An introduction to spectra

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In this talk I'll introduce spectra and show how to reframe a good deal of classical algebraic topology in their language (homology and cohomology, long exact sequences, the integration pairing, cohomology operations, stable homotopy groups). I'll continue on to say a bit about extraordinary cohomology theories too. Once the right machinery is in place, constructing all sorts of products in (co)homology you may never have even known existed (cup product, cap product, cross product (?!), slant products (??!?)) is as easy as falling off a log!

0 Introduction

Here is a table of some homotopy groups of spheres $\pi_{n+k}(S^n)$:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$\pi_n(S^n)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\pi_{n+1}(S^n)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_{n+2}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_{n+3}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}\oplus\mathbb{Z}_{12}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$
$\pi_{n+4}(S^n)$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0
$\pi_{n+5}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
$\pi_{n+6}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_{n+7}(S^n)$	0	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{120}$	$\mathbb{Z}\oplus\mathbb{Z}_{120}$	$\mathbb{Z}_{240}$	$\mathbb{Z}_{240}$

There are many patterns here. The most important one for us is that the values $\pi_{n+k}(S^k)$ eventually stabilize. The stable value is called the n^{th} *stable homotopy group of spheres* and is denoted π_n^S.

This is true much more generally. Given any map $f : X \longrightarrow Y$ of based spaces[1], we can define its *suspension* $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ to be the obvious map which preserves the suspension coordinate and applies f on each copy of X. Then we have the following general theorem.

Theorem (Freudenthal suspension theorem)**.** *The sequence*

$$[X, Y] \xrightarrow{\Sigma} [\Sigma X, \Sigma Y] \xrightarrow{\Sigma} [\Sigma^2 X, \Sigma^2 Y] \xrightarrow{\Sigma} \cdots$$

always stabilizes.

In fact, given any two maps $f, g : \Sigma X \to Y$, we can add them by pinching ΣX in the middle to get the wedge sum $\Sigma X \vee \Sigma X$ and then applying f on the upper copy and g on the lower copy.[2] This ends up making $[\Sigma X, Y]$ into a group[3], and moreover $[\Sigma^{\geq 2} X, Y]$ will always be abelian.[4] The map $\Sigma : [\Sigma^k X, \Sigma^k Y] \longrightarrow [\Sigma^{k+1} X, \Sigma^{k+1} Y]$ is a homomorphism, and so the limit is an abelian group.

Our main goal is to find a general framework for talking about stable homotopy classes of maps, though the framework we find will be far richer than this modest goal would suggest.

[1] Everything for us will be based.
[2] In particular, this gives one way of defining addition in homotopy groups, since $S^k = \Sigma S^{k-1}$ for $k \geq 1$.
[3] Inverses are given by $(-f)([t, x]) = f([-t, x])$.
[4] These are true for the exact same reasons that π_1 is a group and $\pi_{\geq 2}$ is an abelian group.

1 The category of spectra

1.1 Definitions

Definition. A *CW-spectrum*[5] is a sequence of based CW-complexes $\{E_0, E_1, E_2, \dots\}$ (or sometimes $\{E_n\}_{n\in\mathbb{Z}}$) with *structure maps* $\Sigma E_n \longrightarrow E_{n+1}$.

For example, the *suspension spectrum* of a space X is denoted $\Sigma^\infty X$. This has $E_n = \Sigma^n X$, with structure maps the identity. Of particular importance is the *sphere spectrum* $\mathbb{S} = \Sigma^\infty S^0$.

We begin defining morphisms of spectra with the following definition, which ends up being too restrictive for reasons we will see in a moment.

Definition. A *function* $f : E \longrightarrow F$ of spectra of degree r is a collection of functions $f_n : E_n \longrightarrow F_{n-r}$ such that

$$\begin{tikzcd}
\Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d] & \Sigma F_{n-r} \arrow[d] \\
E_{n+1} \arrow[r, "f_{n+1}"] & F_{n-r+1}
\end{tikzcd}$$

commutes (on the nose) for all n.

Here is the problem. Consider the Hopf map $\eta : S^3 \longrightarrow S^2$. We'd like to have a corresponding function $\mathbb{S} \longrightarrow \mathbb{S}$, but η does not desuspend to a map $S^2 \longrightarrow S^1$ or a map $S^1 \longrightarrow S^0$. So we refine our definition.

Definition. Note that k-cells of E_n become $(k+1)$-cells of E_{n+1}. A *cofinal subspectrum* $E' \subseteq E$ is a subspectrum that eventually contains (the images of) all the cells of E. For our purposes a function from a subspectrum is as good as a function from the spectrum itself, because we only care about the stable result. Thus we define a *map* $\overline{f} : E \longrightarrow F$ of CW-spectra to be an equivalence class of functions from cofinal subspectra of E, where $f' : E' \longrightarrow F$ and $f'' : E'' \longrightarrow F$ are equivalent if they agree on $E' \cap E''$ (which is also a cofinal subspectrum of E).

This means that on any particular "stable cell" of E we can always wait until a later E_n to define the map, although of course for each stable cell we do eventually need to say where we want it to go. The slogan (à la Adams) is: "cells now, maps later".

We now make our final definition.

Definition. A *morphism* $\tilde{f} : E \longrightarrow F$ of spectra is a homotopy class of maps of spectra. (Homotopies are defined in the (semi-)obvious way.) We denote the morphisms from E to F of degree n by $[E, F]_n$. By $[E, F]$ we will mean the collection of all morphisms from E to F (of any degree).

And now we have that $\pi_n^S = [\mathbb{S}, \mathbb{S}]_n$, just as we might have hoped! More generally, $[\Sigma^\infty X, \Sigma^\infty Y]_0$ recovers the stable homotopy classes of maps $X \longrightarrow Y$ in a really clean way: a class is no longer represented by a bunch of different maps with different domains and ranges, but is now just a single morphism of spectra.

Let us agree that applying the suspension functor Σ to a spectrum just shifts its indices by 1 (i.e. $(\Sigma E)_n = E_{n+1}$). What's cool is that in our category of spectra we have a totally obvious definition of the inverse suspension function Σ^{-1}: it just shifts the indices the other way. Then, a morphism $E \longrightarrow F$ of degree r (for any $r \in \mathbb{Z}$) can equivalently be written as a morphism $\Sigma^r E \longrightarrow F$ of degree 0 (or a morphism $E \longrightarrow \Sigma^{-r} F$ of degree 0). In particular, the *(stable) homotopy groups* $\pi_*(E)$ of a spectrum E are given by $\pi_n(E) = [\mathbb{S}, E]_n$; for $E = \Sigma^\infty X$, this coincides with the usual definition of stable homotopy groups of X.

1.2 Basic application: (co)homology

Now that we have our category of spectra, let's look for some things to do with it. The obvious thing to do is to search for spectra that aren't just suspension spectra. Of course, it's easy to find a sequence of spaces, so the real task is to look for sequences of spaces that admit meaningful structure maps.

[5] We'll stick to CW-spectra since they're the easiest to work with – in fact, beyond this point we'll probably just say "spectrum" when we mean "CW-spectrum" – but the general case is not entirely different. When we say "space" we may often mean "CW-complex", too.

A very important class of spaces, Eilenberg-MacLane spaces $K(G,n)$, are related by

$$K(G,n) \xrightarrow{\cong} \Omega K(G,n+1).^6$$

These maps are adjoint[7] to maps $\Sigma K(G,n) \longrightarrow K(G,n+1)$ (though these are no longer homeomorphisms), and so the Eilenberg-MacLane spaces $K(G,n)$ fit together to form the *Eilenberg-MacLane spectrum* HG.

The most important property of these Eilenberg-MacLane spaces is that they represent cohomology: that is,

$$H^n(X;G) \cong [X_+, K(G,n)].^8$$

It's not too hard to see that we can also write this as $[\Sigma^\infty X_+, HG]_{-n} = [\Sigma^{\infty-n} X_+, HG]_0$. Despite the slight confusion with indices, we often just write $H^*(X;G) \cong [\Sigma^\infty X_+, HG]$.

As it turns out (and this is harder to see), we can write homology in terms of spectra too:

$$H_n(X;G) \cong [\mathbb{S}, HG \wedge X_+]_n = \pi_n(HG \wedge X_+).^9$$

We say that HG *corepresents* the homology functor. Similarly, we often just write $H_*(X;G) \cong \pi_*(HG \wedge X_+)$.

Note that $\pi_*(HG) = HG_*(\mathrm{pt}) = G$ is the coefficient group, concentrated in degree 0.

1.3 Digression on extraordinary theories

Recall that the Eilenberg-Steenrod axioms[10] completely characterize the singular (co)homology functors. In fact, given one of these functors we can determine its coefficient group by applying it to a point and looking in degree 0. However, once we relax the dimension axiom, we get what are known as *extraordinary (co)homology theories*. The following theorem gives us a lot of traction on understanding their general theory.

Theorem (Adams' Brown representability theorem)**.** *Every cohomology (resp. homology) theory is represented (resp. corepresented) by a spectrum.*

Of course, the way we do this is exactly the same as before. For any spectrum E, the E-cohomology of X is defined to be $E^*X = [\Sigma^\infty X_+, E]$ and the E-homology of X is defined to be $E_*X = \pi_*(E \wedge X_+)$. We call $E_*(\mathrm{pt}) = \pi_*(E) = [\mathbb{S}, E] = [\Sigma^\infty \mathrm{pt}_+, E] = E^*(\mathrm{pt})$ the *coefficient group* for the theory, just as before, which we often just denote by E_*. (Actually this will usually be a graded ring, but we'll get to that later.) In general (and indeed for any spectrum which doesn't represent some singular cohomology theory, by its uniqueness), this will not be concentrated in degree 0.

To give a sense of extraordinary theories, we will briefly mention two important ones below. We note here though that by what we have just said, stable homotopy groups form a homology theory which is corepresented by $\mathbb{S}$, and that similarly stable cohomotopy groups form a cohomology theory.

1.3.1 K-theory

One extraordinary cohomology theory is *K-theory*. Of course, when people write $K(X)$ they generally mean the group completion of the monoid $\mathrm{Vect}_*(X)$ of complex vector bundles over X (under Whitney sum). This is represented by $BU \times \mathbb{Z}$ (the second factor remembers the virtual dimension). But if instead we write this as $K^0(X)$, we can take a nod from the usual suspension isomorphism $H^n(X) \cong H^{n+1}(\Sigma X)$[11] and make the definition $K^{-1}(X) = K^0(\Sigma X)$. This means that $K^{-1}(X) \cong [\Sigma X, BU \times \mathbb{Z}] = [X, \Omega(BU \times \mathbb{Z})]$; that is, the functor K^{-1} is represented by $\Omega(BU \times \mathbb{Z})$. Then of course we define $K^{-2}(X) = K^{-1}(\Sigma X) = K^0(\Sigma^2 X)$, so K^{-2} is represented by $\Omega^2(BU \times \mathbb{Z})$. One might wonder whether this goes on forever. Luckily, we have an excellent theorem that dramatically simplifies the situation.

Theorem ((complex) Bott periodicity)**.** $\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}$.

In other words, the spectrum K is 2-periodic. In other words, complex K-theory is 2-periodic. And now that we have a K-theory spectrum, we can define K-homology as above, too.

[6] This can easily be seen from the long exact sequence in homotopy groups for the path fibration over $K(G,n+1)$.

[7] In general, there is an adjunction $Map(X, \Omega Y) \cong Map(\Sigma X, Y)$. If you think about this, it's pretty obvious. (Start with a map on the left side and try to construct one on the right.)

[8] Here X_+ is X with a disjoint basepoint; otherwise, if X is already based, then $[X, K(G,n)] \cong \tilde{H}^n(X;G)$.

[9] For two spaces (or a spectrum and a space), we define the *smash product* by $X \wedge Y = X \times Y / X \vee Y$. Note that $S^0 \wedge X = X$, $\Sigma X = S^1 \wedge X$, and $\mathbb{S} \wedge X = \Sigma^\infty X$; these will be important later.

[10] The axioms are: (i) homotopy (homotopic maps give the same induced map); (ii) excision; (iii) additivity; (iv) long exact sequence of a pair; and (v) dimension (the value on a point is concentrated in degree 0).

[11] Think about cellular cohomology.

1.3.2 Cobordism

Another extraordinary cohomology theory is *(real) cobordism*, whose spectrum is MO (the O is for "orthogonal", as in $O(n)$). This name comes from the fact that its coefficient ring $\pi_*(MO)$[12] is precisely the *(real) cobordism ring*.[13][14] The elements of this graded ring are cobordism classes of closed manifolds; addition is given by disjoint union, and multiplication is given by Cartesian product.

We construct the spectrum MO as follows. Over reasonable spaces, any real rank-n vector bundle is pulled back from the tautological bundle $\gamma_n \longrightarrow BO(n)$. We take $MO_n = T(\gamma_n)$, the Thom space of this bundle. To get structure maps, we make two observations. First, $\gamma_n \oplus \underline{\mathbb{R}} \longrightarrow BO(n)$ is a rank-$(n+1)$ vector bundle over $BO(n)$, and so there is some map $BO(n) \longrightarrow BO(n+1)$ for which it is given as a pullback in the diagram

$$\begin{tikzcd} \gamma_n \oplus \underline{\mathbb{R}} \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] \\ BO(n) \arrow[r] & BO(n+1). \end{tikzcd}$$

This gives us an induced map $T(\gamma \oplus \underline{\mathbb{R}}) \longrightarrow T(\gamma_{n+1})$ on Thom spaces, and our second observation is that

$$T(\gamma_n \oplus \underline{\mathbb{R}}) = T(\gamma_n) \wedge T(\underline{\mathbb{R}}) = T(\gamma_n) \wedge S^1 = \Sigma MO_n.$$

So the structure maps are just the induced maps on Thom spaces.

2 More structure

We have so far defined a spectrum. If we want to do much more, we will need more structure on our spectra. A *ring spectrum* (of which the prototypical example is HR for R a ring) is a spectrum E along with a "unit map" $\eta : \mathbb{S} \longrightarrow E$ and a "multiplication map" $\mu : E \wedge E \longrightarrow E$.[15] So from now on, we will assume we are working with a ring spectrum. We also Note here that $\mathbb{S}$ is a unit for the smash product.

2.1 Long exact sequences

The long exact sequence in cohomology is easy. Given a map of spaces $A \longrightarrow X$ (which up to homotopy is an inclusion, so we will consider it one), we can take its cofiber $A \longrightarrow X \longrightarrow X/A$.[16] When we iterate this process, (up to homotopy) we get $A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \cdots$. This gives us a cofiber sequence of suspension spectra $\Sigma^\infty A_+ \longrightarrow \Sigma^\infty X_+ \longrightarrow \Sigma^\infty (X/A)_+ \longrightarrow \Sigma^{\infty+1} A_+ \longrightarrow \cdots$, and applying the maps-to-a-spectrum functor $[-_+, E]_{-n}$ gives us a long exact sequence

$$[\Sigma^{\infty+1} A_+, E]_{-n} \longrightarrow [\Sigma^\infty (X/A)_+, E]_{-n} \longrightarrow [\Sigma^\infty X_+, E]_{-n} \longrightarrow [\Sigma^\infty A_+, E]_{-n} \longrightarrow \cdots,$$

which is just

$$E^n(\Sigma A) = E^{n-1} A \longrightarrow E^n(X/A) \longrightarrow E^n X \longrightarrow E^n A \longrightarrow \cdots.$$

[12] In fact, $\pi_*(MO) = \mathbb{Z}/2[\{x_n : n \neq 2^t - 1\}] = \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, x_9, \ldots]$. Note that this is in degee 3. Using geometric topology, one can show that any closed 3-manifold is the boundary of some 4-manifold. However, this method of constructing a *Thom spectrum* (after René Thom, who introduced them in his thesis) allows one to (theoretically) compute the entire cobordism ring in one shot!

[13] There are other types of cobordism, too, and these all have corresponding Thom spectra whose coefficients are the desired ring. (This is by something called the *Thom-Pontryagin construction*, which is pretty easy to understand. But actually computing the coefficients is much more difficult, and uses some pretty heavy machinery called the *Adams spectral sequence*.) Notable examples include the real oriented cobordism (MSO), complex cobordism (MU), symplectic cobordism (MSp), and spin cobordism ($MSpin$).

[14] Generalizing the fact that $\pi_*(MO)$ is the cobordism ring is the fact that MO-cocycles on X are geometrically realized by smooth families of manifolds parametrized by X. (For other Thom spectra, these manifolds have additional structure.)

[15] We have avoided talking about the smash product of spectra thus far because it is a very, very, very hairy issue. The naive definition ends up not having the properties we'd like (associativity, commutativity, etc.), and so people have gone to great lengths to amend the definitions, in many different directions and with varied results. For example, there are A_∞-, E_∞-, H_∞-, symmetric, orthogonal, and parametrized spectra. We'll just pretend that we do have a smash product on spectra and that it shares all the properties of the smash product on spaces.

[16] For the purposes of homotopy theory, X/A is the same thing as $X \cup_A CA$. It is easy to see that a map $X \longrightarrow Y$ induces a nullhomotopic composition $A \longrightarrow X \longrightarrow Y$ if and only if $X \longrightarrow Y$ extends over CA. So, this is exactly the sort of sequence of spaces that becomes exact if we apply the maps-to-a-space functor $[-, Y]$.

The long exact sequence in homology is also easy, but it uses an incredible fact about the category of spectra: a cofiber sequence is the same thing as a fibration sequence[17]! Thus the cofiber sequence of spaces $A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \cdots$ gives us a fibration sequence of spectra $E \wedge A_+ \longrightarrow E \wedge X_+ \longrightarrow E \wedge (X/A)_+ \longrightarrow E \wedge \Sigma A_+ \longrightarrow \cdots$, and applying the maps-from-a-spectrum functor $\pi_n = [\mathbb{S}, -]_n$ gives us a long exact sequence

$$\pi_n(E \wedge A_+) \longrightarrow \pi_n(E \wedge X_+) \longrightarrow \pi_n(E \wedge (X/A)_+) \longrightarrow \pi_n(E \wedge \Sigma A_+) \longrightarrow \cdots,$$

which is just

$$E_n A \longrightarrow E_n X \longrightarrow E_n(X/A) \longrightarrow E_n(\Sigma A) = E_{n-1} A \longrightarrow \cdots.$$

2.2 Integration pairing

Homology is supposed to be something like "chains", and cohomology is supposed to be something like "cochains". Thus, there should be a good notion of applying a cohomology class to a homology class. Indeed, this will be our first taste of the "falling off a long" mantra. For $f \in E^*X = [\Sigma^\infty X_+, E]$ and $\sigma \in E_*X = [\mathbb{S}, E \wedge X_+]$, we define

$$\langle f, \sigma \rangle : \mathbb{S} \xrightarrow{\sigma} E \wedge X_+ = E \wedge \mathbb{S} \wedge X_+ = E \wedge \Sigma^\infty X_+ \xrightarrow{1 \wedge f} E \wedge E \xrightarrow{\mu} E,$$

an element of the coefficient ring $E_* = \pi_*(E)$.[18]

2.3 Cohomology operations

An (unstable) cohomology operation is a natural transformation of functors $E^n(-) \longrightarrow F^m(-)$. Thus, it is classified by a map $E_n \longrightarrow F_m$, i.e. an element of $F^m(E_n)$. The first nontrivial examples people usually seen are the *Steenrod squares*, $Sq^i \in H^{n+i}(K(\mathbb{Z}/2, n); \mathbb{Z}/2) \cong [K(\mathbb{Z}/2, n), K(\mathbb{Z}/2, n+i)]$ for $n \geq i$. These acts on cohomology as follows. If $x \in H^n(X; \mathbb{Z}/2) \cong [X, K(\mathbb{Z}/2, n)]$, then $Sq^i x = Sq^i \circ x : X \longrightarrow K(\mathbb{Z}/2, n+i)$.

But these Steenrod squares have a special additional property: for all $n \geq i$ we can define this element/map Sq^i, and then

$$\begin{tikzcd}
K(G,n) \arrow[r, "Sq^i"] \arrow[d, "\cong"'] & K(G,n+i) \arrow[d, "\cong"] \\
\Omega K(G,n+1) \arrow[r, "\Omega Sq^i"] & \Omega K(G,n+i+1).
\end{tikzcd}$$

Through the adjunction, this implies that

$$\begin{tikzcd}
\Sigma K(G,n) \arrow[r, "\Sigma Sq^i"] \arrow[d] & \Sigma K(G,n+i) \arrow[d] \\
K(G,n+1) \arrow[r, "Sq^i"] & K(G,n+i+1).
\end{tikzcd}$$

In other words, these are *stable cohomology operations*: they live in $H\mathbb{Z}/2^* H\mathbb{Z}/2$.

In general, the algebra of stable cohomology operations for a cohomology theory E is given by E^*E. The action $E^*E \otimes_{E_*} E^*X \longrightarrow E^*X$ is defined by $\theta \cdot f = \theta \circ f : \Sigma^\infty X_+ \longrightarrow E$.[19]

2.4 Products

There are many products on (co)homology. In decreasing order of popularity we have the cup product, the cap product, the cross products, and the slant products. But what's great about spectra is that once you know what your product is supposed to do, there's really only one way to go about constructing it.

[17]Dually, a fibration sequence of spaces is exactly the sort of sequence of spaces that becomes exact if we apply the maps-from-a-space functor $[Y, -]$.

[18]Henceforth we will use this trick we have just employed implicitly, that $E \wedge X_+ = E \wedge \Sigma^\infty X_+$.

[19]For spectra satisfying an additional assumption there are also *(stable) homology cooperations*, the coalgebra of which is given by E_*E. But even when this exists it is harder to construct.

2.4.1 Cup product

The cup product is supposed to be $\smile: E^m X \otimes_{E_*} E^n X \longrightarrow E^{m+n} X$. This is implemented by

$$f \smile g : \Sigma^\infty X_+ \xrightarrow{\Delta} \Sigma^\infty X_+ \wedge \Sigma^\infty X_+ \xrightarrow{f \wedge g} E \wedge E \xrightarrow{\mu} E.$$

2.4.2 Cap product

The cap product is supposed to be $\frown: E_m X \otimes_{E_*} E^n X \longrightarrow E_{m-n} X$. This is implemented by

$$\sigma \frown f : \mathbb{S} \xrightarrow{\sigma} E \wedge X_+ \xrightarrow{1 \wedge \Delta} E \wedge X_+ \wedge X_+ \xrightarrow{1 \wedge f \wedge 1} E \wedge E \wedge X_+ \xrightarrow{\mu \wedge 1} E \wedge X_+.$$

2.4.3 Cross products

The cohomology cross product is supposed to be $\times : E^m X \otimes_{E_*} E^n Y \longrightarrow E^{m+n}(X \times Y)$. This is implemented by

$$f \times g : \Sigma^\infty (X \times Y)_+ = \Sigma^\infty (X_+ \wedge Y_+) = \Sigma^\infty X_+ \wedge \Sigma^\infty Y_+ \xrightarrow{f \times g} E \wedge E \xrightarrow{\mu} E.$$

The homology cross product is supposed to be $\times : E_m X \otimes_{E_*} E_n Y \longrightarrow E_{m+n}(X \times Y)$. This is implemented by

$$\sigma \times \rho : \mathbb{S} = \mathbb{S} \wedge \mathbb{S} \xrightarrow{\sigma \wedge \rho} (E \wedge X_+) \wedge (E \wedge Y_+) \xrightarrow{1 \wedge T \wedge 1} E \wedge E \wedge X_+ \wedge Y_+ \xrightarrow{\mu \wedge 1 \wedge 1} E \wedge X_+ \wedge Y_+ = E \wedge (X \times Y)_+.$$

2.4.4 Slant products

The slant products are supposed to be $/ : E^m(X \times Y) \otimes_{E_*} E_n Y \longrightarrow E^{m-n} X$ and $\backslash : E^m X \otimes_{E_*} E_n(X \times Y) \longrightarrow E_{n-m} Y$. These are implemented by

$$f/\sigma : \Sigma^\infty X_+ = \mathbb{S} \wedge X_+ \xrightarrow{\sigma \wedge 1} E \wedge Y_+ \wedge X_+ = E \wedge (X \times Y)_+ \xrightarrow{1 \wedge f} E \wedge E \xrightarrow{\mu} E$$
$$f\backslash\sigma : \mathbb{S} \xrightarrow{\sigma} E \wedge (X \times Y)_+ = E \wedge X_+ \wedge Y_+ \xrightarrow{1 \wedge f \wedge 1} E \wedge E \wedge Y_+ \overset{\mu \wedge 1}{E} \wedge Y_+.$$

3 Conclusion

Of course, this is just the beginning of the theory of spectra. One last surprise, though. Remember that $\mathbb{S}$ is the unit for smash product. Suppose $\psi \in [\mathbb{S}, \mathbb{S}] = \mathbb{S}^* \mathbb{S} = \mathbb{S}_* = \pi_*(\mathbb{S}) = \pi_*^S$ is a stable element of the homotopy groups of spheres. Then for any morphism of spectra $f : E \longrightarrow F$, we have an action

$$\psi \cdot f : E = \mathbb{S} \wedge E \xrightarrow{\psi \wedge f} \mathbb{S} \wedge F = F.$$

We may encapsulate this distinguished role that π_*^S plays by saying:

$$\boxed{\text{The stable homotopy groups of spheres act on absolutely everything in stable homotopy theory.}}$$

References: Adams, *Stable Homotopy and Generalised Homology.* Switzer, *Algebraic Topology – Homology and Homotopy.*