The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (arXiv:1710.06416)
2. *Factorization homology of enriched ∞-categories* (arXiv:1710.06414)
3. *The geometry of the cyclotomic trace* (arXiv:1710.06409)

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the ***Dennis trace*** map

$$\mathrm{K}(X) \longrightarrow \mathrm{THH}(X)$$

$$(E \downarrow X) \longmapsto \left((S^1 \xrightarrow{\gamma} X) \longmapsto \begin{array}{l} \text{trace of} \\ \text{monodromy} \\ \text{of } \gamma^* E \end{array} \right)$$

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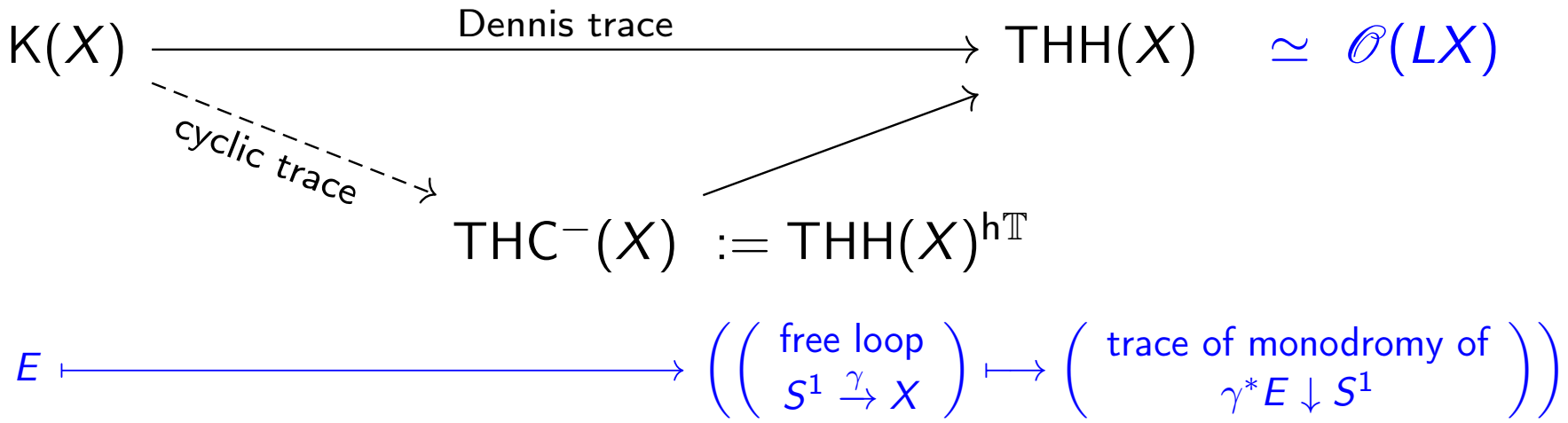

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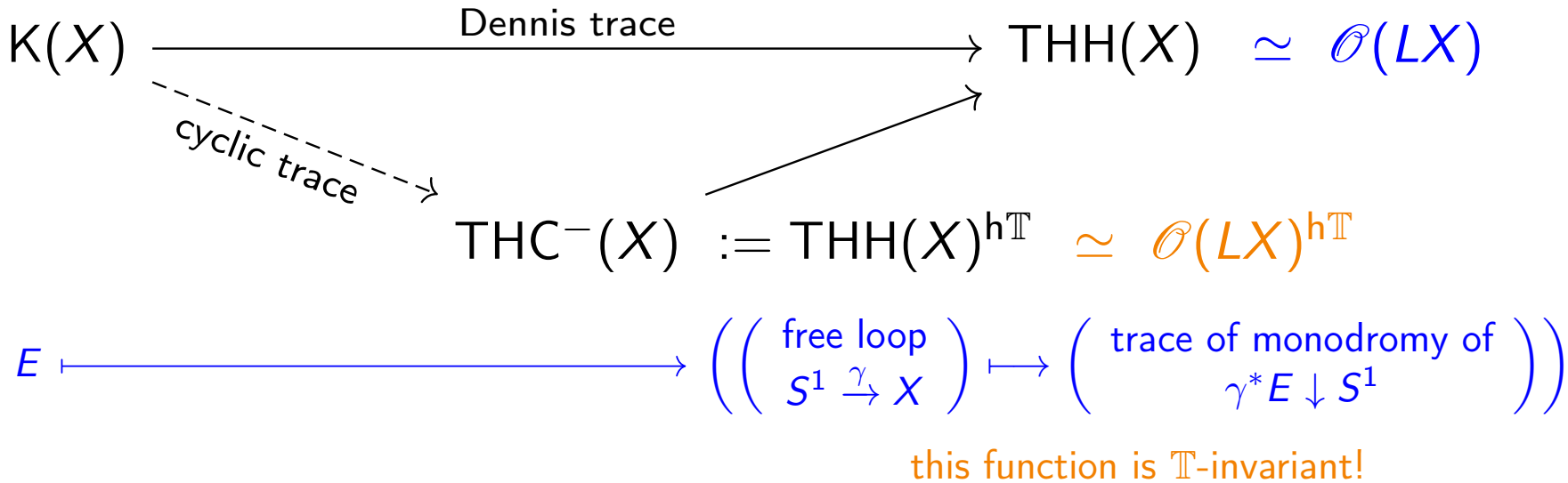

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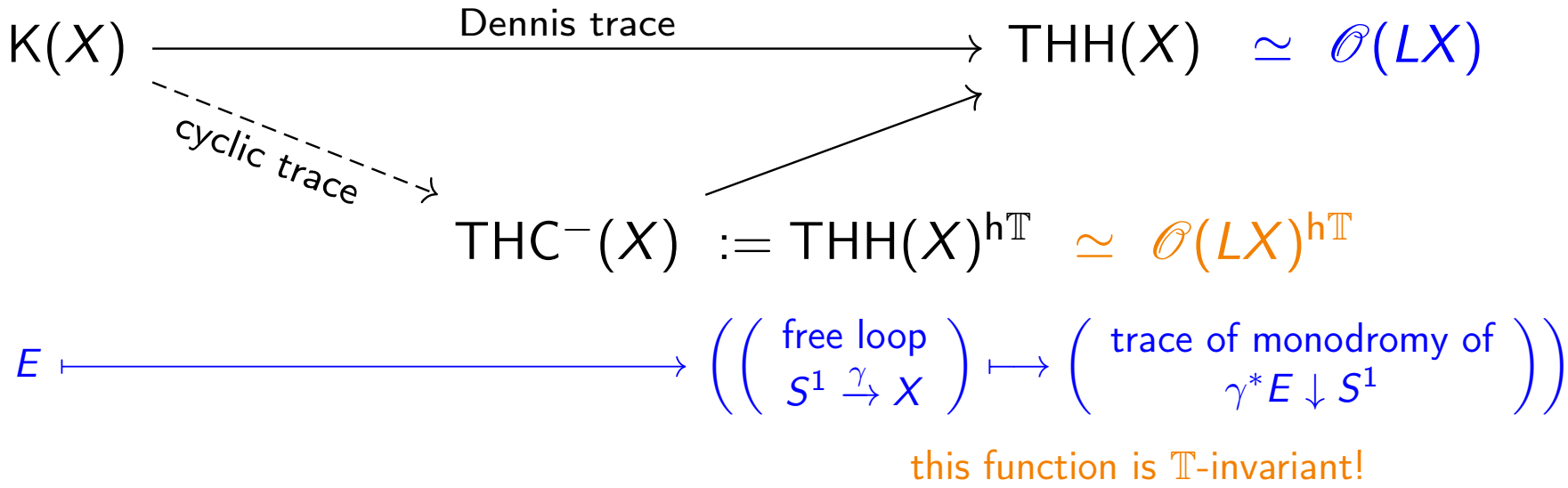

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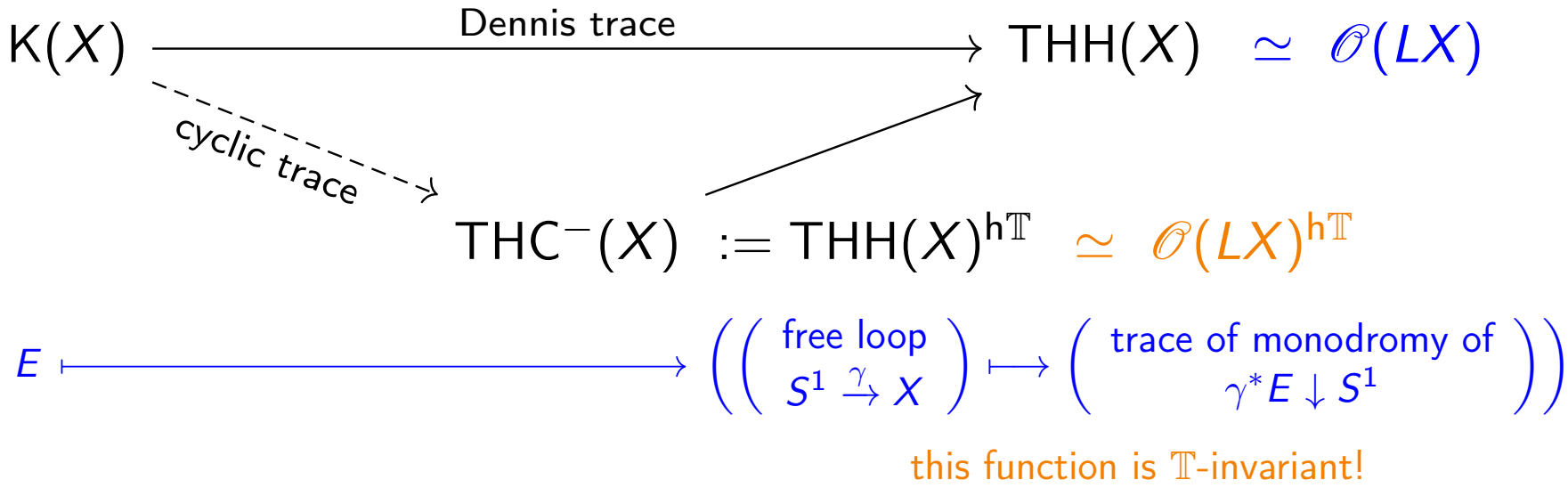

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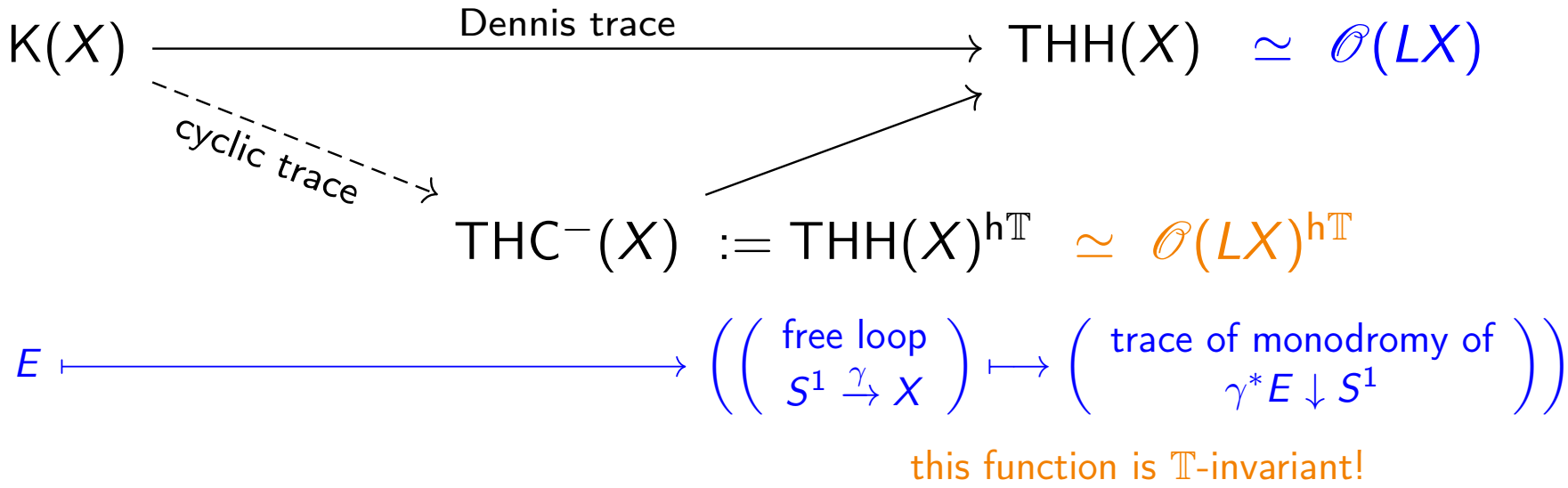

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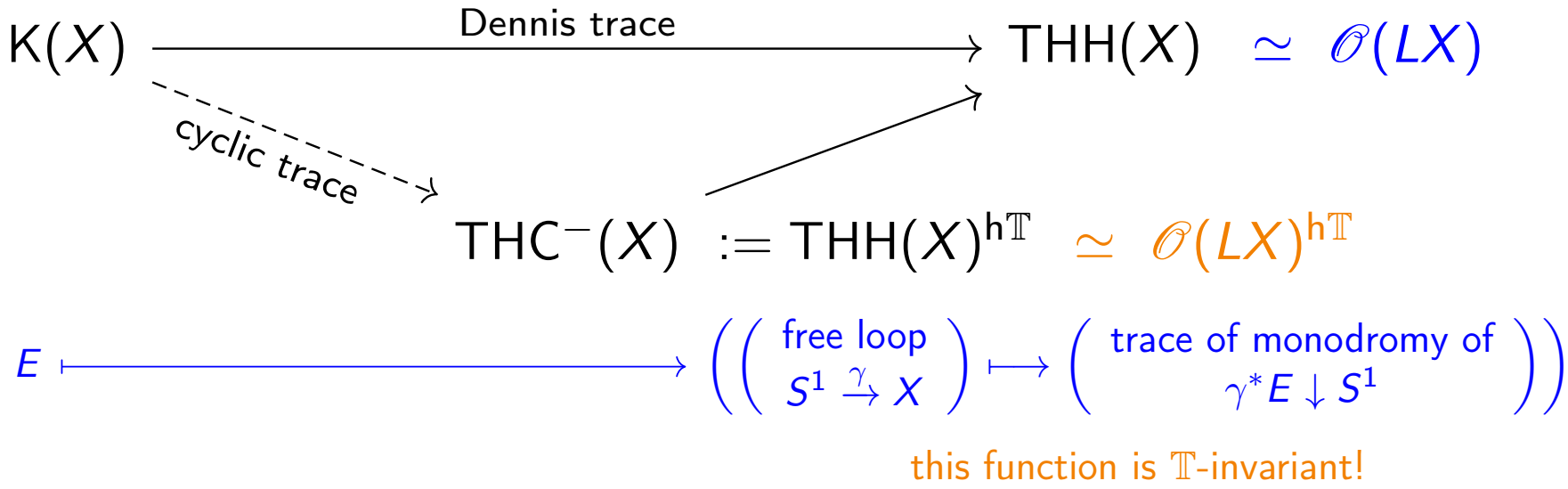

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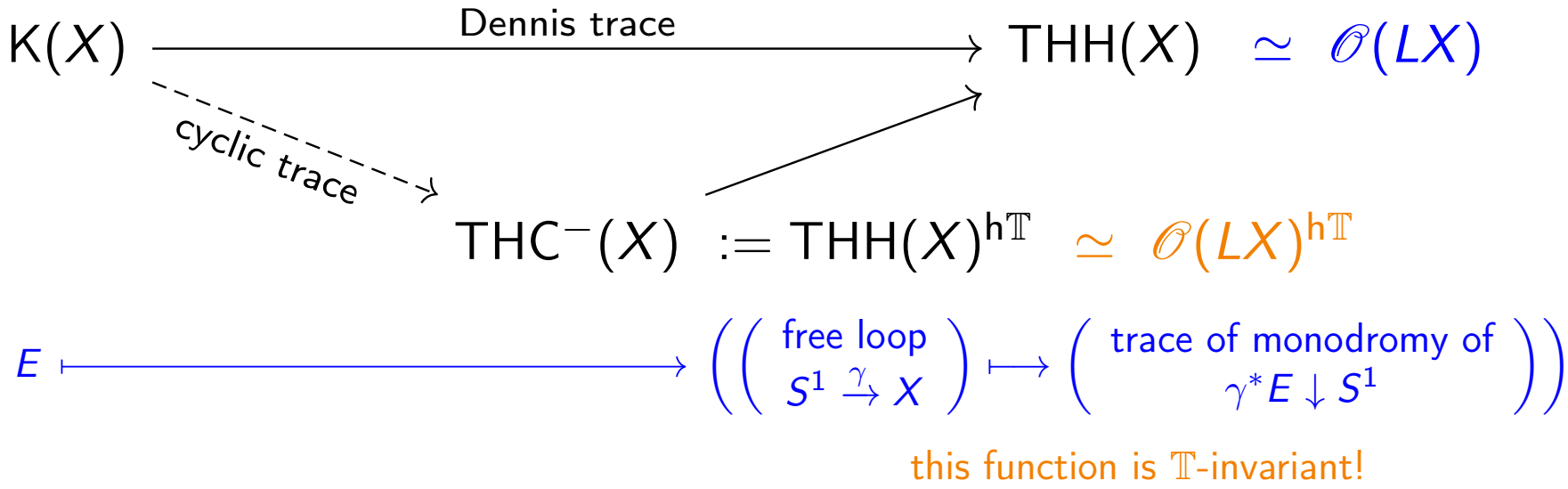

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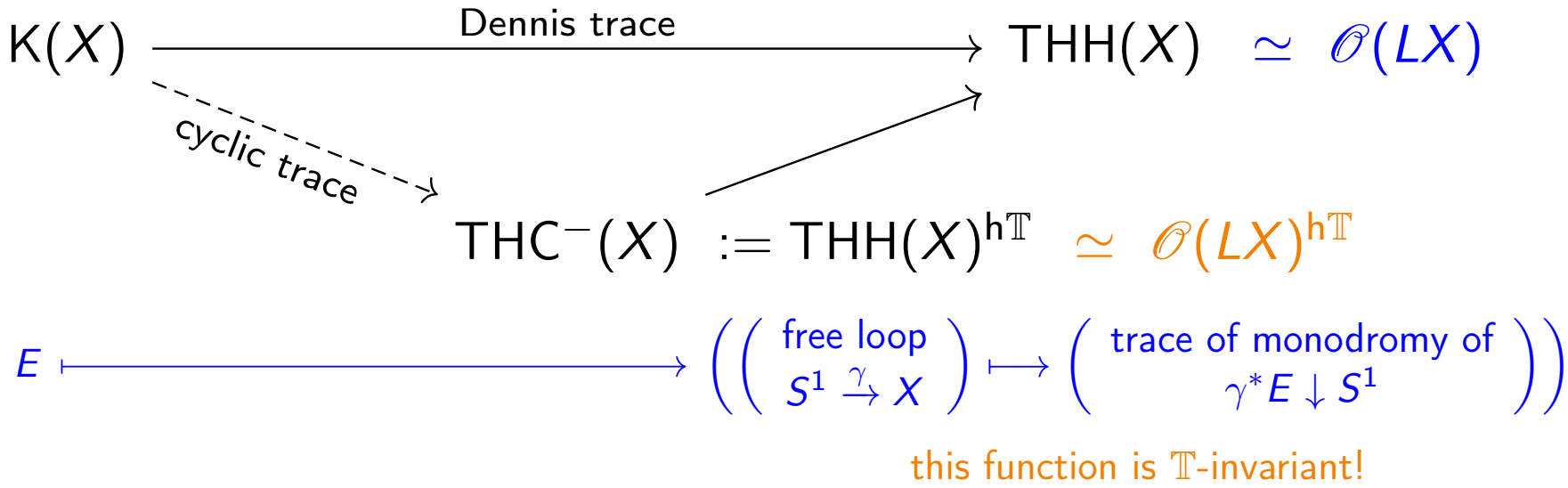

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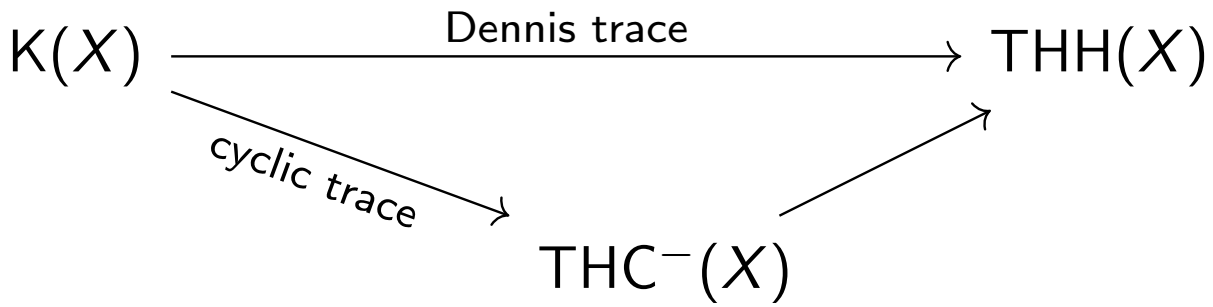

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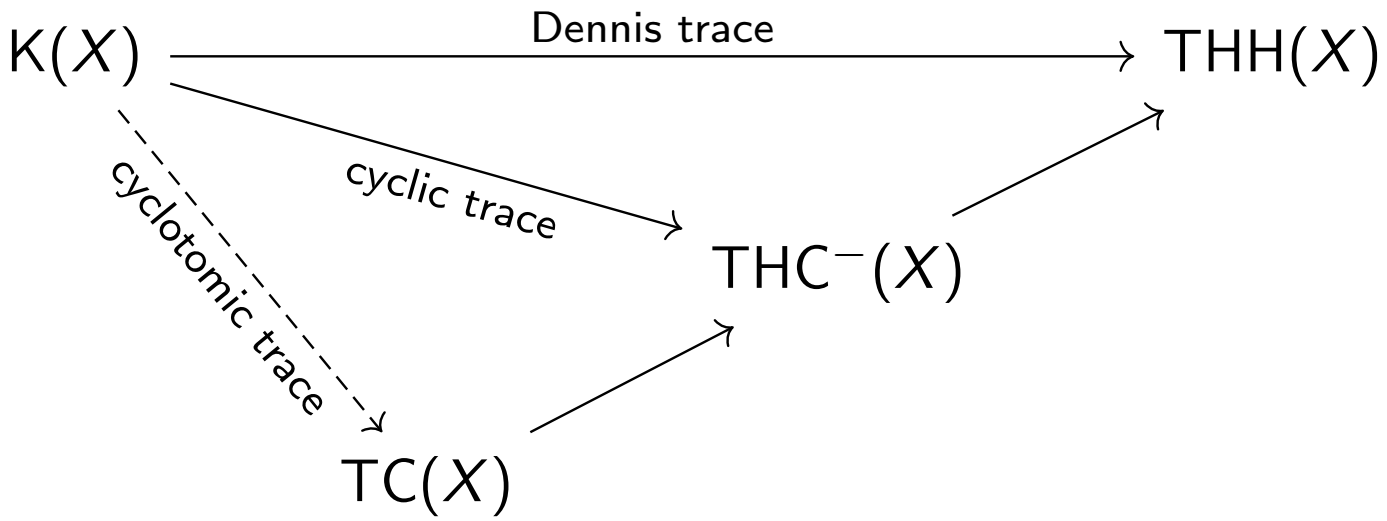

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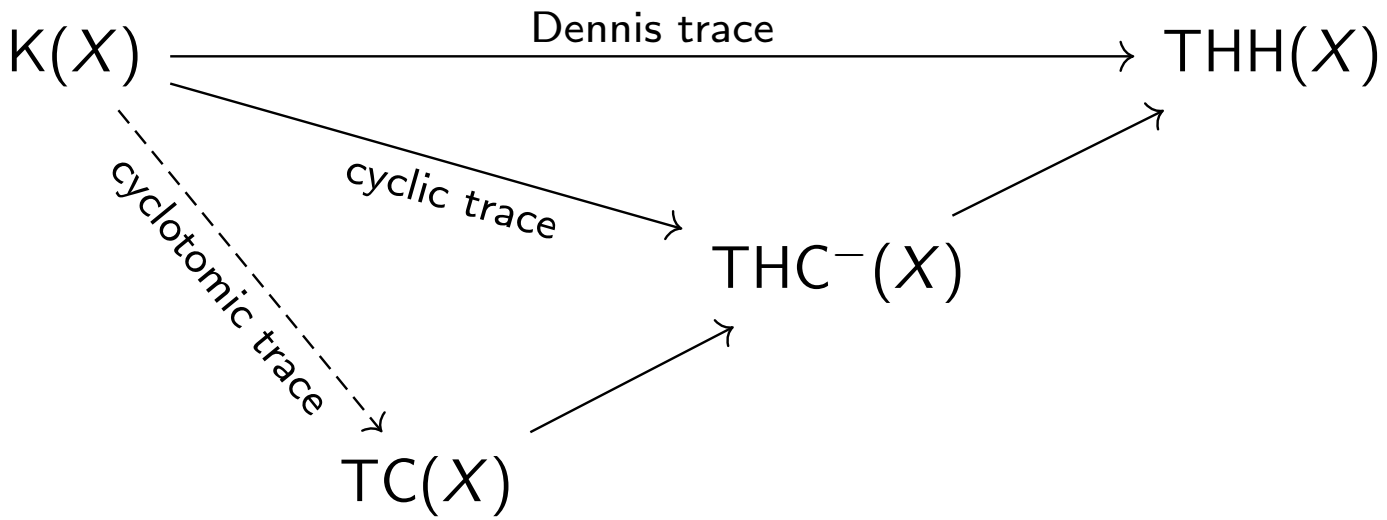

construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

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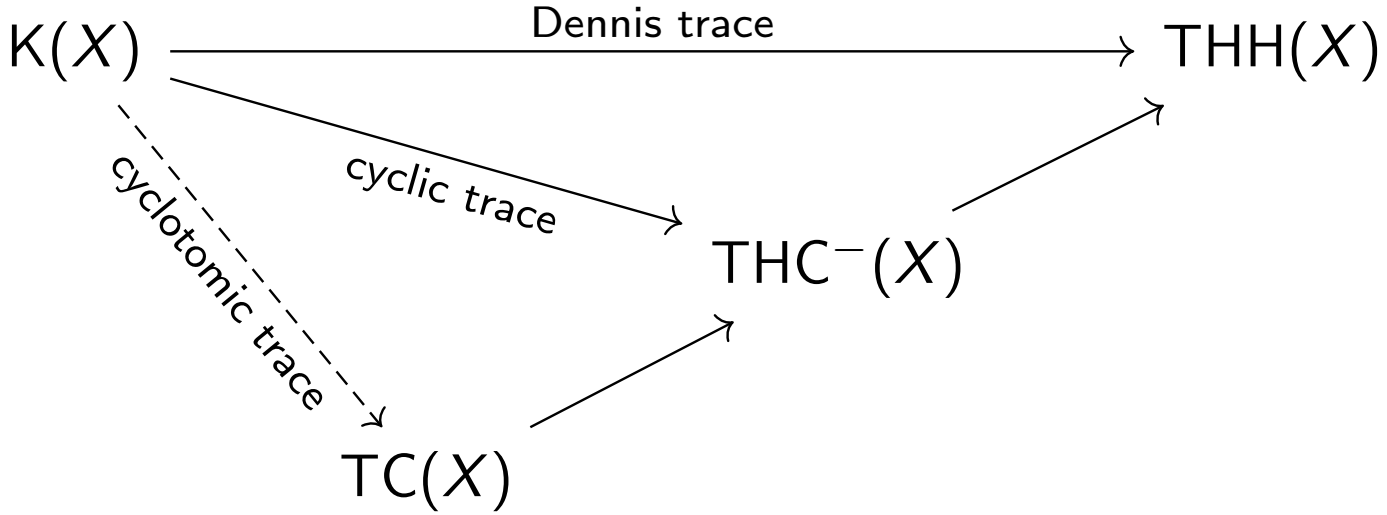

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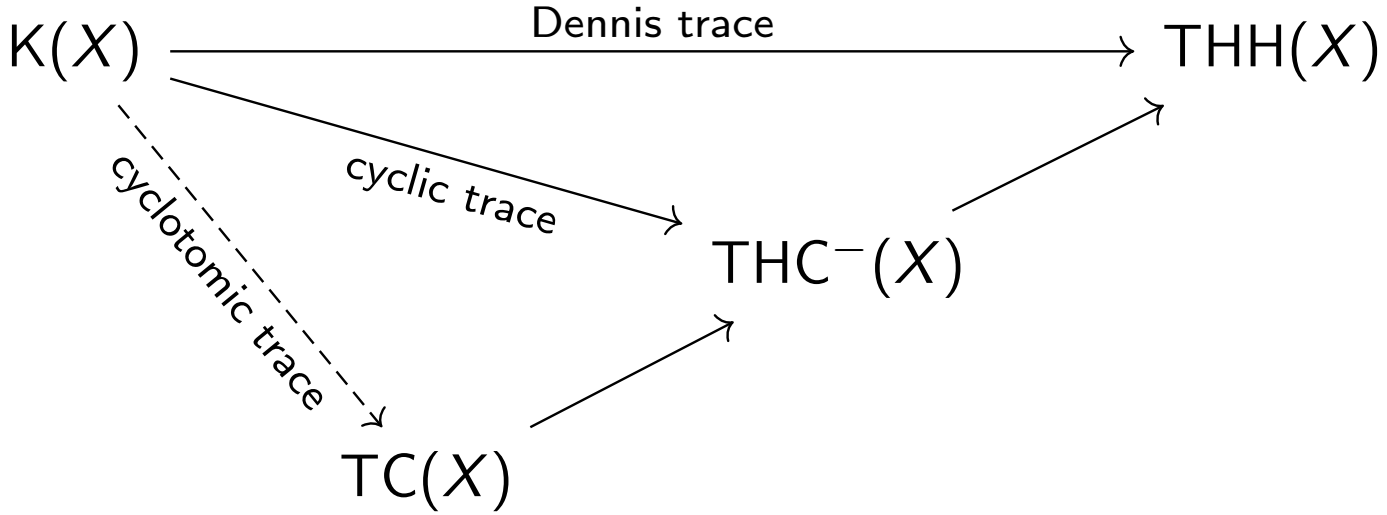

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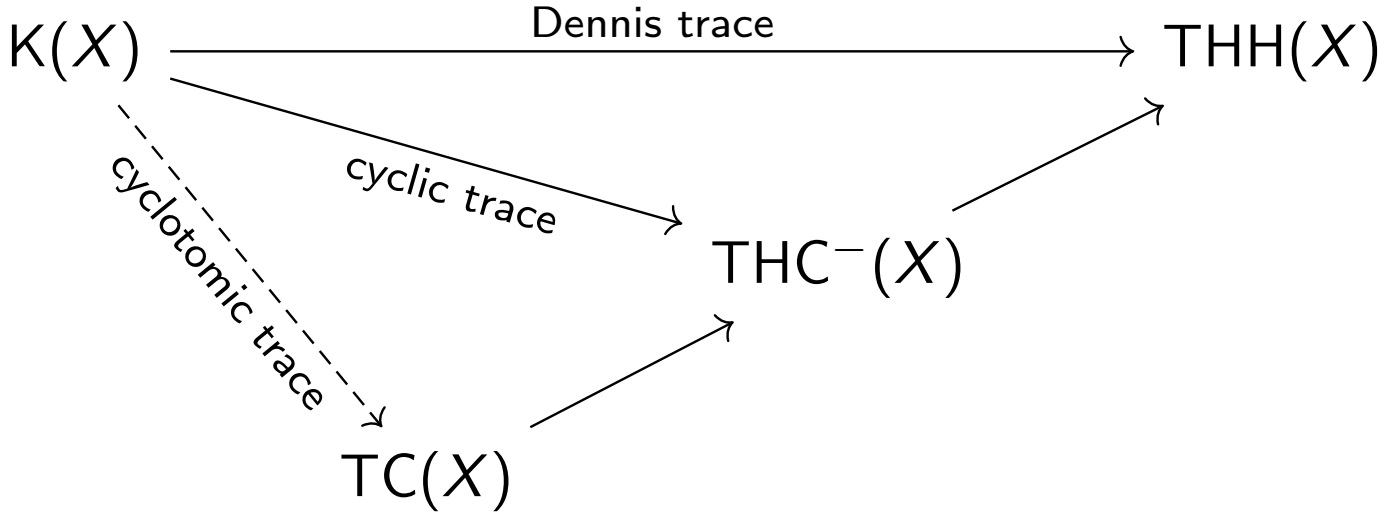

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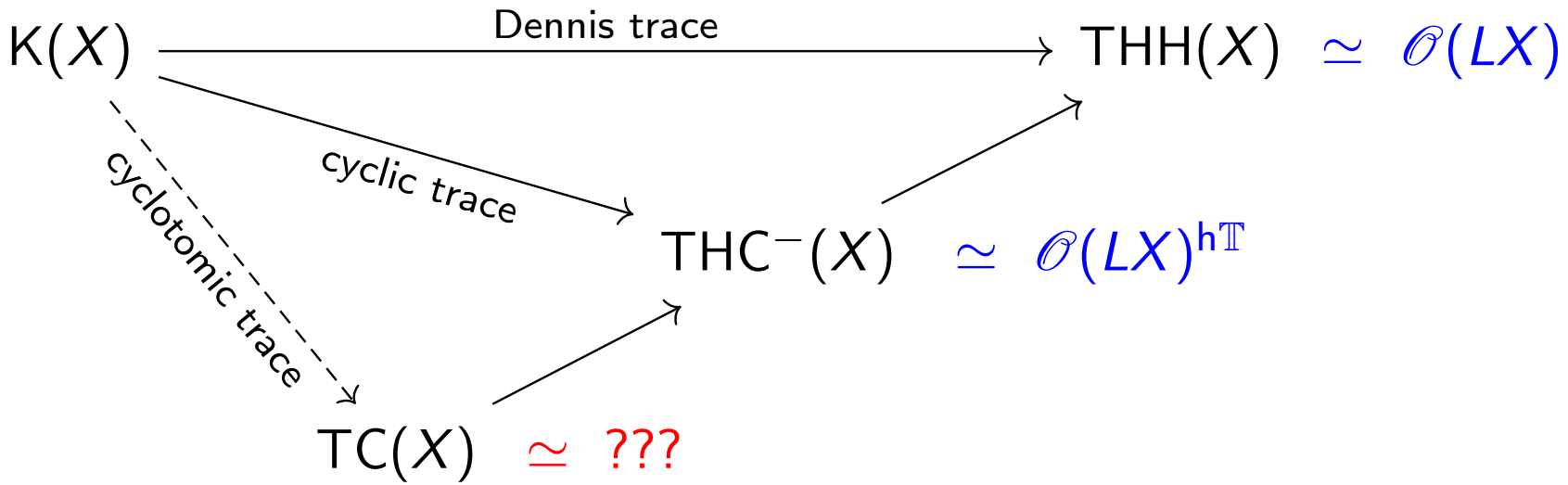

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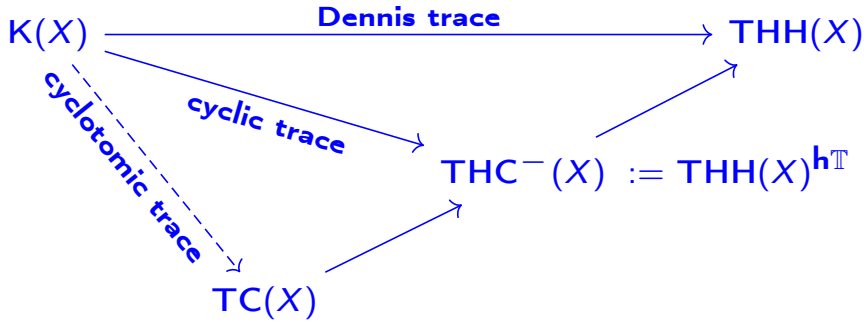
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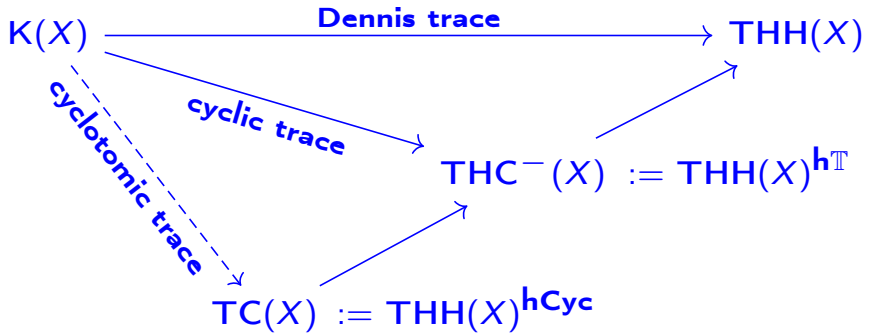

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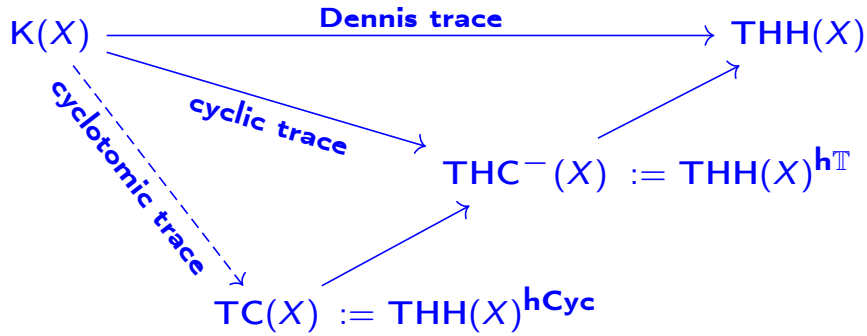

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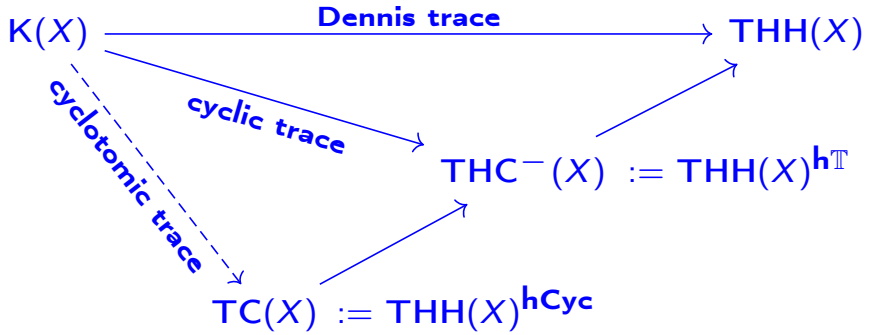

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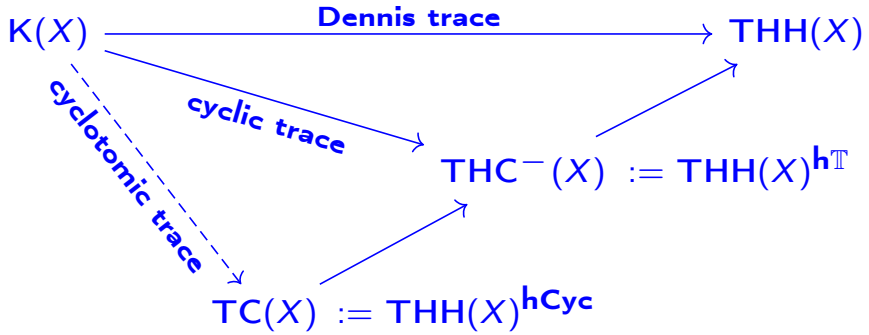

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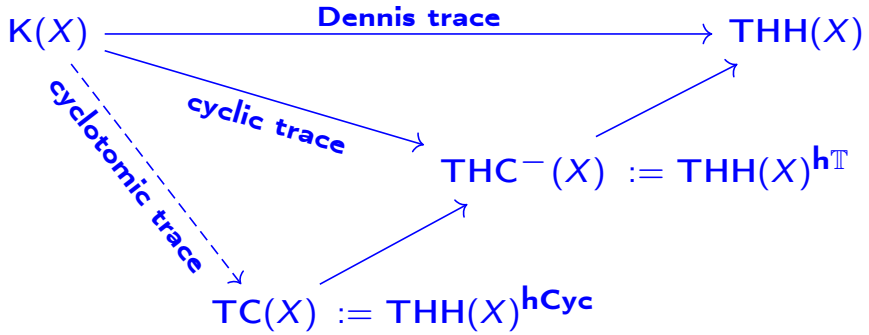

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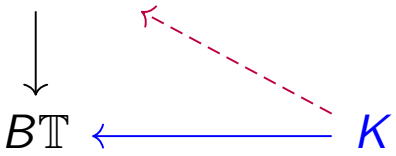

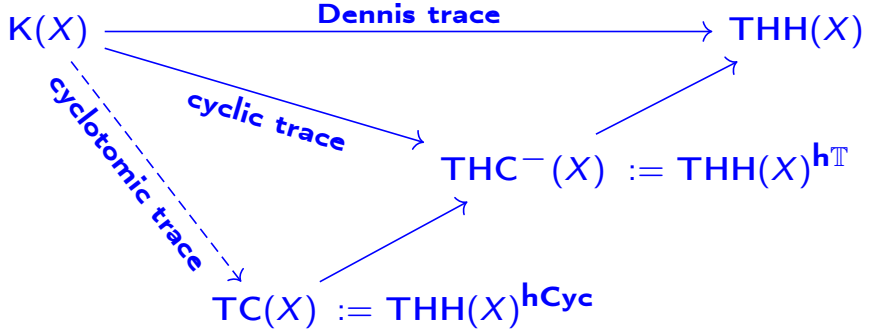

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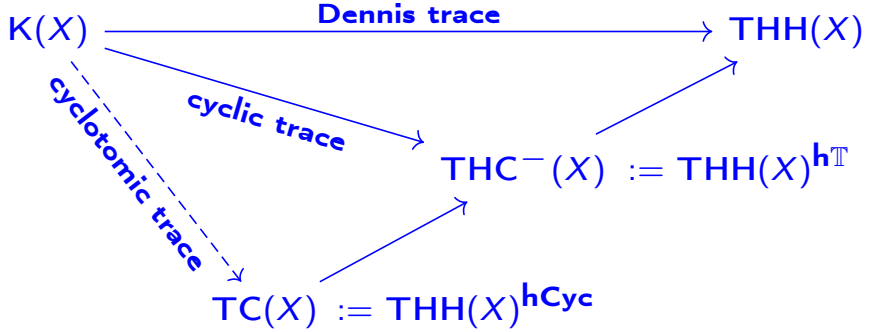

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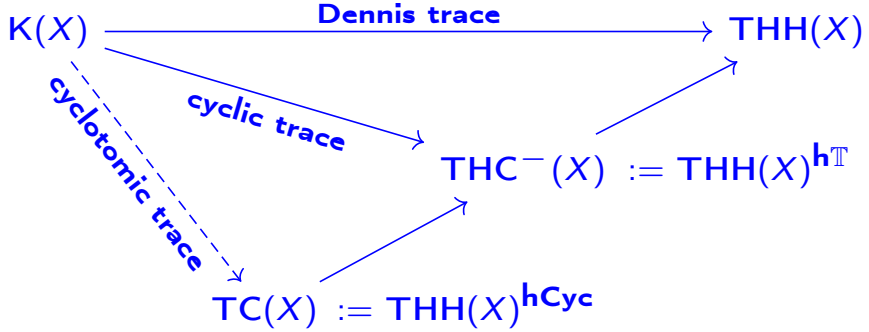

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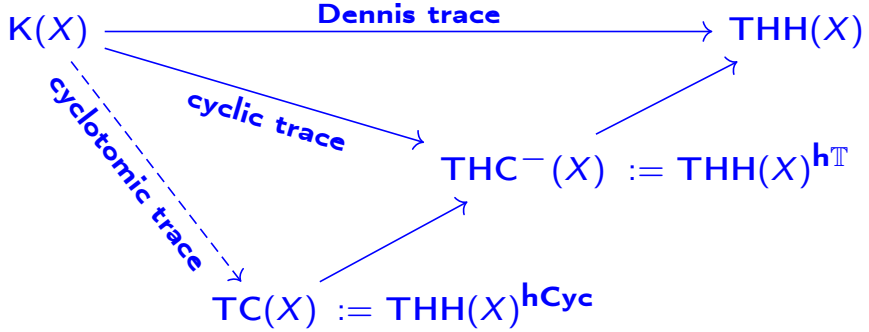

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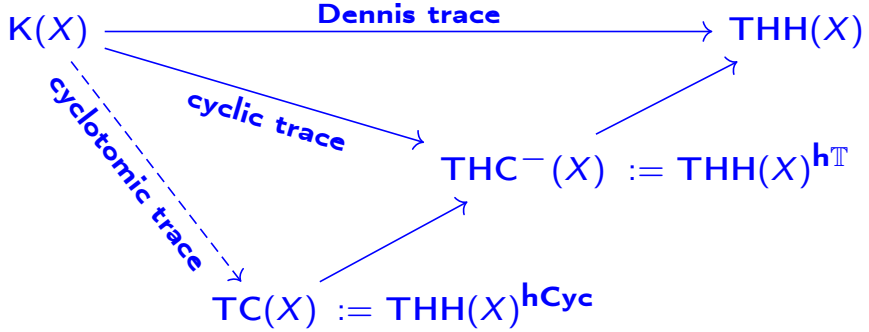

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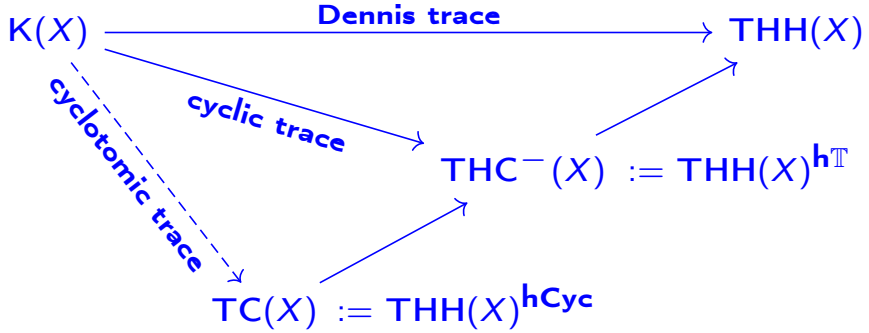

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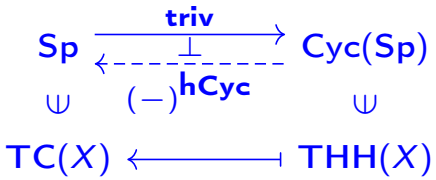
Q.: analogous story for $\mathrm{TC}(X)$?


Sp
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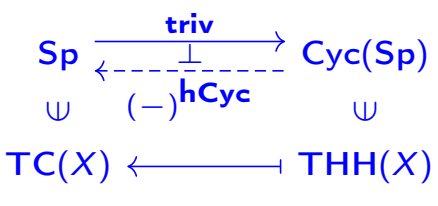

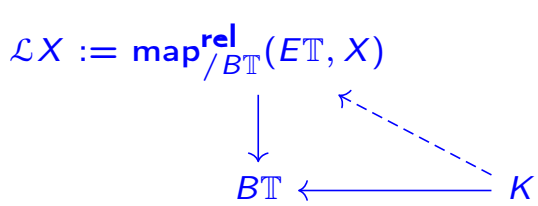

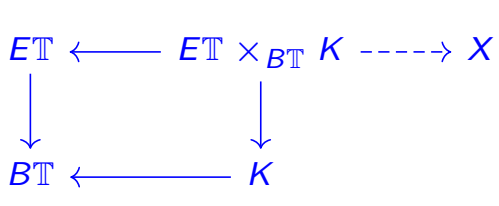

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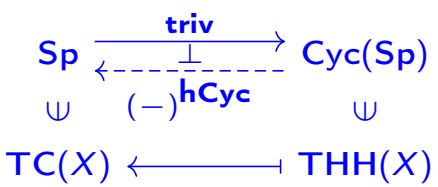

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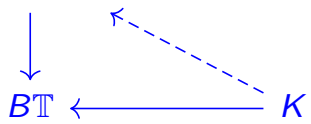

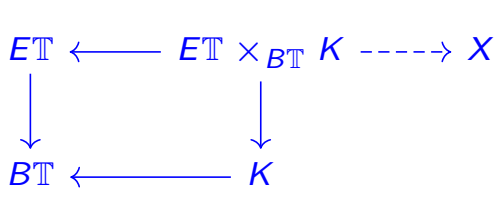

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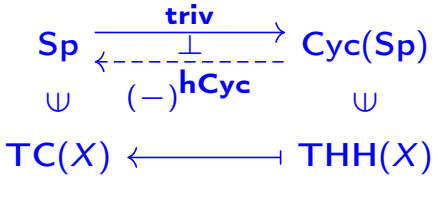
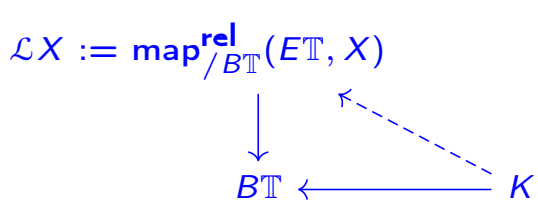
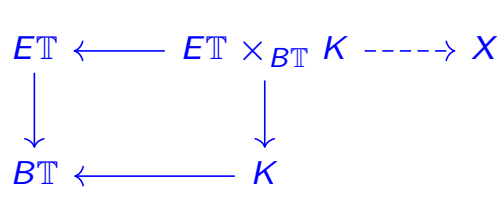
cyclotomic structure records simultaneous $\mathbb{T}$- and $\mathbb{N}^{\times}$-actions...







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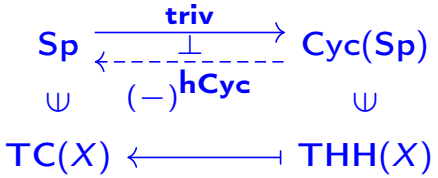

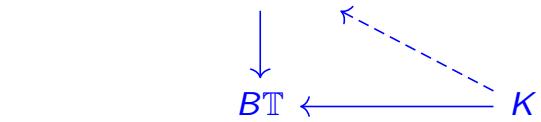

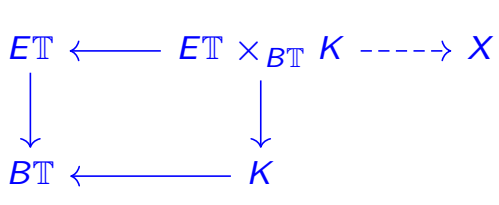

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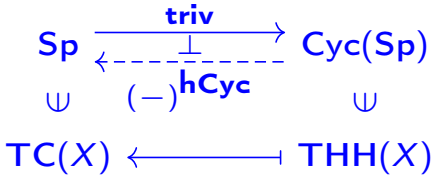

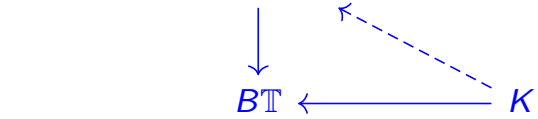

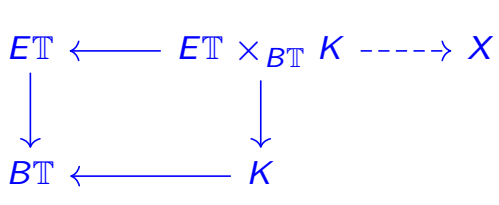

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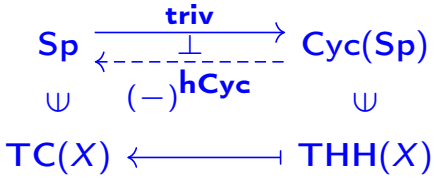

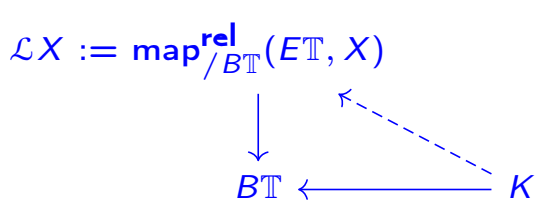

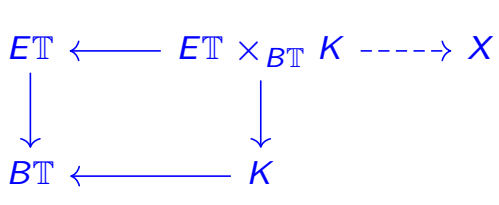

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$\rightsquigarrow \quad \mathrm{TC}(X) \simeq \Gamma^{h\mathbb{N}^\times}(\mathscr{O}_{\mathscr{L}X}) := \{\mathbb{T}\text{- and } \mathbb{N}^\times\text{-equivariant functions on } \mathscr{L}X\}$

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questions 1 & 2: §2

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questions 1 & 2: §2

question 3: §3

§2: stratified stacks and generalized recollements

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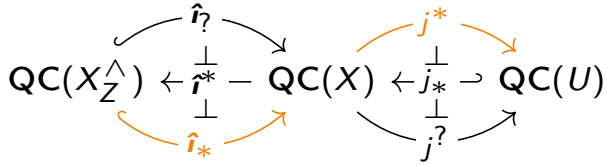
recollement: decomposition/reconstruction of a stable ∞-category

§2: stratified stacks and generalized recollements

recollement: decomposition/reconstruction of a stable ∞-category

main example: $Z \overset{\text{cls}}{\hookrightarrow} X \overset{\text{open}}{\hookleftarrow} U \quad \rightsquigarrow$



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§2: stratified stacks and generalized recollements

recollement: decomposition/reconstruction of a stable ∞-category

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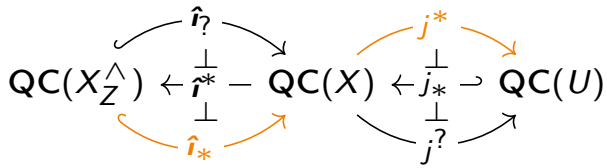

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Microcosm thm for recollements:

§2: stratified stacks and generalized recollements

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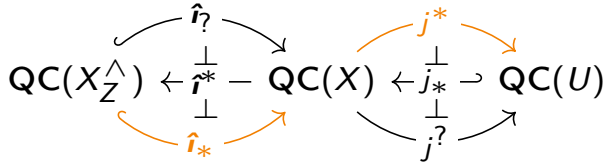
main example: $Z \overset{\text{cls}}{\hookrightarrow} X \overset{\text{open}}{\hookleftarrow} U \quad \rightsquigarrow$ 

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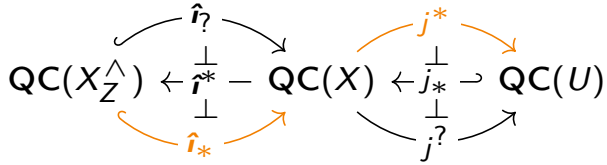
main example: $Z \overset{\text{cls}}{\hookrightarrow} X \overset{\text{open}}{\hookleftarrow} U \quad \rightsquigarrow$ 

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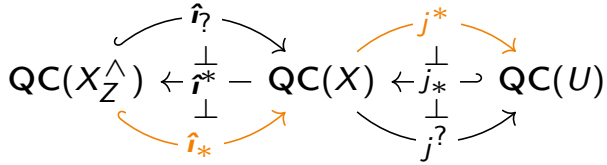
main example: $Z \overset{\text{cls}}{\hookrightarrow} X \overset{\text{open}}{\hookleftarrow} U \quad \rightsquigarrow$ 

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pullback

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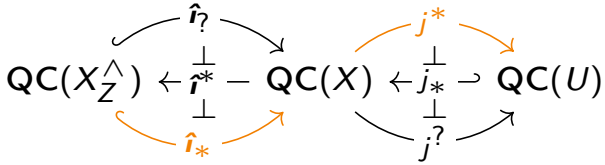

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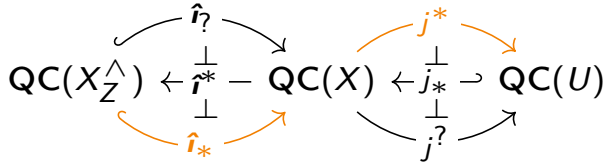

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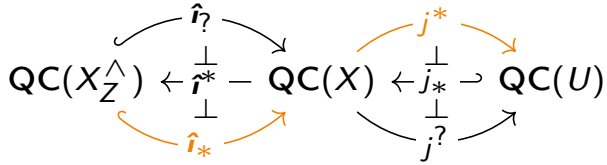

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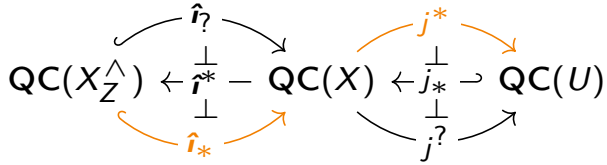

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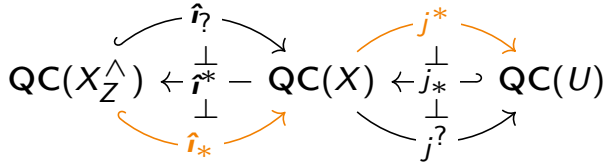

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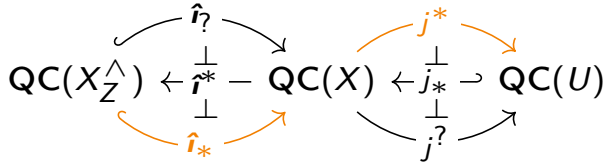

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the ***Tate construction***: the cofiber of the *norm* map

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slogan: a stratified stack* $\mathcal{Z} \xhookrightarrow{\mathrm{cls}} \mathcal{X} := \mathcal{X}_{\mathrm{gC}_p}$ with

- $\mathcal{X}^{\wedge}_{\mathcal{Z}} \simeq \mathrm{BC}_p$,
- $\mathcal{X} \backslash \mathcal{Z} \simeq \mathrm{pt}$,
- $j^* \hat{\imath}_* \simeq (-)^{\mathrm{tC}_p}$.

$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\mathrm{QC}(X^{\wedge}_{Z}) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right)$

another example (Greenlees–May): recollement of *genuine* C_p*-spectra*: $\mathcal{S}\mathrm{p}^{\mathrm{gC}_p} \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\mathcal{S}\mathrm{p}^{\mathrm{hC}_p} \xrightarrow{(-)^{\mathrm{tC}_p}} \mathcal{S}\mathrm{p} \right)$

the ***Tate construction***: the cofiber of the *norm* map

$$\begin{array}{ccccc} E_{\mathrm{hC}_p} & \xrightarrow{\mathrm{Nm}} & E^{\mathrm{hC}_p} & \xrightarrow{\mathrm{cofib}} & E^{\mathrm{tC}_p} \\ \cup & & \cup & & \\ [x] & \longmapsto & \sum_{\sigma \in \mathrm{C}_p} \sigma x & & \end{array}$$

(arises as the composite $\mathcal{S}\mathrm{p}^{\mathrm{hC}_p} \xrightarrow{\mathrm{Borel}} \mathcal{S}\mathrm{p}^{\mathrm{gC}_p} \xrightarrow{\Phi^{\mathrm{C}_p}} \mathcal{S}\mathrm{p}$)

slogan: a stratified stack* $\mathcal{Z} \xhookrightarrow{\mathrm{cls}} \mathcal{X} := \mathcal{X}_{\mathrm{gC}_p}$ with

- $\mathcal{X}^{\wedge}_{\mathcal{Z}} \simeq \mathrm{BC}_p$,
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[smallest example of ☹]

Defn (Glasman): a ***stratification*** of a stable ∞-category $\mathcal{C}$ over a poset P (subcategories $\mathcal{C}_p \subset \mathcal{C}$ for $p \in \mathrm{P}$, ...).

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Reconstruction thm (A–M-G–R): given a stratification of $\mathcal{C}$ over P, get left-lax diagram

$$\begin{array}{ccc} P & \xrightarrow{\text{l.lax}} & \mathrm{Cat} \\ \cup & & \cup \\ p & \longmapsto & \mathcal{C}_p \end{array}$$

and equivalence $\mathcal{C} \xrightarrow{\sim} \lim^{\text{r.lax}}(P \xrightarrow{\text{l.lax}} \mathrm{Cat})$ with right-lax limit.

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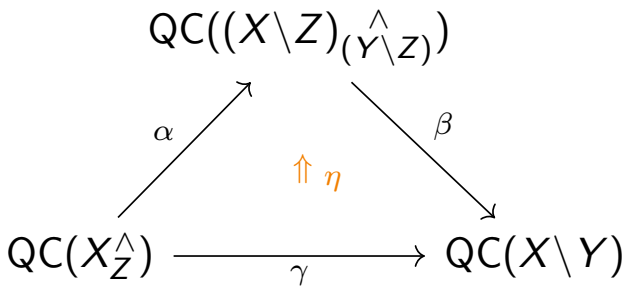

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add pix here!!!

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$\star$ above, need μ_{012} because laxness of diagram $\neq$ laxness of limit

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note: $\mathrm{B}\mathbb{N}^{\times} \simeq (\mathbb{N}^{\mathbf{div}})_{\mathbf{h}\mathbb{N}^{\times}}$

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$\leftrightarrow$ pullback of $\lim^{\mathrm{r.lax}}$ along action $\mathbb{N}^{\times} \curvearrowright \mathbb{N}^{\mathrm{div}}$ $\quad n \mapsto (i \mapsto ni)$

Defn (Blumberg–Mandell, Barwick–Glasman):

cyclotomic spectra := $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) := (\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}})^{\mathrm{h}\mathbb{N}^{\times}} \overset{\textbf{Cor.}}{\simeq} \lim^{\mathrm{r.lax}}(\ B\mathbb{N}^{\times} \xrightarrow{\text{l.lax}} \mathrm{Cat} \ )$.

note: $B\mathbb{N}^{\times} \simeq (\mathbb{N}^{\mathbf{div}})_{\mathbf{h}\mathbb{N}^{\times}}$

i.e., cyclotomic spectra $\simeq$ right-lax limit of left-lax $\mathbb{N}^{\times}$-action on $\mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}}$.

Thm: $\mathrm{Cyc}(\mathbb{S}\mathrm{p}) \simeq \lim^{\mathbf{r.lax}}(\mathrm{B}\mathbb{N}^{\times} \xrightarrow{\mathbf{l.lax}} \mathrm{Cat}) =: \lim^{\mathbf{r.lax}} \left(\mathbb{S}\mathrm{p}^{\mathbf{h}\mathbb{T}} \overset{\mathbf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times} \right)$.

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§3: the geometry of the cyclotomic trace

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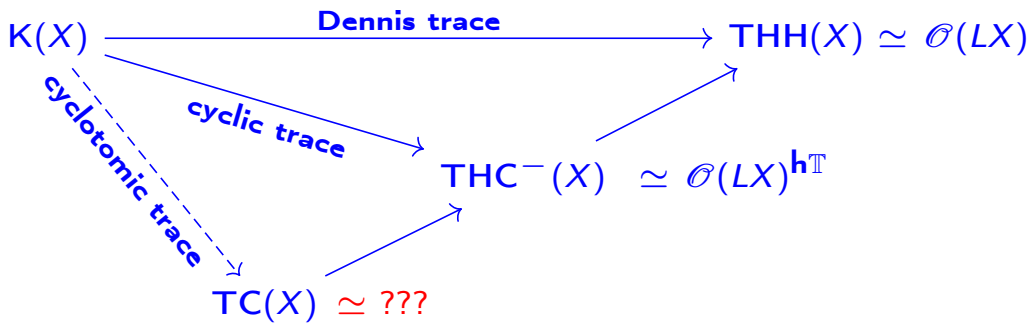

$$E \longmapsto \left(\begin{pmatrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{pmatrix} \longmapsto \begin{pmatrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{pmatrix} \right)$$

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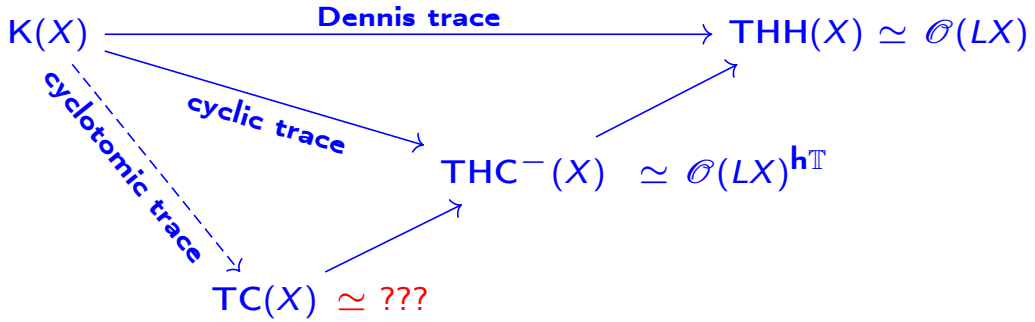

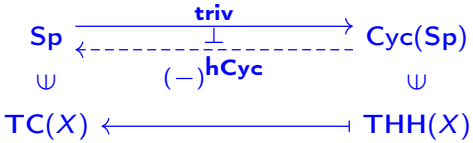

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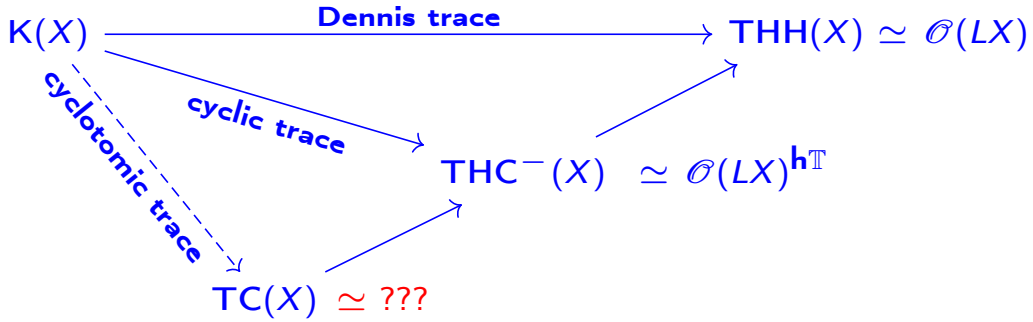

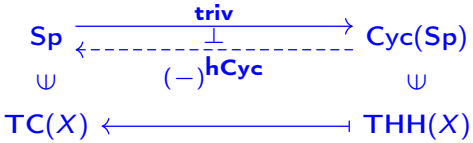

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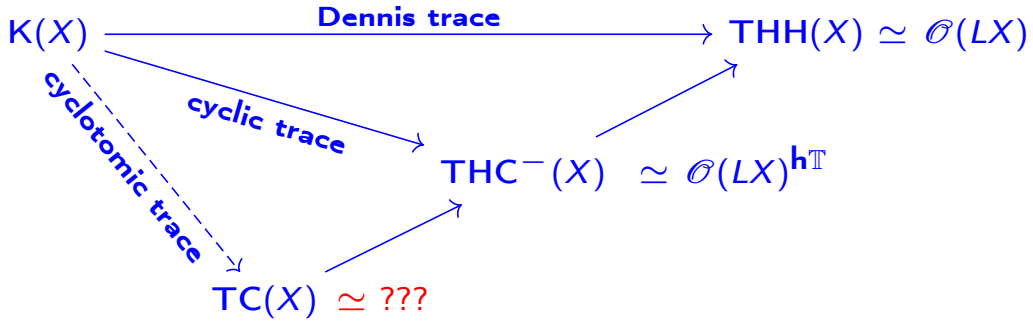

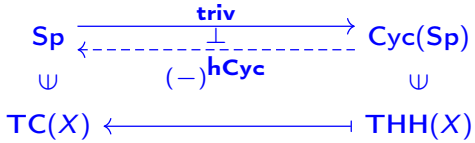

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right adjoint (limit-type construction): *imposing conditions*

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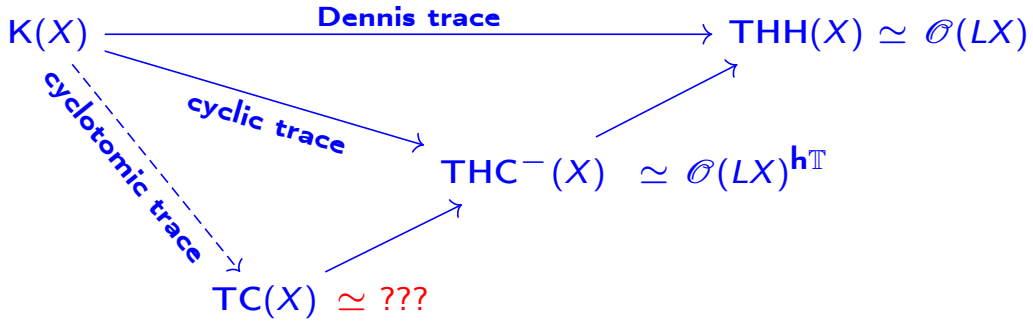

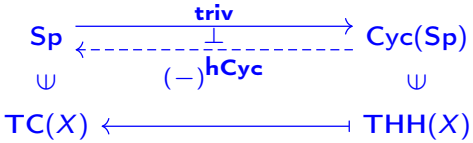

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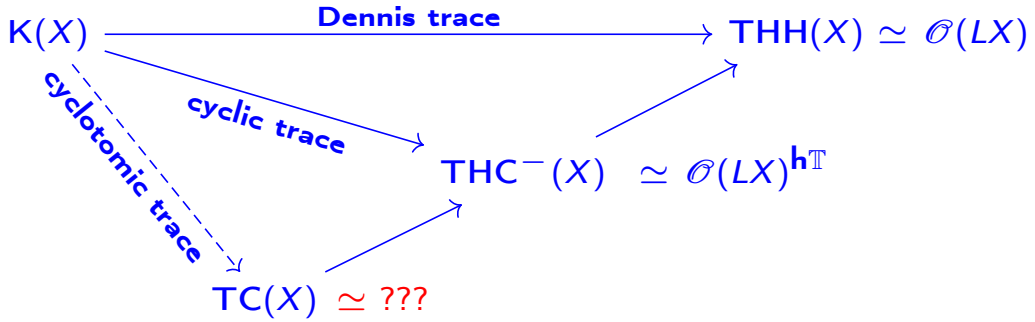
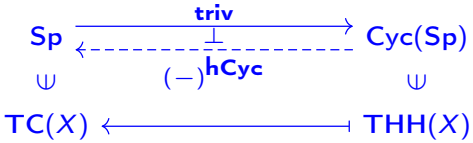
right adjoint (limit-type construction): *imposing conditions*

main idea:

§3: the geometry of the cyclotomic trace





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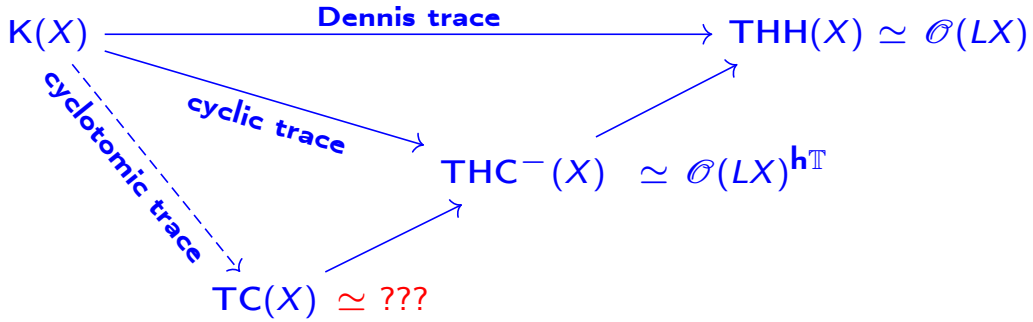

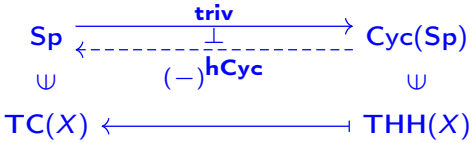

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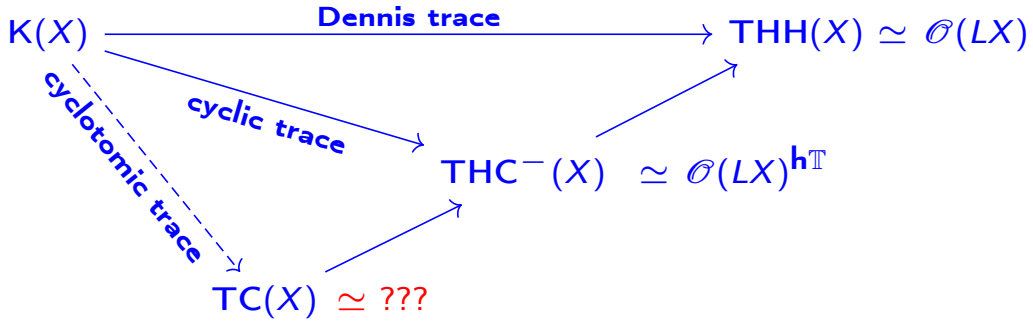

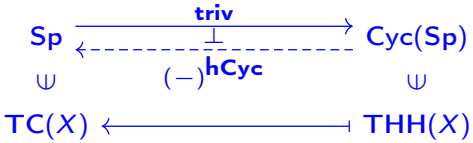

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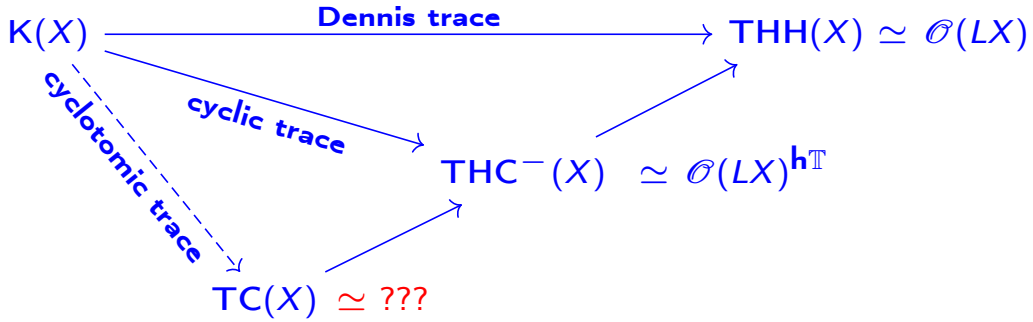

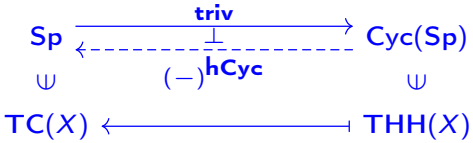

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relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

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over $\mathbb{Q}$, norm an iso! $\rightsquigarrow$ $(R \otimes R)^{\mathrm{t}C_2} = 0$, assertion is vacuous

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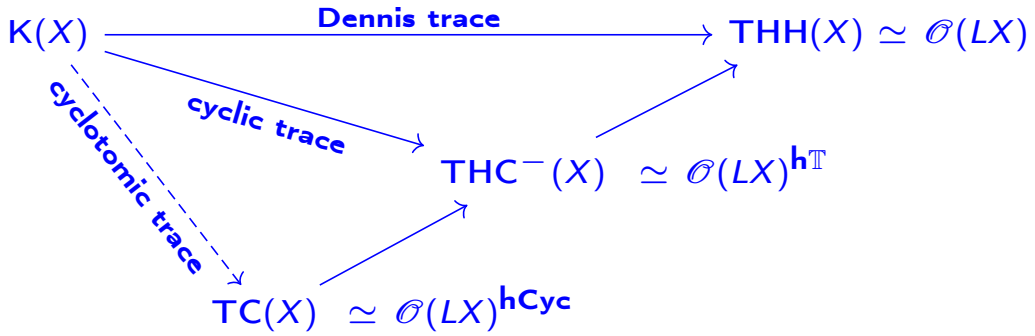
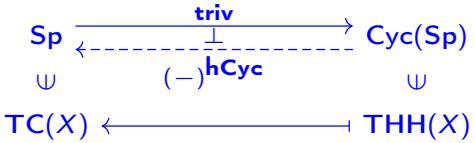
doesn't exist over $\mathbb{Z}$! only have "Tate diagonal" in $\mathcal{S}\mathrm{p}$, not $\mathscr{D}(\mathbb{Z})$.

the geometry of the cyclotomic trace:





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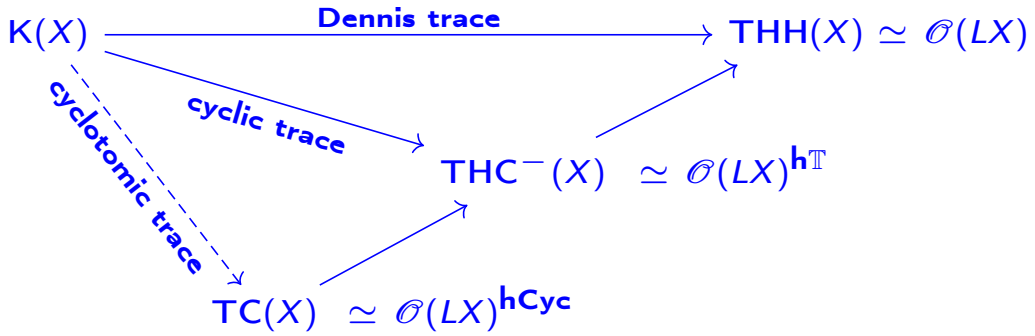

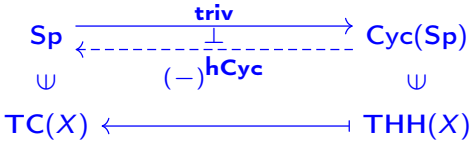

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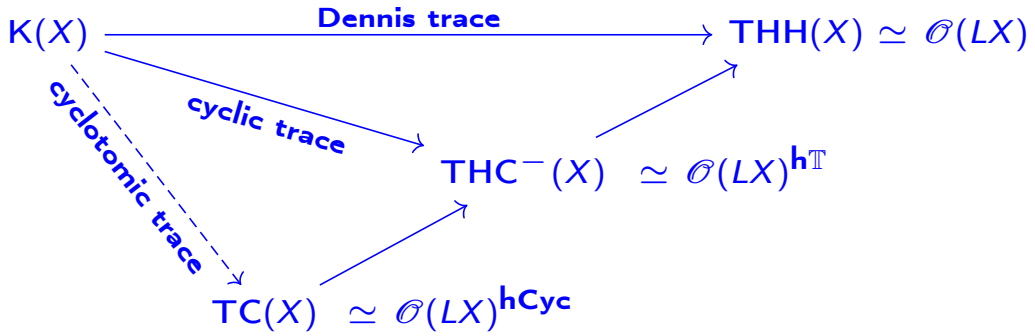

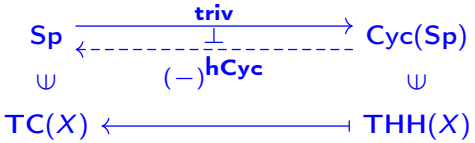

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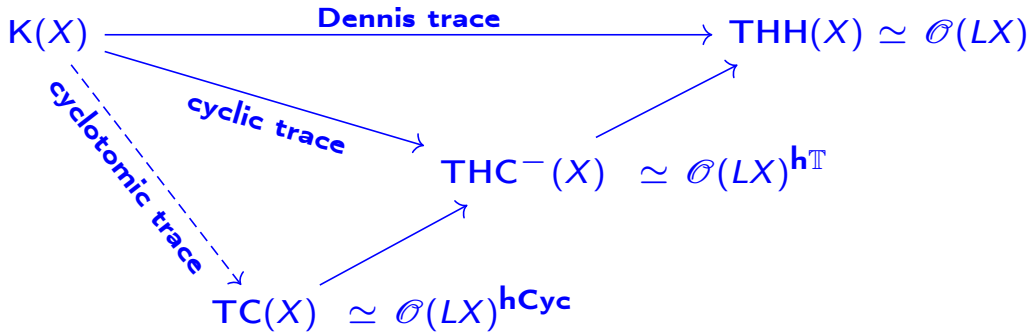

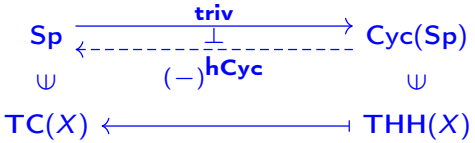

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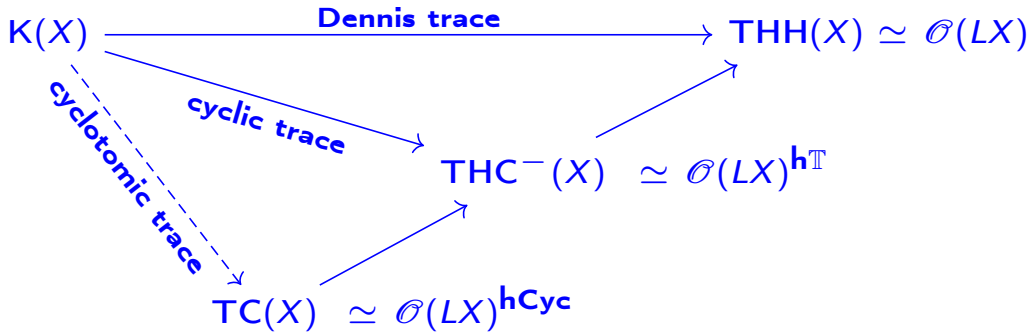

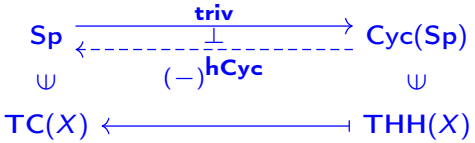

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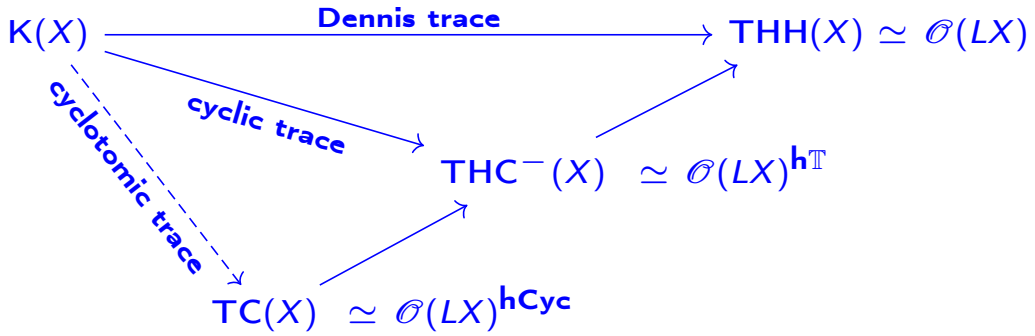

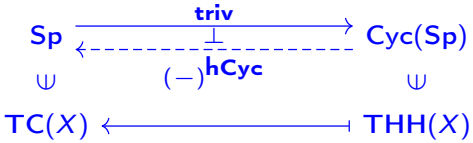

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...which is precisely the structure present on trace-of-monodromy functions of vector bundles! $\mathrm{tr}(M)^r \equiv \mathrm{tr}(M^r)$, ...

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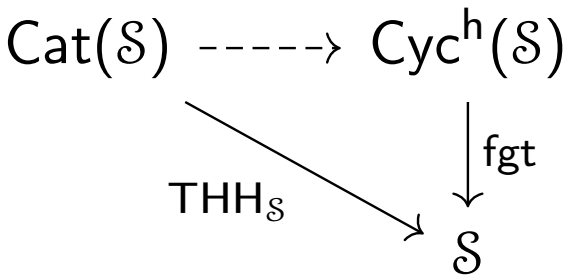

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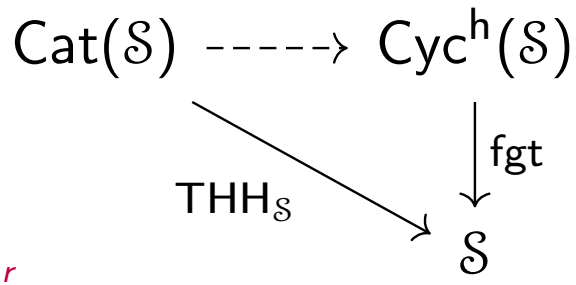

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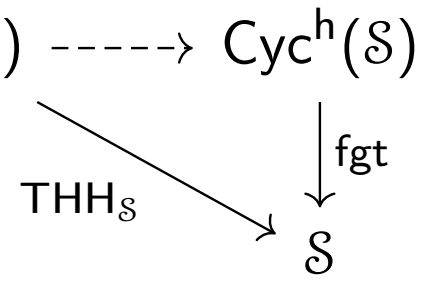

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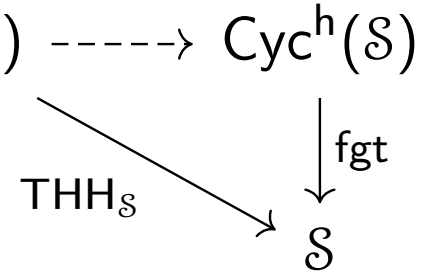

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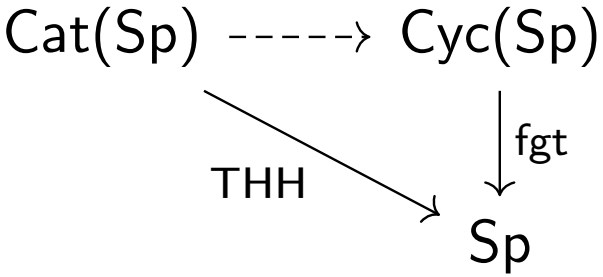

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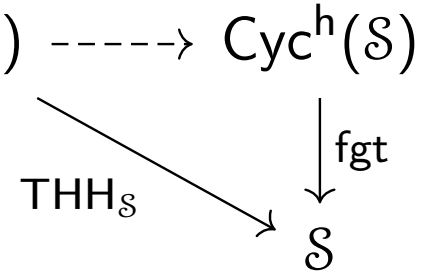

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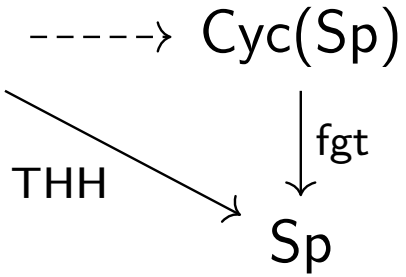

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Cyc(Sp) := cyclotomic spectra

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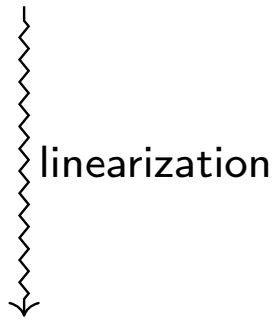

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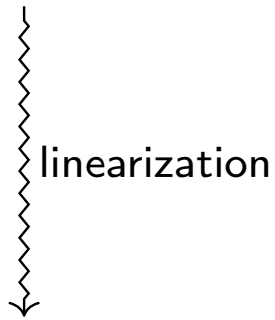

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input: K-theory is the ***universal additive invariant*** of stable ∞-categories (Blumberg–Gepner–Tabuada)