M}_{A/k}(M,n)$$

in $\mathcal{S}$.

Proof. For $n \geq 1$, an inverse is provided by the functor

$$(Z \to W) \mapsto \delta_n(W \to P_0^{\mathrm{alg}}(W)).$$

For $n = 0$, an inverse is provided by the functor taking the pair

$$\left(W \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \ , \ Z \in \mathrm{Ab}(\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{/W})\right),$$

say with structure map $Z \xrightarrow{\varphi} W$, to the map

$$K_{\pi_0 W} \to K_{\pi_0 W}(\ker(\pi_0(\varphi)), 1)$$

(which is evidently of type $\vec{K}_A(M, 1)$). □

Proposition 7.5.22 ([GHb, Lemma 2.5.18])**.** *Let $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, let $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, let $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$, and let $n \geq 0$. Then there exists a natural isomorphism*

$$[X, K_A(M, n)]_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}} \cong \coprod_{\hom_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}}(\pi_0 X, A)} H^n_{\tilde{T}_E}(X/k; M)$$

in Ab *(where the implicit structure map $X \to A = \mathrm{const}(A)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{k/}$ necessary for defining the cohomology of X varies over the indexing set).* □

Notation 7.5.23. Given an ∞-category $\mathcal{D}$ and objects $d_1, d_2 \in \mathcal{D}$, we write $\hom_{\mathcal{D}}^{\simeq}(d_1, d_2) \subset \hom_{\mathcal{D}}(d_1, d_2)$ for the subspace of equivalences. For any other sort of decoration denoting a certain property of a morphism, we use corresponding exponent notation to denote the subspace of the hom-space corresponding to morphisms having this property.

Notation 7.5.24. For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, we write $\mathrm{Aut}_k(A) = \mathrm{Aut}_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}}(A)$. Moreover, for any $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, we write $\mathrm{Aut}_k(A, M)$ for the group of pairs

$$\left(\varphi \in \mathrm{Aut}_k(A) \ , \ \psi \in \hom^{\cong}_{\mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})}(M, \varphi^*(M))\right).$$

Proposition 7.5.25 ([GHb, Proposition 2.5.19(1)])**.** *For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, we have an equivalence $\mathscr{M}_{A/k} \simeq B\mathrm{Aut}_k(A)$ in $\mathcal{S}$.*

Proof. This is the assertion that the canonical map

$$\mathrm{Aut}_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}}(A) \to \mathrm{Aut}_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}}(\mathrm{const}(A))$$

induced by the functor

$$\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{const}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$$

is an equivalence, which follows from Lemma 7.4.19 since it implies that this functor is a full inclusion. □

Proposition 7.5.26 ([GHb, Proposition 2.5.19(2)])**.** *Suppose that* $A \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{k/}$ *and that* $M \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$*. Then for any* $n \geq 0$ *we have an equivalence* $\mathscr{M}_{A/k}(M, n) \simeq B\mathrm{Aut}_k(A, M)$*.*

Proof. This follows from combining Proposition 7.5.21 with the essentially definitional equivalence $\mathscr{M}_{A/k}(M, 0) \simeq B\mathrm{Aut}_k(A, M)$. □

Notation 7.5.27. Given an object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$, we write

$$\mathscr{M}_k(X) \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$$

for the full subgroupoid generated by it.

Lemma 7.5.28 ([GHb, Proposition 2.5.22])**.** *For any* $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$*, there exists a canonical pullback square*

$$\begin{array}{ccc} \coprod_{\hom^{\cong}_{\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})}(\pi_0 X, A)} \mathscr{H}^n_{\tilde{T}_E}(X/k; M) & \longrightarrow & \mathscr{M}_k(X \twoheadrightarrow K_A(M, n) \hookleftarrow A) \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[(X, \mathrm{id}_{(A,M)})]{} & \mathscr{M}_k(X) \times B\mathrm{Aut}_k(A, M) \end{array}$$

in $\mathcal{S}$*.*

Proof. This is immediate from the definitions. □

Notation 7.5.29. We write

$$\widehat{\mathscr{H}}^n_{\tilde{T}_E}(A/k; M) = \left(\mathscr{H}^n_{\tilde{T}_E}(A/k; M)\right)_{\mathrm{Aut}_k(A,M)} \in \mathcal{S}_*$$

for the based space of coinvariants of the canonical action of $\mathrm{Aut}_k(A, M)$ on $\mathscr{H}^n_{\tilde{T}_E}(A/k; M) \in \mathcal{S}_*$.

Corollary 7.5.30. *There exists a canonical pullback square*

$$\begin{array}{ccc} \mathscr{H}^n_{\tilde{T}_E}(A/k;M) & \longrightarrow & \mathscr{M}_k(A \looparrowright K_A(M,n) \hookleftarrow A) \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \longrightarrow & B\mathrm{Aut}_k(A,M) \end{array}$$

in $\mathcal{S}$*, whose induced action of* $\mathrm{Aut}_k(A,M)$ *on* $\mathscr{H}^n_{\tilde{T}_E}(A/k;M)$ *is the natural one, and which induces an equivalence*

$$\mathscr{M}_k(A \looparrowright K_A(M,n) \hookleftarrow A) \simeq \widehat{\mathscr{H}^n_{\tilde{T}_E}}(A/k;M)$$

in $\mathcal{S}$*.*

Proof. First of all, applying Lemma 7.5.28 in the case that $X = A$ yields a pullback square

$$\begin{array}{ccc} \coprod_{\hom^{\cong}_{\mathrm{Alg}_\Phi(\tilde{A})}(A,A)} \mathscr{H}^n_{\tilde{T}_E}(A/k;M) & \longrightarrow & \mathscr{M}_k(A \looparrowright K_A(M,n) \hookleftarrow A) \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[(X,\mathrm{id}_{(A,M)})]{} & \mathscr{M}_k(A) \times B\mathrm{Aut}_k(A,M) \end{array}$$

in $\mathcal{S}$. By Proposition 7.5.25, we have $\mathscr{M}_k(A) \simeq B\mathrm{Aut}_k(A) = \mathrm{Aut}_{\mathrm{Alg}_\Phi(\tilde{A})_{k/}}(A)$, and the action on the fibers is clearly the canonical one and is hence free on its path components. Thus, pulling back along the map

$$B\mathrm{Aut}_k(A,M) \simeq \{A\} \times B\mathrm{Aut}_k(A,M) \to \mathscr{M}_k(A) \times B\mathrm{Aut}_k(A,M)$$

yields a pullback square

$$\begin{array}{ccc} \mathscr{H}^n_{\tilde{T}_E}(A/k;M) & \longrightarrow & \mathscr{M}_k(A \looparrowright K_A(M,n) \hookleftarrow A) \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow[\mathrm{id}_{(A,M)}]{} & B\mathrm{Aut}_k(A,M) \end{array}$$

in $\mathcal{S}$. The claim now follows readily from Proposition 3.3.1. □

7.6 Homotopical topology

7.6.1 Postnikov towers in topology

We now study the homotopy theory of the ∞-category $\mathrm{Alg}_T(s\mathcal{C})$ of simplicial T-algebras; we will mostly work in its localization $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$, but we will ultimately be interested in deducing results about its further localization $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]$ (recall Observation 7.3.15).

Definition 7.6.1. For any $n \geq 0$, an object $X \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ is called ***n-truncated*** if $\pi^\natural_{>n,\varepsilon} X = 0$ for all $\varepsilon \in \mathcal{G}_{\mathcal{C}}^{E^\delta}$. Such objects form a full subcategory $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]^{\leq n} \subset \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$, and as n varies these subcategories are evidently nested as

$$\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}] \hookleftarrow \cdots \hookleftarrow \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]^{\leq 1} \hookleftarrow \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]^{\leq 0}.$$

By presentability considerations, these inclusions admit left adjoints, and we denote the corresponding left localization adjunctions by

$$P_n^{\mathrm{top}} : \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]^{\leq n} : \mathrm{U}_n^{\mathrm{top}}.$$

We therefore obtain a tower of functors

$$\mathrm{id}_{\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]} \to \cdots \to P_1^{\mathrm{top}} \to P_0^{\mathrm{top}}.$$

We refer to its value on an object of $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ as its ***Postnikov tower***. We write

$$\mathrm{id}_{\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]} \xrightarrow{\tau_n^{\mathrm{top}}} P_n^{\mathrm{top}}$$

for the natural transformation (or its for composite with $\mathrm{U}_m^{\mathrm{top}}$ for any $m \geq 0$), which we refer to as the ***n-truncation map***.

Observation 7.6.2. By a colimit argument, if $X \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ is n-truncated then $E^\natural_{\leq n,*} X = 0$ as well.

7.6.2 Topological Eilenberg–Mac Lane objects

We now define certain objects of $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ which will represent the various functors "apply E_*^{lw}, then take cohomology".

Definition 7.6.3. Let $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, let $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, and let $n \geq 1$.

(1) We say that an object $X \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ is of ***type*** B_A if there exists a universal map $E^{\mathrm{lw}}_* X \to K_A$ inducing natural equivalences

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]}(Z, X) \xrightarrow{\sim} \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]}(E^{\mathrm{lw}}_* Z, K_A)$$

for all $Z \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$.

(2) We say that an object $Y \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ is of ***type*** $B_A(M, n)$ if there exists a universal map $E^{\mathrm{lw}}_* Y \to K_A(M, n)$ inducing natural equivalences

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]}(Z, X) \xrightarrow{\sim} \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}_{\pi_*}^{-1}]}(E^{\mathrm{lw}}_* Z, K_A(M, n))$$

for all $Z \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$.

(3) We say that a map $X \to Y$ in $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ is of ***type*** $\vec{B}_A(M, n)$ if X is of type B_A, Y is of type $B_A(M, n)$ and the map $\pi_0 E^{\mathrm{lw}}_* X \to \pi_0 E^{\mathrm{lw}}_* Y$ is an isomorphism in $\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$.

We refer to objects of type B_A and $B_A(M, n)$ collectively as ***topological Eilenberg–Mac Lane objects***, and to morphisms of type $\vec{K}_A(M, n)$ collectively as ***topological Eilenberg–Mac Lane morphisms***.

Lemma 7.6.4. *For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, and $M \in \mathrm{Mod}^\Phi_A(\tilde{\mathcal{A}})$, and any $n \geq 1$, there exist objects of type B_A and $B_A(M, n)$, and there exists a morphism of type $\vec{B}_A(M, n)$.*

Proof. By the presentability of $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ (which follows from Theorem 7.1.22 and the derived monadic adjunction underlying the monadic Quillen adjunction $\mathrm{F}_T \dashv \mathrm{U}_T$), this follows from Corollary 7.4.10. □

Notation 7.6.5. Justified by Lemma 7.6.4, we may simply write B_A or $B_A(M, n)$ for convenience when referring to a topological Eilenberg–Mac Lane object of the indicated type.

Observation 7.6.6. If $X \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}_{\mathrm{res}}^{-1}]$ is an object of type B_A, it follows immediately that $\pi^\natural_{0,*} X \cong \text{ɬ}^{\pi_0 E^{\mathrm{lw}}_*}(A)$ in $\mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_\mathcal{C}))$ and that $\pi^\natural_{>0,*} X = 0$. By the spiral exact sequence, it follows that

$$\pi_i \pi_* X \cong \begin{cases} \text{ɬ}^E(A), & i = 0 \\ \text{ɬ}^E(\Omega A), & i = 2 \\ 0, & i \notin \{0, 2\}. \end{cases}$$

For convenience, we simply write $\pi_* \pi_* X \cong \text{ɬ}^E(A) \times \text{ɬ}^E(\Omega A)[2]$.

Now, suppose that $X \to Y$ is a map of type $\vec{B}_A(M,n)$. It follows from the definition of an object of type $B_A(M,n)$ that $\pi^{\natural}_{0,S^\varepsilon}Y \cong \text{ⱡ}^E(A)$ in $\mathcal{P}^\delta_\Sigma(T(\mathcal{G}^E_{\mathcal{C}}))$ and that for $i \geq 1$,

$$\pi^{\natural}_{i,*}Y \cong \begin{cases} \text{ⱡ}^E(M), & i = n \\ 0, & i \neq n \end{cases}$$

in $\mathrm{Mod}^\Phi_A(\tilde{\mathcal{A}})$. Then, note further that if $X \to Y$ is a map of type $\vec{B}_A(M,n)$, then the composite $X \to Y \to P^{\mathrm{top}}_0(Y)$ is an equivalence; combining this with the spiral exact sequence yields that $\pi_*\pi_*Y \cong \pi_*\pi_*X \times \text{ⱡ}^E(M)[n] \times \text{ⱡ}^E(\Omega M)[n+2]$.

7.6.3 Moduli spaces in topology

We begin by mimicking Construction 7.5.11.

Construction 7.6.7. Let $X \xrightarrow{\varphi} Y$ be a map in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$, and write

$$\mathrm{p}^{\mathrm{top}}_0(\varphi) = Y \coprod_X P^{\mathrm{top}}_0 X = \mathrm{colim} \left(\begin{array}{ccc} X & \xrightarrow{\tau^{\mathrm{top}}_0} & P^{\mathrm{top}}_0(X) \\ {\scriptstyle\varphi}\downarrow & & \\ Y & & \end{array} \right)$$

for the indicated pushout. For any $n \geq 0$ we obtain a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\tau^{\mathrm{top}}_0} & P^{\mathrm{top}}_0(X) & & \\ {\scriptstyle\varphi}\downarrow & & \downarrow & \searrow^{\delta_n(\varphi)} & \\ Y & \longrightarrow & \mathrm{p}^{\mathrm{top}}_0(\varphi) & \xrightarrow[\tau^{\mathrm{top}}_{n+1}]{} & P^{\mathrm{top}}_{n+1}(\mathrm{p}^{\mathrm{top}}_0(\varphi)) \end{array}$$

in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$, and we refer to the map $\delta_n(\varphi)$ as the $n^{\mathbf{th}}$ ***difference construction*** on the map φ. This defines an augmented endofunctor on $\mathrm{Fun}([1], \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!])$. We will generally only apply this in the case that $n \geq 1$, and in the case that $\pi^{\natural}_{<n,*}(\varphi)$ is an isomorphism.

We now employ our assumption that T is homotopically adapted to E, which provides a fundamental link between our computations in homotopical topology and homotopical algebra.

Proposition 7.6.8. *Let $X \xrightarrow{\varphi} Y$ be a map in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$, let $n \geq 1$, and suppose that $E^{\natural}_{<n,*}(\varphi)$ is an isomorphism and that $E^{\natural}_{n,*}(\varphi)$ is surjective. Write $A = E^{\natural}_{0,*}X \cong E^{\natural}_{0,*}Y$ in $\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, and write $M = \mathrm{fib}(E^{\natural}_{n,*}(\varphi)) \in \tilde{\mathcal{A}}$.*

(1) We can canonically consider $M \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$.

(2) The map $\delta_n(\varphi)$ *becomes equivalent to a morphism of type* $\vec{B}_A(M,n)$ *under the localization functor* $\mathrm{L}_{E_*^{\mathrm{lw}}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$.

(3) If $\pi_{i,*}^{\natural}(\mathrm{fib}(\varphi)) = 0$ *for* $i \neq n+1$, *then the square*

$$\begin{array}{ccc} X & \xrightarrow{\tau_0^{\mathrm{top}}} & P_0^{\mathrm{top}}(X) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\delta_n(\varphi)} \\ Y & \longrightarrow & P_{n+1}^{\mathrm{top}}(\mathrm{p}_0^{\mathrm{top}}(\varphi)) \end{array}$$

becomes a pullback under the localization functor $\mathrm{L}_{E_*^{\mathrm{lw}}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$.

Proof. It follows from Corollary 7.4.10 that the functor

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \xrightarrow{E_*^{\mathrm{lw}}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$$

preserves pushouts. Thus, the square

$$\begin{array}{ccc} E_*^{\mathrm{lw}}X & \xrightarrow{E_*^{\mathrm{lw}}(\tau_0^{\mathrm{top}})} & E_*^{\mathrm{lw}}(P_0^{\mathrm{top}}(X)) \\ {\scriptstyle E_*^{\mathrm{lw}}(\varphi)}\downarrow & & \downarrow \\ E_*^{\mathrm{lw}}Y & \longrightarrow & E_*^{\mathrm{lw}}(\mathrm{p}_0^{\mathrm{top}}(\varphi)) \end{array}$$

is a pushout in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. From here, the proof is essentially identical to that of [GHb, Proposition 3.2.9]. □

In order to work in a relative setting, we fix the following.

Notation 7.6.9. We assume we are given an object $Y \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ equipped with an isomorphism $E_*^{\mathrm{lw}}Y \cong k$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ for some chosen object $k \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ (specialized via the derived right adjoint $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{const}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ from our previous assumption from Notation 7.5.15 that $k \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$). A map $k \to A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ gives rise to a composite

$$E_*^{\mathrm{lw}}\mathrm{const}(Y) \xrightarrow{\cong} k \to A$$

in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, via which for any choice of topological Eilenberg–Mac Lane object B_A we obtain a canonical map $\mathrm{const}(Y) \to B_A$. We will simply write $Y = \mathrm{const}(Y) \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$, and we will work in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y//B_A}$.

Observation 7.6.10. Fix any morphism $B_A \to B_A(M,n)$ in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ of type $\vec{B}_A(M,n)$. From Observation 7.6.6 and Notation 7.6.9, we obtain a sequence of composable morphisms

$$Y \to B_A \to B_A(M,n) \to B_A$$

(in which the composite of all but the first map is an equivalence). For any $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y//B_A}$ and as soon as $n \geq 2$, we immediately obtain equivalences

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}}(X, B_A) \xrightarrow{\sim} \hom_{\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{k/}}(\pi_0 E_*^{\mathrm{lw}} X, A)$$

and

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y//B_A}}(X, B_A(M,n)) \xrightarrow{\sim} \mathscr{H}^n_{\tilde{T}_E}(E_*^{\mathrm{lw}}(X)/k; M)$$

in $\mathcal{S}_*$.

Notation 7.6.11. We write $\mathscr{M}_Y(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}$ for the moduli space of objects $Y \to X$ such that X is of type B_A and moreover the map $E_0^{\mathrm{lw}} Y \to E_0^{\mathrm{lw}} X$ is equivalent to the map $k \to A$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. Moreover, we write $\mathscr{M}_{A/Y}(M,n) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}$ for the moduli space of morphisms $Z \to W$ of type $\vec{B}_A(M,n)$ under Y such that $(Y \to Z) \in \mathscr{M}_Y(A)$.

Proposition 7.6.12. *The functor*

$$X \mapsto P_0^{\mathrm{alg}} E_*^{\mathrm{lw}}(X)$$

defines an equivalence

$$\mathscr{M}_Y(A) \xrightarrow{\sim} \mathscr{M}_k(A),$$

and the functor

$$\varphi \mapsto \delta_{n-1}(E_*(\varphi))$$

defines an equivalence

$$\mathscr{M}_{A/Y}(M,n) \xrightarrow{\sim} \mathscr{M}_{A/k}(M,n) \simeq B\mathrm{Aut}_k(A,M).$$

Proof. These assertions both follow immediately from the functors that topological Eilenberg–Mac Lane objects are defined to represent, just as in the proof of [GHb, Proposition 3.2.17]. □

7.7 Decomposition of moduli spaces

7.7.1 Realizations and n-stages

We finally come to our main theorems: these provide an inductive procedure for understanding our moduli space of ultimate interest, which we begin by introducing.

Definition 7.7.1. With respect to

- our fixed base object $Y \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$,
- our chosen morphism $k \to A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, and
- our chosen isomorphism $E_*Y \cong k$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$,

we define a ***realization*** to be an object $(Y \xrightarrow{\varphi} X) \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))_{Y/}$ such that there exists an isomorphism $E_*X \cong A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{k/}$. We write

$$\mathscr{M}_{A/Y} \subset \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))_{Y/}$$

for the moduli space of realizations (and E_*-equivalences between them).

Before diving in, we provide a bit of big-picture intuition.

Remark 7.7.2. Given a simplicial T-algebra Z, a good way to control $E_*|Z|$ is to control its spiral spectral sequence. More to the point, the easiest way to ensure that $|Z|$ be a realization is to demand that $\mathrm{E}^2 = \pi_* E_*^{\mathrm{lw}} Z \cong \pi_0 E_*^{\mathrm{lw}} Z \cong A$, so that the spectral sequence collapses immediately.

However, it is not so straightforward to obtain such an object or understand its automorphisms: the E^2 page consists of *natural* E-homology groups, but it is the *classical* E-homology groups that are more closely connected to the actual homotopy theory of the ∞-category $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$.

Luckily, however, we have a tool that relates these two types of E-homology groups: the localized spiral exact sequence. As it is one-third classical and two-thirds natural, it allows us to exert control over the classical E-homology groups by manipulating the natural E-homology groups.

Thus, our method will be to attempt to interpolate one stage at a time from

- objects which are easy to understand (read: have controlled natural E-homology) but do not have the correct E^2 pages (read: have the wrong classical E-homology), towards
- objects which are somewhat more difficult to understand (read: have more complicated natural E-homology) but have E^2 pages which are closer and closer to collapsing at A (read: their classical E-homology is equivalent to A itself (concentrated in degree 0) in an increasingly large range).

Of course, such interpolation will not always be possible, but in the course of our attempt we will discover the precise cohomological obstructions to their possibility.

We now define certain objects of $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ which, via geometric realization, provide approximations to realizations.

Definition 7.7.3. For $0 \leq n \leq \infty$, we say that an object $Z \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}$ is an n***-stage*** if the following conditions hold:

(1) there exists an isomorphism $\pi_0 E_*^{\mathrm{lw}} Z \cong A$ in $\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$;

(2) $\pi^\natural_{>n,*} Z = 0$; and

(3) $\pi_i E_*^{\mathrm{lw}} Z = 0$ for $1 \leq i \leq n+1$.

We write

$$\mathscr{M}_n(A/Y) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]_{Y/}$$

for the moduli space of n-stages (and E_*-equivalences between them).

Observation 7.7.4. Suppose that $Z \in \mathscr{M}_n(A/Y)$. By condition (3), the tail end of the localized spiral exact sequence degenerates into a sequence of isomorphisms. By induction, this implies that $E^\natural_{i,*} Z \cong \Omega^i A$ for all $i \leq n$: the base case of $i = 0$ follows from condition (1) and Lemma 7.4.22. Then, after a colimit argument, condition (2) implies that we have an isomorphism $\pi_{n+2} E_*^{\mathrm{lw}} Z \xrightarrow{\cong} \Omega(E^\natural_{n,*} Z)$ and that $\pi_{>n+2} E_*^{\mathrm{lw}} Z = 0$. The table of Figure 7.1 summarizes these computations. Moreover,

i	0	1	2	$\cdots$	$n-1$	n	$n+1$	$n+2$	$n+3$	$\cdots$
$\pi_i E_*^{\mathrm{lw}} Z$	A	0	0	$\cdots$	0	0	0	$\Omega^{n+1} A$	0	$\cdots$
$E^\natural_{i,*} Z$	A	ΩA	$\Omega^2 A$	$\cdots$	$\Omega^{n-1} A$	$\Omega^n A$	0	0	0	$\cdots$

Figure 7.1: The classical and natural E-homology groups of an n-stage $Z \in \mathscr{M}_n(A/Y)$.

the same argument shows that if $n = \infty$ then $E^\natural_{i,*} Z \cong \Omega^i A$ for all $i \geq 0$ and that $\pi_* E_*^{\mathrm{lw}} Z \cong \pi_0 E_*^{\mathrm{lw}} Z \cong A$.

We now provide the connection between realizations and n-stages.

Theorem 7.7.5. *Geometric realization induces an equivalence*

$$\mathscr{M}_\infty(A/Y) \xrightarrow{\sim} \mathscr{M}_{A/Y}.$$

Proof. The adjunction $|-| : \mathrm{Alg}_T(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) : \mathrm{const}$ evidently descends (or perhaps rather restricts) to an adjunction $|-| : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C})) : \mathrm{const}$ by the universal property of localization. In turn, the spiral spectral sequence implies that (after taking undercategories of Y) this latter adjunction restricts to give the desired equivalence. □

Remark 7.7.6. Note that we do *not* generally have a pullback square

$$\begin{array}{ccc} \mathscr{M}_\infty(A/Y) & \longrightarrow & \mathscr{M}_{A/Y} \\ \downarrow & & \downarrow \\ \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]_{Y/} & \xrightarrow[|-|]{} & \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))_{Y/}. \end{array}$$

Rather, as alluded to in Remark 7.7.2, an ∞-stage is exactly an object whose spiral spectral sequence has $\mathrm{E}^2 = \pi_* E^{\mathrm{lw}}_* X \cong \pi_0 E^{\mathrm{lw}}_* \cong A$, so that in particular it collapses immediately.

Theorem 7.7.7. *For any $0 \leq n \leq m \leq \infty$, the n-truncation functor*

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \xrightarrow{P^{\mathrm{top}}_n} \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$$

induces a map

$$\mathscr{M}_m(A/Y) \to \mathscr{M}_n(A/Y),$$

and these assemble to give an equivalence

$$\mathscr{M}_\infty(A/Y) \xrightarrow{\sim} \lim\left(\cdots \xrightarrow{P^{\mathrm{top}}_2} \mathscr{M}_2(A/Y) \xrightarrow{P^{\mathrm{top}}_1} \mathscr{M}_1(A/Y) \xrightarrow{P^{\mathrm{top}}_0} \mathscr{M}_0(A/Y)\right).$$

Proof. First of all, it is immediate from the localized spiral exact sequence that the n-truncation of an m-stage is an n-stage. From here, the asserted equivalence follows from an (∞-categorical but otherwise) identical argument to that of [DK84a, 4.6]. □

Theorem 7.7.8. *The functor*

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] \xrightarrow{\pi_0 E^{\mathrm{lw}}_*} \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$$

induces an equivalence

$$\mathscr{M}_0(A/Y) \xrightarrow{\sim} \mathscr{M}_{A/k}.$$

Proof. Inspection of the definitions reveals an equivalence $\mathscr{M}_0(A/Y) \simeq \mathscr{M}_Y(A)$ with the moduli space of objects under Y of type B_A, and from here the claim follows from Proposition 7.6.12. □

7.7.2 Climbing the tower

We now come to the essential result, which explains how to move up the tower of moduli spaces.

Theorem 7.7.9. *For any $n \geq 1$, there is a natural pullback square*

$$\begin{array}{ccc} \mathscr{M}_n(A/Y) & \longrightarrow & B\mathrm{Aut}_k(A, \Omega^n A) \\ {\scriptstyle P^{\mathrm{top}}_{n-1}}\downarrow & & \downarrow \\ \mathscr{M}_{n-1}(A/Y) & \longrightarrow & \widehat{\mathscr{H}}^{n+2}_{\tilde{T}_E}(A/k; \Omega^n A) \end{array}$$

in $\mathcal{S}$.

In order to prove this, we will first develop an understanding of the object-by-object passage between $(n-1)$-stages and n-stages, and then we will analyze how this behaves in families.

Observation 7.7.10. Directly from the definitions, topological Eilenberg–Mac Lane objects are local with respect to the left localization adjunction $\mathrm{L}_{E^{\mathrm{lw}}_*} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] : \mathrm{U}_{E^{\mathrm{lw}}_*}$. Nevertheless, we will often keep the localization functor in the notation for clarity.

Observation 7.7.11. Suppose first that $Z \in \mathscr{M}_n(A/Y)$. Then $P^{\mathrm{top}}_{n-1}(Z) \in \mathscr{M}_{n-1}(A/Y)$ by Theorem 7.7.7, and moreover Proposition 7.6.8(3) implies that we have a pullback square

$$\begin{array}{ccc} \mathrm{L}_{E^{\mathrm{lw}}_*}(Z) & \longrightarrow & \mathrm{L}_{E^{\mathrm{lw}}_*}(B_A) \\ {\scriptstyle \mathrm{L}_{E^{\mathrm{lw}}_*}(\tau^{\mathrm{top}}_{n-1})}\downarrow & & \downarrow \\ \mathrm{L}_{E^{\mathrm{lw}}_*}(P^{\mathrm{top}}_{n-1}(Z)) & \longrightarrow & \mathrm{L}_{E^{\mathrm{lw}}_*}(B_A(\Omega^n A, n+1)) \end{array}$$

in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$.

Let us attempt to reverse this process. Suppose that $W \in \mathscr{M}_{n-1}(A/Y)$, and suppose that we form a pullback

$$\begin{array}{ccc} \mathrm{L}_{E^{\mathrm{lw}}_*}(\widetilde{W}) & \longrightarrow & \mathrm{L}_{E^{\mathrm{lw}}_*}(B_A) \\ \downarrow & & \downarrow \\ \mathrm{L}_{E^{\mathrm{lw}}_*}(W) & \xrightarrow{\varphi} & \mathrm{L}_{E^{\mathrm{lw}}_*}(B_A(\Omega^n A, n+1)) \end{array}$$

in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]$. Then, $\mathrm{L}_{E^{\mathrm{lw}}_*}(\widetilde{W}) \in \mathscr{M}_n(A/Y)$ if and only if the induced composite

$$E^{\mathrm{lw}}_* W \xrightarrow{E_*(\varphi)} E^{\mathrm{lw}}_*(B_A(\Omega^n A, n+1)) \to K_A(\Omega^n A, n+1)$$

with the universal map is an equivalence in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$: this follows from the long exact sequence in classical E-homology induced by a pullback square.

Observation 7.7.12. We can interpret the conclusion of Observation 7.7.11 as follows. By Observation 7.7.4, the object $E^{\mathrm{lw}}_* W \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ has homotopy concentrated in degrees 0 and $n+1$ and moreover $P_n^{\mathrm{alg}}(E^{\mathrm{lw}}_* W) \simeq A$. By Proposition 7.5.18, this object therefore corresponds to a unique pullback square

$$\begin{array}{ccc} E^{\mathrm{lw}}_* W & \longrightarrow & K_A \\ \downarrow & & \downarrow \\ A & \xrightarrow[\chi]{} & K_A(\Omega^n A, n+2) \end{array}$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$.

Recall from Observation 7.5.4 that we have a pullback square

$$\begin{array}{ccc} K_A(\Omega^n A, n+1) & \longrightarrow & K_A \\ \downarrow & & \downarrow \\ K_A & \longrightarrow & K_A(\Omega^n A, n+2) \end{array}$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. Now, we claim that there exists an equivalence $E^{\mathrm{lw}}_* W \xrightarrow{\sim} K_A(\Omega^n A, n+1)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ if and only if χ represents the zero element $0 \in H^{n+2}_{\tilde{T}_E}(A/k; \Omega^n A)$.

- Indeed, if $[\chi] = 0$, then the existence of an equivalence is manifest.
- Conversely, if such an equivalence exists, then by Proposition 7.5.18 there exists an equivalence between these two pullback squares, implying that $[\chi] = 0$.

Thus, the obstructions to a given $(n-1)$-stage lifting to an n-stage are given by elements of $H^{n+2}_{\tilde{T}_E}(A/k; \Omega^n A)$. In particular, if this group vanishes then *every* $(n-1)$-stage lifts to an n-stage.

We now provide the key piece of input to the proof of Theorem 7.7.9: in effect, we work with $\mathscr{M}_{n-1}(A/Y)$ one path component at a time.

Notation 7.7.13. For any $Z \in \mathscr{M}_{n-1}(A/Y)$, we write $\mathscr{M}_{n/Z}(A/Y) \subset \mathscr{M}_n(A/Y)$ for the subspace of those n-stages $W \in \mathscr{M}_n(A/Y)$ such that there exists an equivalence $P_{n-1}(W) \simeq Z$ in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E^{\mathrm{lw}}_*}^{-1}]\!]_{Y/}$.

Observation 7.7.14. Note that the space $\mathscr{M}_{n/Z}(A/Y)$ may well be empty; indeed, by Observation 7.7.12 it will be empty if and only if $\mathscr{M}_k(E^{\mathrm{lw}}_* Z \twoheadrightarrow K_A(\Omega^n A, n+1))$ is empty.

Notation 7.7.15. For any $Z \in \mathscr{M}_{n-1}(A/Y)$, we write $Z \xrightarrow{\approx} B_A(\Omega^n A, n)$ for a morphism in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}$ which classifies an equivalence $E^{\mathrm{lw}}_* Z \xrightarrow{\sim} K_A(\Omega^n A, n)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$.

Lemma 7.7.16. *Suppose that $Z \in \mathscr{M}_{n-1}(A/Y)$ for some $n \geq 1$. Then there is a natural pullback square*

$$\begin{array}{ccc} \mathscr{M}_{n/Z}(A/Y) & \longrightarrow & \mathscr{M}_k(E^{\mathrm{lw}}_* Z \twoheadrightarrow K_A(\Omega^n A, n+1) \hookleftarrow K_A) \\ \Big\downarrow{\scriptstyle P^{\mathrm{top}}_{n-1}} & & \Big\downarrow \\ \mathscr{M}_Y(Z) & \xrightarrow[E^{\mathrm{lw}}_*]{} & \mathscr{M}_k(E^{\mathrm{lw}}_* Z) \end{array}$$

in $\mathcal{S}$.

Proof. The difference construction provides a map $\mathscr{M}_{n/Z}(A/Y) \to \mathscr{M}_Y(Z \xrightarrow{\approx} B_A(\Omega^n A, n+1) \hookleftarrow B_A)$, which is an equivalence by Observation 7.7.11. Thus we obtain a commutative diagram

$$\begin{array}{ccccc} \mathscr{M}_{n/Z}(A/Y) & \xrightarrow{\sim} & \mathscr{M}_Y(Z \xrightarrow{\approx} B_A(\Omega^n A, n+1) \hookleftarrow B_A) & \longrightarrow & \mathscr{M}_k(E^{\mathrm{lw}}_* Z \twoheadrightarrow K_A(\Omega^n A, n+1) \hookleftarrow K_A) \\ \Big\downarrow{\scriptstyle P^{\mathrm{top}}_{n-1}} & & \Big\downarrow & & \Big\downarrow \\ \mathscr{M}_Y(Z) & \xrightarrow[\sim]{} & \mathscr{M}_Y(Z) & \xrightarrow[E^{\mathrm{lw}}_*]{} & \mathscr{M}_k(E^{\mathrm{lw}}_* Z) \end{array}$$

in $\mathcal{S}$, in which

- the right square is obtained by applying E^{lw}_* and using the universal characterization of topological Eilenberg–Mac Lane objects,
- the left square is tautologically a pullback, and

- our goal is to show that the outer rectangle is a pullback;

thus, it suffices to show that the right square is a pullback.

In the right square, both downwards maps are obtained by forgetting certain data: a morphism of type $\vec{B}_A(\Omega^n A, n+1)$ on the left, and a morphism of type $\vec{K}_A(\Omega^n A, n+1)$ on the right. Thus, it is convenient to use the equivalence $\mathscr{M}_{A/Y}(\Omega^n A, n+1) \xrightarrow{\sim} \mathscr{M}_{A/k}(\Omega^n A, n+1)$ of Proposition 7.6.12 (between the moduli spaces of such Eilenberg–Mac Lane morphisms) to obtain a larger commutative square

$$\begin{array}{ccc} \mathscr{M}_Y(Z \xrightarrow{\approx\approx} B_A(\Omega^n A, n+1) \hookleftarrow B_A) & \longrightarrow & \mathscr{M}_k(E^{\mathrm{lw}}_* Z \twoheadrightarrow K_A(\Omega^n A, n+1) \hookleftarrow K_A) \\ \downarrow & & \downarrow \\ \mathscr{M}_Y(Z) \times \mathscr{M}_{A/Y}(\Omega^n A, n+1) & \longrightarrow & \mathscr{M}_k(E^{\mathrm{lw}}_* Z) \times \mathscr{M}_{A/k}(\Omega^n A, n+1) \end{array}$$

which it then suffices to show is a pullback.

Now, observe that both spaces on the bottom row are connected (by definition and by Propositions 7.5.26 and 7.6.12). So for any basepoint of $\mathscr{M}_Y(Z) \times \mathscr{M}_{A/Y}(\Omega^n A, n+1)$, it suffices to check that the induced map on fibers is an equivalence. Unwinding the definitions, we see that this is the map

$$\hom^{\approx\approx}_{\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]}(Z, B_A(\Omega^n A, n+1)) \to \hom^{\simeq}_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}^{-1}_{\pi_*}]}(E^{\mathrm{lw}}_* Z, K_A(\Omega^n A, n+1)).$$

As $\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}] \subset \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}^{-1}_{\mathrm{res}}]$ is a full subcategory, we see that this is by definition an equivalence of subspaces of the equivalence

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}^{-1}_{\mathrm{res}}]}(Z, B_A(\Omega^n A, n+1)) \xrightarrow{\sim} \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\mathbf{W}^{-1}_{\pi_*}]}(E^{\mathrm{lw}}_* Z, K_A(\Omega^n A, n+1))$$

characterizing the object $B_A(\Omega^n A, n+1) \in \mathrm{Alg}_T(s\mathcal{C})[\mathbf{W}^{-1}_{\mathrm{res}}]$. □

We can now prove our main decomposition theorem.

Proof of Theorem 7.7.9. We begin with the commutative square

$$\begin{array}{ccc} \mathscr{M}_k(K_A(\Omega^n A, n+1) \hookleftarrow K_A) & \xrightarrow{\sim} & \mathscr{M}_k(K_A(\Omega^n, n+2) \hookleftarrow K_A) \\ \downarrow & & \downarrow \\ \mathscr{M}_k(K_A \oplus (\Omega^n A, n+1)) & \xrightarrow[\sim]{} & \mathscr{M}_k(K_A \twoheadrightarrow K_A(\Omega^n A, n+2) \hookleftarrow K_A) \end{array}$$

in $\mathcal{S}$, in which

- the upper horizontal map is (the inverse of) the equivalence of Proposition 7.5.21,
- the left vertical map is forgetful,
- the right vertical map repeats the given morphism,
- the lower horizontal map is the equivalence of Proposition 7.5.18.

This is tautologically a pullback square.

Now, suppose that $Z \in \mathscr{M}_{n-1}(A/Y)$. We claim that there exists a pullback square

$$\begin{array}{ccc} \mathscr{M}_{n/Z}(A/Y) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n A, n+2) \twoheadleftarrow K_A) \\ \downarrow & & \downarrow \\ \mathscr{M}_Y(Z) & \longrightarrow & \mathscr{M}_k(K_A \twoheadrightarrow K_A(\Omega^n A, n+2) \twoheadleftarrow K_A) \end{array}$$

in $\mathcal{S}$. To see this, we separate the argument into two cases, depending on whether or not there exists an equivalence $E^{\mathrm{lw}}_* Z \xrightarrow{\sim} K_A(\Omega^n A, n+1)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\widetilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\pi_*}]\!]$.

- Suppose that no such equivalence exists. Then $\mathscr{M}_{n/Z}(A/Y)$ is empty by Observation 7.7.14. In this case, the subspace $\mathscr{M}_k(E^{\mathrm{lw}}_* Z) \subset \mathscr{M}_k(K_A \oplus (\Omega^n A, n+1))$ is not in the image of the left vertical map of our original tautological pullback square. These facts imply that the above square is indeed (equally tautologically) a pullback.

- Suppose that such an equivalence exists. In this case, we obtain an evident forgetful equivalence

  $$\mathscr{M}_k(E^{\mathrm{lw}}_* Z \twoheadrightarrow K_A(\Omega^n A, n+1) \twoheadleftarrow K_A) \xrightarrow{\sim} \mathscr{M}_k(K_A(\Omega^n A, n+1) \twoheadleftarrow K_A)$$

  in $\mathcal{S}$, which reduces the pullback square of Lemma 7.7.16 to a pullback square

  $$\begin{array}{ccc} \mathscr{M}_{n/Z}(A/Y) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n A, n+1) \twoheadleftarrow K_A) \\ \downarrow & & \downarrow \\ \mathscr{M}_Y(Z) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n A, n+1)). \end{array}$$

  The right vertical arrow of this pullback square includes as a subobject of the left vertical arrow of our original tautological pullback square, yielding the claim.

Now, assembling this pullback square over all $Z \in \mathscr{M}_{n-1}(A/Y)$, we obtain a pullback square

$$\begin{array}{ccc} \mathscr{M}_n(A/Y) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n, n+2) \leftarrow\!\!\shortmid K_A) \\ \downarrow & & \downarrow \\ \mathscr{M}_{n-1}(A/Y) & \longrightarrow & \mathscr{M}_k(K_A \shortmid\!\!\rightarrow K_A(\Omega^n A, n+2) \leftarrow\!\!\shortmid K_A). \end{array}$$

From here, the equivalence

$$\mathscr{M}_k(K_A(\Omega^n A, n+2) \leftarrow\!\!\shortmid K_A) = \mathscr{M}_{A/k}(\Omega^n A, n+2) \simeq B\mathrm{Aut}_k(A, \Omega^n A)$$

of Proposition 7.5.26 and the equivalence

$$\mathscr{M}_k(K_A \shortmid\!\!\rightarrow K_A(\Omega^n A, n+2) \leftarrow\!\!\shortmid K_A) \simeq \widehat{\mathscr{H}}^{n+2}_{\tilde{T}_E}(A/k; \Omega^n A)$$

of Corollary 7.5.30 allow us to rewrite this as a pullback square

$$\begin{array}{ccc} \mathscr{M}_n(A/Y) & \longrightarrow & B\mathrm{Aut}_k(A, \Omega^n A) \\ \downarrow & & \downarrow \\ \mathscr{M}_{n-1}(A/Y) & \longrightarrow & \widehat{H}^{n+2}_{\tilde{T}_E}(A/k; \Omega^n A), \end{array}$$

which completes the proof. □

Chapter 8

$\mathbb{E}_\infty$ automorphisms of motivic Morava E-theories

In this chapter, we show that the motivic Morava E-theories always admit $\mathbb{E}_\infty$ structures, but that these may admit "exotic" $\mathbb{E}_\infty$ automorphisms not coming from the usual Morava stabilizer group.

8.0 Introduction

8.0.1 Overview

In this short chapter, we equip the motivic Morava E-theory spectra with canonical $\mathbb{E}_\infty$ structures, and compute their automorphisms as $\mathbb{E}_\infty$ ring spectra. We find that these automorphism groups are (homotopically) discrete, but that they are apparently distinct from the usual Morava stabilizer group. We refer the reader to Theorem 8.1.1 for a precise statement of our main result, and to Remark 8.1.4 for a discussion of these automorphism groups. In Remark 8.1.5, we explain the precise relationship between our work and that of Naumann–Spitzweck–Østvær [NSØ15] on motivic algebraic K-theory (i.e. in the height-1 case).

Our proof is patterned directly on that of Goerss–Hopkins [GH04, GHb] for the ordinary (i.e. non-motivic) Morava E-theory spectra (which is based on much prior work, notably that of Hopkins–Miller [Rez98]). Whereas their proof is based in Goerss–Hopkins obstruction theory for ordinary spectra, our proof uses our generalization of Goerss–Hopkins obstruction theory to an arbitrary presentably symmetric monoidal stable ∞-category obtained in Chapter 7.

The most immediate consequence of the present work is that it endows the mo-

tivic cohomology theories represented by the motivic Morava E-theories with the rich algebraic structure of power operations. However, we also view it as a first step towards a moduli-theoretic construction of a motivic spectrum *mmf* of *motivic modular forms*, in analogy with the ordinary spectrum *tmf* of topological modular forms [DFHH14].[1] As the construction of *tmf* has been highly influential in chromatic homotopy theory, so would the construction of *mmf* significantly advance the chromatic approach to motivic homotopy theory, which is a highly active area of research [Voe98, HK01, Vez01, Bor03, Hor, LM07, PPR08, NSØ09b, NSØ09a, Bal10, Isa09, Isa, And, Hoy15a, Hor18, Joa, HO, Ghe].

There has been much recent interest in "genuine" operadic structures, e.g. genuine G-spectra with multiplications indexed by maps of finite G-sets (instead of just finite sets) [BH15, HH, BHb, BHc, Rub, BP, GW18], as well as analogous structures in motivic homotopy theory [BHa]. We do not contend with such structures here. However, we are optimistic that our generalization of Goerss–Hopkins obstruction theory admits a fairly direct enhancement to one that would handle them in a formally analogous way. Thereafter, it seems quite plausible that the present work would admit a straightforward extension to give "motivically genuine" $\mathbb{E}_\infty$ structures on the motivic Morava E-theory spectra.

8.0.2 Conventions

- We write $\mathcal{S}\mathrm{p}^{\mathrm{mot}}$ for the (presentably symmetric monoidal stable) ∞-category of motivic spectra.[2] This comes equipped with a distinguished group of invertible objects
$$\mathcal{G} = \{S^{i,j}\}_{i,j\in\mathbb{Z}} \cong \mathbb{Z}\times\mathbb{Z},$$
the *motivic sphere spectra*: the unit object is $\mathbf{1} = S^{0,0}$, its categorical suspension is $\Sigma\mathbf{1} = S^{1,0}$, and then by definition we have $\Sigma^\infty\mathbb{G}_m = S^{1,1}$. In particular, it follows that $S^{2,1} = \Sigma^\infty\mathbb{P}^1$.

- For any $X \in \mathcal{S}\mathrm{p}^{\mathrm{mot}}$, we write $X_{**} = \pi_{**}X$ for its bigraded homotopy groups, i.e. $X_{i,j} = \pi_{i,j}X = [S^{i,j}, X]_{\mathcal{S}\mathrm{p}^{\mathrm{mot}}}$. Additionally, we write $X_* = \pi_*X$ for its $(2,1)$-line of homotopy groups, i.e. $X_i = \pi_iX = [S^{2i,i}, X]_{\mathcal{S}\mathrm{p}^{\mathrm{mot}}}$.

[1] The works [Ric, GIKR] take a different approach, producing motivic spectra over $\mathbb{R}$ and $\mathbb{C}$ whose cohomologies coincide with that expected of *mmf* (in analogy with *tmf*). These constructions are indirect, and relatively specific to the chosen base fields; in particular, the resulting motivic spectra are not manifestly related to any theory of elliptic motivic spectra.

[2] We implicitly work over a regular noetherian base scheme of finite Krull dimension, but this is only in order to employ the results of [NSØ09b]. We will additionally use a result of [GS09], which requires a (not necessarily regular) noetherian base scheme of finite Krull dimension.

- We write $\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}} \subset \mathcal{S}\mathrm{p}^{\mathrm{mot}}$ for the coreflective subcategory of *cellular* motivic spectra. This is the subcategory generated under colimits by the motivic sphere spectra. It can also be characterized as the subcategory of colocal objects for the "bigraded homotopy groups" functor; in particular, within this subcategory, bigraded homotopy groups detect equivalences.

- We fix a finite field k of characteristic $p > 0$, and we fix a formal group law $\mathbb{G}_0$ over k of finite height n.

- We write $E(k, \mathbb{G}_0)$ for the corresponding Lubin–Tate deformation ring, we write $\mathfrak{m} \subset E(k, \mathbb{G}_0)$ for its unique maximal ideal, and we fix an isomorphism $E(k, \mathbb{G}_0)/\mathfrak{m} \cong k$.

- We fix a versal deformation $\mathbb{G}$ of $\mathbb{G}_0$ over $E(k, \mathbb{G}_0)$. To be precise, $\mathbb{G}$ is a formal group law over $E(k, \mathbb{G}_0)$, and pushes forward to $\mathbb{G}_0$ along the now-canonical map $E(k, \mathbb{G}_0) \to k$. Geometrically, this corresponds to a pullback

$$\begin{array}{ccc} \mathbb{G}_0 & \longrightarrow & \mathbb{G} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spf}(E(k, \mathbb{G}_0)) \end{array}$$

  of formal groups (where we notationally identify formal group laws with their underlying formal groups).

- We write

$$E^{\mathrm{top}} = E^{\mathrm{top}}_{k,\mathbb{G}_0} \in \mathcal{S}\mathrm{p}$$

  for the (ordinary) Morava E-theory spectrum corresponding to the pair $(k, \mathbb{G}_0)$, coming from the Landweber exact functor theorem (see e.g. [Rez98, Theorem 6.4 and 6.9]) applied to the formal group law $\mathbb{G}$ over $E(k, \mathbb{G}_0)$.[3] To be precise, we have a chosen isomorphism

$$E^{\mathrm{top}}_* \cong E(k, \mathbb{G}_0)[u^{\pm}]$$

  (with $|u| = 2$), and the degree-(-2) formal group law $\overline{\mathbb{G}}$ on E^{top}_* coming from its complex orientation corresponds to $\mathbb{G}$ via the unit $u \in E^{\mathrm{top}}_2$, considered as a degree-0 formal group law on E^{top}_*.

[3] This is known to be $\mathbb{E}_\infty$, by [GH04, Corollary 7.6] (which is precisely the result we generalize here).

- We write
$$E = E^{\mathrm{mot}} = E^{\mathrm{mot}}_{k,\mathbb{G}_0} \in \mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}$$
for the motivic Morava E-theory spectrum corresponding to the pair $(k, \mathbb{G}_0)$ coming from the motivic Landweber exact functor theorem of [NSØ09b, Theorem 8.7]. This is cellular by construction, and comes equipped with a quasi-multiplication (i.e. a multiplication up to phantom maps). Moreover, writing $MGL \in \mathcal{S}\mathrm{p}^{\mathrm{mot}}$ for the algebraic bordism spectrum and $MU \in \mathcal{S}\mathrm{p}$ for the complex bordism spectrum, we have isomorphisms
$$E_{**} \cong MGL_{**} \otimes_{MU_*} E^{\mathrm{top}}_*$$
and
$$E_{**}E \cong E_{**} \otimes_{E^{\mathrm{top}}_*} E^{\mathrm{top}}_* E^{\mathrm{top}},$$
and the structure maps of the Hopf algebroid $(E_{**}, E_{**}E)$ are tensored up from those of (E_*, E_*E).

8.0.3 Acknowledgments

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8.1 $\mathbb{E}_\infty$ automorphisms of motivic Morava E-theories

We now state the main result.

Theorem 8.1.1. *The motivic Morava E-theory spectrum $E = E^{\mathrm{mot}}_{k,\mathbb{G}_0}$ has a unique $\mathbb{E}_\infty$ structure refining the ring structure on its bigraded homotopy groups, and as such generates a subgroupoid of* $\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}})$ *equivalent to*
$$B(\mathrm{Aut}_{\mathrm{CAlg}(\mathrm{Comod}_{(E_{**},E_{**}E)})}(E_{**}E)).$$
In particular, its space of automorphisms is discrete.

Lemma 8.1.2. *Any Landweber exact motivic spectrum satisfies Adams's condition.*

Proof. The proof is almost identical to that of [Rez98, Proposition 15.3]. First of all, the general statement follows from the universal case of MGL. In turn, we can present MGL as a filtered colimit of Thom spectra over finite Grassmannians, which are then dualizable. Let us write this as $MGL \simeq \operatorname{colim}_\alpha MGL_\alpha$.[4] So, it only remains to verify that $MGL_{**}(\mathrm{D}(MGL_\alpha))$ is projective as an MGL_{**}-module. In bidegree $(0,0)$, we observe that $MGL_{**}(\mathrm{D}(MGL_\alpha)) \cong (MGL^{**}MGL_\alpha)^\vee$, so that here the claim follows from the algebra presentation of [GS09, Proposition 2.19], which in particular implies (by inducting on the dimension of the Grassmannians) that this algebra itself is actually free as an MGL_{**}-module. From here, in an arbitrary bidegree (i,j) we then compute that

$$\begin{aligned} MGL_{i,j}(\mathrm{D}(MGL_\alpha)) &\cong MGL_{0,0}(S^{-i,-j} \otimes \mathrm{D}(MGL_\alpha)) \\ &\cong MGL_{0,0}(S^{-i,-j}) \otimes_{MGL_{0,0}} MGL_{0,0}(\mathrm{D}(MGL_\alpha)) \end{aligned}$$

(using the Künneth theorem). □

Observation 8.1.3. By definition, E_{**}-localization in $\mathcal{S}\mathrm{p}^{\mathrm{mot}}$ is the localization determined by the E_{**}-acyclics, i.e. those objects Z such that $E_{**}Z \cong 0$. Note that such motivic spectra Z may not be E-acyclic, i.e. it might still be the case that $E \otimes Z \not\simeq 0$. On the other hand, if Z is also cellular, since E is cellular then so is $E \otimes Z$ (since $\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}$ is a colocalization of $\mathcal{S}\mathrm{p}^{\mathrm{mot}}$ and the symmetric monoidal structure commutes with colimits in each variable). Thus, when restricted to cellular motivic spectra, the localizations L_E and $\mathrm{L}_{E_{**}}$ agree. This is summarized by the diagram

$$\begin{array}{ccc} L_{E_{**}}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})) & \longrightarrow & L_{E_{**}}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}})) \\ \downarrow\wr & & \downarrow \\ L_E(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})) & \longrightarrow & L_E(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}})) \\ \downarrow & & \downarrow \\ \mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}) & \longrightarrow & \mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}) \end{array}$$

of ∞-categories.

Proof of Theorem 8.1.1. The proof is formally identical to that of [GH04, Corollary 7.6], only we work in the ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}$: the key pieces of input are Theorems 7.7.5, 7.7.8, and 7.7.9, which are respectively generalizations of [GH04, Proposition 5.2, Proposition 5.5, and Theorem 5.8]. The passage from the ordinary case to the motivic case runs as follows.

[4] Explicitly, $\mathrm{D}(MGL_\alpha)$ is also a Thom spectrum via the formula $\mathrm{D}(X^\xi) \simeq X^{-\xi}$.

First of all, a priori we only have a quasi-multiplication on $E \in \mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}$. However, this suffices to give all the required structure on its bigraded E-homology groups: these are by definition homotopy classes of maps out of bigraded spheres, which by definition cannot detect phantom maps.

Next, a priori, Goerss–Hopkins obstruction theory in $\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}$ using the homology theory E_{**} computes a moduli space in $\mathrm{L}_{E_{**}}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}))$. However, as explained in Observation 8.1.3, we have an equivalence

$$\mathrm{L}_{E_{**}}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})) \simeq \mathrm{L}_{E}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})),$$

and the usual proof that E is E-local then applies (see e.g. [Rav84, Proposition 1.17]). Thus we have $E \in \mathrm{L}_{E_{**}}\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}$, and hence the moduli space that we construct inside of $\mathrm{CAlg}(\mathrm{L}_{E_{**}}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})) \simeq \mathrm{L}_{E_{**}}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}))$ is that of an object whose underlying motivic spectrum is indeed E itself.

Now, let us turn to the remainder of the proof of [GH04, Corollary 7.6] and its ingredients. We do *not* carry over the last line (which identifies the relevant automorphism group with an automorphism group in a category of formal group laws).[5] However, everything else used there is entirely algebraic, and works equally well replacing ordinary gradings with bigradings. Note that the gradings appearing in [GH04, §6] arise from the external simplicial direction (and the internal gradings play no real role); note too that the "Dyer–Lashof operations" arising there arise from the algebraic theory given in [May70] (and in particular have nothing whatsoever to do with operations in motivic homology). □

Remark 8.1.4. Using various adjunctions as well as the fact that all morphisms respect bigradings, one can identify the endomorphism monoid

$$\mathrm{End}_{\mathrm{CAlg}(\mathrm{Comod}_{(E_{**},E_{**}E)})}(E_{**}E)$$

(the classifying space of whose maximal subgroup appears in the statement of Theorem 8.1.1) with the hom-set

$$\hom_{\mathrm{CAlg}(\mathrm{Mod}_{E^{\mathrm{top}}_*})}(E^{\mathrm{top}}_* E^{\mathrm{top}}, MGL_* \otimes_{MU_*} E^{\mathrm{top}}_*).$$

This appears to fall under the auspices of [Rez98, §17], and thus ought to have a moduli-theoretic interpretation.

A reasonable guess would be that, if we define the map χ via the pullback diagram

$$\begin{array}{ccc} \mathrm{Spec}(E^{\mathrm{mot}}_*) & \xrightarrow{\chi} & \mathrm{Spec}(E^{\mathrm{top}}_*) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(MGL_*) & \longrightarrow & \mathrm{Spec}(MU_*), \end{array}$$

[5] However, see Remark 8.1.4.

then the group in question should be the group of (strict) automorphisms of the formal group law $\chi^*\mathbb{G}$ over

$$E^{\mathrm{mot}}_* = MGL_* \otimes_{MU_*} E^{\mathrm{top}}_*.$$

However, we have not managed to verify this claim. If it holds, however, it would be in keeping with the general philosophy that motivic homotopy theory should be thought of as a flavor of parametrized homotopy theory: the pullback of a sheaf over a small space to a larger one will generally admit more automorphisms than the original sheaf itself.

In any case, there is an evident map to this automorphism group from the Morava stabilizer group, which therefore acts on the object $E^{\mathrm{mot}} \in \mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}})$ as well. Moreover, this map should be an inclusion whenever the map $MU_* \to MGL_*$ is (indeed, in certain cases the latter is even an isomorphism (see [Hoy15b])).

Remark 8.1.5. in [NSØ15], Naumann–Spitzweck–Østvær prove that the motivic algebraic K-theory spectrum KGL (over a noetherian base scheme of finite Krull dimension) admits a unique $\mathbb{E}_\infty$ structure refining the canonical multiplication on its represented motivic cohomology theory. Meanwhile, Goerss–Hopkins obstruction theory takes a commutative algebra in comodules and returns the moduli space of realizations. These are not directly comparable: the former addresses the question of $\mathbb{E}_\infty$ structures on a *given* object, while the latter addresses the question of the ∞-groupoid of objects that *realize* some chosen algebraic datum. Moreover, [NSØ15] addresses KGL as an integral object, whereas Theorem 8.1.1 only applies to $\mathcal{E}^{\mathrm{mot}}_{k,\hat{\mathbb{G}}_m} \simeq KGL^\wedge_p$.

To clarify, for a variable object $X \in \mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}$ we locate both the main theorem of [NSØ15] as well as Theorem 8.1.1 in the diagram

$$\begin{array}{ccc} \hom_{\mathrm{Op}}(\mathrm{Comm}, \mathcal{E}\mathrm{nd}_{\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}}(X)) & & \\ \downarrow & & \\ \hom_{\mathrm{Op}}(\mathrm{Comm}, \mathcal{E}\mathrm{nd}_{\mathrm{ho}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})}(X))) & \xrightarrow{E_{**}} & \mathrm{CAlg}(\mathrm{Comod}_{(E_{**},E_{**}E)})^{\simeq} \\ & & \downarrow{\scriptstyle \mathscr{M}(-)} \\ & & \mathcal{S}_{/\mathrm{L}_{E_{**}}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}))} \end{array}$$

(where $\mathcal{E}\mathrm{nd}$ denotes the endomorphism operad): the two downwards arrows are the settings for the respective theorems.

- On the one hand, taking $X = KGL$, there is a canonical point in the set $\hom_{\mathrm{Op}}(\mathrm{Comm}, \mathcal{E}\mathrm{nd}_{\mathrm{ho}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})}(KGL))$ which selects the standard multiplication

on KGL in $\mathrm{ho}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})$. The main theorem of [NSØ15] can then be interpreted as saying that the fiber over this point is nonempty and contractible.

- On the other hand, Goerss–Hopkins obstruction takes an *algebraic* object in $\mathrm{CAlg}(\mathrm{Comod}_{(E_{**},E_{**}E)})^{\simeq}$ and provides a spectral sequence converging to the homotopy groups of its moduli space of realizations (which in our case collapses), considered as a subgroupoid of the ∞-category $\mathrm{L}_{E_{**}}(\mathrm{CAlg}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}}))$. The inclusion of this subgroupoid is the target of this algebraic object under the lower vertical map.

A toy example illustrating the difference between these two approaches is the difference between $\mathbb{E}_\infty$ structures on a fixed two-element set (of which there are four) and the moduli space of such objects in $\mathrm{CAlg}(\mathcal{S}\mathrm{et})$ (which consists of two discrete components).[6] These two approaches are both explored in the more sophisticated setting of algebras over an operad in [Rez96].

Note that the horizontal map in this diagram may not be injective: it is a priori possible that distinct multiplications on X in $\mathrm{ho}(\mathcal{S}\mathrm{p}^{\mathrm{mot}}_{\mathrm{cell}})$ might induce the same commutative algebra object structure on $E_{**}X \in \mathrm{Comod}_{(E_{**},E_{**}E)}$. This represents a further obstruction to a direct comparison of these two approaches to the realization problem.

[6] However, this analogy fails in that the upper vertical map is already an equivalence since $\mathcal{S}\mathrm{et} \xrightarrow{\sim} \mathrm{ho}(\mathcal{S}\mathrm{et})$.

Appendix A

Notation, terminology, and conventions

In this appendix we spell out all the precise foundations on which this thesis is built.

A.1 On ∞-categories

We begin with our philosophy surrounding the semantics of the signifier "∞-category".

(1) For definiteness, we ground ourselves in the theory of *quasicategories*: an ∞-category is a quasicategory. We will refer to these as "quasicategories" only when we mean to make specific reference to their properties or manipulation as such, which we will avoid doing to the largest extent possible. We use [Lur09b] as our primary reference, but we note that many of the ideas given there have their origins in [Joyc, Joyb, Joya].

In order to proceed with the enumeration of our foundations, we must immediately lay out the following basic conventions.

(a) We will be ignoring all set-theoretic issues. They are irrelevant to our aims, and in any case can be dispensed with by appealing to the usual device of *Grothendieck universes* (see e.g. §T.1.2.15).

(b) If an ∞-category $\mathcal{C}$ has an initial (resp. terminal) object, we will write $\varnothing_{\mathcal{C}}$ (resp. $\mathrm{pt}_{\mathcal{C}}$) for any such object, or we will simply write $\varnothing$ (resp. pt) if the ambient ∞-category $\mathcal{C}$ is clear from the context. For ∞-categories of co/pointed objects, we will make the abbreviations $\mathcal{C}_{\varnothing} = \mathcal{C}_{/\varnothing}$ and $\mathcal{C}_{*} = \mathcal{C}_{\mathrm{pt}/}$.

(c) We write $\mathcal{S}$ for the ∞-category of spaces. Up to equivalence, we can take this to be either $\mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]$ or $s\mathcal{S}\mathrm{et}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]$, where in both cases the symbol $\mathbf{W}_{\mathrm{w.h.e.}}$ denotes the *weak homotopy equivalences*.[1,2] In particular, by "space" we will mean an object of $\mathcal{S}$; when we mean to refer to an object of $\mathcal{T}\mathrm{op}$, we will instead use the term "topological space". The ∞-category $\mathcal{S}$ of spaces plays the same fundamental role in the theory of ∞-categories that the category $\mathcal{S}\mathrm{et}$ of sets plays in the theory of categories: whereas categories are naturally enriched in sets, ∞-categories are naturally enriched in spaces (see item (21)).

We adopt the following conventions regarding $\mathcal{S}$.

- A map in $\mathcal{S}$ is called
  - *étale* if it induces a $\pi_{\geq 1}$-isomorphism for every basepoint of the source;
  - a *monomorphism* (or, more informally, the inclusion of a *subspace*) if it is étale and additionally induces a π_0-monomorphism;
  - a *surjection* if it induces a π_0-surjection.[3]
- There is an evident adjunction $\pi_0 : \mathcal{S} \rightleftarrows \mathcal{S}\mathrm{et} : \mathrm{disc}$, and we call a space *discrete* if it is in the image of the right adjoint (see item (24)). We will only include this right adjoint in the notation if we mean to emphasize it.
- More generally, for any $n \geq 0$ we have a *truncation* adjunction $\tau_{\leq n} : \mathcal{S} \rightleftarrows \mathcal{S}^{\leq n} : \mathrm{U}_{\leq n}$ and a *cotruncation* adjunction $\mathrm{U}_{\geq n} : \mathcal{S}_*^{\geq n} \rightleftarrows \mathcal{S}_*^{\geq 1} : \tau_{\geq n}$. (In the special case that $n = 0$, the truncation adjunction reduces to the adjunction $\pi_0 : \mathcal{S} \rightleftarrows \mathcal{S}\mathrm{et} : \mathrm{disc}$ given above.)
- We write $\mathcal{S}^{\mathrm{fin}} \subset \mathcal{S}$ for the full subcategory on the *finite* spaces.[4]
- We will refer to spaces as ∞-*groupoids* when we mean to emphasize the fact that they are just particular examples of ∞-categories.

[1] Of course, by $\mathcal{T}\mathrm{op}$ we mean to denote any "convenient" category of topological spaces.

[2] More invariantly, one can also characterize the ∞-category of spaces as the free cocompletion of the terminal ∞-category (see item (23)).

[3] Surjections are also called "effective epimorphisms", but note that they are *not* generally epimorphisms in $\mathcal{S}$ (see item (22)).

[4] A space is finite exactly when it can be presented either by a finite CW complex or by a finite simplicial set (i.e. a simplicial set with finitely many nondegenerate simplices). More invariantly, one can also characterize $\mathcal{S}^{\mathrm{fin}}$ as the initial ∞-category admitting all *finite* colimits.

(d) We write $\mathcal{C}\mathrm{at}_\infty$ for the ∞-category of ∞-categories. We adopt the following conventions regarding $\mathcal{C}\mathrm{at}_\infty$.

- A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ of ∞-categories is an *equivalence* precisely if it is (homotopically) fully faithful and surjective: that is,
  - for all $x, y \in \mathcal{C}$, the induced map
  $$\hom_{\mathcal{C}}(x, y) \to \hom_{\mathcal{D}}(F(x), F(y))$$
  is an equivalence in $\mathcal{S}$, and moreover
  - for every $z \in \mathcal{D}$ there is some $w \in \mathcal{C}$ and an equivalence $F(w) \simeq z$ in $\mathcal{D}$.[5]

- To say that an ∞-category $\mathcal{C}$ is a *subcategory* of some other ∞-category $\mathcal{D}$ means, in the most invariant possible language, that we have a chosen functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ which is (homotopically) faithful: that is, for all $x, y \in \mathcal{C}$, the induced map
$$\hom_{\mathcal{C}}(x, y) \to \hom_{\mathcal{D}}(F(x), F(y))$$
is a monomorphism in $\mathcal{S}$. We will call the functor F the *inclusion* of a subcategory, but we will usually suppress it from the notation and simply write $\mathcal{C} \subset \mathcal{D}$ as shorthand.[6] A subcategory $\mathcal{C} \subset \mathcal{D}$ is uniquely specified by the resulting subcategory $\mathrm{ho}(\mathcal{C}) \subset \mathrm{ho}(\mathcal{D})$ of its homotopy category.

- More generally, if I is a class of maps in $\mathcal{S}$, then a functor in $\mathcal{C}\mathrm{at}_\infty$ is called a *local* I if all the induced maps on hom-spaces are in I. (So for instance, the inclusion of a subcategory might otherwise be called a *local monomorphism*.)

- An ∞-category will be called a *category*, or sometimes a *1-category* for emphasis, if its hom-spaces are discrete, i.e. they lie in the full subcategory $\mathcal{S}\mathrm{et} \subset \mathcal{S}$. These form a full subcategory $\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty$,

[5] This is essentially Definition T.1.1.5.14, which makes use of the left Quillen equivalence $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \to (\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ of Theorem T.2.2.5.1. (Note that all objects of $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ are cofibrant.)

[6] Note that these are not quite the monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ (see item (22)). Rather, the monomorphisms are precisely the *pseudomonic* functors, i.e. the inclusions of subcategories which are full on equivalences. This is perhaps most easily seen by appealing to the equivalence $\mathcal{C}\mathrm{at}_\infty \simeq \mathcal{CSS}$ of item (2)(c) below: as the inclusion $\mathcal{CSS} \subset s\mathcal{S}$ preserves limits (being a right adjoint), a map in $\mathcal{CSS}$ is a monomorphism precisely if it is a monomorphism when considered in $s\mathcal{S}$.

the inclusion of which we will denote by $\mathrm{U}_{\mathcal{C}\mathrm{at}} : \mathcal{C}\mathrm{at} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$. This inclusion is the right adjoint in an adjunction

$$\mathrm{ho} : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{C}\mathrm{at} : \mathrm{U}_{\mathcal{C}\mathrm{at}}$$

whose left adjoint is given by the *homotopy category* functor. Given an ∞-category $\mathcal{C}$ and any pair of objects $c, d \in \mathcal{C}$, we will sometimes write

$$[c, d]_{\mathcal{C}} = \hom_{\mathrm{ho}(\mathcal{C})}(c, d)$$

for the corresponding hom-set in the homotopy category $\mathrm{ho}(\mathcal{C})$ of $\mathcal{C}$. By definition, the map

$$\hom_{\mathcal{C}}(c, d) \to \hom_{\mathrm{ho}(\mathcal{C})}(c, d)$$

in $\mathcal{S}$ induced by projection $\mathcal{C} \to \mathrm{ho}(\mathcal{C})$ (i.e. the unit of the adjunction $\mathrm{ho} \dashv \mathrm{U}_{\mathcal{C}\mathrm{at}}$) is precisely the projection to the set of path components (i.e. the unit of the adjunction $\pi_0 \dashv \mathrm{disc}$).

- We write $\mathrm{U}_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ for the inclusion of spaces as ∞-groupoids.
  - This inclusion is the left adjoint in an adjunction

    $$\mathrm{U}_{\mathcal{S}} : \mathcal{S} \rightleftarrows \mathcal{C}\mathrm{at}_\infty : (-)^{\simeq}$$

    whose right adjoint is given by the *maximal subgroupoid* functor.[7]
  - This inclusion is the right adjoint in an adjunction

    $$(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{S} : \mathrm{U}_{\mathcal{S}}$$

    whose left adjoint is given by the (∞-)*groupoid completion* functor.[8]
- We write $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \in \mathcal{C}\mathrm{at}_\infty$ for the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$. This is the internal hom in $(\mathcal{C}\mathrm{at}_\infty, \times)$, and admits a canonical equivalence $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}, \mathcal{D})$ in $\mathcal{S}$.[9]

[7] The adjunction $\mathrm{U}_{\mathcal{S}} : \mathcal{S} \rightleftarrows \mathcal{C}\mathrm{at}_\infty : (-)^{\simeq}$ is presented by an adjunction of $s\mathcal{S}\mathrm{et}$-enriched categories between that of Kan complexes and that of quasicategories, whose left adjoint is the canonical inclusion and whose right adjoint takes a quasicategory to the largest Kan complex that it contains (see Proposition T.1.2.5.3 and Corollary T.5.2.4.5).

[8] The adjunction $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{S} : \mathrm{U}_{\mathcal{S}}$ is presented by the Quillen adjunction $\mathrm{id}_{s\mathcal{S}\mathrm{et}} : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} : \mathrm{id}_{s\mathcal{S}\mathrm{et}}$ (see item (34) (and Remark T.1.2.5.6)).

[9] As the model category $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is cartesian (as can easily be seen from Corollary T.2.2.5.4), the ∞-category of functors is presented therein by the internal hom in $(s\mathcal{S}\mathrm{et}, \times)$.

- We write $(-)^{op} : \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$ for the involution given by taking opposites.

(2) Despite our grounding declared in item (1), our notion of "∞-category" is nevertheless a rather flexible one: over the course of this thesis, we interchange fluidly between a number of distinct but essentially equivalent notions thereof. In accordance with current best practices, those that we will employ all appear naturally as objects in various model categories. For the reader's convenience, we itemize these notions and their ambient model categories here, and we give some indication of the roles that they play in this thesis.

(a) The notion of a *quasicategory* plays a distinguished role in this thesis, as indicated in item (1). These are precisely the bifibrant objects in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, the category of simplicial sets equipped with the *Joyal model structure* of Theorem T.2.2.5.1. We view these as the most convenient of the notions to employ as an ambient framework, which advantage is surely in large part due to the abundance of theory that has been built up around them.

(b) The notion which most closely adheres to the intuition of a "category enriched in spaces" is that of a *category enriched in simplicial sets*, or simply a $s\mathcal{S}$et-*enriched category* for short. These organize into the model category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ under the *Bergner model structure* of [Ber07, Theorem 1.1] (or see Proposition T.A.3.2.4 for a generalization). Just as when one uses simplicial sets to present spaces one should generally be working with Kan complexes, when considering a $s\mathcal{S}$et-enriched category as an ∞-category one should generally be working with a category which is in fact enriched in Kan complexes: indeed, these are precisely the fibrant objects of $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ (which sits in a Quillen equivalence $\mathfrak{C} : s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows (\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}} : \mathrm{N}^{\mathrm{hc}}$ (see Theorem T.2.2.5.1)), though note that not all objects are cofibrant. This model category provides an explicit bridge from $\mathfrak{Rel}\mathrm{Cat}_{\mathrm{BK}}$ to $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ (see subitem (2)(d)).

(c) The notion which is most "homotopy invariant" is that of a *complete Segal space*. These are actually bisimplicial sets, thought of as simplicial spaces via choices of distinguished "simplicial" and "geometric" directions. They are precisely the bifibrant objects in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$, the category of bisimplicial sets equipped with the *Rezk model structure* of [Rez01, Theorem 7.2] (there called the "complete Segal space" model structure).

However, it is also fruitful to consider a theory of complete Segal spaces *internally* to the world of ∞-categories, i.e. to define them as a subcategory $\mathcal{CSS} \subset s\mathcal{S}$ of the ∞-category of simplicial spaces.[10] From this viewpoint, a complete Segal space can be thought of as a homotopical analog of the *nerve* of a category: the equivalence $\mathrm{N}_\infty : \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{CSS}$ takes an ∞-category $\mathcal{C}$ to its ∞-*categorical nerve*, namely the simplicial space

$$\mathrm{N}_\infty(\mathcal{C})_\bullet = \hom^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}_\infty}([\bullet], \mathcal{C})$$

(i.e. the levelwise hom-space from the standard cosimplicial category $[\bullet] : \mathbf{\Delta} \hookrightarrow \mathcal{C}\mathrm{at}$).[11,12,13] The inverse equivalence takes a complete Segal space $Y_\bullet \in \mathcal{CSS}$ to the ∞-category

$$\int^{[n]\in\mathbf{\Delta}} Y_n \times [n]$$

(where we implicitly consider $Y_n \in \mathcal{C}\mathrm{at}_\infty$ via the inclusion $\mathrm{U}_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$).[14] In fact, this inclusion is the right adjoint in an adjunction $\mathrm{L}_{\mathcal{CSS}} : s\mathcal{S} \rightleftarrows \mathcal{CSS} : \mathrm{U}_{\mathcal{CSS}}$.[15] It is fruitful to think of the resulting composite adjunction

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\rightleftarrows}} \mathcal{CSS} \underset{\mathrm{N}_\infty}{\overset{\mathrm{N}_\infty^{-1}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty$$

as being a homotopical analog of the usual "nerve/homotopy category" adjunction

$$s\mathcal{S}\mathrm{et} \underset{\mathrm{N}}{\overset{\mathrm{L}_{\mathcal{C}\mathrm{at}}}{\rightleftarrows}} \mathcal{C}\mathrm{at}$$

(see subitem (4)(c)).

[10] This perspective is explored in detail (and in greater generality) in [Lur09c, §1].

[11] Indeed, a simplicial set is the nerve of a category precisely if it satisfies the Segal condition.

[12] Note that the ∞-categorical nerve of a 1-category does *not* generally coincide with its 1-categorical nerve.

[13] Throughout §A, for the sake of clarity we will exclusively refer to this construction as the "∞-categorical nerve". However, it appears quite frequently in the main body of this thesis (beginning with its reintroduction in §2.2), and so for brevity we will omit the modifier "∞-categorical" there except when we mean to emphasize the distinction.

[14] This formula follows from [Lur09c, Corollary 4.3.15], but note that this is ultimately just an instance of the "generalized nerve/realization" Quillen equivalence first proved as [DK84b, Theorem 3.1].

[15] This adjunction is presented by the left Bousfield localization $\mathrm{id}_{ss\mathcal{S}\mathrm{et}} : s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} : \mathrm{id}_{ss\mathcal{S}\mathrm{et}}$ (see item (34)).

(d) Finally, the simplest notion is that of a *relative category*. These organize into the model category $\mathscr{R}\mathrm{el}\mathscr{C}\mathrm{at}_{\mathrm{BK}}$ under the *Barwick–Kan model structure* of [BK12b, Theorem 6.1]. We write $\mathscr{L}^H_\delta : \mathscr{R}\mathrm{el}\mathscr{C}\mathrm{at}_{\mathrm{BK}} \to (\mathscr{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ for the hammock localization functor, which is a relative functor (see [BK12a, Theorem 1.8]); in fact, it is even a weak equivalence in $\mathscr{R}\mathrm{el}\mathscr{C}\mathrm{at}_{\mathrm{BK}}$ (see [BK12a, Theorem 1.7] and item (3)).[16,17]

We mainly use relative categories (and the Barwick–Kan model structure) as a technical device that allows us to make rigorous sense of the *underlying* ∞*-category* of a relative category (in particular of a model category). In this situation, we say that the model category gives a *presentation* of its underlying ∞-category. See § A.3 for details regarding our usage of model categories as presentations of ∞-categories.

(3) The assertion made in item (2) that these various notions of ∞-categories are all "essentially equivalent" is rather multifaceted. We therefore give a careful account of this assertion. Our perspective is espoused in a number of relatively recent papers, notably [BSP] (from the introduction of which this item is more or less directly lifted), and seems to represent the emerging consensus among practitioners of higher category theory.

First of all, these four model categories are all connected by a diagram of Quillen equivalences along with the weak equivalence $\mathscr{L}^H_\delta : \mathscr{R}\mathrm{el}\mathscr{C}\mathrm{at}_{\mathrm{BK}} \xrightarrow{\approx} (\mathscr{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ in $\mathscr{R}\mathrm{el}\mathscr{C}\mathrm{at}_{\mathrm{BK}}$ (see [BSP, Figure 1] and the references cited therein). Thus, any homotopically meaningful manipulations that we might make using one of these notions can equally well be made using any other notion.

However, there is still cause for potential concern: the diagram of [BSP, Figure 1] does not commute, even up to natural isomorphism. However, a moment's reflection should reassure us that this is a stronger request than we should really be making: after all, we would generally like to consider objects of a model category up to *weak equivalence*, not up to isomorphism. Thus, it is helpful to reinterpret this diagram *within* one of the given model categories. Rather than choose a particular one, we will simply refer to objects of this model category as '∞-categories' (with scare-quotes) for the remainder of the

[16] In fact, the Rezk nerve functor $\mathrm{N}^{\mathrm{R}} : \mathscr{R}\mathrm{el}\mathscr{C}\mathrm{at}_{\mathrm{BK}} \to ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ (see [Rez01, 3.3], where it is called the "classification diagram" functor) *also* creates the weak equivalences by [BK12b, Theorem 6.1(i)]. However, the objects in its image are not generally fibrant, even up to weak equivalence in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ (see [LMG15]).

[17] The letter δ in the notation $\mathscr{L}^H_\delta$ stands for "discrete": in Chapter 4 we study an ∞-categorical version of this functor, which we denote simply by $\mathscr{L}^H$.

item; as explained in item (34), Quillen equivalences between model categories induce weak equivalences of underlying '∞-categories'.

This conceptual leap leads us to the alternative point of view that what we are looking at is a not-necessarily-commutative diagram of weak equivalences of '∞-categories'. This may not seem like an improvement in and of itself, but in fact we are saved by the following remarkable facts ([Toë05, Théorème 6.3], reproved as [Lur09c, Theorem 4.4.1] and generalized as [BSP, Theorem 8.2]), originally stated within the model category $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ of complete Segal spaces.

- The '∞-category' of complete Segal spaces (i.e. the '∞-category' corresponding to $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$) – and hence any '$\infty$-category' weakly equivalent to it – has a *discrete* derived automorphism space, which is equivalent as a group to $\mathbb{Z}/2$.

- Furthermore, the unique nontrivial derived automorphism of this '∞-category' is given by the involution of taking opposites, and is therefore detected by considering its restriction to the full subcategory generated by the objects $[0], [1] \in \mathcal{C}\mathrm{at}$ (considered as objects of each of these various model categories).

It now follows readily that the diagram of [BSP, Figure 1] commutes *as a diagram in an ∞-category*: more precisely, as a diagram internal to the quasicategory corresponding to the ambient model category of '∞-categories'.

A.2 Conventions regarding ∞-categories

We now establish some conventions surrounding our usage of ∞-categories.

(4) As a rule, the statements we make will generally be invariant under equivalence of ∞-categories. In fact, when we make statements about ∞-categories, we will generally mean to be working *in the ∞-category of ∞-categories*. However, this is only a matter of taste: the sufficiently motivated reader should readily be able to turn our invariant arguments about ∞-categories into simplex-by-simplex arguments about quasicategories and model-categorical arguments in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$.

The choice of such a foundational regime compels us to lay out the following related conventions.

(a) When working ∞-categorically, we will omit the modifier "essential" (and its variants) wherever it might be used in its technical capacity. For instance, we simply say *unique* where one might otherwise say "essentially unique": in the invariant world, the adjective "unique" has no other possible meaning.

(b) We reserve the symbol $=$ to indicate nothing other than

- that some equivalence holds by definition, or
- the equality of two elements of a set, and in particular
  - the equivalence of two subobjects of a given object (see item (22)).

Along these same lines, whereas we generally use the symbol $\simeq$ to denote an equivalence in an arbitrary ∞-category, if that ∞-category is in fact a 1-category then we may instead write $\cong$ (and refer to the equivalence as an *isomorphism*).

(c) We will have to be slightly careful with our definition of ordinary categories. For instance, we will sometimes want to refer to the nerve of a 1-category, but this is not a well-defined operation on the full subcategory $\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty$: for example, the notion of "the set of objects" is not invariant under equivalence of categories.

Thus, we will use the term *strict category* (or even *strict 1-category*) to mean a simplicial set satisfying the Segal condition. These assemble into a full subcategory $\mathfrak{cat} \subset s\mathcal{S}\mathrm{et}$. This 1-category of categories now admits a nerve functor

$$\mathfrak{cat} \xrightarrow{\mathrm{N}} s\mathcal{S}\mathrm{et},$$

although this is entirely cosmetic: according to our definition, it is simply the defining inclusion. Note that this inclusion sits as the right adjoint in a left localization adjunction

$$s\mathcal{S}\mathrm{et} \underset{\mathrm{N}}{\overset{\mathrm{L}_{\mathfrak{cat}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathfrak{cat},$$

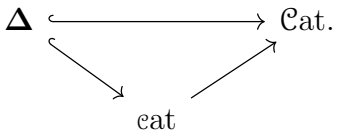
and hence commutes with limits.

The 1-category of strict categories also admits a functor $\mathfrak{cat} \xrightarrow{\mathrm{U}_{\mathfrak{cat}}} \mathcal{C}\mathrm{at}$, namely the factorization of the composite functor

$$\mathfrak{cat} \xhookrightarrow{\mathrm{N}} s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}[\![(\mathbf{W}_{\mathrm{Joyal}})^{-1}]\!] \simeq \mathcal{C}\mathrm{at}_\infty$$

through its image, but this is *not* the inclusion of a subcategory. On the other hand, the *gaunt* objects of $\mathfrak{cat}$ – that is, those in which every isomorphism is in fact an identity morphism – do include as a full subcategory

of $\mathcal{C}\mathrm{at}$. In fact, the map $\hom_{\mathcal{C}\mathrm{at}}(\mathcal{C},\mathcal{D}) \to \hom_{\mathcal{C}\mathrm{at}}(\mathrm{U}_{\mathcal{C}\mathrm{at}}(\mathcal{C}),\mathrm{U}_{\mathcal{C}\mathrm{at}}(\mathcal{D}))$ is an equivalence in $\mathcal{S}$ whenever $\mathcal{D}$ is gaunt. Note in particular that we obtain full inclusions



In fact, note further that we can consider $\mathbf{\Delta}$ *itself* as a strict category: in such situations, we will take $\mathbf{\Delta}$ to be *skeletal*, i.e. to be the full subcategory $\mathbf{\Delta} \subset \mathcal{C}\mathrm{at}$ on the gaunt categories $[n] \in \mathcal{C}\mathrm{at}$ (rather than the full subcategory on *all* finite nonempty totally ordered sets).

The functor $\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ does not preserve monomorphisms. At the risk of confusion, we will nevertheless use the same notation $\mathcal{C} \subset \mathcal{D}$ to indicate a monomorphism in $\mathcal{C}\mathrm{at}$.

In contrast with subitem (4)(a), we *will* use terms such as "essentially surjective" to refer to maps in $\mathcal{C}\mathrm{at}$, since otherwise the meaning would be ambiguous.

We will similarly speak of *strict groupoids*, *strict relative categories*, etc., likewise shrinking the capital letters in their names as $\mathcal{G}\mathrm{pd}$, $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$, etc. to indicate the 1-categories of these. (For us, $s\mathcal{S}\mathrm{et}$-enriched categories will *always* be strict; the 1-category of these is correspondingly denoted by $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$.) We will also write

$$\mathcal{F}\mathrm{un}(-,-) : \mathcal{C}\mathrm{at}^{op} \times \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$$

for the internal hom bifunctor in $(\mathcal{C}\mathrm{at},\times)$, which can be computed by the internal hom bifunctor in $(s\mathcal{S}\mathrm{et},\times)$; since $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is cartesian, for any $\mathcal{C},\mathcal{D} \in \mathcal{C}\mathrm{at}$ we have a canonical equivalence

$$\mathrm{U}_{\mathcal{C}\mathrm{at}}(\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{U}_{\mathcal{C}\mathrm{at}}(\mathcal{C}),\mathrm{U}_{\mathcal{C}\mathrm{at}}(\mathcal{D}))$$

in $\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty$.

However, now that we have carefully clarified this distinction, we will often simply ignore it (e.g. referring to strict categories just as "categories") rather than overburden our terminology, unless it warrants emphasis. Our meaning should always be clear from context.

(5) An object $c \in \mathcal{C}$ defines a functor $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \to \mathcal{C}$. For convenience, we will generally denote this functor by $\{c\} \hookrightarrow \mathcal{C}$ (even though it will not generally be a monomorphism!); that is, we take the notation $\{c\}$ to denote a terminal object of $\mathcal{C}\mathrm{at}_\infty$ which is equipped with a preferred map to $\mathcal{C}$.

(6) We will generally write $s, t : \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$ for the *source* and *target* maps, i.e. for the evaluation maps at $0 \in [1]$ and $1 \in [1]$ respectively. Relatedly, as various flavors of categories will sometimes be defined as simplicial objects (recall e.g. subitems (2)(c) and (4)(c)), for such a simplicial object $Y_\bullet \in s\mathcal{C}$ we may write the two structure maps $Y_1 \rightrightarrows Y_0$ in $\mathcal{C}$ as $\delta_1 = s$ and $\delta_0 = t$.

(7) In general, given an object $c \in \mathcal{C}$, we will write $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\mathrm{ev}_c} \mathcal{D}$ for the functor given by evaluation at c (i.e. the pullback along the functor $\{c\} \hookrightarrow \mathcal{C}$).

(8) For ∞-categories $\mathcal{I}$ and $\mathcal{C}$, we will generally write $\mathrm{const}_{\mathcal{I}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ for the *constant diagram* functor. However, when it is clear from context, we may omit the subscript and simply write $\mathrm{const} : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$.

(9) For an object $c \in \mathcal{C}$, we write $\mathrm{diag} : c \to c \times c$ for the diagonal map (if it exists). This will usually be applied in the case that $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$.

(10) We will sometimes want to identify a bifunctor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ with its adjunct $\mathcal{C} \to \mathrm{Fun}(\mathcal{D}, \mathcal{E})$. For clarity, if the original bifunctor is denoted by $F(-,-)$, then this adjunct will be denoted by $F(-,=)$. That is, we will use the symbols $-$ and $=$ to respectively indicate the slot being filled first and the slot being considered as a free variable. We will use similar notation for adjuncts of multivariable functors.

(11) Given a functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$, we will generally denote its colimit (if it exists), an object of $\mathcal{C}$, by

- $\mathrm{colim}(\mathcal{I} \xrightarrow{F} \mathcal{C})$, or
- $\mathrm{colim}_{\mathcal{I}}(F)$, or
- $\mathrm{colim}^{\mathcal{C}}_{\mathcal{I}}(F)$ if we'd like to emphasize the ∞-category $\mathcal{C}$ in which the colimit is being taken, or
- $\mathrm{colim}_{i \in \mathcal{I}} F(i)$ if we'd like to emphasize the functoriality of F for $i \in \mathcal{I}$, or
- $\mathrm{colim}^{\mathcal{C}}_{i \in \mathcal{I}} F(i)$ to combine the previous two notations.

Dually, we will denote its limit (if it exists), also an object of $\mathcal{C}$, by $\lim(\mathcal{I} \xrightarrow{F} \mathcal{C})$, or $\lim_{\mathcal{I}}(F) \in \mathcal{C}$, or $\lim^{\mathcal{C}}_{\mathcal{I}}(F) \in \mathcal{C}$, or $\lim_{i \in \mathcal{I}} F(i)$, or $\lim^{\mathcal{C}}_{i \in \mathcal{I}} F(i)$.

For convenience, we will often write $|-| = \mathrm{colim}_{\mathbf{\Delta}^{op}}(-)$ for any colimit functor $s\mathcal{C} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C}) \to \mathcal{C}$ and refer to it as *geometric realization*. Similarly, we will often write $\|-\| = \mathrm{colim}_{\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}}(-)$ for any colimit functor $ss\mathcal{C} = \mathrm{Fun}(\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}, \mathcal{C}) \to \mathcal{C}$.

(12) Given a span $d \xleftarrow{\varphi} c \xrightarrow{\psi} e$ in an ∞-category $\mathcal{C}$, we may denote by

$$d \coprod_{\varphi,c,\psi} e$$

its colimit (i.e. its pushout). Dually, given a cospan $d \xrightarrow{\varphi} c \xleftarrow{\psi} e$ in an ∞-category $\mathcal{C}$, we may denote by

$$d \underset{\varphi,c,\psi}{\times} e$$

its limit (i.e. its limit). On the other hand, we may omit either or both of the maps from the subscript if they are clear from context. Meanwhile, in the absolute cases, given a set of objects $\{c_i \in \mathcal{C}\}_{i \in I}$, we may write

$$\coprod_{i \in I} c_i$$

for their coproduct and

$$\prod_{i \in I} c_i$$

for their product.

(13) Given a functor $\mathcal{I} \xrightarrow{F} \mathcal{J}$ and an ∞-category $\mathcal{C}$, restriction along F induces a functor $\mathrm{Fun}(\mathcal{J}, \mathcal{C}) \xrightarrow{F^*} \mathrm{Fun}(\mathcal{I}, \mathcal{C})$. In many cases (for instance if $\mathcal{C}$ is cocomplete) this admits a left adjoint, which we denote by $F_! : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ and refer to as the *left Kan extension* (along F) functor. (See §T.4.3.)

(14) We will occasionally use the theory of *coends* and *ends*: given a functor $\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{F} \mathcal{D}$, we will denote its coend by

$$\int^{c \in \mathcal{C}} F(c,c) \in \mathcal{D}$$

and its end by

$$\int_{c \in \mathcal{C}} F(c,c) \in \mathcal{D}.$$

(We refer the reader to [GHN, §2] for a brief review of the theory of co/ends in the ∞-categorical setting.)

(15) Suppose we are given a bifunctor $\mathcal{I} \times \mathcal{J} \xrightarrow{F} \mathcal{C}$. Then, *Fubini's theorem for colimits* asserts that we have a canonical equivalence

$$\operatorname{colim}^{\mathcal{C}}_{(i,j)\in\mathcal{I}\times\mathcal{J}} F(i,j) \simeq \operatorname{colim}^{\mathcal{C}}_{i\in\mathcal{I}} \left(\operatorname{colim}^{\mathcal{C}}_{j\in\mathcal{J}} F(i,j)\right)$$

in $\mathcal{C}$, if either side exists. This can be proved by the juggling of iterated coends (which explains the name), but it is really just a consequence of the observation that the composite

$$\mathcal{C} \xhookrightarrow{\mathrm{const}_{\mathcal{J}}} \mathrm{Fun}(\mathcal{J},\mathcal{C}) \xhookrightarrow{\mathrm{const}_{\mathcal{I}}} \mathrm{Fun}(\mathcal{I},\mathrm{Fun}(\mathcal{J},\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C})$$

coincides with the functor

$$\mathcal{C} \xhookrightarrow{\mathrm{const}_{\mathcal{I}\times\mathcal{J}}} \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C})$$

(at least when $\mathcal{C}$ is cocomplete, or else embedding $\mathcal{C}$ into its free cocompletion for precisely those colimits that it lacks).

(16) We will often implicitly use the fact (which is proved as [Joyb, Chapter 5, Theorem C] (combined with [Joyb, Proposition 4.8])) that a natural transformation between functors of ∞-categories is an equivalence precisely if it is a componentwise equivalence.

(17) We will often implicitly use the fact (which is proved as Corollary T.5.1.2.3 and its dual) that co/limits in a functor ∞-category are computed objectwise.

(18) Given a functor $\mathcal{I} \to \mathcal{C}$ that factors through the inclusion $\mathcal{C}^{\simeq} \subset \mathcal{C}$ of the maximal subgroupoid (i.e. that takes all maps in $\mathcal{I}$ to equivalences in $\mathcal{C}$), there exists a unique induced extension

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathcal{C} \\ \downarrow & \searrow & \uparrow \\ \mathcal{I}^{\mathrm{gpd}} & \dashrightarrow & \mathcal{C}^{\simeq} \end{array}$$

in $\mathcal{C}\mathrm{at}_\infty$ over the canonical projection $\mathcal{I} \to \mathcal{I}^{\mathrm{gpd}}$ to the groupoid completion. In other words, restriction induces an equivalence

$$\mathrm{Fun}(\mathcal{I}^{\mathrm{gpd}},\mathcal{C}^{\simeq}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{I},\mathcal{C}^{\simeq}) \hookrightarrow \mathrm{Fun}(\mathcal{I},\mathcal{C})$$

onto the (non-full) subcategory of $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ on such functors (and natural equivalences between them). In particular, if $\mathcal{I}^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$, then this subcategory

can be canonically identified with the subcategory of *constant* functors (and the natural equivalences between them), which of course is canonically equivalent to $\mathcal{C}^{\simeq}$ itself.

(19) An ∞-category $\mathcal{I}$ is called *sifted* if it is nonempty and its diagonal map $\mathrm{diag} : \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is final (see Definition T.5.5.8.1, Definition 3.4.4, and Remark 3.4.7). The most important single example of a sifted ∞-category is $\mathbf{\Delta}^{op}$ (see Lemma T.5.5.8.4), but note too that all filtered ∞-categories are also sifted (see Example T.5.5.8.3).

The following facts regarding sifted ∞-categories will be important to us.

- If $\mathcal{I}$ is a sifted ∞-category, then $\mathcal{I}^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$ (see Proposition T.5.5.8.7).
- If
  - $\mathcal{I}$ is a sifted ∞-category,
  - $\mathcal{C}$ is an ∞-category admitting finite products and $\mathcal{I}$-indexed colimits, and
  - the product bifunctor $\mathcal{C} \times \mathcal{C} \xrightarrow{- \times -} \mathcal{C}$ preserves $\mathcal{I}$-indexed colimits separately in each variable,

  then $\mathrm{colim} : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ preserves finite products (see Lemma T.5.5.8.11). In particular, this holds when $\mathcal{C} = \mathcal{S}$, or more generally when $\mathcal{C}$ is an ∞-topos (see Remark T.5.5.8.12).

(20) Given a relative ∞-category $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty}$, its localization $\mathcal{R}[\![\mathbf{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_{\infty}$ might be more carefully termed its *free* localization. This construction is left adjoint to the functor $\mathcal{C}\mathrm{at}_{\infty} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty}$ taking an ∞-category $\mathcal{C}$ to its corresponding *minimal* relative ∞-category $(\mathcal{C}, \mathcal{C}^{\simeq})$, and can hence be constructed explicitly as the pushout

$$\mathcal{R}[\![\mathbf{W}^{-1}]\!] \simeq \mathrm{colim} \left(\begin{array}{ccc} \mathbf{W} & \longrightarrow & \mathbf{W}^{\mathrm{gpd}} \\ \downarrow & & \\ \mathcal{R} & & \end{array} \right).$$

(These notions are all discussed in detail in §2.1.)

We warn the reader that this notion does *not* generally agree with the definition of "localization" studied in §T.5.2.7 (see Warning T.5.2.7.3), namely a functor admitting a fully faithful right adjoint. When we discuss it, we will refer to

this latter notion as a *left* localization; its right adjoint may then be referred to as the inclusion of a *reflective* subcategory. We will denote general such adjunctions by $\mathrm{L} \dashv \mathrm{U}$ (with additional decorations in specific instances). These are actually a special case of free localizations (see Proposition T.5.2.7.12 or Remark 2.1.25).[18]

Of course, there is the dual notion of a *right* localization (into a *coreflective* subcategory), although due to the overall handedness of mathematics (boiling down to the fact that we're generally more comfortable thinking about $\mathcal{S}\mathrm{et}$ than about $\mathcal{S}\mathrm{et}^{op}$), this arises less frequently in practice and in particular does not appear anywhere in [Lur09b] (hence the unambiguity of the terminology "localization" used there). We will similarly denote general such adjunctions by $\mathrm{U} \dashv \mathrm{R}$.

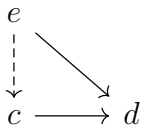
Note that a (free) localization which is neither a left nor a right localization can nevertheless admit a section; see for instance Example 1.2.19.

(21) Many of our arguments will implicitly rely on the existence of a hom bifunctor

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\hom_{\mathcal{C}}(-,-)} \mathcal{S}$$

for an arbitrary ∞-category $\mathcal{C}$. This is achieved by the *twisted arrow* ∞-*category* construction (see Proposition A.5.2.1.11). If $\mathcal{C}$ is an enriched ∞-category, we will write $\underline{\hom}_{\mathcal{C}}(-,-)$ for the enriched hom-object and continue to write $\hom_{\mathcal{C}}(-,-)$ for its underlying hom-spaces. (For the most part, the enriched categories we will encounter will be of the particularly special sort described in item (28).)

(22) A morphism $c \to d$ in an ∞-category $\mathcal{C}$ is called a *monomorphism* if for any other object $e \in \mathcal{C}$, the induced map $\hom_{\mathcal{C}}(e,c) \to \hom_{\mathcal{C}}(e,d)$ is a monomorphism in $\mathcal{S}$: these are precisely the morphisms for which it is merely a condition (as opposed to requiring additional data) for there to exist a factorization



[18] Somewhat confusingly, accessible left localizations of presentable ∞-categories additionally satisfy a universal property among *left adjoint* functors (see Proposition T.5.5.4.20).

of a given map $e \to d$ in $\mathcal{C}$. This is equivalent to the requirement that the commutative square

$$\begin{array}{ccc} c & \longrightarrow & c \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

in $\mathcal{C}$ is a pullback.

Dually, a morphism $c \to d$ in an ∞-category $\mathcal{C}$ is called an *epimorphism* if for any other object $e \in \mathcal{C}$, the induced map $\hom_{\mathcal{C}}(d, e) \to \hom_{\mathcal{C}}(c, e)$ is a monomorphism in $\mathcal{S}$: similarly, these are precisely the morphisms for which it is merely a condition for there to exist an extension

$$\begin{array}{ccc} c & \longrightarrow & d \\ & \searrow & \vdots \\ & & e \end{array}$$

of a given map $c \to e$ in $\mathcal{C}$. This is equivalent to the requirement that the commutative square

$$\begin{array}{ccc} c & \longrightarrow & d \\ \downarrow & & \downarrow \\ d & \longrightarrow & d \end{array}$$

in $\mathcal{C}$ is a pushout.

(23) For any ∞-category $\mathcal{C}$, we will write

$$よ_{\mathcal{C}} = \hom_{\mathcal{C}}(=, -) : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

for the *Yoneda functor*, namely the indicated adjunct to the hom bifunctor of item (21); we may simply write よ if the ∞-category $\mathcal{C}$ is clear from context.[19] (If $\mathcal{C}$ is a 1-category, we may also write $よ_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$ for its factorization through the subcategory of *discrete* objects (see item (24)).) We generally write

$$\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

for the ∞-category of *presheaves* (of spaces) on $\mathcal{C}$, the target of the Yoneda functor. An ∞-categorical version of *Yoneda's lemma* (see Proposition T.5.1.3.1) asserts that, just as in ordinary category theory, this functor is fully faithful;

[19] Pronounced "yo", the character よ is the first letter of "Yoneda" in Hiragana.

we therefore will also refer to it as the *Yoneda embedding*. We will also use the fact (proved as Theorem T.5.1.5.6) that the Yoneda embedding models the *free cocompletion*, i.e. that for any cocomplete ∞-category $\mathcal{D}$, restriction along the Yoneda embedding defines an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow[\sim]{(\text{ɤ}_{\mathcal{C}})^*} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

(where we write $\mathrm{Fun}^{\mathrm{colim}}$ to denote the full subcategory of the functor ∞-category on those functors which preserve colimits), with inverse given by left Kan extension along $\text{ɤ}_{\mathcal{C}}$.

(24) An object $c \in \mathcal{C}$ of an ∞-category $\mathcal{C}$ is called *discrete* if the functor

$$\text{ɤ}_{\mathcal{C}}(c) = \hom_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \to \mathcal{S}$$

factors through the subcategory $\mathcal{S}\mathrm{et} \subset \mathcal{S}$.[20] For instance, $\mathcal{S}\mathrm{et} \subset \mathcal{S}$ is the inclusion of the full subcategory of discrete objects. It is not hard to see that in a presheaf ∞-category $\mathcal{P}(\mathcal{D}) = \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$, an object $F \in \mathcal{P}(\mathcal{D})$ is discrete if and only if it itself factors through $\mathcal{S}\mathrm{et}$: that is, discreteness is determined objectwise. Thus, for instance, $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ is likewise the inclusion of the full subcategory of discrete objects.

(25) Given two ∞-categories $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$, an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is uniquely determined by a bifunctor

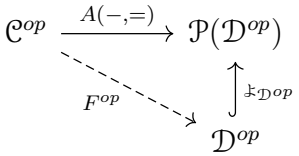
$$\mathcal{C}^{op} \times \mathcal{D} \xrightarrow{A} \mathcal{S},$$

where $A \simeq \hom_{\mathcal{C}}(-, G(-)) \simeq \hom_{\mathcal{D}}(F(-), -)$.[21] In fact, we can *define* an adjunction to be an arbitrary bifunctor $\mathcal{C}^{op} \times \mathcal{D} \xrightarrow{A} \mathcal{S}$ which is "co/representable in each slot". More precisely, this means that for any $c \in \mathcal{C}$ the functor $\mathcal{D} \xrightarrow{A(c,-)} \mathcal{S}$ must be corepresentable, while for any $d \in \mathcal{D}$ the functor $\mathcal{C}^{op} \xrightarrow{A(-,d)} \mathcal{S}$ must be representable. Since the Yoneda embedding is fully faithful, we recover the

[20] This condition is equivalent to requiring that the diagonal map $\text{ɤ}_{\mathcal{C}}(c) \to \text{ɤ}_{\mathcal{C}}(c) \times \text{ɤ}_{\mathcal{C}}(c)$ in $\mathcal{P}(\mathcal{C})$ be a monomorphism. If the product $c \times c$ exists in $\mathcal{C}$, then we have an equivalence $\text{ɤ}_{\mathcal{C}}(c \times c) \simeq \text{ɤ}_{\mathcal{C}}(c) \times \text{ɤ}_{\mathcal{C}}(c)$ in $\mathcal{P}(\mathcal{C})$ and it follows easily that this condition can also be checked in $\mathcal{C}$.

[21] That this agrees with Definition T.5.2.2.1 follows easily from the formalism of *correspondences* (see §T.2.3.1 and §T.5.2.1 (and the model-independent theory of co/cartesian fibrations laid out in [MG])).

adjoint functors via the unique factorizations



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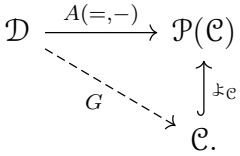


Following standard conventions, in our diagrams that involve adjunctions, we keep left adjoints above and/or to the left of their right adjoints to whatever extent possible. (In-line adjunctions will *always* have their left adjoints on top.) For added clarity, we often use the "turnstile" symbol $\perp$, which sits on the right adjoint and points towards the left adjoint. Even in the absence of an ambient diagram, we write $F \dashv G$ to indicate that F is left adjoint to G.

We define the ∞-category of *adjunctions from* $\mathcal{C}$ *to* $\mathcal{D}$ to be the full subcategory

$$\mathrm{Adjn}(\mathcal{C};\mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{D}, \mathcal{S})$$

on those bifunctors that define adjunctions. If the objects $A, A' \in \mathrm{Adjn}(\mathcal{C};\mathcal{D})$ determine adjunctions $F \dashv G$ and $F' \dashv G'$, then a map $A \to A'$ in $\mathrm{Adjn}(\mathcal{C};\mathcal{D})$ is uniquely determined by either datum of a morphism $F' \to F$ in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ or a morphism $G \to G'$ in $\mathrm{Fun}(\mathcal{D},\mathcal{C})$.[22] We will write $\mathrm{LAdjt}(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{C},\mathcal{D})$ for the full subcategory on those functors which are left adjoints, and we will write $\mathrm{RAdjt}(\mathcal{D},\mathcal{C}) \subset \mathrm{Fun}(\mathcal{D},\mathcal{C})$ for the full subcategory on those functors which are right adjoints. We therefore have equivalences

$$\mathrm{LAdjt}(\mathcal{C},\mathcal{D})^{op} \xleftarrow{\sim} \mathrm{Adjn}(\mathcal{C};\mathcal{D}) \xrightarrow{\sim} \mathrm{RAdjt}(\mathcal{D},\mathcal{C})$$

by the uniqueness of adjoints.

We also note here that given an adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ and any ∞-category $\mathcal{E}$, applying the functor $\mathrm{Fun}(\mathcal{E},-) : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ yields a canonical adjunction $\mathrm{Fun}(\mathcal{E},\mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{E},\mathcal{D})$. (This follows easily by combining Proposition T.5.2.2.8 with [Gla, Proposition 2.3] (or with [GHN, Proposition 5.1]).)

[22] One might say that these two natural transformations are *conjugates* with respect to the given adjunctions (as in [ML98, Chapter IV, §7], except that our variance is reversed).

(26) More generally, we define an *adjunction of i contravariant variables and j covariant variables* to be a multifunctor

$$(\mathcal{C}_1)^{op} \times \cdots \times (\mathcal{C}_i)^{op} \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_j \xrightarrow{A} \mathcal{S}$$

satisfying the condition that fixing all but any one of the slots yields a co/representable functor.[23] Note that by definition, fixing any number of slots in such an adjunction yields an adjunction in the remaining free variables.[24] We similarly define a full subcategory

$$\mathrm{Adjn}(\mathcal{C}_1, \ldots, \mathcal{C}_i; \mathcal{D}_1, \ldots, \mathcal{D}_j) \subset \mathrm{Fun}((\mathcal{C}_1)^{op} \times \cdots (\mathcal{C}_i)^{op} \times \mathcal{D}_1 \times \cdots \mathcal{D}_j, \mathcal{S})$$

on such multivariable adjunctions.

Aside from ordinary adjunctions, we will mainly be interested in (what have come to be called) *adjunctions of two variables* (or simply *two-variable adjunctions*), namely the case $i = 2$ and $j = 1$. A two-variable adjunction is thus a trifunctor

$$\mathcal{C}^{op} \times \mathcal{D}^{op} \times \mathcal{E} \xrightarrow{A} \mathcal{S},$$

in which the co/representability condition furnishes three bifunctors denoted in general as

$$\left(\mathcal{C} \times \mathcal{D} \xrightarrow{-\otimes-} \mathcal{E} \ , \ \mathcal{C}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_l(-,-)} \mathcal{D} \ , \ \mathcal{D}^{op} \times \mathcal{E} \xrightarrow{\underline{\hom}_r(-,-)} \mathcal{C} \right),$$

which come equipped with uniquely determined natural equivalences

$$A(c,d,e) \simeq \hom_{\mathcal{C}}(c, \underline{\hom}_r(d,e)) \simeq \hom_{\mathcal{D}}(d, \underline{\hom}_l(c,e)) \simeq \hom_{\mathcal{E}}(c \otimes d, e)$$

in $\mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{D}^{op} \times \mathcal{E}, \mathcal{S})$ (for $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $e \in \mathcal{E}$). Just as with ordinary adjunctions, we will often denote a two-variable adjunction simply by listing its constituent bifunctors, leaving the natural equivalences implicit.

(27) Given an ∞-category $\mathcal{C}$, an object $c \in \mathcal{C}$, and a space $Y \in \mathcal{S}$, a *tensor* of c over Y is an object $c \odot Y \in \mathcal{C}$ equipped with an equivalence

$$\hom_{\mathcal{C}}(c \odot Y, -) \simeq \hom_{\mathcal{S}}(Y, \hom_{\mathcal{C}}(c, -))$$

[23] The 1-categorical version of this notion (in the case $i = 1$ and $j = n$, called there an "adjunction of n variables") is defined in [CGR14, Definition 2.1] (see [CGR14, Theorem 2.2]).

[24] Thus, by convention, an adjunction in a single variable in just a functor to $\mathcal{S}$, and an adjunction in zero variables is just an object of $\mathcal{S}$.

in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$. (We may sometimes write this as $Y \odot c \in \mathcal{C}$ for notational convenience.) Dually, a *cotensor* of Y with c is an object $Y \pitchfork c \in \mathcal{C}$ equipped with an equivalence

$$\hom_{\mathcal{C}}(-, Y \pitchfork c) \simeq \hom_{\mathcal{S}}(Y, \hom_{\mathcal{C}}(-, c))$$

in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$. (As this is a sort of generalized mapping object, we will never reverse the order of the objects.) We will say that $\mathcal{C}$ is *tensored* (over $\mathcal{S}$) if it admits all tensors, and that it is *cotensored* (over $\mathcal{S}$) if it admits all cotensors. We will say that $\mathcal{C}$ is *bitensored* if it is both tensored and cotensored.

If we denote by $(\mathcal{S} \times \mathcal{C})^{\odot} \subset \mathcal{S} \times \mathcal{C}$ the full subcategory on those pairs admitting a tensoring, then by definition we can construct the tensoring as a bifunctor via the factorization

$$\begin{array}{ccc} (\mathcal{S} \times \mathcal{C})^{\odot} & \xrightarrow{\hom_{\mathcal{S}}(-, \hom_{\mathcal{C}}(-,=))} & \mathcal{P}(\mathcal{C}^{op}) \\ & \underset{-\odot-}{\searrow} & \uparrow \wr \\ & & \mathcal{C} \end{array}$$

through the fully faithful Yoneda embedding, and we can similarly construct the (maximal) cotensoring as a bifunctor

$$(\mathcal{S}^{op} \times \mathcal{C})^{\pitchfork} \xrightarrow{-\pitchfork-} \mathcal{C}.$$

Using the same argument, if $\mathcal{C}$ is tensored (resp. cotensored), it is not hard to extend the tensoring (resp. cotensoring) bifunctor to an action of the symmetric monoidal ∞-category $(\mathcal{S}, \times) \in \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty})$ (resp. $(\mathcal{S}^{op}, \times) \in \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty})$) on the ∞-category $\mathcal{C} \in \mathcal{C}\mathrm{at}_{\infty}$.[25] If $\mathcal{C}$ is bitensored, then we obtain a two-variable adjunction

$$\left(\mathcal{C} \times \mathcal{S} \xrightarrow{-\odot-} \mathcal{C} \ , \ \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\hom_{\mathcal{C}}(-,-)} \mathcal{S} \ , \ \mathcal{S}^{op} \times \mathcal{C} \xrightarrow{-\pitchfork-} \mathcal{C} \right).$$

By Corollary T.4.4.4.9, considering $Y \in \mathcal{S} \subset \mathcal{C}\mathrm{at}_{\infty}$, we have an equivalence

$$c \odot Y \simeq \mathrm{colim}^{\mathcal{C}}_{Y} \mathrm{const}(c)$$

in $\mathcal{C}$ (assuming either side exists). Thus, a tensoring is a sort of colimit, and hence a cocomplete ∞-category is in particular tensored. Dually, a cotensoring

[25] If $\mathcal{C}$ is additionally presentable (and hence in particular cocomplete), we can alternatively recover the tensoring action from the symmetric monoidal structure on the ∞-category of presentable ∞-categories, for which $\mathcal{S}$ is the unit object (see Proposition A.4.8.1.14 and Example A.4.8.1.19).

is a sort of limit, and hence a complete ∞-category is in particular cotensored. Similarly, an ∞-category which is finitely co/complete is in particular co\tensored over $\mathcal{S}^{\mathrm{fin}} \subset \mathcal{S}$.

(28) More generally, suppose that $(\mathcal{V}, \otimes) \in \mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)$ is a closed monoidal ∞-category, and suppose that $\mathcal{C} \in \mathrm{RMod}_{\mathcal{V}}(\mathcal{C}\mathrm{at}_\infty)$ is a right $\mathcal{V}$-module.[26] Writing

$$\mathcal{C} \times \mathcal{V} \xrightarrow{-\odot-} \mathcal{C}$$

for the underlying bifunctor of the right action of $\mathcal{V}$ on $\mathcal{C}$, let us suppose further that this extends to a two-variable adjunction. Such an extension is precisely the data of an *enrichment* and *bitensoring* of $\mathcal{C}$ over $\mathcal{V}$: the action defines the tensoring, and we write

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} \mathcal{V}$$

and

$$\mathcal{V}^{op} \times \mathcal{C} \xrightarrow{-\pitchfork-} \mathcal{C}$$

for the other two constituent bifunctors; as the notation indicates, these come with natural equivalences

$$\hom_{\mathcal{C}}(c \odot v, d) \simeq \hom_{\mathcal{V}}(v, \underline{\hom}_{\mathcal{C}}(c, d)) \simeq \hom_{\mathcal{C}}(c, v \pitchfork d)$$

in $\mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{V}^{op} \times \mathcal{C}, \mathcal{S})$ (for $c, d \in \mathcal{C}$ and $v \in \mathcal{V}$).[27]

To see that this gives an enrichment of $\mathcal{C}$, observe first that we have equivalences

$$\hom_{\mathcal{V}}(\mathbf{1}_{\mathcal{V}}, \underline{\hom}_{\mathcal{C}}(c, d)) \simeq \hom_{\mathcal{C}}(c \odot \mathbf{1}_{\mathcal{V}}, d) \simeq \hom_{\mathcal{C}}(c, d)$$

by the unitality of the action of $\mathcal{V}$ on $\mathcal{C}$. The enriched composition maps are obtained from the evaluation maps

$$\hom_{\mathcal{C}}(c \odot \underline{\hom}_{\mathcal{C}}(c, d), d) \simeq \hom_{\mathcal{V}}(\underline{\hom}_{\mathcal{C}}(c, d), \underline{\hom}_{\mathcal{C}}(c, d)) \xleftarrow{\mathrm{id}_{\underline{\hom}_{\mathcal{C}}(c,d)}} \mathrm{pt}_{\mathcal{S}}$$

as the composites

$$\hom_{\mathcal{V}}(\underline{\hom}_{\mathcal{C}}(c_0, c_1) \otimes \underline{\hom}_{\mathcal{C}}(c_1, c_2), \underline{\hom}_{\mathcal{C}}(c_0, c_2))$$

[26] For us, a monoidal ∞-category being *closed* by definition means that it is both left closed and right closed. Note that this is actually just a property, not additional structure: left/right closure only demands the existence of certain adjoints.

[27] If we are simply given a two-variable adjunction of this signature *without* an extension of the first bifunctor to an action of $\mathcal{V}$ on $\mathcal{C}$, then there will not be any compatibility between the symmetric monoidal structure on $\mathcal{V}$ and the bifunctors comprising the two-variable adjunction.

$$\begin{aligned}
&\simeq \hom_{\mathcal{C}}(c_0 \odot (\underline{\hom}_{\mathcal{C}}(c_0, c_1) \otimes \underline{\hom}_{\mathcal{C}}(c_1, c_2)), c_2) \\
&\simeq \hom_{\mathcal{C}}((c_0 \odot \underline{\hom}_{\mathcal{C}}(c_0, c_1)) \odot \underline{\hom}_{\mathcal{C}}(c_1, c_2), c_2) \\
&\leftarrow \hom_{\mathcal{C}}(c_1 \odot \underline{\hom}_{\mathcal{C}}(c_1, c_2), c_2) \\
&\leftarrow \hom_{\mathcal{C}}(c_2, c_2) \\
&\xleftarrow{\mathrm{id}_{c_2}} \mathrm{pt}_{\mathcal{S}},
\end{aligned}$$

and the higher composition maps are obtained by essentially this same construction. It is not hard to see that applying the functor $\hom_{\mathcal{V}}(\mathbf{1}_{\mathcal{V}}, -) : \mathcal{V} \to \mathcal{S}$, which is canonically lax monoidal, recovers the original composition maps in $\mathcal{C}$.

Then, to see that these functors define *enriched* co/tensors, we check that for an arbitrary test object $w \in \mathcal{V}$,

$$\begin{aligned}
\hom_{\mathcal{V}}(w, \underline{\hom}_{\mathcal{C}}(c \odot v, d)) &\simeq \hom_{\mathcal{C}}((c \odot v) \odot w, d) \\
&\simeq \hom_{\mathcal{C}}(c \odot (v \otimes w), d) \\
&\simeq \hom_{\mathcal{V}}(v \otimes w, \underline{\hom}_{\mathcal{C}}(c, d)) \\
&\simeq \hom_{\mathcal{V}}(w, \underline{\hom}_{\mathcal{V}}(v, \underline{\hom}_{\mathcal{C}}(c, d)))
\end{aligned}$$

and similarly

$$\hom_{\mathcal{V}}(w, \underline{\hom}_{\mathcal{C}}(c, v \pitchfork d)) \simeq \hom_{\mathcal{V}}(w, \underline{\hom}_{\mathcal{V}}(v, \underline{\hom}_{\mathcal{C}}(c, d)));$$

by Yoneda's lemma, we obtain the desired natural equivalences

$$\underline{\hom}_{\mathcal{C}}(c \odot v, d) \simeq \underline{\hom}_{\mathcal{V}}(v, \underline{\hom}_{\mathcal{C}}(c, d)) \simeq \underline{\hom}_{\mathcal{C}}(c, v \pitchfork d)$$

of enriched hom-objects in $\mathcal{V}$.[28]

Finally, we observe that the cotensoring bifunctor can be canonically extended to a *left* action of $(\mathcal{V}^{op}, \otimes^{op}) \in \mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)$ on $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ (i.e. we can consider $\mathcal{C} \in \mathrm{LMod}_{\mathcal{V}^{op}}(\mathcal{C}\mathrm{at}_\infty)$), simply by passing the tensoring action through the adjunction; for instance, for any $c, d \in \mathcal{C}$ and any $v, w \in \mathcal{V}$ we have a natural string of equivalences

$$\begin{aligned}
\hom_{\mathcal{C}}(c, w \pitchfork (v \pitchfork d)) &\simeq \hom_{\mathcal{C}}(c \odot w, v \pitchfork d) \simeq \hom_{\mathcal{C}}((c \odot w) \odot v, d) \\
&\simeq \hom_{\mathcal{C}}(c \odot (w \otimes v), d) \simeq \hom_{\mathcal{C}}(c, (w \otimes v) \pitchfork d),
\end{aligned}$$

[28] This is the only place we have used that $\mathcal{V}$ is closed; without this assumption, the given data still define a $\mathcal{V}$-enrichment of $\mathcal{C}$ along with *unenriched* co/tensors over $\mathcal{V}$ (in the evident sense).

which by Yoneda's lemma provides a canonical natural equivalence

$$w \pitchfork (v \pitchfork d) \simeq (w \otimes v) \pitchfork d$$

in $\mathrm{Fun}(\mathcal{V}^{op} \times \mathcal{V}^{op} \times \mathcal{C}, \mathcal{C})$.

Of course, this same discussion goes through without essential change in the special case that $\mathcal{V}$ is in fact *symmetric* monoidal.

(29) Given a finitely complete ∞-category $\mathcal{C}$, its corresponding *generalized matching object* bifunctor

$$(s\mathcal{S}^{\mathrm{fin}})^{op} \times s\mathcal{C} \xrightarrow{\mathrm{M}_{(-)}(-)} \mathcal{C}$$

is given by

$$\mathrm{M}_K(Y) = \int^{[n] \in \mathbf{\Delta}^{op}} K_n \pitchfork Y_n.$$

By construction, this comes equipped with an equivalence

$$\hom_{\mathcal{C}}(-, \mathrm{M}_K(Y)) \simeq \hom_{s\mathcal{S}}(K, \hom^{\mathrm{lw}}_{\mathcal{C}}(-, Y))$$

in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$, so that in particular when $\mathcal{C}$ is in fact bicomplete we obtain a two-variable adjunction

$$\left(\mathcal{C} \times s\mathcal{S} \xrightarrow{(-\odot-)^{\mathrm{lw}}} s\mathcal{C} \ , \ \mathcal{C}^{op} \times s\mathcal{C} \xrightarrow{\hom^{\mathrm{lw}}_{\mathcal{C}}(-,-)} s\mathcal{S} \ , \ (s\mathcal{S})^{op} \times s\mathcal{C} \xrightarrow{\mathrm{M}_{(-)}(-)} \mathcal{C} \right).$$

Dually, given a finitely cocomplete ∞-category $\mathcal{C}$, its corresponding *generalized latching object* bifunctor

$$s\mathcal{S}^{\mathrm{fin}} \times c\mathcal{C} \xrightarrow{\mathrm{L}_{(-)}(-)} \mathcal{C}$$

is given by

$$\mathrm{L}_K(Z) = \int_{[n] \in \mathbf{\Delta}^{op}} Z^n \odot K_n.$$

By construction, this comes equipped with an equivalence

$$\hom_{\mathcal{C}}(\mathrm{L}_K(Z), -) \simeq \hom_{s\mathcal{S}}(K, \hom^{\mathrm{lw}}_{\mathcal{C}}(Z, -))$$

in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$, so that in particular when $\mathcal{C}$ is in fact bicomplete we obtain a two-variable adjunction

$$\left(s\mathcal{S} \times c\mathcal{C} \xrightarrow{\mathrm{L}_{(-)}(-)} \mathcal{C} \ , \ (s\mathcal{S})^{op} \times \mathcal{C} \xrightarrow{(-\pitchfork-)^{\mathrm{lw}}} c\mathcal{C} \ , \ (c\mathcal{C})^{op} \times \mathcal{C} \xrightarrow{\hom^{\mathrm{lw}}_{\mathcal{C}}(-,-)} s\mathcal{S} \right).$$

These notions are extensions of the usual theory of matching and latching objects in Reedy categories, and correspondingly we make the abbreviations $\mathrm{M}_n(-) = \mathrm{M}_{\partial\Delta^n}(-)$ and $\mathrm{L}_n(-) = \mathrm{L}_{\partial\Delta^n}(-)$.

A.3 On model categories as presentations of ∞-categories

Note that we are considering model categories as objects of study in their own right: they are nothing more than model ∞-categories whose hom-spaces are discrete. However, we will also be using model categories as *presentations* of their underlying ∞-categories (as indicated in subitem (2)(d)). Thus, we must also establish our conventions regarding their manipulation in this capacity.

For historical context, we will make some attempt to reference the primary sources for results concerning model categories. However, the body of literature is vast, and so as catch-all resources we will generally refer to [Hir03] and [GJ99], especially for the more classical results.[29]

(30) As a consistency check, we observe that our consideration of objects of $\mathfrak{relcat}_{\mathrm{BK}}$ as presentations of ∞-categories does indeed identify a relative category $(\mathcal{R}, \mathbf{W}) \in \mathfrak{relcat}$ with its ∞-categorical localization $\mathcal{R}[\![\mathbf{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_\infty$ (as defined in item (20)). More precisely, the natural commutative diagram

$$\begin{array}{ccc} \mathbf{W} & \longrightarrow & \mathscr{L}^H_\delta(\mathbf{W}, \mathbf{W}) \\ \downarrow & & \downarrow \\ \mathcal{R} & \longrightarrow & \mathscr{L}^H_\delta(\mathcal{R}, \mathbf{W}) \end{array}$$

is a homotopy pushout square in $(\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ (and hence presents a pushout in $\mathcal{C}\mathrm{at}_\infty$ (see item (36))), and moreover the object $\mathscr{L}^H_\delta(\mathbf{W}, \mathbf{W}) \in (\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ presents the groupoid completion $\mathbf{W}^{\mathrm{gpd}} \in \mathcal{C}\mathrm{at}_\infty$.[30,31] More succinctly, we can

[29] The former requires that its model categories have functorial factorizations, whereas we do not. We will never use general results on model categories that depend on functorial factorizations.

[30] The proof of this assertion is mostly contained in [BK12a, 3.4]. However, note that there, they do not work in $(\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$, but rather work in the Dwyer–Kan model structure on "simplicial O-categories" (see [DK80c, Proposition 7.2], though note that their citation for this model structure should actually be to [Qui67, Chapter II, §4, Theorem 4]). However, the characterization [DK80c, Proposition 7.6] of the cofibrations implies that the forgetful functor to $(\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ preserves cofibrations, so that it also preserves homotopy pushouts (since it also preserves ordinary pushouts). Moreover, the assertion that $\mathscr{L}^H_\delta(\mathbf{W}, \mathbf{W}) \in (\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ presents $\mathbf{W}^{\mathrm{gpd}} \in \mathcal{C}\mathrm{at}_\infty$ follows from [DK80c, 5.5] and [DK80a, Proposition 2.2].

[31] As described in item (36), it is only known that homotopy co/limits in *combinatorial simplicial* model categories present co/limits in their underlying ∞-categories. However, even though $(\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ is not a combinatorial simplicial model category, it is easy enough to show that homotopy pushouts therein coincide up to a zigzag of natural weak equivalences with those com-

(apparently circularly but now in fact soundly) summarize this assertion by saying that the localization functor

$$\mathfrak{relcat} \to \mathfrak{relcat}[\mathbf{W}_{\mathrm{BK}}^{-1}] \simeq \mathcal{C}\mathrm{at}_\infty$$

is *itself* given by localization.

(31) In keeping with our general desire for our language to remain independent of any noncanonical choices, when we choose a *representative* in a model category of an object or a map in its underlying ∞-category, we will only mean a representative up to equivalence in the underlying ∞-category. When doing so, we indicate this noncanonical choice using "typewriter text", so that for instance, given an ∞-category $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, we might write $\mathtt{C} \in s\mathcal{S}\mathrm{et}^f_{\mathrm{Joyal}}$ to denote a quasicategory representing it.

(32) Given a *simplicial* model category $\mathcal{M}_\bullet$ (with underlying model category $\mathcal{M}$), another notion of "underlying ∞-category" is given by the full simplicial subcategory $\mathcal{M}_\bullet^{cf} \subset \mathcal{M}_\bullet$ on the bifibrant objects. By [DK80b, Proposition 4.8], this is weakly equivalent to $\mathscr{L}^H_\delta(\mathcal{M}, \mathbf{W})$ in $(\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$.[32,33] In making connections between model categories and ∞-categories, the results of [Lur09b] generally assume that the given model categories are simplicial. As a result, some of the connections that we make will carry this same caveat.

(33) As we have indicated in (1), the primary model category we will use to present the ∞-category $\mathcal{C}\mathrm{at}_\infty$ will be $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. Unfortunately, this does not enjoy all the nice properties that one might hope; in particular, it is not a simplicial model category. However, all is not lost: there exist both left and right Quillen equivalences to the combinatorial simplicial model category $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ given by [JT07, Theorems 4.11 and 4.12]. These allow us to port many convenient features of $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ over to $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ (such as in items (36) and (38) below).

(34) Suppose that $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen adjunction. Note that the functors F and G do not define functors of underlying relative categories: they do

puted in the combinatorial simplicial model category $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ (using the Quillen equivalence $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} \rightleftarrows (\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ and the functorial factorizations guaranteed by cofibrant generation).

[32]In the diagram in the statement of [DK80b, Proposition 4.8], the right arrow should also be labeled as a weak equivalence (in $(\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$), as indicated by its proof.

[33]This is also proved directly to present the ∞-categorical localization $\mathcal{M}^c[\![(\mathbf{W}^c)^{-1}]\!]$ as Theorem A.1.3.4.20 (and there is a canonical equivalence $\mathcal{M}^c[\![(\mathbf{W}^c)^{-1}]\!] \xrightarrow{\sim} \mathcal{M}[\![\mathbf{W}^{-1}]\!]$ e.g. by [MG16, Lemma 2.8]).

not generally preserve weak equivalences. Nevertheless, we prove as [MG16, Theorem 2.1] that a Quillen adjunction between model categories induces an associated adjunction of quasicategories.[34] By Kenny Brown's lemma (or rather its immediate consequence [Hir03, Corollary 7.7.2]), the composites

$$\mathcal{M}^c \hookrightarrow \mathcal{M} \xrightarrow{F} \mathcal{N}$$

and

$$\mathcal{M} \xleftarrow{G} \mathcal{N} \hookleftarrow \mathcal{N}^f$$

do preserve weak equivalences, and these respectively present the left and right adjoint functors. As a particular case, we immediately obtain that left Bousfield localizations present left localizations (and dually).

(35) If $\mathcal{M}$ is a model category and $x \xrightarrow{f} y$ is any map in $\mathcal{M}$, it is easy to check that the induced adjunction $\mathcal{M}_{x/} \rightleftarrows \mathcal{M}_{y/}$ is automatically a Quillen adjunction. If f is additionally a weak equivalence, we might hope that this is then a Quillen equivalence. For this to hold, however, we need for every pushout of f along a cofibration to be a weak equivalence. This will be true either

- if f is an *acyclic* cofibration, or
- if $\mathcal{M}$ is left proper.

This observation allows us to partially address the question of when the induced model structure on $\mathcal{M}_{x/}$ present the undercategory $\mathcal{M}[\![\mathbf{W}^{-1}]\!]_{x/}$ (or dually, when the induced model structure on $\mathcal{M}_{/y}$ presents the overcategory $\mathcal{M}[\![\mathbf{W}^{-1}]\!]_{/y}$): we only establish the connection for *simplicial* model categories, though this will suffice for our purposes. Namely, let $\mathcal{M}_\bullet$ be a simplicial model category.

- Suppose that $x \in \mathcal{M}^c$. If we choose any factorization $x \xrightarrow{\approx} x' \twoheadrightarrow \mathrm{pt}_{\mathcal{M}}$, then we obtain a Quillen equivalence $(\mathcal{M}_{x/})_\bullet \rightleftarrows (\mathcal{M}_{x'/})_\bullet$ with $x' \in \mathcal{M}^{cf}$. Since Quillen equivalences induce equivalences of underlying ∞-categories, the dual result to Lemma T.6.1.3.13 implies that $(\mathcal{M}_{x/})_\bullet$ (and hence also the underlying model category $\mathcal{M}_{x/}$) presents the undercategory of the object of the underlying ∞-category of $\mathcal{M}_\bullet$ corresponding to x.

- On the other hand, if $\mathcal{M}_\bullet$ is left proper, then this statement holds for any $x \in \mathcal{M}$. Indeed, if we choose any factorization $\varnothing_{\mathcal{M}} \rightarrowtail x'' \xrightarrow{\approx} x$, then we

[34] In the case of a Quillen adjunction of *simplicial* model categories, this result is proved as Proposition T.5.2.4.6.

obtain a Quillen equivalence $(\mathcal{M}_{x''/})_\bullet \rightleftarrows (\mathcal{M}_{x/})_\bullet$, which reduces us to the previous case.

(36) We will use the term *homotopy co/limit* in a model category $\mathcal{M}$ to refer to a (not necessarily commutative) diagram which becomes a (commutative) co/limit diagram in $\mathcal{M}[\mathbf{W}^{-1}]$.

If $\mathcal{M}_\bullet$ is a combinatorial simplicial model category, then it follows from Remark T.A.3.3.11, Proposition T.A.3.3.12, Remark T.A.3.3.13, and Theorem T.4.2.4.1 that homotopy co/limits in $\mathcal{M}_\bullet$ (in the classical sense) compute co/limits in its underlying ∞-category. (See those results for a precise statement.)

Homotopy co/limits are generally computed using *model structures on functor categories*, of which there are three main examples.[35]

- An *injective model structure* on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{inj}}$, has its weak equivalences and cofibrations determined objectwise. This is guaranteed to exist when $\mathcal{M}$ is combinatorial.
- A *projective model structure* on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{proj}}$, has its weak equivalences and fibrations determined objectwise. This is guaranteed to exist when $\mathcal{M}$ is cofibrantly generated.
- Given a category $\mathcal{C}$ endowed with a Reedy structure, the corresponding *Reedy model structure* on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{M})_{\mathrm{Reedy}}$, has its weak equivalences determined objectwise (but its cofibrations and fibrations depend on the Reedy structure), and exists without any additional assumptions on $\mathcal{M}$.[36]

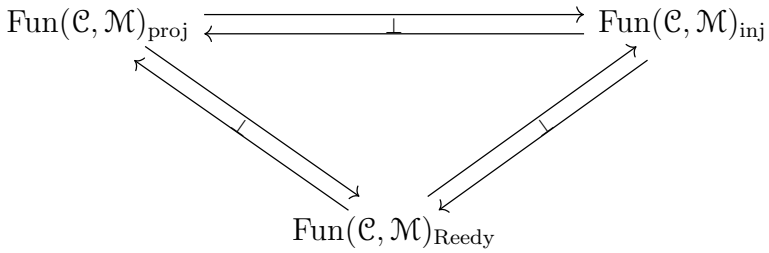
These enjoy the following properties.

- Whenever these various model structures exist, the identity adjunction

[35] For details on these, see respectively: §T.A.2.8; [Hir03, §11.6] and §T.A.2.8; [Hir03, Chapter 15] and §T.A.2.9.

[36] If the Reedy structure on $\mathcal{C}$ has $\overleftarrow{\mathcal{C}} = \mathcal{C}$, then the Reedy and injective model structures on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ coincide (and both always exist). Dually, if the Reedy structure on $\mathcal{C}$ has $\overrightarrow{\mathcal{C}} = \mathcal{C}$, then the Reedy and projective model structures on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ coincide (and both always exist). Thus, in general, the Reedy model structure can be seen as a "mixture" of the injective and projective model structures (see Example T.A.2.9.22).

gives rise to Quillen equivalences



between them.

- Applying $\mathrm{Fun}(\mathcal{C},-)$ to a Quillen adjunction (resp. Quillen equivalence) $\mathcal{M} \rightleftarrows \mathcal{N}$ gives rise to another Quillen adjunction (resp. Quillen equivalence) with respect to any of these model structures that exist on both $\mathrm{Fun}(\mathcal{C},\mathcal{M})$ and $\mathrm{Fun}(\mathcal{C},\mathcal{N})$.
- These various model structures participate in Quillen adjunctions as follows.
  - We have a Quillen adjunction

    $$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{C},\mathcal{M})_{\mathrm{inj}} : \lim$$

    whenever the injective model structure and the limit functor both exist.
  - We have a Quillen adjunction

    $$\mathrm{colim} : \mathrm{Fun}(\mathcal{C},\mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathcal{M} : \mathrm{const}$$

    whenever the projective model structure and the colimit functor both exist.
  - If $\mathcal{C}$ is endowed with a Reedy model structure with *cofibrant constants* then we are guaranteed a Quillen adjunction

    $$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{C},\mathcal{M})_{\mathrm{Reedy}} : \lim,$$

    while if $\mathcal{C}$ is endowed with a Reedy model structure with *fibrant constants* then we are guaranteed a Quillen adjunction

    $$\mathrm{colim} : \mathrm{Fun}(\mathcal{C},\mathcal{M})_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const}.$$

However, these adjunctions (if they exist) can still be Quillen adjunctions even without these restrictions on $\mathcal{C}$, albeit (necessarily by definition) only for specific choices of $\mathcal{M}$.

At least when $\mathcal{M}_\bullet$ is a combinatorial simplicial model category, any of these model structures on $\mathrm{Fun}(\mathcal{C},\mathcal{M})$ presents the ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{M}[\mathbf{W}^{-1}])$ by Proposition T.4.2.4.4 and Remark T.4.2.4.5.[37] As the functor $\mathrm{const} : \mathcal{M} \to \mathrm{Fun}(\mathcal{C},\mathcal{M})$ in $\mathfrak{Rel}\mathfrak{Cat}_{\mathrm{BK}}$ clearly presents the functor $\mathrm{const} : \mathcal{M}[\mathbf{W}^{-1}] \to \mathrm{Fun}(\mathcal{C},\mathcal{M}[\mathbf{W}^{-1}])$, combining item (34) with the uniqueness of adjoints shows that the derived functors of these various Quillen adjunctions do indeed compute homotopy co/limits. (In particular, as foreshadowed in item (33), from here it is straightforward to see that homotopy co/limits in the combinatorial model category $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ do indeed compute co/limits in $\mathcal{C}\mathrm{at}_\infty$.)

(37) As a particular case of item (36), there is a Reedy structure on the walking span category

$$\mathrm{N}^{-1}(\Lambda^2_0) = (\bullet \leftarrow \bullet \to \bullet)$$

determined by the degree function described by the picture $(0 \leftarrow 1 \to 2)$. This has fibrant constants (see e.g. the proof of [Hir03, Proposition 15.10.10]), so that for any model category $\mathcal{M}$ we obtain a Quillen adjunction

$$\mathrm{colim} : \mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_0),\mathcal{M})_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const}.$$

Moreover, the cofibrant objects of $\mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_0),\mathcal{M})_{\mathrm{Reedy}}$ are precisely the diagrams of the form

$$x \leftarrow y \rightarrowtail z$$

for $x, y, z \in \mathcal{M}^c \subset \mathcal{M}$.

Dually, there is a Reedy structure on the walking cospan category

$$\mathrm{N}^{-1}(\Lambda^2_2) = (\bullet \to \bullet \leftarrow \bullet)$$

determined by the degree function described by the picture $(0 \to 1 \leftarrow 2)$. This has cofibrant constants, so that for any model category $\mathcal{M}$ we obtain a Quillen adjunction

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_2),\mathcal{M})_{\mathrm{Reedy}} : \lim.$$

[37] Moreover, the results of [Dug01] and [RSS01] can sometimes be used to replace a model category (via a Quillen equivalence) with a combinatorial simplicial one.

Moreover, the fibrant objects of $\mathrm{Fun}(\mathrm{N}^{-1}(\Lambda^2_2), \mathcal{M})_{\mathrm{Reedy}}$ are precisely the diagrams of the form

$$x \to y \twoheadleftarrow z$$

for $x, y, z \in \mathcal{M}^f \subset \mathcal{M}$.

We will refer to either of these dual techniques simply as *the Reedy trick*.

(38) We will at times make computations in functor ∞-categories using model structures on functor categories; that these present the desired functor ∞-categories will always follow from the observations of items (36) and (33). We also recall here that the model structures $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ and $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{inj}}$ coincide by Example T.A.2.9.21. From this, it follows that the model structures $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ and $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{inj}}$ also coincide: they have the same weak equivalences by definition, and their cofibrations coincide since those of $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ coincide with those of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

A.4 Miscellanea

We end §A by laying out a few other miscellaneous conventions.

(39) Whenever we draw a diagram which takes place in a model (∞-)category, we explicitly mention the ambient model structure for emphasis. However, we will only decorate those aspects of the diagram (e.g. a morphism as a co/fibration) which are relevant to the argument.

(40) Given an ∞-category $\mathcal{C}$ and an object $c \in \mathcal{C}$, for emphasis we may denote the corresponding object by $c^\circ \in \mathcal{C}^{op}$: explicitly, if the object $c \in \mathcal{C}$ is selected by a morphism $[0] \xrightarrow{\chi} \mathcal{C}$ in $\mathcal{C}\mathrm{at}_\infty$, then the object $c^\circ \in \mathcal{C}^{op}$ is selected by the composite

$$[0] \xrightarrow{\cong} [0]^{op} \xrightarrow{\chi^{op}} \mathcal{C}^{op}$$

in $\mathcal{C}\mathrm{at}_\infty$. Similarly, if a morphism f in $\mathcal{C}$ is selected by a morphism $[1] \xrightarrow{\varphi} \mathcal{C}$ in $\mathcal{C}\mathrm{at}_\infty$, then we may denote by f° the morphism in $\mathcal{C}^{op}$ selected by the composite

$$[1] \xrightarrow{\cong} [1]^{op} \xrightarrow{\varphi^{op}} \mathcal{C}^{op}$$

in $\mathcal{C}\mathrm{at}_\infty$, where the isomorphism is determined by the assignments $0 \mapsto 1^\circ$ and $1 \mapsto 0^\circ$. On the other hand, we will sometimes omit these decorations in order not to overburden our notation.[38]

(41) Given a set I of homotopy classes of maps in an ∞-category $\mathcal{C}$, we write $\mathrm{llp}(I)$ and $\mathrm{rlp}(I)$ for the sets of (homotopy classes of) maps that have the left or right lifting property with respect to I, respectively. (A lifting property with respect to a subcategory by definition means a lifting property with respect to the homotopy classes of maps contained in that subcategory.) To be explicit, note that a commutative square in an ∞-category is presented by a map from $\Delta^1 \times \Delta^1$ to a quasicategory. To obtain a lift through that commutative square is then to obtain an extension over the map

$$\Delta^1 \times \Delta^1 \cong \Delta^{\{013\}} \coprod_{\Delta^{\{03\}}} \Delta^{\{023\}} \rightarrowtail \Delta^3$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. Alternatively (and invariantly), given a pair of maps $x \xrightarrow{i} y$ and $z \xrightarrow{p} w$, to say that $i \in \mathrm{llp}(\{p\})$ (or equivalently that $p \in \mathrm{rlp}(\{i\})$) is precisely to say that the induced map

$$\hom_{\mathcal{C}}(y,z) \to \lim \left(\begin{array}{ccc} & & \hom_{\mathcal{C}}(y,w) \\ & & \downarrow{\scriptstyle i^*} \\ \hom_{\mathcal{C}}(x,z) & \xrightarrow[p_*]{} & \hom_{\mathcal{C}}(x,w) \end{array} \right)$$

in $\mathcal{S}$ is a surjection.

(42) When working in the cosimplicial indexing category $\mathbf{\Delta}$, we will often indicate an inclusion simply by specifying its image, so that for instance the notation $[0] \xrightarrow{\{i\}} [n]$ refers to the map given by $0 \mapsto i$. (In particular, we will therefore denote by $\Delta^{\{i_0,\dots,i_j\}} \subset \Delta^n$ the evident subobject in $s\mathcal{S}\mathrm{et}$.) We will also employ the standard notations

[38] One might naively hope to simply write e.g. c^{op} for the object of $\mathcal{C}^{op}$ corresponding to the object $c \in \mathcal{C}$, but then one would run into trouble as soon as different "category levels" begin to mix: for example, the notation $[n]^{op}$ could then either refer to an object of $\mathcal{C}\mathrm{at}$ (which is in fact (equivalent to) the object $[n] \in \mathbf{\Delta} \subset \mathcal{C}\mathrm{at}$) or to an object of $\mathbf{\Delta}^{op}$. Thus, we reserve the superscript $(-)^{op}$ to denote the involution of $\mathcal{C}\mathrm{at}_\infty$. Note, however, that this does not just induce a covariant action on the objects and morphisms of $\mathcal{C}\mathrm{at}_\infty$, but also induces a contravariant action on its *2-morphisms*: for any $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ we have a canonical identification $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D}^{op}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{op}$, so that a pair of functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ and a natural transformation $\alpha : F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ corresponds to a pair of functors $F^{op}, G^{op} : \mathcal{C}^{op} \rightrightarrows \mathcal{D}^{op}$ and a natural transformation $\alpha^{op} : G^{op} \to F^{op}$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D}^{op})$.

- $\delta^i_m \in \hom_{\mathbf{\Delta}}([m-1],[m])$ for the *coface* maps (for $0 \leq i \leq m$), and
- $\sigma^j_n \in \hom_{\mathbf{\Delta}}([n+1],[n])$ for the *codegeneracy* maps (for $0 \leq j \leq n$),

or we may simply write δ^i or σ^j (resp.) if the source and/or target are clear from the context.

(43) Given any ∞-category $\mathcal{C}$, we write $c\mathcal{C} = \mathrm{Fun}(\mathbf{\Delta}, \mathcal{C})$ for the ∞-category of *cosimplicial objects* in $\mathcal{C}$, and we write $s\mathcal{C} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C})$ for the ∞-category of *simplicial objects* in $\mathcal{C}$. For any objects $Y \in c\mathcal{C}$ and $Z \in s\mathcal{C}$,

- we denote their constituent objects of $\mathcal{C}$ by $Y^n = Y([n])$ and $Z_n = Z([n]^\circ)$, and
- we variously denote their structure maps as follows:
  - a coface map $[m] \xrightarrow{\delta^i_m} [m+1]$ in $\mathbf{\Delta}$ induces
    - a coface map $Y^m \xrightarrow{\delta^i_m} Y^{m+1}$ and
    - a face map $Z_{m+1} \xrightarrow{\delta^m_i} Z_m$

    (or simply δ^i and δ_i, resp.);
  - a codegeneracy map $[n+1] \xrightarrow{\sigma^j_n} [n]$ induces
    - a codegeneracy map $Y^{n+1} \xrightarrow{\sigma^j_n} Y^n$ and
    - a degeneracy map $Z_n \xrightarrow{\sigma^n_j} Z_{n+1}$ (or simply σ^j and σ_j, resp.);
  - an arbitrary map $[m] \xrightarrow{\varphi} [n]$ (not explicitly identified as a coface or codegeneracy) induces
    - a map $Y^m \xrightarrow{\varphi} Y^n$ and
    - a map $Z_n \xrightarrow{\varphi} Z_m$

    (or $Y(\varphi)$ and $Z(\varphi)$ (or even $Z(\varphi^\circ)$), resp., if we wish to emphasize the functoriality of $Y : \mathbf{\Delta} \to \mathcal{C}$ or $Z : \mathbf{\Delta}^{op} \to \mathcal{C}$).

(44) There are certain decorations which are sometimes useful to include for emphasis or clarity but are at other times useful to exclude for simplicity. For instance, we may write $(-)^\bullet$ to emphasize that an object is cosimplicial, but we may omit this decoration if we are considering the entire cosimplicial object at once and have no plans to extract its constituents. We list these here.

decoration	**meaning**

$(-)^\bullet$	cosimplicial object
$(-)_\bullet$	simplicial object
$(-)^{\mathrm{lw}}$	functor being taken levelwise
$(-)^\circ$	corresponding object or morphism in the opposite ∞-category

(Given a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, we will sometimes (but not always) write $c\mathcal{C} \xrightarrow{F^{\mathrm{lw}}} c\mathcal{D}$ and $s\mathcal{C} \xrightarrow{F^{\mathrm{lw}}} s\mathcal{D}$ to denote the induced functors on ∞-categories of co/simplicial objects given by postcomposition with F (instead of cF or sF, resp.).)

(45) We will at times refer to various "named" results, both within this thesis and in external citations. For the reader's convenience, we will always refer to these both by name and by number. We take the conventions that

- if the name of the result includes the *type* of result (e.g. "theorem", "lemma", etc.) then we won't repeat it – so for instance we'll simply refer to "Kenny Brown's lemma (5.3.5)" –, whereas
- if the name of the result does not include its type, then we will include it – so for instance we'll refer to "the small object argument (Proposition 1.3.6)".

Appendix B

Index of notation

For the reader's convenience, in this appendix we provide an index of all (potentially not-completely-standard) mathematical symbols that we use throughout this thesis. We list them in alphabetical order (to the greatest extent possible) and indicate where they are defined or first appear. We generally list multi-use decorations as separate entries (e.g. the subscript indicating a "named" model ∞-category) so as to minimize repetition.

$|-|$: A(11), 4.1.15
$\|-\|$: A(11)
$\rightarrowtail$: 1.1.1
$\twoheadrightarrow$: 1.1.1
$\xrightarrow{\approx}$: 1.1.1
$\rightsquigarrow$: 3.2.1
$\overset{op}{\rightsquigarrow}$: 3.2.1
$(-)_*$: A(1)(b)
$(-)_{**}$: 4.2.1, 6.4.6
$(-)_{(**)}$: 4.2.3, 6.4.8
$\square$: 5.4.1
$\widehat{\boxtimes}$: 1.7.3
$\odot$: A(27), A(28), 4.2.4, 6.4.14
$\otimes$: A(26)
$\pitchfork$: A(27), A(28)
$(-)^\bullet$: A(44)
$(-)^{\times(\bullet+1)}$: 4.1.2
$(-)_\bullet$: A(44)
$\coprod$: A(12)
$(-)^\circ$: A(40)
$\diamond$: T.4.2.2.1
$\varnothing$: A(1)(d)
$(-)_\varnothing$: A(1)(b)
$=$: A(4)(b)
$(=)$: A(10)
$\cong$: A(4)(b)
$(-)^\simeq$: A(1)(d)
$\approx$: 1.1.1
$(-)_!$: A(13)
$(-)^\flat$: §T.3.1
$\int$: A(14)
$(-)[\![(-)^{-1}]\!]$: §1.0.1, 2.1.8
$(-)[(-)^{-1}]$: §1.0.1
$(-)^\natural$: T.3.1.1.8
$\prod$: A(12)
$(-)^\sharp$: §1.8, §T.3.1
$\dashv$: A(25)
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