The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (arXiv:1710.06416)
2. *Factorization homology of enriched ∞-categories* (arXiv:1710.06414)
3. *The geometry of the cyclotomic trace* (arXiv:1710.06409)

§1 traces in differential geometry

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§2 traces in algebraic geometry

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its ***complex topological K-theory***: the group-completion

$$KU(X) := (\mathrm{VBdl}_{\mathbb{C}}(X), \oplus)^{\mathrm{gp}} ,$$

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$$KU(X) \xrightarrow{\quad \mathsf{ch} \quad} H^{even}(X; \mathbb{Q})$$

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chromatic homotopy theory: over $\mathbb{Q}$, have $\widehat{\mathbb{G}}_a \cong \widehat{\mathbb{G}}_m$ (via exp/log)

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its ***trace maps*** (to be explained):

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\begin{tikzcd}
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DAG

HKR theorem

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can define $\mathrm{K}(\mathcal{C})$ for *any* $\mathcal{C}$ with "exact sequences"

enforce relations *derivedly*: record relations, relations between relations, ... $\rightsquigarrow$ $\mathrm{K}(X)$ a *spectrum* $\approx$ chain complex

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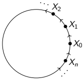

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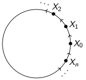

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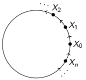

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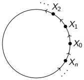

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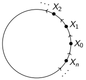

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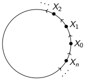

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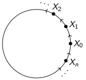

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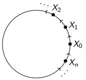

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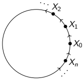

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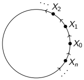

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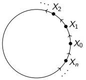

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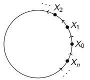

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simplicial	paracyclic	cyclic	
Δ^{op}	$\Delta^{\mathrm{op}}_{\circlearrowleft}$	Λ^{op}	
S^1_*	S^1		

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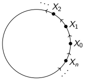
Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



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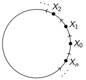

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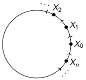

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Goodwillie '86: cyclic trace a local $\mathbb{Q}$-equivalence

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construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

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Main Question: What is the geometry of $\mathrm{TC}(X)$?

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relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

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over $\mathbb{Q}$, norm an iso! $\rightsquigarrow$ $(R \otimes R)^{\mathrm{t}C_2} = 0$, assertion is vacuous

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doesn't exist over $\mathbb{Z}$! only have "Tate diagonal" in $\mathcal{S}\mathrm{p}$, not $\mathscr{D}(\mathbb{Z})$.

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$$\mathrm{Cyc}(\mathrm{Sp}) \simeq$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \qquad \mathrm{Sp}^{\mathrm{h}\mathbb{T}} := \mathrm{Fun}(\mathrm{B}\mathbb{T}, \mathrm{Sp})$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \qquad \mathrm{Sp}^{\mathrm{h}\mathbb{T}}$$

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$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \left(\mathrm{Sp}^{\mathrm{h}\mathbb{T}} \underset{\tau}{\circlearrowleft} \mathbb{N}^{\times} \right)$$

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$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{\mathrm{h}\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times} \right)$$

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- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{tikzcd} T \arrow[r, "\sigma_r"] \arrow[d, "\sigma_{rs}"'] & T^{\tau C_r} \arrow[d, "(\sigma_s)^{\tau C_r}"] \\ T^{\tau C_{rs}} \arrow[r, "\mathrm{can.}"'] & (T^{\tau C_s})^{\tau C_r} \end{tikzcd}$$

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Thm. [Nikolaus–Scholze]
for T connective and $r = s = p$ prime
Cor.: suff to specify just σ_p
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Thms. sufficient conditions + reconstruction theorem for *stratifications* of stable ∞-categories (a.k.a. generalized recollements), after Glasman. in general: $\mathcal{C} \simeq \lim^{\mathbf{r.lax}}(\ldots\mathrm{l.lax}\ldots)$.

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⋆ $\mathrm{strat}^{\mathbf{n}}$ of a scheme/stack Y (e.g. closed-open decomposition) $\rightsquigarrow$ $\mathrm{strat}^{\mathbf{n}}$ of $\mathrm{QC}(Y)$ [add pix here!]

Thm (A–M-G–R).

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⋆ strat$^{\mathbf{n}}$ of a scheme/stack Y (e.g. closed-open decomposition) $\rightsquigarrow$ strat$^{\mathbf{n}}$ of QC(Y) [add pix here!]

$\rightsquigarrow$ suggests that $\mathrm{THH}(X) \leftrightarrow \mathscr{O}_{\mathscr{L}X}$ for $\mathscr{L}X$ a stratified "cyclotomic" enhancement of $\mathcal{L}X$,
$\mathrm{TC}(X) \leftrightarrow$ equivariant global functions on $\mathscr{L}X$.

the geometry of the cyclotomic trace:

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$$
\begin{tikzcd}
\mathsf{K}(X) \arrow[rrr, "\text{Dennis trace}"] \arrow[rrd, "\text{cyclic trace}"] \arrow[rdd, dashed, "\text{cyclotomic trace}"'] & & & \mathsf{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) \\
& & \mathsf{THC}^{-}(X) \simeq \mathscr{O}(\mathcal{L}X)^{\mathsf{h}\mathbb{T}} \arrow[ru] & \\
& \mathsf{TC}(X) \simeq \mathscr{O}(\mathcal{L}X)^{\mathsf{hCyc}} \arrow[ru] & &
\end{tikzcd}
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$$

$$
\begin{tikzcd}
\mathrm{Sp} \arrow[r, shift left, "\mathrm{triv}"] \arrow[r, phantom, "\perp"] & \mathrm{Cyc}(\mathrm{Sp}) \arrow[l, shift left, dashed, "(-)^{\mathrm{hCyc}}"] \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\mathrm{TC}(X) & \mathrm{THH}(X) \arrow[l, mapsto]
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\end{tikzcd}$$

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$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

the geometry of the cyclotomic trace:

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\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sp} \arrow[r, shift left, "\mathrm{triv}"] \arrow[r, phantom, "\perp"] & \mathrm{Cyc}(\mathrm{Sp}) \arrow[l, shift left, dashed, "(-)^{\mathrm{hCyc}}"] \\
\mathrm{TC}(X) \arrow[u, phantom, sloped, "\in"] & \mathrm{THH}(X) \arrow[l, mapsto] \arrow[u, phantom, sloped, "\in"]
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$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;

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\end{tikzcd}
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$\mathsf{TC}(X)$ is built from $\mathsf{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible" (+ higher coherences)...

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\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, shift left, "\mathrm{triv}"] \arrow[r, phantom, "\perp"] & \mathrm{Cyc}(\mathrm{Sp}) \arrow[l, shift left, dashed, "(-)^{\mathrm{hCyc}}"] \\
\Cup & \Cup \\
\mathrm{TC}(X) & \mathrm{THH}(X) \arrow[l, mapsto]
\end{tikzcd}$$

$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T \,,\ \left\{ \begin{tikzcd} T \arrow[d, "\sigma_r"] \\ T^{\tau C_r} \end{tikzcd} \right\}_{r \in \mathbb{N}^\times} ,\ \left\{ \begin{tikzcd} T \arrow[r, "\sigma_r"] \arrow[d, "\sigma_{rs}"'] & T^{\tau C_r} \arrow[d, "(\sigma_s)^{\tau C_r}"] \\ T^{\tau C_{rs}} \arrow[r, "\text{can.}"'] & (T^{\tau C_s})^{\tau C_r} \end{tikzcd} \right\}_{r,s \in \mathbb{N}^\times} ,\ \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible" (+ higher coherences)...

...which is precisely the structure present on trace-of-monodromy functions of vector bundles! $\mathbf{tr}(M)^r \equiv \mathbf{tr}(M^r)$, ...

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input: K-theory is the ***universal additive invariant*** of stable ∞-categories (Blumberg–Gepner–Tabuada)