The geometry of the cyclotomic trace

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with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (arXiv:1710.06416)
2. *Factorization homology of enriched ∞-categories* (arXiv:1710.06414)
3. *The geometry of the cyclotomic trace* (arXiv:1710.06409)

§1: motivation and big picture

X a scheme (derived)

$\mathrm{K}(X)$ = ***algebraic K-theory*** of X

$:= \mathrm{K}(\mathrm{Perf}_X)$

$\approx$ group-completion of $(\mathrm{VBdl}(X)/\mathrm{iso.}, \oplus)$

[hard to compute!]

$\mathrm{THH}(X)$ = ***topological Hochschild homology*** of X

$:= \mathrm{THH}(\mathrm{Perf}_X) := \int_{S^1} \mathrm{Perf}_X$ $\quad \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}_{\circlearrowleft} \to \Lambda^{\mathrm{op}} \to \widetilde{\Lambda}^{\mathrm{op}}$

$\approx \mathscr{O}(LX)$, functions on the free loopspace of X

[easier to compute]

the ***Dennis trace*** map

$$\mathrm{K}(X) \longrightarrow \mathrm{THH}(X)$$

$$(E \downarrow X) \longmapsto \left((S^1 \xrightarrow{\gamma} X) \longmapsto \begin{array}{l} \text{trace of} \\ \text{monodromy} \\ \text{of } \gamma^* E \end{array} \right)$$

traces approximate *rationalized* *rationalized* algebraic K-theory:

$$\begin{array}{ccc}
\mathrm{K}(X) & \xrightarrow{\text{Dennis trace}} & \mathrm{THH}(X) \simeq \mathscr{O}(LX) \\
 & \overset{\text{cyclic trace}}{\searrow} \quad \nearrow & \\
 & \mathrm{THC}^{-}(X) := \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}} \simeq \mathscr{O}(LX)^{\mathrm{h}\mathbb{T}} &
\end{array}$$

$$E \longmapsto \left(\left(\begin{array}{c} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{array} \right) \longmapsto \left(\begin{array}{c} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{array} \right) \right)$$

this function is $\mathbb{T}$-invariant!

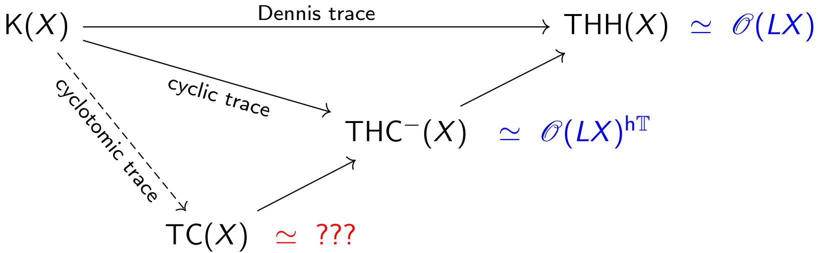
Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc}
\mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\
\downarrow & & \downarrow \\
\mathrm{THC}^{-}(R) & \longrightarrow & \mathrm{THC}^{-}(R_0)
\end{array}$$

pullback after $- \otimes \mathbb{Q}$.

slogan: vbdl/Spec(R) $\overset{\mathbb{Q}}{\approx}$ restriction to Spec(R_0)
+ compatible trace-of-monodromy function
+ *data* of $\mathbb{T}$-invariance of this function

the integral story...

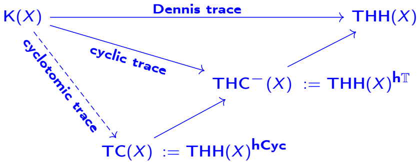


construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

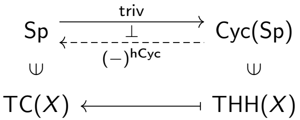
Thm (Dundas–McCarthy '97). the cyclotomic trace is a *local equivalence* (without rationalization).

"This is how people other than Quillen compute algebraic K-theory."
$\sim$ *A. Blumberg*, algebraic K-theorist

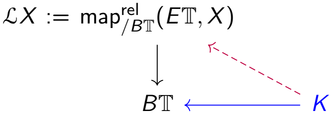
Main Question: In terms of DAG, what is $\mathrm{TC}(X)$**?**

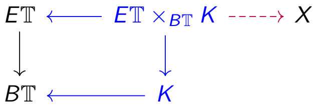


THH is a ***cyclotomic spectrum***



the $\mathbb{T}$**-action on** THH**, geometrically:** using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,





$\rightsquigarrow$ $\mathscr{O}_{\mathcal{L}X} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1, X)} = \mathscr{O}_{LX}$

$\rightsquigarrow$ $\mathrm{THC}^-(X) \simeq \Gamma(\mathscr{O}_{\mathcal{L}X})$, global sections of this sheaf over $\mathrm{B}\mathbb{T}$

Q.: analogous story for $\mathrm{TC}(X)$?

$$\begin{array}{ccc} \mathsf{Sp} & \underset{(-)^{\mathsf{hCyc}}}{\overset{\mathsf{triv}}{\rightleftarrows}} & \mathsf{Cyc(Sp)} \\ \Cup & & \Cup \\ \mathsf{TC}(X) & \longleftarrow & \mathsf{THH}(X) \end{array} \qquad \begin{array}{ccc} \mathcal{L}X := \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \quad \Bigg| \quad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$$\mathbf{THC}^{-}(X) \simeq \Gamma(\mathscr{O}_{\mathcal{L}X})$$

cyclotomic structure records simultaneous $\mathbb{T}$- and $\mathbb{N}^{\times}$-actions...

$\approx$ action of the *Witt monoid* $\mathbb{W} \simeq \mathbb{T} \rtimes \mathbb{N}^{\times}$

action $\mathbb{N}^{\times} \curvearrowright \mathbb{N}^{\mathrm{div}}$ by multiplication $\qquad n \mapsto (i \mapsto ni)$

a stack* $\mathscr{B}\mathbb{T}$, stratified over $\mathbb{N}^{\mathrm{div}}$ with fibers $\mathrm{B}\mathbb{T}$, compatible $\mathbb{N}^{\times}$-action by dilation

*not really ☹ $\qquad$ (to be explained...)

$$\begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^{\times}} \\ {\scriptstyle (-)^{\mathrm{hCyc}}} \searrow & & \swarrow {\scriptstyle \Gamma^{\mathrm{h}\mathbb{N}^{\times}}} \\ & \mathcal{S}\mathrm{p} & \end{array}$$

$\mathbb{N}^{\times}$-equivariant universal bundle $\mathscr{E}\mathbb{T} \downarrow \mathscr{B}\mathbb{T}$

$$\mathscr{L}X := \mathrm{map}^{\mathrm{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \xrightarrow{\mathbb{N}^{\times}\text{-eq}} \mathscr{B}\mathbb{T} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^{\times}} \\ \Cup & & \Cup \\ \mathrm{THH}(X) & \longleftrightarrow & \mathscr{O}_{\mathscr{L}X} \end{array}$$

$\rightsquigarrow \quad \mathrm{TC}(X) \simeq \Gamma^{\mathrm{h}\mathbb{N}^{\times}}(\mathscr{O}_{\mathscr{L}X}) := \{\mathbb{T}\text{- and } \mathbb{N}^{\times}\text{-equivariant functions on } \mathscr{L}X\}$

$\mathscr{L}X := \mathbf{map}^{\mathbf{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \longrightarrow \mathscr{B}\mathbb{T}$

$$\begin{array}{ccc} \mathsf{Cyc}(\mathcal{S}\mathsf{p}) & \simeq & \mathsf{QC}(\mathscr{B}\mathbb{T})^{\mathsf{h}\mathbb{N}^{\times}} \\ \Cup & & \Cup \\ \mathsf{THH}(X) & \longleftrightarrow & \mathscr{O}_{\mathscr{L}X} \end{array}$$

$\mathsf{TC}(X) \simeq \Gamma^{\mathsf{h}\mathbb{N}^{\times}}(\mathscr{O}_{\mathscr{L}X})$

questions:

1. what's up with the stack* $\mathscr{B}\mathbb{T}$?
2. what's the connection with the original definition of TC?
3. how & why does TC receive the cyclotomic trace map from K-theory?

questions 1 & 2: §2

question 3: §3

§2: stratified stacks and generalized recollements

recollement: decomposition/reconstruction of a stable ∞-category

main example: $Z \overset{\text{cls}}{\hookrightarrow} X \overset{\text{open}}{\hookleftarrow} U \quad \rightsquigarrow \quad \mathrm{QC}(X^\wedge_Z) \underset{\hat{\imath}_*}{\overset{\hat{\imath}_?}{\leftarrow \hat{\imath}^* -}} \mathrm{QC}(X) \underset{j^?}{\overset{j^*}{\leftarrow j_* -}} \mathrm{QC}(U)$

add pix here!!!

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ {\scriptstyle \eta_{\hat{\imath}}(F)}\downarrow & \text{pullback} & \downarrow {\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F)) \,=\, j_*(\mu)} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements: μ = ***monodromy*** = "gluing data"

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim{}^{\text{r.lax}} \left(\mathrm{QC}(X^\wedge_Z) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right) := \left\{ \left(F_0 \in \mathrm{QC}(X^\wedge_Z) \ , \ F_1 \in \mathrm{QC}(U) \ , \ \begin{array}{c} F_1 \\ {\scriptstyle \mu}\downarrow \\ j^* \hat{\imath}_* F_0 \end{array} \right) \right\} .$$

given $\mathcal{I} \xrightarrow{\mathcal{C}_\bullet} \mathbf{Cat}$, an obj of $\left\{ \begin{array}{l} \text{strict} \\ \text{left-lax} \\ \text{right-lax} \end{array} \right.$ limit is: $\{x_i \in \mathcal{C}_i\}_{i \in \mathcal{I}} \quad + \quad \forall \ i \xrightarrow{\varphi} j$ in $\mathcal{I}$, $\begin{cases} \mathcal{C}_\varphi(x_i) \simeq x_j \\ \mathcal{C}_\varphi(x_i) \rightarrow x_j \\ \mathcal{C}_\varphi(x_i) \leftarrow x_j \end{cases}$

$+$ higher coherence data

$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\mathrm{r.lax}}\left(\mathrm{QC}(X^{\wedge}_{Z}) \xrightarrow{j^*\hat{i}_*} \mathrm{QC}(U)\right)$

another example (Greenlees–May): recollement of *genuine* C_p*-spectra*: $\mathcal{S}\mathrm{p}^{\mathrm{gC}_p} \xrightarrow{\sim} \lim^{\mathrm{r.lax}}\left(\mathcal{S}\mathrm{p}^{\mathrm{hC}_p} \xrightarrow{(-)^{\mathrm{tC}_p}} \mathcal{S}\mathrm{p}\right)$

the ***Tate construction***: the cofiber of the *norm* map

$$\begin{array}{ccccc} E_{\mathrm{hC}_p} & \xrightarrow{\mathrm{Nm}} & E^{\mathrm{hC}_p} & \xrightarrow{\mathrm{cofib}} & E^{\mathrm{tC}_p} \\ \Uparrow & & \Uparrow & & \\ [x] & \longmapsto & \displaystyle\sum_{\sigma \in \mathrm{C}_p} \sigma x & & \end{array}$$

(arises as the composite $\mathcal{S}\mathrm{p}^{\mathrm{hC}_p} \xrightarrow{\mathrm{Borel}} \mathcal{S}\mathrm{p}^{\mathrm{gC}_p} \xrightarrow{\Phi^{\mathrm{C}_p}} \mathcal{S}\mathrm{p}$)

slogan: a stratified stack* $\mathcal{Z} \xhookrightarrow{\mathrm{cls}} \mathcal{X} := \mathcal{X}_{\mathrm{gC}_p}$ with

- $\mathcal{X}^{\wedge}_{\mathcal{Z}} \simeq \mathrm{BC}_p$,
- $\mathcal{X} \backslash \mathcal{Z} \simeq \mathrm{pt}$,
- $j^* \hat{i}_* \simeq (-)^{\mathrm{tC}_p}$.

[smallest example of ☹]

Defn (Glasman): a ***stratification*** of a stable ∞-category $\mathcal{C}$ over a poset P (subcategories $\mathcal{C}_p \subset \mathcal{C}$ for $p \in \mathrm{P}$, ...).

ordinary recollement when $\mathrm{P} = [1] = \{0 \to 1\}$

notation: for G a compact Lie group, write P_G for its poset of closed subgroups & subconjugacy

Thm (Glasman): for G finite, stratification of $\mathcal{S}\mathrm{p}^{\mathrm{g}G}$ over P_G.

recover Greenlees–May when $G = \mathsf{C}_p$ (note that $\mathrm{P}_{\mathsf{C}_p} = \{\{e\} \to \mathsf{C}_p\} \cong [1]$, so ordinary recollement)

Thm (A–M-G–R): general criteria for stratifications.

Cor: for G compact Lie, stratification of $\mathcal{S}\mathrm{p}^{\mathrm{g}G}$ over P_G.
uses *generalized* Tate $(-)^{\tau G}$: quotient by norms from *all* proper subgps.

Reconstruction thm (A–M-G–R): given a stratification of $\mathcal{C}$ over P, get left-lax diagram

$$\begin{array}{ccc} \mathrm{P} & \xrightarrow{\text{l.lax}} & \mathrm{Cat} \\ \Uparrow\!\!\!\!\cup & & \cup \\ p & \longmapsto & \mathcal{C}_p \end{array}$$

and equivalence $\mathcal{C} \xrightarrow{\sim} \lim^{\text{r.lax}}(\mathrm{P} \xrightarrow{\text{l.lax}} \mathrm{Cat})$ with right-lax limit.

reconstruction thm: $\mathcal{C} \xrightarrow{\sim} \lim^{\mathrm{r.lax}}(\mathrm{P} \xrightarrow{\mathrm{l.lax}} \mathrm{Cat})$

Ex.: $Z \underset{\text{closed}}{\hookrightarrow} Y \underset{\text{closed}}{\hookrightarrow} X \quad \rightsquigarrow \quad [2] \xrightarrow{\mathrm{l.lax}} \mathrm{Cat}$

Thm.

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X \backslash Z)^{\wedge}_{(Y \backslash Z)}) & \\ \alpha \nearrow & \Uparrow \eta & \searrow \beta \\ \mathrm{QC}(X^{\wedge}_{Z}) & \xrightarrow{\gamma} & \mathrm{QC}(X \backslash Y) \end{array} \right) .$$

$$\begin{array}{c} F_1 \\ \downarrow \mu_{01} \\ \alpha F_0 \end{array} \qquad F_0 \qquad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ \mu_{02} \downarrow & \circlearrowleft \ \mu_{012} & \downarrow \beta(\mu_{01}) \\ \gamma F_0 & \xrightarrow[\eta(F_0)]{} & \beta \alpha F_0 \end{array}$$

add pix here!!!

simpler: $\lim^{\mathrm{l.lax}} \left(\begin{array}{ccc} & \mathcal{C}_1 & \\ \alpha \nearrow & \Uparrow \eta & \searrow \beta \\ \mathcal{C}_0 & \xrightarrow{\gamma} & \mathcal{C}_2 \end{array} \right) \simeq \left\{ \left(\begin{array}{ccc} & \begin{array}{c} x_1 \\ \uparrow \mu_{01} \\ \alpha x_0 \end{array} & \\ x_0 & & x_2 \xleftarrow{\mu_{12}} \beta x_1 \end{array} \right) \right\}$ because now $\begin{array}{ccc} x_2 & \xleftarrow{\mu_{12}} & \beta x_1 \\ \exists! \ \mu_{02} \uparrow & & \uparrow \beta(\mu_{01}) \\ \gamma x_0 & \xrightarrow[\eta(x_0)]{} & \beta \alpha x_0 \end{array}$

$\star$ above, need μ_{012} because laxness of diagram $\neq$ laxness of limit *cyclotomic spectra* built from ***cyclonic spectra*** := genuine-proper $\mathbb{T}$-spectra := $\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}}$.

Thm (A–M-G–R).

$$\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}} \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\begin{array}{ccc} \mathbb{N}^{\mathrm{div}} & \xrightarrow{\mathrm{l.lax}} & \mathrm{Cat} \\ \Cup & & \Cup \\ i & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_i)} \xleftarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \\ {\scriptstyle j}\downarrow & \longmapsto & \downarrow {\scriptstyle (-)^{\tau C_j}} \\ ij & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_{ij})} \xrightarrow[\sim]{} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \end{array} \right) .$$

$\mathbb{N}^{\mathbf{div}} \cong$ poset of proper closed subgroups of $\mathbb{T}$

$i \leftrightarrow C_i$

geometric fixedpoints: $\Phi^{C_n} : \mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{\mathrm{g}<(\mathbb{T}/C_n)} \xrightarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}} \quad \rightsquigarrow \quad \mathbb{N}^{\times} \curvearrowright \mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}}$

$\leftrightarrow$ pullback of $\lim^{\mathrm{r.lax}}$ along action $\mathbb{N}^{\times} \curvearrowright \mathbb{N}^{\mathrm{div}}$ $\quad n \mapsto (i \mapsto ni)$

Defn (Blumberg–Mandell, Barwick–Glasman):

cyclotomic spectra := $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) := (\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}})^{\mathrm{h}\mathbb{N}^{\times}} \overset{\textit{Cor.}}{\simeq} \lim^{\mathrm{r.lax}}(\mathrm{B}\mathbb{N}^{\times} \xrightarrow{\mathrm{l.lax}} \mathrm{Cat})$.

note: $\mathrm{B}\mathbb{N}^{\times} \simeq (\mathbb{N}^{\mathbf{div}})_{\mathbf{h}\mathbb{N}^{\times}}$

i.e., cyclotomic spectra $\simeq$ right-lax limit of left-lax $\mathbb{N}^{\times}$-action on $\mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}}$.

Thm: $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \lim^{\mathbf{r.lax}}(\mathrm{B}\mathbb{N}^\times \xrightarrow{\ \mathbf{l.lax}\ } \mathrm{Cat}) =: \lim^{\mathbf{r.lax}} \left(\mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \overset{\mathbf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times \right)$.

⋆ an object of $\lim^{\text{r.lax}}$ is given by $T \in \mathrm{Sp}^{\mathrm{h}\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^\times$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^\times$, the *data* of a commutative square

Thm. [Nikolaus–Scholze]
for T connective and $r = s = p$ prime
Cor.: suff to specify just σ_p
(since $\sigma_{p^n} = (\sigma_p)^{\circ n}$, and n-cubes
canonically commute $\forall\ n \geq 2$)

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \underset{\text{can.}}{\xrightarrow{\sim}} & (T^{\tau C_s})^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^\times$, the *data* of a commutative n-cube...

thus, **slogan:** there's a stack* $\mathscr{B}\mathbb{T} := \mathfrak{X}_{\mathsf{g}<\mathbb{T}}$, $\mathbb{N}^\times$-equivariantly stratified over $\mathbb{N}^{\mathrm{div}} \cong \mathrm{P}_{<\mathbb{T}}$, such that $\mathcal{S}\mathrm{p}^{\mathsf{g}<\mathbb{T}} \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})$, $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times}$, and

$$\begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} \\ {\scriptstyle (-)^{\mathrm{hCyc}}}\searrow & & \swarrow{\scriptstyle \Gamma^{\mathrm{h}\mathbb{N}^\times}} \\ & \mathcal{S}\mathrm{p} & \end{array} .$$

§3: the geometry of the cyclotomic trace

$$\begin{array}{ccc}
\mathrm{K}(X) & \xrightarrow{\text{Dennis trace}} & \mathrm{THH}(X) \simeq \mathscr{O}(LX) \\
 & \searrow^{\text{cyclic trace}} & \nearrow \\
 & & \mathrm{THC}^{-}(X) \simeq \mathscr{O}(LX)^{\mathrm{h}\mathbb{T}} \\
 \text{cyclotomic trace} \searrow & & \nearrow \\
 & \mathrm{TC}(X) \simeq \text{???} &
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Sp} & \underset{(-)^{\mathrm{hCyc}}}{\overset{\text{triv}}{\rightleftarrows}} & \mathrm{Cyc}(\mathrm{Sp}) \\
\Psi & & \Psi \\
\mathrm{TC}(X) & \longleftarrow & \mathrm{THH}(X)
\end{array}$$

$$E \longmapsto \left(\left(\begin{array}{c}\text{free loop} \\ S^1 \xrightarrow{\gamma} X\end{array} \right) \longmapsto \left(\begin{array}{c}\text{trace of monodromy of} \\ \gamma^* E \downarrow S^1\end{array} \right) \right)$$

Thm: $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} \simeq \left\{ \left(\mathbb{T} \curvearrowright T \, , \left\{ \begin{array}{c} T \\ \downarrow \sigma_r \\ T^{\tau C_r} \end{array} \right\}_{r \in \mathbb{N}^\times} , \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ \sigma_{rs} \downarrow & & \downarrow (\sigma_s)^{\tau C_r} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s \in \mathbb{N}^\times} , \ \cdots \right) \right\}$.

right adjoint (limit-type construction): *imposing conditions*

main idea: $\mathrm{TC}(X) \; = \;$ functions on LX that are:

- invariant under $\mathbb{T}$-action on LX;
- "sensitive" to relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$.   **Q.** What does "sensitive" mean?

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \cdot \otimes m_j \quad , \quad \mathrm{tr}(M^{\otimes 2}) = \sum_k m_k \cdot \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is *norms*: image of $\sum_{i<j} [m_i \otimes m_j]$ under

$$\begin{array}{ccc} (R \otimes R)_{C_2} & \xrightarrow{\mathrm{Nm}} & (R \otimes R)^{C_2} \\ [x \otimes y] & \longmapsto & \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{array}$$

$\rightsquigarrow$ become equal in the *Tate construction*, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{\mathrm{t}C_2}$$

over $\mathbb{Q}$, norm an iso! $\rightsquigarrow$ $(R \otimes R)^{\mathrm{t}C_2} = 0$, assertion is vacuous

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

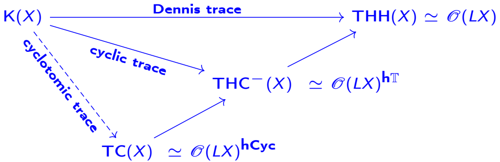
$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

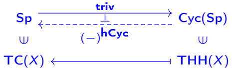
main point: for $\mathcal{C}$ a spectrally enriched ∞-category,
$S^1_b \xleftarrow{r} S^1_a$ a covering map of framed circles,
get ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

doesn't exist over $\mathbb{Z}$! only have "Tate diagonal" in $\mathcal{S}\mathrm{p}$, not $\mathscr{D}(\mathbb{Z})$.

the geometry of the cyclotomic trace:





$$E \longmapsto \left(\left(\begin{matrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{matrix} \right) \longmapsto \left(\begin{matrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{matrix} \right) \right)$$

Thm: $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} \simeq \left\{ \left(\mathbb{T} \curvearrowright T \, , \left\{ \begin{matrix} T \\ \downarrow \sigma_r \\ T^{\tau C_r} \end{matrix} \right\}_{r \in \mathbb{N}^\times} , \left\{ \begin{matrix} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ \sigma_{rs} \downarrow & & \downarrow (\sigma_s)^{\tau C_r} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{matrix} \right\}_{r,s \in \mathbb{N}^\times} , \ \cdots \right) \right\} .$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(LX)$ by selecting just those functions on LX such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible" (+ higher coherences)...

...which is precisely the structure present on trace-of-monodromy functions of vector bundles! $\mathrm{tr}(M)^r \equiv \mathrm{tr}(M^r)$, ...

Q.: Where does the cyclotomic structure on THH come from?

Thm (A–M-G–R).

1. ***diagonal package*** for spaces $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathcal{S}) & \dashrightarrow & \mathrm{Cyc}^{\mathsf{h}}(\mathcal{S}) \\ & {\scriptstyle \mathrm{THH}_{\mathcal{S}}} \searrow & \downarrow {\scriptstyle \mathrm{fgt}} \\ & & \mathcal{S} \end{array}$$

packages *diagonal maps* for spaces: $X \to (X^{\times r})^{\mathsf{h}\mathsf{C}_r}$

$\mathrm{Cyc}^{\mathsf{h}}(\mathcal{S}) := \mathrm{Fun}(\mathrm{B}\mathbb{W}, \mathcal{S}) :=$ "unstable cyclotomic spaces"

$B\mathbb{W} :=$ (objects: framed S^1's , morphisms: covering maps)

2. ***diagonal package*** for spaces

$\rightsquigarrow$ linearization

Tate package for spectra $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathrm{Sp}) & \dashrightarrow & \mathrm{Cyc}(\mathrm{Sp}) \\ & {\scriptstyle \mathrm{THH}} \searrow & \downarrow {\scriptstyle \mathrm{fgt}} \\ & & \mathrm{Sp} \end{array}$$

packages *Tate diagonal maps* for spectra: $E \to (E^{\otimes r})^{\tau \mathsf{C}_r}$

$\mathrm{Cyc}(\mathrm{Sp}) :=$ cyclotomic spectra

Q.: Where does the cyclotomic trace come from?

Thm (A–M-G–R).

the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

$$\max^{\ell} \text{ subgpd of } \mathcal{C} := \iota\mathcal{C} \simeq \int_{\mathbb{D}^0} \mathcal{C} \longrightarrow \left(\int_{S^1} \mathcal{C}\right)^{h\mathbb{W}} =: \mathsf{THH}_{\mathcal{S}}(\mathcal{C})^{h\mathbb{W}} =: \mathsf{TC}^{h}_{\mathcal{S}}(\mathcal{C})$$

input: the fiber bundle $S^1 \downarrow \mathbb{D}^0$ is invariant for the $\mathbb{W}$-action on S^1

$$\Big\downarrow \text{ linearization}$$

the ***cyclotomic trace***: for $\mathcal{C}$ a stable ∞-category,

$$\mathsf{K}(\mathcal{C}) \longrightarrow \mathsf{THH}(\mathcal{C})^{h\mathsf{Cyc}} =: \mathsf{TC}(\mathcal{C})$$

input: K-theory is the ***universal additive invariant*** of stable ∞-categories (Blumberg–Gepner–Tabuada)