

$An(\infty, 4)$ -CATEGORY: SALES PITCH FOR OPEN DIRECTIONS

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ABSTRACT. I gave the last lecture of my career in academic mathematics at the conference [\$\(\infty, n\)\$ -Categories and Their Applications](#) at Utrecht University on April 16, 2024. Therein, I offered some open directions for exploration stemming from the material in the talk itself, which I’ve also detailed here.

1. TITLE AND ABSTRACT

The talk was entitled $An(\infty, 4)$ -Category. Its abstract was as follows.

In this talk, I will introduce an $(\infty, 4)$ -category of which I’m both fond and proud, constructed in [recent joint work](#) with Leon Liu, David Reutter, Catharina Stroppel, and Paul Wedrich. It arises as a braided monoidal $(\infty, 2)$ -category, which can very roughly be thought of as that of “categorified representations of categorified quantum $\mathfrak{sl}_\infty$ ”. We intend for it and its cousins to play an analogous role for knot homologies (e.g. Khovanov homology) that ordinary braided monoidal categories of quantum group representations play for knot polynomials (e.g. the Jones polynomial), inspired by the longstanding open problem of generalizing knot homology theories from knots in $\mathbb{R}^3$ to knots in arbitrary 3-manifolds.

2. PITCH FOR HOMOTOPY THEORISTS

I began my lecture with a brief [farewell speech](#), and then introduced the mathematical content as follows.

As I emphasized with my title, this is a talk about an *application* of (∞, n) -category theory. We as a community are getting to a point where we have a decent handle on universal properties in (∞, n) -categories, and that’s wonderful. But not everything good in math has a universal property, and even if it does it might be unclear what that is – and that’ll be the case here.

I think it’s incumbent on us to get mathematicians in other fields excited about what we’re doing. But in order to do that, we need to find out what *they* are excited about, and do the work of showing how our toys can be applied to theirs.¹

I hope this talk is inspiring; there’s a lot more to do here. I’ll mention some open directions and inspiration at the end of the talk.

3. OPEN DIRECTIONS

3.1. Categorical Context for Triply-Graded Knot Homology. Our braided $(\infty, 2)$ -category (which is enriched in dimension 2 in the graded derived ∞ -category of a $\mathbb{Q}$ -algebra $\mathbb{k}$) only affords two of the three gradings in triply-graded homology. Of course, one should expect that the third grading is also visible at this level.

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¹This perspective is inspired – as am I – by Clark Barwick’s open letter [The Future of Homotopy Theory](#).

3.2. Tannakian Reconstruction of $\mathcal{U}_q(\mathfrak{sl}_\infty)$. We do not construct the categorified quantum group $\mathcal{U}_q(\mathfrak{sl}_\infty)$ as such, but only construct what should “morally” be considered as $\text{Rep}(\mathcal{U}_q(\mathfrak{sl}_\infty))$ – or at least the portion tensor-generated by the fundamental (categorified) representation. On the other hand, this should be sufficient for a sufficiently robust form of Tannakian reconstruction to recover the object $\mathcal{U}_q(\mathfrak{sl}_\infty)$ itself – which is likely something along the lines of “a categorified quantum group with a quasitriangular Hopf structure”. The extensive literature on categorified quantum groups strongly suggests that this would be an extremely interesting object in its own right.

3.3. Recovering $\text{Rep}(\mathcal{U}_q(\mathfrak{sl}_N))$ for $N < \infty$. For any integer $N \in \mathbb{N}$, we believe there should be an $\mathbb{E}_2$ -monoidal functor $\text{Rep}(\mathcal{U}_q(\mathfrak{sl}_\infty)) \rightarrow \text{Rep}(\mathcal{U}_q(\mathfrak{sl}_N))$ which witnesses the latter as an $\mathbb{E}_2$ -monoidal quotient of the former – probably simply that by (the “ideal” generated by) the 2-endomorphism $x^{N+1} \in \mathbb{k}[x]$ of the fundamental representation. In particular, we expect this to recover the spectral sequence from triply-graded homology to $\mathfrak{sl}_N$ -homology as that arising from interchanging the order of taking a colimit (i.e. factorization homology) and the passage to homology groups. Whereas triply-graded homology is not expected to extend to links in arbitrary 3-manifolds (corresponding to the infinite-dimensionality of $\mathbb{k}[x]$ as a $\mathbb{k}$ -module), this should enjoy the requisite dualizability properties to give a robust definition of $\mathfrak{sl}_N$ -homology for links in arbitrary 3-manifolds.

3.4. Lifting from $\mathbb{Q}$ to $\mathbb{S}$ or to MU . As indicated above, we work over a $\mathbb{Q}$ -algebra $\mathbb{k}$. This is in order to leverage standard results about chain complexes of Soergel bimodules. However, it seems likely that this requirement can be substantially relaxed, and indeed our early explorations suggested that it might be possible to lift this entire story to the sphere spectrum $\mathbb{S}$ using flag varieties and their Schubert stratifications. Leon (one of my collaborators on the original paper) accomplished such a lift to MU (the complex cobordism spectrum) in [this](#) subsequent paper, albeit using our same deformation-theoretic techniques rather than any stratified geometry. It seems very possible that MU (and not $\mathbb{S}$) is actually the universal base where such a lift exists, as noted there.

3.5. Deformation-Theoretic Context. We construct our $\mathbb{E}_2$ -monoidal $(\infty, 2)$ -category using a sort of “fiber functor”, which might be heuristically denoted “ $\mathbf{K}(\text{SBim}) \rightarrow \mathbf{D}(\text{SBim})$ ”. (As with more classical fiber functors, the target is $\mathbb{E}_\infty$ while the former is only $\mathbb{E}_2$.) This seems to be a sort of categorified analog of the specialization homomorphism $H_n \rightarrow \mathbb{k}[S_n]$ from the Hecke algebra to the group ring of the symmetric group, via which we consider the Hecke algebra as a quantum deformation of the group algebra. This suggests a general deformation-theoretic formalism, which should be closely connected to the Lurie–Dold–Kan correspondence as well.

4. FURTHER RESOURCES

You can find my lecture slides [here](#), and a video of a similar talk is available [here](#) (starting a minute or two in, due to a technical glitch).

Of course there is no substitute for reading the original [paper](#) – or at least the relevant aspects thereof.

This paper is the primary research product of my NSF grant, the proposal for which I am making freely available [here](#). That document contains many beautiful ideas that remain open for further exploration. (Aside from that of §3.4, the open directions described here are not contained therein.)

For a thorough introduction to the relevant representation theory, I highly recommend the book *Introduction to Soergel Bimodules* by Elias–Makisumi–Thiel–Williamson.