The chromatic tower
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Abstract
Much of chromatic homotopy theory organizes around the chromatic tower, a tower of certain Bousfield localizations of a given spectrum; the chromatic convergence theorem asserts that the limit of the tower recovers the original spectrum in many cases. After reviewing this background, we’ll discuss the proof of the chromatic convergence theorem and then examine the individual layers of the tower.

0 Notation
Everything is ∞-categorical, and S denotes the category of spectra.

1 Introduction
Recall that \( \langle E(n) \rangle = \langle K(0) \lor \cdots \lor K(n) \rangle \); this implies that there are natural transformations \( \text{id} \to \cdots \to L_2 \to L_1 \to L_0 \), and applying these to a space \( X \) to a space gives the chromatic tower for the space \( X \). As the following theorem makes precise, this tower encodes a good deal of information about \( X \).

Theorem 1 (Chromatic convergence). If \( X \) is a finite \( p \)-local spectrum, then \( X \xrightarrow{\sim} \lim L_n X \).

Remark 2. The chromatic convergence theorem closely mirrors a corresponding phenomenon in algebraic geometry, namely that we can understand a sheaf on a variety via its behavior on a nested sequence of open subvarieties. In this analogy, the variety corresponds to the moduli stack \( \mathcal{M}_{FG,p}^{\text{typ}} \) of \( p \)-typical formal group laws, and the open subvarieties correspond to the open substacks \( \mathcal{M}_{FG,p}^{<n} \) of formal groups with height less than \( n \).

Remark 3. It’s unambiguous to say “\( E \)-local finite spectrum” versus “finite \( E \)-local spectrum” (i.e. a complex built out of finitely many \( E \)-local spheres and disks) if and only if \( L_E \) is an arithmetic localization. Otherwise, \( L_E S \) has new self-maps, which already give rise to “exotic Moore spectra”.

Remark 4. The chromatic tower might be thought of as a “Postnikov-type tower”, only with respect to a much subtler filtration than dimension, namely the chromatic filtration. In fact, for all \( n < \infty \) the Bousfield class \( \langle K(n) \rangle \) is minimal. This implies that the chromatic tower is unrefineable: there is no localization functor that sits between \( L_n \) and \( L_{n-1} \).

It seems that we’ve all made our peace with the fact that stable homotopy theory is generally done “one prime at a time”. If we’re additionally willing to grant that rational homotopy theory is some sort of “0th degree approximation” to stable homotopy theory (in the sense that \( \pi_* S_0 \) is concentrated in dimension 0), then the chromatic tower gives a maximal factorization of the rationalization map, and the convergence theorem may be interpreted as saying that this assignment is faithful on finite \( p \)-local spectra.

In this talk, we’ll first discuss the proof of the chromatic convergence theorem and then explore in some depth the levels of the chromatic tower.

2 Proof of chromatic convergence

Reduction 1. The class \( \text{ChromConv} \) of finite \( p \)-local spectra for which the chromatic convergence theorem holds is thick. Thus, to prove the chromatic convergence theorem it suffices to prove that \( S_{(p)} \in \text{ChromConv} \).
Proof. Clearly ChromConv is closed under weak equivalences.

To see that ChromConv is closed under taking cofibers, suppose $X, Y \in \text{ChromConv}$ and suppose we have some $X \xrightarrow{f} Y$. Then we have

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\lim L_nX & \xrightarrow{\lim L_n f} & \lim L_nY
\end{array}
$$

cofib(f),

and the dashed arrow is an equivalence by comparing long exact sequences in homotopy. Moreover,

$$
\text{cofib}(\lim L_n f) \simeq \Sigma \text{fib}(\lim L_n f) \simeq \Sigma \lim \text{fib}(L_n f) \simeq \lim \text{cofib}(L_n f) \simeq \lim L_n \text{cofib}(f).
$$

To see that ChromConv is closed under retracts, suppose that $A \hookrightarrow X$ is the inclusion of a retract and suppose that $X \in \text{ChromConv}$. Since any functor preserves retracts, we have the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\sim} & X \\
\downarrow & & \downarrow \\
\lim L_n A & \xrightarrow{\sim} & \lim L_n X
\end{array}
$$

where both rows are retract diagrams. Choosing a homotopy inverse for the right vertical arrow and chasing the resulting diagram yields that $A \xrightarrow{\sim} \lim L_n A$. \hfill \Box

Remark 5. Actually, we don’t need to rely on the thick subcategory theorem to prove Reduction 1. The localizations $L_n$ are smashing (although all localizations are smashing on for finite spectra anyways), and moreover smashing with finite spectrum commutes with limits. Of course, this is just the essence of the argument used above to show that ChromConv is closed under cofibers, which proves Reduction 1 by itself anyways.

Now, we will actually rephrase the chromatic convergence theorem slightly. Let us define the functor $C_n = \text{fib}(\text{id} \to L_n)$, so that for any $X$ we have a commutative diagram of fiber sequences

$$
\begin{array}{ccccccc}
\lim C_nX & \to & \cdots & \to & C_2X & \to & C_1X & \to & C_0X \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
X & \to & \cdots & \to & X & \to & X & \to & X \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\lim L_n X & \to & \cdots & \to & L_2X & \to & L_1X & \to & L_0X
\end{array}
$$

Then we have the following reduction.

Reduction 2. To prove that $S(p) \in \text{ChromConv}$, it suffices to prove that for all $m \geq 0$, $\{\pi_m C_n S(p)\}_{n \geq 0} \to \{0\}_{n \geq 0}$ is a pro-isomorphism of towers of abelian groups.

Proof. This pro-isomorphism implies that $\lim C_n S(p)$ is weakly contractible, which implies that $S(p) \xrightarrow{\sim} \lim L_n S(p)$. \hfill \Box

Remark 6. Note that a tower may be pro-trivial even if it is nontrivial at all levels. For instance, to prove that the tower of groups $\cdots \to A_2 \to A_1 \to A_0$ is pro-trivial it suffices to prove that for any $n$, the map $A_{n+s} \to A_n$ is trivial for all sufficiently large $s$. This is what we will show to hold for the tower $\{\pi_m C_n S(p)\}_{n \geq 0}$.

We now make a further reduction. We take as a black box the following theorem, which marks the entrance of $MU$ into the story.

Theorem 7. The natural map $C_n S(p) \to C_{n-1} S(p)$ induces the zero map $MU_* C_n S(p) \to MU_* C_{n-1} S(p)$.
Suppose we are given the solid arrows in the diagram

Proof sketch. Writing $L$ for the Lazard ring, inductively define the $L$-modules $M_n$ by $M_{-1} = L(p)$ and $M_n = v_n^{-1}M_{n-1}/M_{n-1}$. One computes directly that there is a canonical isomorphism $MU_*C_nS(p) \cong M_n$, and that the induced map $MU_*C_nS(p) \to MU_*C_{n-1}S(p)$ is indeed zero.

\[\text{Remark 8.}\] In fact, this computation is roughly the starting point for the chromatic spectral sequence. We’ll discuss this later.

To show why this implies the pro-isomorphism statement of Reduction 2, we introduce the following terminology.

Definition 1. Write $\overline{MU} = \text{fib}(S \xrightarrow{\eta} MU)$. The natural map $\overline{MU} \xrightarrow{\zeta} S$ induces a sequence

$$\cdots \xrightarrow{\varepsilon \text{id}_{\overline{MU}} \wedge^3} \overline{MU} \wedge^3 \xrightarrow{\varepsilon \text{id}_{\overline{MU}} \wedge^2} \overline{MU} \wedge^2 \xrightarrow{\varepsilon \text{id}_{\overline{MU}}} \overline{MU} \xrightarrow{\zeta} S.$$ 

Then, we define the Adams–Novikov filtration on $\pi_*X$, denoted

$$\cdots \subseteq F_3\pi_*X \subseteq F_2\pi_*X \subseteq F_1\pi_*X \subseteq F_0\pi_*X = \pi_*X,$$

by defining $F_s\pi_*X$ to be the image of $\pi_*\overline{MU}^{\wedge s} \wedge X \xrightarrow{\varepsilon^{\wedge s}} \pi_*X$. (This is the filtration coming from the normalized $MU$-Adams resolution of $X$.)

Lemma 1. If $X \xrightarrow{\xi} Y$ induces the zero map $MU_*X \xrightarrow{MU_*\xi} MU_*Y$, then $\pi_*X \xrightarrow{\pi_*\xi} \pi_*Y$ raises Adams–Novikov filtration degree.

Proof. Suppose we are given the solid arrows in the diagram

\[
\begin{array}{ccc}
S^m \\
\downarrow \pi \\
X & \xleftarrow{f} & \overline{MU}^{\wedge s} \wedge X \\
\downarrow \pi_* \wedge Y & \xrightarrow{id} & MU \wedge \overline{MU}^{\wedge s} \wedge X \\
Y & \xleftarrow{f} & \overline{MU}^{(s+1)} \wedge Y \\
\end{array}
\]

We claim that $\pi_*(\text{id}_{MU \wedge \overline{MU}^{\wedge s} \wedge f})$ is the zero map, from which it follows that the dashed arrow exists. This is equivalent to the statement that the map $\overline{MU}^{\wedge s} \wedge X \xrightarrow{\text{id}_{MU} \wedge \pi_* \wedge f} \overline{MU}^{\wedge s} \wedge Y$ is $MU_*$-null. From the $MU_*$-long exact sequence for $\overline{MU} \to S \to MU$, we see that $MU_*\overline{MU}$ is a free $MU_*$-module, and so we get Künneth isomorphisms $MU_*(\overline{MU}^{\wedge s} \wedge X) \cong (MU_*\overline{MU})^{\otimes s} \otimes_{MU_*} MU_*X$ and $MU_*(\overline{MU}^{\wedge s} \wedge Y) \cong (MU_*\overline{MU})^{\otimes s} \otimes_{MU_*} MU_*Y$. Since we assumed that $MU_*f = 0$, the claim follows.

This allows us the following further reduction.

Reduction 3. To prove that $\{\pi_mC_nS(p)\}_{n \geq 0} \to \{0\}_{n \geq 0}$ is a pro-isomorphism, it suffices to prove that the Adams–Novikov filtration on each $\pi_mC_nS(p)$ is finite (i.e. there is some $s = s(m, n)$ such that $F_s\pi_mC_nS(p) = 0$).

Proof. By Theorem 7, the image of $\pi_*C_{n+s}S(p) \to \pi_*C_nS(p)$ lands in $F_s\pi_*C_nS(p)$. Hence, the only endomorphism of the pro-object $\{\pi_*C_nS(p)\}_{n \geq 0}$ is the zero map, which implies the claim.

To reduce this even further, we introduce a bit more terminology.

Definition 2. A map $X \xrightarrow{f} Y$ is called phantom below dimension $n$, or $n$-phantom, if for any finite spectrum $F$ with $\text{dim } F \leq n$, $[F, X] \xrightarrow{f} [F, Y]$ is zero. (So by definition, a phantom map is a map which is $n$-phantom for all $n$.)
Definition 3. $X$ is called $MU$-convergent if for every $n$ there is some $s = s(n)$ such that $\overline{MU}^s \wedge X \to X$ is $n$-phantom. (This implies that $\{\pi_s \overline{MU}^s \wedge X\}_{s \geq 0} \to \{\pi_s X\}_{s \geq 0}$ is a pro-isomorphism.) We will write $MU$-Conv for the class of $MU$-convergent spectra.

We can now give our final reduction.

Reduction 4. To prove that the Adams–Novikov filtration on each $\pi_n C_m S(p)$ is finite, it suffices to prove that if $X$ is connected (i.e. $\pi_i X = 0$ for $i < 0$), then $C_m X \in MU$-Conv for all $m$.

Proof. Of course $S(p)$ is connected, and so if $C_m S(p) \in MU$-Conv and we choose $s = s(n)$ as in the definition of $MU$-convergence then we obtain that $F_s \pi_n C_m S(p) = 0$.

Finally, we can prove the hypothesis of Reduction 4. It is an immediate consequence of the following three facts.

1. If $X$ is connective (i.e. $\pi_i X = 0$ for $i < 0$), then $X \in MU$-Conv.
2. $MU$-Conv is thick.
3. For any $X$, $L_n X \in MU$-Conv.

We prove these facts in turn.

Fact 1. If $X$ is connective, then $X \in MU$-Conv.

Proof. Suppose $X$ is connective, and choose any integer $n$. Since $MU$ is connected, it follows that $\overline{MU}^{\wedge(n+1)} \wedge X$ is $n$-connected. So if $F$ is finite and dim $F \leq n$, then $[F, \overline{MU}^{\wedge(n+1)} \wedge X] = 0$, so certainly $[F, \overline{MU}^{\wedge(n+1)} \wedge X] \to [F, X]$ is zero. Thus we can take $s = n + 1$.

To prove Fact 2, we need the following lemma.

Lemma 2. If $X \to Y$ is $n$-phantom and $W$ is connective, then $X \wedge W \to Y \wedge W$ is also $n$-phantom.

Proof. Suppose that $F$ is finite with dim $F \leq n$ and that we are given a map $F \to X \wedge W$. If we write $W$ as a filtered colimit of finite connective spectra $W_\alpha$, then there exists a factorization

$$
\begin{array}{c}
F \longrightarrow X \wedge W \\
\downarrow \quad \downarrow \\
X \wedge W_\alpha \longrightarrow Y \wedge W_\alpha.
\end{array}
$$

Thus, it suffices to show that $X \wedge W_\alpha \to Y \wedge W_\alpha$ is $n$-phantom; that is, we may additionally assume that $W$ is finite. Now, since $W$ is connective then dim $DW \leq 0$, so dim $DW \wedge F \leq n$. Thus in the commutative diagram

$$
\begin{array}{ccc}
[F, X \wedge W] & \cong & [DW \wedge F, X] \\
\downarrow & & \downarrow \\
[F, Y \wedge W] & \cong & [DW \wedge F, Y],
\end{array}
$$

the right map is zero since $X \to Y$ is $n$-phantom, so the left map is also zero.

Fact 2. $MU$-Conv is thick.

Proof. Clearly $MU$-Conv is closed under weak equivalences.

To see that $MU$-Conv is closed under cofibers, clearly it is closed under suspension, so it suffices to show that if $X \to Y \to Z$ is a cofiber sequence with $X, Z \in MU$-Conv then $Y \in MU$-Conv. So, fix any integer $n$, and choose $s$ such that $\overline{MU}^s \wedge X \to X$ and $\overline{MU}^s \wedge Z \to Z$ are both $n$-phantom. We will show that $\overline{MU}^{2s} \wedge Y \to Y$ is
n-phantom. First, by Lemma 2, since $\overline{MU}$ is connective then $\overline{MU}^\wedge 2s \wedge Z \to \overline{MU}^\wedge s \wedge Z$ is also n-phantom. Now for any finite $F$ with dim $F \leq n$, applying $[F,-]$ to the diagram

$$
\begin{array}{c}
\overline{MU}^\wedge 2s \wedge Y \\
\downarrow \\
\overline{MU}^\wedge s \wedge X \\
\downarrow \\
X
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
\overline{MU}^\wedge 2s \wedge Z \\
\overline{MU}^\wedge s \wedge Y \\
\overline{MU}^\wedge s \wedge Z \\
Y \\
\end{array}
$$

yields the diagram

$$
\begin{array}{c}
[F,\overline{MU}^\wedge 2s \wedge Y] \\
\downarrow \\
[F,\overline{MU}^\wedge s \wedge X] \\
\downarrow \\
[F,X]
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
[F,\overline{MU}^\wedge 2s \wedge Z] \\
[F,\overline{MU}^\wedge s \wedge Y] \\
[F,\overline{MU}^\wedge s \wedge Z] \\
[F,Y]
\end{array}
$$

The middle row is exact, and both the top right vertical arrow and the bottom left vertical arrow are zero by assumption; hence, the claim that the composition of the middle two vertical arrows is indeed zero follows from an easy diagram chase. So $Y \in MU\text{-Conv}$.

To see that $MU\text{-Conv}$ is closed under retracts, suppose that $A \hookrightarrow X$ is the inclusion of a retract and suppose that $X \in MU\text{-Conv}$. Given any $n$, choose $s$ such that $\overline{MU}^\wedge s \wedge X \to X$ is $n$-phantom. Then so is $\overline{MU}^\wedge s \wedge A \to A$, since if $F$ is finite with dim $F \leq n$ then we have the commutative diagram

$$
\begin{array}{c}
[F,\overline{MU}^\wedge s \wedge A] \\
\downarrow \\
[F,A]
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
[F,\overline{MU}^\wedge s \wedge X] \\
[F,X]
\end{array}
$$

and a retract of a zero map is zero. So $A \in MU\text{-Conv}$. \hfill \Box

**Fact 3.** For any $X$, $L_n X \in MU\text{-Conv}$.

**Proof.** By the smash product theorem, the natural map $\{L_n X\}_{m \geq 0} \to \{\text{Tot}^n E(n)^\wedge (\wedge 1) \wedge X\}_{m \geq 0}$ is an equivalence of towers. So there exists some $m$ such that $L_n X \hookrightarrow \text{Tot}^n E(n)^\wedge (\wedge 1) \wedge X$ is the inclusion of a retract. Since $MU\text{-Conv}$ is thick, it suffices to show that $\text{Tot}^n E(n)^\wedge (\wedge 1) \wedge X \in MU\text{-Conv}$. This is defined as a finite limit of $E(n)$-modules, hence of $MU$-modules, so again since $MU\text{-Conv}$ is thick it suffices to show that if $M$ is an $MU$-module, then $M \in MU\text{-Conv}$. In fact, we can see that $\overline{MU} \wedge M \to M$ is null via the diagram

$$
\begin{array}{c}
\overline{MU} \wedge M \\
\downarrow \\
M
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
M \\
M
\end{array}
$$

so in the definition of $MU$-convergent we can take $s(n) = 1$ (since null implies n-phantom for all $n$). \hfill \Box

3 The levels of the tower

There are a number of interrelated ways to try to understand the levels of the chromatic tower.
3.1 The chromatic fracture square

First, the map \( L_n X \to L_{n-1} X \) fits into a natural pullback square

\[
\begin{array}{ccc}
L_n X & \longrightarrow & L_{K(n)} X \\
\downarrow & & \downarrow \\
L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X,
\end{array}
\]

called the chromatic fracture square. This also mirrors a corresponding phenomenon in algebraic geometry, namely that there is a Mayer–Vietoris principle to reconstruct a sheaf from its restrictions to a formal neighborhood of a closed subvariety and its open complement along with the appropriate gluing data.

Remark 9. As a converse to the chromatic fracture square, it’s not hard to see that if \( Y \) is \( E(n-1) \)-local, \( Z \) is \( K(n) \)-local, and we are given a map \( Y \to L_{n-1} Z \), then \( X = \lim(Y \to L_{n-1} Z \leftarrow Z) \) is \( E(n) \)-local (and the pullback square is indeed the chromatic fracture square for \( X = L_n X \)). This implies that we can recover the category \( L_n \mathcal{S} \) via the pullback diagram

\[
\begin{array}{ccc}
L_n \mathcal{S} & \longrightarrow & L_{K(n)} \mathcal{S} \\
\downarrow & & \downarrow \\
L_{n-1} \eta_{L_{K(n)}} & \longrightarrow & L_{n-1} \mathcal{S},
\end{array}
\]

where \( \mathcal{A}rr \) denotes the arrow category. (We’re keeping track of the map \( Y \to L_{n-1} Z \), and the choice of localization map \( Z \to L_{n-1} Z \) is swept into the definition of a homotopy pullback.) One might say that this gives a “semi-orthogonal decomposition” \( L_n \mathcal{S} \simeq L_{n-1} \mathcal{S} \times L_{K(n)} \mathcal{S} \). All together, this gives us the diagram

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
L_{n-2} \mathcal{S}^{\text{fin}} & \longrightarrow & L_{K(n-1)} \mathcal{S}^{\text{fin}} \\
\downarrow & & \downarrow \\
L_{n-3} \mathcal{S}^{\text{fin}} & \longrightarrow & L_{K(n-2)} \mathcal{S}^{\text{fin}} \\
\downarrow & & \downarrow \\
\cdots & & \cdots
\end{array}
\]

fully faithful (and essentially surjective???)
in which the parallelograms are pullbacks. (It seems plausible that $S_{(p)}^{fin} \to \lim L_n S_{(p)}^{fin}$ might be an equivalence, but a priori there might be some infinite $p$-local spectrum whose $E(n)$-localizations are all equivalent to $E(n)$-localizations of various finite spectra.)

3.2 The monochromatic and $K(n)$-local categories

If we define $M_n = \text{fib}(L_n \to L_{n-1})$, then $M_n X$ is called the $n^{th}$ chromatic layer of $X$. Since $L_n L_{K(n)} \simeq L_{K(n)}$, the right vertical arrow in the chromatic fracture square is actually a map in the chromatic tower for $L_{K(n)}X$; the fact that it’s a pullback square implies that $M_n L_{K(n)} X \simeq M_n X$.

On the other hand, since $L_{K(n)} L_{n-1} \simeq pt$ and $L_{K(n)} L_n \simeq L_{K(n)}$, then the fiber sequence obtained by applying $L_{K(n)}$ to the fiber sequence $M_n X \to L_n X \to L_{n-1} X$ yields that $L_{K(n)} X \simeq L_{K(n)} M_n X$.

So, $M_n L_{K(n)} \simeq M_n$ and $L_{K(n)} M_n \simeq L_{K(n)}$. It is in this sense that monochromatic layers and $K(n)$-localizations determine each other; indeed, $L_{K(n)} : M_n S \to L_{K(n)} S : M_n$ defines an equivalence of categories. (Note that $M_n$ – while not a localization – is a smashing functor, since it’s defined as the fiber of two smashing functors. It follows that it is indeed an idempotent functor.) In fact, with a little fussing one can even check that this is a monoidal equivalence. [the “internal” homotopy groups are equivalent, but the internal ones in $L_{K(n)} S$ are the same as the external ones, while the internal ones in $M_n S$ are distinct.]

Remark 10. Recall that $\pi_* S$ contains periodic families of elements coming from $v_n$-self maps of finite spectra of type $n$. In fact, this same method generalizes very cleanly to any $\pi_* M_n X$.

Let us write $I = (i_0, \ldots, i_{n-1}) \in \mathbb{N}^n$. By the periodicity theorem, for some cofinal set of $I \in \mathbb{N}^n$ there exist “generalized Moore spectra” $M(I) = M(p^{i_0}, \ldots, v_n^{i_{n-1}})$ (with top cell in degree 0), defined inductively by the cofiber sequence

$$M(p^{i_0}, \ldots, v_n^{i_{n-1}}) \to M(p^{i_0}, \ldots, v_n^{i_{n-2}}) \xrightarrow{v_n^{i_{n-2}}} \Sigma^{-i_{n-1}} M(p^{i_0}, \ldots, v_n^{i_{n-2}}).$$

Then we can present the $n^{th}$ monochromatic layer of $X$ as $M_n X \simeq \text{colim}_I \in \mathbb{N}^n L_n X \wedge M(I)$.

Now, if we are given some $\alpha \in \pi_m M_n X$, then by the small object argument there exists a factorization

$$S^m \xrightarrow{\alpha} \xrightarrow{\alpha} \xrightarrow{\alpha} M_n X$$

for some $I = (i_0, \ldots, i_{n-1})$. Hence for any $v_n$-self map $\Sigma^{i_n} |v_n| M(I) \xrightarrow{v_n^{i_n}} M(I)$ we can define a family of elements $\alpha_{s |v_n|} \in \pi_{m + s |v_n| |v_n|} M_n X$ (beginning with $\alpha_0 = \alpha$), where $\alpha_s$ is given as the composite

$$S^{m + s |v_n| |v_n|} \xrightarrow{\alpha} \Sigma^{s |v_n| |v_n|} L_n X \wedge M(I) \xrightarrow{(v_n^{i_n})^{s |v_n|}} L_n X \wedge M(I) \xrightarrow{(v_n^{i_n})^{s |v_n|}} M_n X.$$

One can check that the maps $\Sigma^{s |v_n| |v_n|} L_n X \wedge M(I) \to L_n X \wedge M(I)$ are all equivalences (since $v_n^{i_n}$ is a $K(n)$-equivalence and the smash product of an $E(n)$-local spectrum with a finite spectrum of type $n$ is $K(n)$-local); this doesn’t imply that nontriviality of $\alpha$ guarantees nontriviality of the $\alpha_t$, but at least it’s some indication that this isn’t too naive of an operation. Note also that by the asymptotic uniqueness and centrality of $v_n$-self maps, the $\alpha_t$ are asymptotically independent of the choices of $I$ and $v_n^{i_n}$.

4 All the spectral sequences

There are a bunch of spectral sequences running around, all computing related things. In this section we attempt to organize them.

4.1 The chromatic spectral sequence

As we mentioned above, the computation that $MU_* C_n S_{(p)} \to MU_* C_n S_{(p)}$ is the zero map is roughly the starting point for the chromatic spectral sequence, aside from the small point that rather than apply $MU_*$ to $p$-local spectra we simply apply $BP_*$ instead.
To construct the chromatic spectral sequence, we will want to use the chromatic tower, but we must amend it slightly so that it gives a resolution of $S_p$ instead of $L_0S_p$. Actually, so that we don’t have to keep making exceptions at the low levels, we replace the bottom of the chromatic tower with

$$
\cdots \rightarrow M_2S_p \rightarrow L_2S_p \rightarrow M_1S_p \rightarrow L_1S_p \rightarrow M_0S_p \rightarrow S_p/p^\infty \rightarrow \Sigma S_p
$$

(Note that $M_0S_p \simeq L_0S_p \simeq S_\mathbb{Q} = p^{-1}S_p$.) For notational convenience, we denote the levels of this tower by $\mathcal{L}_nS_p$; that is, we set $\mathcal{L}_nS_p = L_nS_p$ for $n \geq 1$, and we then set $\mathcal{L}_0S_p = S_p/p^\infty$ and $\mathcal{L}_{-1}S_p = S_p$. We retain the $M_nS_p$, although note that $M_1S_p \rightarrow \mathcal{L}_1S_p \rightarrow \mathcal{L}_0S_p$ is not a fiber sequence.

We begin with the fact that the fiber sequence $S_p \rightarrow L_nS_p \rightarrow \Sigma \mathcal{L}_nS_p$ induces a short exact sequence in $BP_\ast$, which is split for $n \geq 1$. Thus in that case, $BP_\ast\mathcal{L}_nS_p \cong BP_\ast \oplus BP_\ast\mathcal{C}_nS_p$, and in fact the aforementioned computation can be used to show that for $n \geq 2$, the map $BP_\ast\mathcal{L}_nS_p \rightarrow BP_\ast\mathcal{L}_{n-1}S_p$ is an isomorphism on the first summands and zero on the second summands. At the bottom, we have that $BP_\ast\mathcal{L}_0S_p \cong BP_\ast/p^\infty$ and that $BP_\ast\mathcal{L}_{-1}S_p \cong \Sigma BP_\ast$, and that the maps $BP_\ast\mathcal{L}_1S_p \rightarrow BP_\ast\mathcal{L}_0S_p$ and $BP_\ast\mathcal{L}_0S_p \rightarrow BP_\ast\mathcal{L}_{-1}S_p$ are both zero.

Now, returning to the modified chromatic tower, this all implies that for $n \geq 2$ the fiber sequence $\Sigma^{-1}\mathcal{L}_{n-1}S_p \rightarrow M_nS_p \rightarrow L_nS_p$ induces a short exact sequence $0 \rightarrow \Sigma BP_\ast\mathcal{L}_{n-1}S_p \rightarrow BP_\ast M_nS_p \rightarrow BP_\ast\mathcal{L}_nS_p \rightarrow 0$ (where by an abuse of notation we write $BP_\ast\mathcal{L}_nS_p$ to mean $BP_\ast\mathcal{C}_nS_p$, despite the fact that the homology of a spectrum isn’t ever really unredced). However, letting $BP_\ast\mathcal{L}_0S_p = BP_\ast\mathcal{L}_0S_p$ and $BP_\ast\mathcal{L}_{-1}S_p = BP_\ast\mathcal{L}_{-1}S_p$ for notational convenience, we actually have this short exact sequence for all $n$.

Then, taking cohomology (i.e. Ext in $BP_\ast BP_\ast$-comodules) yields the exact couple

$$
\bigoplus_{i,j,n} H^{i,j}(BP_\ast\mathcal{L}_nS_p) \xrightarrow{(i,j,n) \mapsto (i+1,j+1,n-1)} \bigoplus_{i,j,n} H^{i,j}(\tilde{BP}_\ast\mathcal{L}_nS_p)) \xrightarrow{(i,j,n) \mapsto (i,j,n)} \bigoplus_{i,j,n} H^{i,j}(BP_\ast M_nS_p),
$$

and the associated spectral sequence – the chromatic spectral sequence – abuts to $H^{\ast,\ast}(\tilde{BP}_\ast\mathcal{L}_{-1}S_p)$, i.e. to the $E_2$-page of the Adams–Novikov spectral sequence for $\pi_\ast S_p$. (This exact couple gives the $E_1$-page of the chromatic spectral sequence, and so we have $d_r : E_r^{i,j,n} \rightarrow E_r^{i,j-r,n+r}$.)

### 4.2 Another route to $\pi_\ast S_p$

If we apply $\pi_\ast$ to the chromatic tower, we get a spectral sequence $\pi_\ast M_\ast S_p \Rightarrow \pi_\ast S_p$. This also might reasonably be called the “chromatic spectral sequence” – though it isn’t, so we’ll refer to the usual chromatic spectral sequence
as the *algebraic chromatic spectral sequence* and the present spectral sequence as the *topological spectral sequence* (even though these aren’t exactly analogs of each other in the algebraic and topological worlds or anything like that).

To get at the topological chromatic spectral sequence, we have Adams–Novikov spectral sequences \( H^{*,*}(BP, M_n S(p)) \Rightarrow \pi_\ast M_n S(p) \). Note that the input to these is the same as the input to the chromatic spectral sequence.

**Remark 11.** Let \( E_n \) denote the Morava \( E \)-theory spectrum obtained from the height-\( n \) Honda formal group over \( \mathbb{F}_p \), and let \( G_n = \text{Aut}(H_n) \times \text{Gal}(\mathbb{F}_p/\mathbb{F}_p) \) denote the \( n \)th extended Morava stabilizer group. By the Morava change-of-rings theorem, the Adams–Novikov spectral sequences for \( M_n S \) and \( L_{K(n)} S \) take the form

\[
H^*_{cts}(G_n; (E_n)_*)/(p^\infty, \ldots, u_{n-1}^\infty) \Rightarrow \pi_\ast M_n S \\
H^*_{cts}(G_n; (E_n)_*) \Rightarrow \pi_\ast L_{K(n)} S.
\]

In fact, there is a topological refinement of the second statement: \( G_n \) can be made to act by \( E_\infty \)-ring maps on \( E_n \), and then the unit map \( S \to E_n \) induces an equivalence \( L_{K(n)} S \cong E_n^{hG_n} \). That is, the spectrum \( L_{K(n)} S \) itself is the (hyper)cohomology of the group \( G_n \) with coefficients in the spectrum \( E_n \). (Given this, the above spectral sequence for \( \pi_\ast L_{K(n)} S \) can also be viewed as a descent spectral sequence.)

[mention Galois extensions?]