The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (`arXiv:1710.06416`)
2. *Factorization homology of enriched ∞-categories* (`arXiv:1710.06414`)
3. *The geometry of the cyclotomic trace* (`arXiv:1710.06409`)

§1 traces in differential geometry

§2 traces in algebraic geometry

§3 the geometry of the cyclotomic trace

§1: TRACES IN DIFFERENTIAL GEOMETRY

X a (nice) topological space

its ***complex topological K-theory***: the group-completion

$$KU(X) := (\mathrm{VBdl}_{\mathbb{C}}(X), \oplus)^{\mathrm{gp}} ,$$

a commutative ring via $[E] \cdot [F] := [E \otimes F]$.

the ***Chern character***: a ring homomorphism

Thm.

$$\begin{array}{ccc} KU(X) & \xrightarrow{\mathrm{ch}} & H^{even}(X; \mathbb{Q}) \\ & \searrow \quad\quad \nearrow_{\cong} & \\ & KU(X) \otimes \mathbb{Q} & \end{array}$$

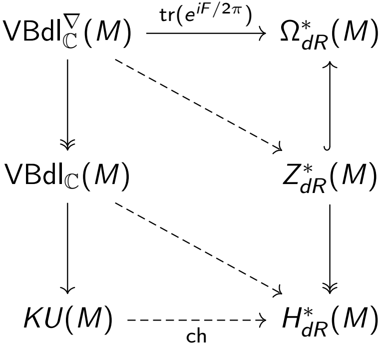
idea: $H^{even}(X; \mathbb{Q})$ is an approximation to $KU(X)$ (loses torsion)

chromatic homotopy theory: over $\mathbb{Q}$, have $\widehat{\mathbb{G}}_a \cong \widehat{\mathbb{G}}_m$ (via exp/log) for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_p M$ and section s, $\nabla_v(s) \in E_p \;\approx\;$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_p M$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\quad$ $\tilde{v}, \tilde{w}$ **any extensions of** v, w

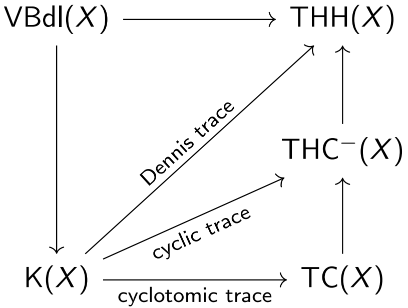
$\rightsquigarrow \quad F^\nabla \;\approx\;$ "monodromy around infinitesimal parallelograms"



§2: TRACES IN ALGEBRAIC GEOMETRY

X a scheme (variety / scheme / stack / derived stack)

its ***trace maps*** (to be explained):

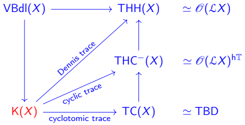


$\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) \approx \Omega^*_{dR}(X)$

$\mathrm{THC}^-(X) \simeq \mathscr{O}(\mathcal{L}X)^{\mathrm{h}\mathbb{T}} \approx H^*_{dR}(X)$

$\mathrm{TC}(X)$: **TODAY** $\approx ???^*_{dR}(X)$

DAG | HKR theorem



X a scheme $\rightsquigarrow$ $\mathrm{VBdl}(X) \subset \mathrm{QC}(X)$

affine case: $X = \mathrm{Spec}(R)$ $\rightsquigarrow$ $\mathrm{Proj}_R^{\mathrm{f.g.}} \subset \mathrm{Mod}_R$

derived version: $\mathrm{Perf}(\mathrm{X}) \subset \mathscr{D}(X) := \mathscr{D}(\mathrm{QC}(X))$ (triangulated category / stable ∞-category)

noncommutative version: R an associative ring $\rightsquigarrow$ "VBdl(Spec(R))" $:=$ $\mathrm{Proj}_R^{\mathrm{f.g.}}$

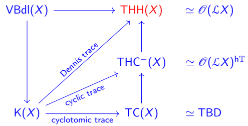
$\mathrm{K}(X) :=$ the ***algebraic K-theory*** of X

$$:= \mathrm{K}(\mathrm{VBdl}(X)) \simeq \mathrm{K}(\mathrm{Perf}(X))$$

$E_0 \rightarrowtail E_1 \twoheadrightarrow E_2$ $\rightsquigarrow$ $[E_1] = [E_0] + [E_2]$ (not all sexseq's / distinguished triangles split!)

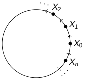
can define $\mathrm{K}(\mathcal{C})$ for *any* $\mathcal{C}$ with "exact sequences"

enforce relations *derivedly*: record relations, relations between relations, ... $\rightsquigarrow$ $\mathrm{K}(X)$ a *spectrum* $\approx$ chain complex



Def. $(\mathcal{V}, \boxtimes)$ a monoidal ∞-category, $\mathcal{C}$ a $\mathcal{V}$-enriched ∞-category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

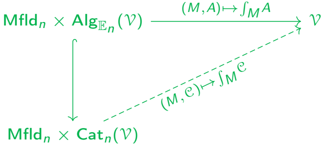
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



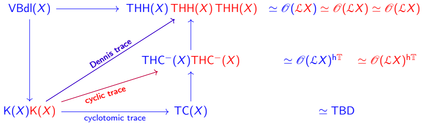
$\mathcal{V} = \mathcal{S}\mathrm{p} =$ spectra $\rightsquigarrow$ *topological* Hochschild homology (THH)

$\mathrm{THH}(X) := \mathrm{THH}(\mathrm{VBdl}(X)) \simeq \mathrm{THH}(\mathrm{Perf}(X))$

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathcal{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathcal{D}(\mathcal{S}\mathrm{et})$!)



simplicial	paracyclic	cyclic	epicyclic
Δ^{op}	$\Delta_{\circlearrowleft}^{\mathrm{op}}$	Λ^{op}	$\widetilde{\Lambda}^{\mathrm{op}}$
S^1_*	S^1	S^1 & auto's	S^1 & endo's



$\mathcal{L}X :=$ the *free loopspace* of X (necessarily derived)

$\mathbb{T} :=$ the circle group

$$\mathrm{K}(X) \xrightarrow{\text{Dennis trace}} \mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X)$$

$$E \longmapsto \left(\left(\begin{array}{c} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{array} \right) \longmapsto \left(\begin{array}{c} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{array} \right) \right)$$

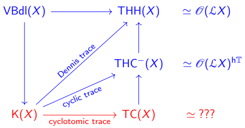
$\star$ trace-of-mdrmy is invariant under $\mathbb{T}$-action (rotation of loops) $\rightsquigarrow$ *cyclic trace*

Thm (Goodwillie '86). The cyclic trace is a local $\mathbb{Q}$-equivalence: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{THC}^-(R) & \longrightarrow & \mathrm{THC}^-(R_0) \end{array}$$

is a pullback after rationalization.

slogan: vbdl/Spec(R) $\overset{\mathbb{Q}}{\approx}$ restriction to Spec(R_0)



Goodwillie '86: cyclic trace a local $\mathbb{Q}$-equivalence

slogan: vbdl/Spec(R) $\overset{\mathbb{Q}}{\approx}$ restriction to Spec($R_{\mathbf{0}}$)
+ compatible trc-of-mdrmy function
+ data of $\mathbb{T}$-invariance of this function

construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

Thm (Dundas–McCarthy '97). the cyclotomic trace is a local equivalence (without rationalization!).

"This is how people other than Quillen compute algebraic K-theory."
$\sim$ *A. Blumberg*, algebraic K-theorist

Main Question: What is the geometry of TC(X)?

§3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$\mathsf{K}(X) \xrightarrow{\text{cyclotomic trace}} \mathsf{TC}(X) := \mathsf{THH}(X)^{\mathsf{hCyc}}$$

THH is a ***cyclotomic spectrum***; TC is the *homotopy invariants* of its cyclotomic structure

old defn via "genuine-equivariant" homotopy theory (useful (e.g. Poincaré duality), but no DAG meaning)

$$\begin{array}{ccc} \mathsf{Sp} & \underset{(-)^{\mathsf{hCyc}}}{\overset{\mathsf{triv}}{\rightleftarrows}} \ (\bot) & \mathsf{Cyc}(\mathsf{Sp}) \\ \Cup & & \Cup \\ \mathsf{TC}(X) & \longleftarrow\!\shortmid & \mathsf{THH}(X) \end{array}$$

right adjoint (limit-type construction): *imposing conditions*

recall: $\mathsf{THH}(X) \ = \$ functions on $\mathcal{L}X$

main idea: $\mathsf{TC}(X) \ = \$ functions on $\mathcal{L}X$ that are:

- invariant under $\mathbb{T}$-action on $\mathcal{L}X$;
- "sensitive" to relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$. **Q.** What does "sensitive" mean?

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\operatorname{tr}(M)^r$ and $\operatorname{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n\times n}(R)$

$$\operatorname{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \cdot \otimes m_j \quad , \quad \operatorname{tr}(M^{\otimes 2}) = \sum_k m_k \cdot \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is ***norms***: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{array}{ccc} (R \otimes R)_{C_2} & \xrightarrow{\quad \mathrm{Nm} \quad} & (R \otimes R)^{C_2} \\ [x \otimes y] & \longmapsto & \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{array}$$

$\rightsquigarrow$ become equal in the ***Tate construction***, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{\mathrm{t}C_2}$$

over $\mathbb{Q}$, norm an iso! $\rightsquigarrow$ $(R \otimes R)^{\mathrm{t}C_2} = 0$, assertion is vacuous

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

main point: for $\mathcal{C}$ a spectrally enriched ∞-category,
$S^1_b \xleftarrow{r} S^1_a$ a covering map of framed circles,
get ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

doesn't exist over $\mathbb{Z}$! only have "Tate diagonal" in $\mathcal{S}\mathrm{p}$, not $\mathscr{D}(\mathbb{Z})$.

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{\mathrm{h}\mathbb{T}} \underset{\tau}{\overset{\mathrm{l.lax}}{\circlearrowleft}} \mathbb{N}^{\times} \right)$$

$:= \mathrm{Fun}(\mathrm{B}\mathbb{T}, \mathrm{Sp})$

⋆ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{\mathrm{h}\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

Thm. [Nikolaus–Scholze]
for T connective and $r = s = p$ prime
Cor.: suff to specify just σ_p
(since $\sigma_{p^n} = (\sigma_p)^{\circ n}$, and n-cubes
canonically commute $\forall\ n \geq 2$)

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \underset{\text{can.}}{\xrightarrow{\sim}} & (T^{\tau C_s})^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

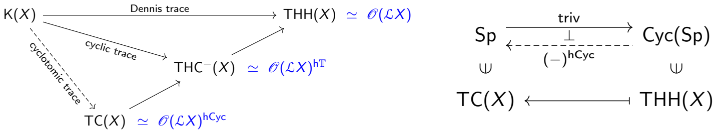
Thms. sufficient conditions + reconstruction theorem for *stratifications* of stable ∞-categories (a.k.a. generalized recollements), after Glasman. in general: $\mathcal{C} \simeq \lim^{\mathbf{r.lax}}(\ldots\mathrm{l.lax}\ldots)$. key examples:

⋆ genuine G-spectra (G cpct Lie), e.g. $\mathcal{S}\mathrm{p}^{\mathbf{g}\mathbf{C}_p} \simeq \lim^{\mathbf{r.lax}}(\mathcal{S}\mathrm{p}^{\mathbf{h}\mathbf{C}_p} \xrightarrow{(-)^{\mathbf{t}\mathbf{C}_p}} \mathcal{S}\mathrm{p})$ [Greenlees–May]

⋆ strat$^{\mathbf{n}}$ of a scheme/stack Y (e.g. closed-open decomposition) $\rightsquigarrow$ strat$^{\mathbf{n}}$ of $\mathrm{QC}(Y)$ [add pix here!]

$\rightsquigarrow$ suggests that $\mathrm{THH}(X) \leftrightarrow \mathcal{O}_{\mathscr{L}X}$ for $\mathscr{L}X$ a stratified "cyclotomic" enhancement of $\mathcal{L}X$,
$\mathrm{TC}(X) \leftrightarrow$ equivariant global functions on $\mathscr{L}X$.

the geometry of the cyclotomic trace:



$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T \,,\, \left\{ \begin{array}{c} T \\ \downarrow \sigma_r \\ T^{\tau C_r} \end{array} \right\}_{r \in \mathbb{N}^\times} ,\, \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ \sigma_{rs} \downarrow & & \downarrow (\sigma_s)^{\tau C_r} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s \in \mathbb{N}^\times} ,\, \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible" (+ higher coherences)...

...which is precisely the structure present on trace-of-monodromy functions of vector bundles! $\mathbf{tr}(M)^r \equiv \mathbf{tr}(M^r)$, ...

Q: where does the cyclotomic structure on THH come from?

Thm (A–M-G–R).

1. ***diagonal package*** for spaces $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathcal{S}) & \dashrightarrow & \mathrm{Cyc}^{\mathsf{h}}(\mathcal{S}) \\ & \searrow^{\mathrm{THH}_{\mathcal{S}}} & \downarrow \mathrm{fgt} \\ & & \mathcal{S} \end{array}$$

packages *diagonal maps* for spaces: $X \to (X^{\times r})^{\mathsf{h}\mathsf{C}_r}$

$\mathrm{Cyc}^{\mathsf{h}}(\mathcal{S}) := \mathrm{Fun}(\mathrm{B}\mathbb{W}, \mathcal{S}) :=$ "unstable cyclotomic spaces"

$B\mathbb{W} :=$ (objects: framed S^1's , morphisms: covering maps)

2. ***diagonal package*** for spaces

$$\Big\downarrow \rightsquigarrow \text{ linearization}$$

Tate package for spectra $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathrm{Sp}) & \dashrightarrow & \mathrm{Cyc}(\mathrm{Sp}) \\ & \searrow^{\mathrm{THH}} & \downarrow \mathrm{fgt} \\ & & \mathrm{Sp} \end{array}$$

packages *Tate diagonal maps* for spectra: $E \to (E^{\otimes r})^{\tau \mathsf{C}_r}$

$\mathrm{Cyc}(\mathrm{Sp}) := \mathrm{Cyc}^{\tau}(\mathrm{Sp}) :=$ cyclotomic spectra

Q: where does the cyclotomic trace $\mathsf{K} \to \mathsf{TC}$ come from?

Thm (A–M-G–R).

the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

$$\max^{\ell} \text{ subgpd of } \mathcal{C} := \iota\mathcal{C} \simeq \int_{\mathbb{D}^0} \mathcal{C} \longrightarrow \left(\int_{S^1} \mathcal{C}\right)^{\mathsf{h}\mathbb{W}} =: \mathsf{THH}_{\mathcal{S}}(\mathcal{C})^{\mathsf{h}\mathbb{W}} =: \mathsf{TC}^{\mathsf{h}}_{\mathcal{S}}(\mathcal{C})$$

input: the fiber bundle $S^1 \downarrow \mathbb{D}^0$ is invariant for the $\mathbb{W}$-action on S^1

$$\Big\downarrow \text{ linearization}$$

the ***cyclotomic trace***: for $\mathcal{C}$ a stable ∞-category,

$$\mathsf{K}(\mathcal{C}) \longrightarrow \mathsf{THH}(\mathcal{C})^{\mathsf{hCyc}} =: \mathsf{TC}(\mathcal{C})$$

input: K-theory is the ***universal additive invariant*** of stable ∞-categories (Blumberg–Gepner–Tabuada)