The geometry of the cyclotomic trace

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with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (`arXiv:1710.06416`)
2. *Factorization homology of enriched ∞-categories* (`arXiv:1710.06414`)
3. *The geometry of the cyclotomic trace* (`arXiv:1710.06409`)

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 2

§2.1 stratified schemes and generalized recollements (interlude)

§2.2 the geometry of TC (conjectural refinement of §1.3)

§2.3 the construction: factorization homology of enriched ∞-categories

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2}$ ... but in $KO(S^2)$,

$$\begin{aligned} [TS^2] + [\underline{\mathbb{R}^1}] &= [TS^2] + [NS^2] = [TS^2 \oplus NS^2] \\ &= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}] \end{aligned}$$

$S^2 \overset{i}{\hookrightarrow} \mathbb{R}^3$

$\rightsquigarrow [TS^2] = [\underline{\mathbb{R}^2}]$ (TS^2 is *stably trivial*)

Bott periodicity: $\widetilde{KU}(\Sigma^2 X) \cong \widetilde{KU}(X)$, $\widetilde{KO}(\Sigma^8 X) \cong \widetilde{KO}(X)$

$\widetilde{K}(X) := \mathrm{coker}(K(\mathrm{pt}) \to K(X))$

in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,

$[E_0, E_1] \cdot [F_0, F_1] = [(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (F_0 \otimes E_1)]$

the ***Chern character***: a ring homomorphism

$$KU(X) \xrightarrow{\mathrm{ch}} H^*(X;\mathbb{Q}) \ .$$

Thm.

$$\begin{tikzcd}
KU(X) \arrow[rr] \arrow[dr] & & H^{even}(X;\mathbb{Q}) \\
& KU(X) \otimes \mathbb{Q} \arrow[ur, dashed, "\cong"] &
\end{tikzcd}$$

idea: $H^{even}(X;\mathbb{Q})$ is an approximation to $KU(X)$ (loses torsion)

chromatic homotopy theory: over $\mathbb{Q}$, have $\widehat{\mathbb{G}}_a \cong \widehat{\mathbb{G}}_m$ (via exp/log) for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an End(E)-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde v}\nabla_{\tilde w} - \nabla_{\tilde w}\nabla_{\tilde v} - \nabla_{[\tilde v, \tilde w]}$ $\tilde v, \tilde w$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{tikzcd}
\mathrm{VBdl}^\nabla_{\mathbb{C}}(M) \arrow[r, "\mathrm{tr}(e^{iF/2\pi})"] \arrow[d, two heads] \arrow[dr, dashed] & \Omega^*_{dR}(M) \\
\mathrm{VBdl}_{\mathbb{C}}(M) \arrow[d] \arrow[dr, dashed] & Z^*_{dR}(M) \arrow[u, hook] \arrow[d, two heads] \\
KU(M) \arrow[r, dashed, "\mathrm{ch}"'] & H^*_{dR}(M)
\end{tikzcd}$$

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:=$ $\mathrm{Proj}_R^{\mathrm{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...): glued together from affines

to define ***algebraic K-theory*** of X, have a choice: not all exact seq's split $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in $\mathrm{Mod}_{\mathbb{Z}}$: does this enforce $[\mathbb{Z}/4] = [\mathbb{Z}/2] + [\mathbb{Z}/2]$ in $\mathrm{K}(\mathbb{Z})$?

make easier choice: $E_0 \rightarrowtail E_1 \twoheadrightarrow E_2$ $\rightsquigarrow$ $[E_1] = [E_0] + [E_2]$

enforce relations *derivedly*: record relations, relations between relations, ... $\rightsquigarrow$ $\mathrm{K}(X)$ a *spectrum* $\approx$ chain complex

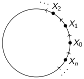
for *any* category $\mathcal{C}$ with "exact sequences", get spectrum $\mathrm{K}(\mathcal{C})$

$$\mathrm{K}(X) := \mathrm{K}(\mathrm{VBdl}(X)) \xrightarrow{\sim} \mathrm{K}(\mathrm{Perf}(X))$$

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal ∞-category, $\mathcal{C}$ a $\mathcal{V}$-enriched ∞-category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

$$\mathrm{HH}(\mathcal{C}) \ := \ \int_{S^1} \mathcal{C} \ :\approx \ \ \operatorname{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right) .$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} =$ spectra $\quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

$\boxed{\S 2.3}$

$$\begin{tikzcd}
\mathrm{Mfld}_n \times \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{V}) \arrow[r, "(M,A) \mapsto \int_M A"] \arrow[d, hook] & \mathcal{V} \\
\mathrm{Mfld}_n \times \mathrm{Cat}_n(\mathcal{V}) \arrow[ur, dashed, "(M,\mathcal{C}) \mapsto \int_M \mathcal{C}"'] &
\end{tikzcd}$$

simplicial	paracyclic	cyclic	epicyclic
Δ^{op}	$\Delta^{\mathrm{op}}_{\circlearrowleft}$	Λ^{op}	$\widetilde{\Lambda}^{\mathrm{op}}$
S^1_*	S^1	S^1 & auto's	S^1 & endo's

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X)\ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong\ k[\varepsilon]/\varepsilon^2\ , \quad |\varepsilon| = 1\ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR}\ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H^*_{dR}(X)$$

if X is a *spectral* scheme (e.g. Spec(R) for R a ring spectrum), define ***topological negative cyclic homology:***

$$\mathrm{THC}^-(X) := \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{THC}^-(X)) \approx H^*_{dR}(X)$$

$\star\star\star$ **from now on, *everything* is spectral** $\star\star\star$

any variety or scheme can be considered as spectral, through the forgetful functor $\mathrm{Mod}_{\mathbb{Z}} \to \mathscr{D}(\mathbb{Z}) \to \mathcal{S}\mathrm{p}$

enter DAG...

Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

$$\mathcal{L}X = \mathrm{map}(S^1, X)\ .$$

$$\begin{tikzcd}
 & S^1 & \mathrm{pt} \arrow[l] & & \mathcal{L}X \arrow[r] \arrow[d] & X \arrow[d] & \\
\text{pushout} & & & {} \arrow[r, mapsto, "\mathrm{map}(-{,}X)"] & {} & & \text{pullback} \\
 & \mathrm{pt} \arrow[uu] & S^0 \arrow[l] \arrow[uu] & & X \arrow[r] & X \times X &
\end{tikzcd}$$

$$\rightsquigarrow \quad \mathscr{O}_{\mathcal{L}X} \simeq \mathscr{O}_X \overset{\mathbb{L}}{\underset{\mathscr{O}_{X\times X}}{\otimes}} \mathscr{O}_X$$

$\rightsquigarrow$ $\mathscr{O}(\mathcal{L}X)$ is computed by the cyclic bar construction for THH!

$\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X)$, $\mathbb{T}$-action ↔ rotation of loops

traces approximate *rationalized* *rationalized* algebraic K-theory...

$$\begin{tikzcd}
\mathrm{K}(X) \arrow[rr, "\text{Dennis trace}"] \arrow[dr, dashed, "\text{cyclic trace}"'] & & \mathrm{THH}(X) & \simeq \mathscr{O}(\mathcal{L}X) \\
& \mathrm{THC}^-(X) \arrow[ur] & \simeq \mathscr{O}(\mathcal{L}X)^{\mathrm{h}\mathbb{T}}
\end{tikzcd}$$

$$E \longmapsto \left(\left(\begin{matrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{matrix} \right) \longmapsto \left(\begin{matrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{matrix} \right) \right)$$

this function is $\mathbb{T}$-invariant!

Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{tikzcd}
\mathrm{K}(R) \arrow[r] \arrow[d] & \mathrm{K}(R_0) \arrow[d] \\
\mathrm{THC}^-(R) \arrow[r] & \mathrm{THC}^-(R_0)
\end{tikzcd}$$

pullback after $- \otimes \mathbb{Q}$.

slogan: vbdl/Spec(R) $\overset{\mathbb{Q}}{\approx}$ restriction to Spec(R_0)
+ compatible trace-of-monodromy function
+ *data* of $\mathbb{T}$-invariance of this function

the integral story...

$$\begin{tikzcd}
\mathrm{K}(X) \arrow[rrr, "\text{Dennis trace}"] \arrow[rrd, "\text{cyclic trace}"] \arrow[rdd, dashed, "\text{cyclotomic trace}"'] & & & \mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) \\
& & \mathrm{THC}^{-}(X) \simeq \mathscr{O}(\mathcal{L}X)^{\mathrm{h}\mathbb{T}} \arrow[ru] & \\
& \mathrm{TC}(X) \simeq ??? \arrow[ru] & &
\end{tikzcd}$$

construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

Thm (Dundas–McCarthy '97). the cyclotomic trace is a *local equivalence* (without rationalization).

"This is how people other than Quillen compute algebraic K-theory."
∼ *A. Blumberg*, algebraic K-theorist

Main Question: In terms of DAG, what is $\mathrm{TC}(X)$?

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$\mathrm{K}(X) \xrightarrow{\text{cyclotomic trace}} \mathrm{TC}(X) := \mathrm{THH}(X)^{\mathrm{hCyc}}$$

THH is a ***cyclotomic spectrum***; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, shift left, "\mathrm{triv}"] \arrow[r, phantom, "\perp"] & \mathrm{Cyc}(\mathrm{Sp}) \arrow[l, shift left, dashed, "(-)^{\mathrm{hCyc}}"] \\
\mathrm{TC}(X) \arrow[u, phantom, "\Cup"] & \mathrm{THH}(X) \arrow[l, mapsto] \arrow[u, phantom, "\Cup"]
\end{tikzcd}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

recall: $\mathrm{THH}(X) \;=\;$ functions on $\mathcal{L}X$

main idea: $\mathrm{TC}(X) \;=\;$ functions on $\mathcal{L}X$ that are:

- invariant under $\mathbb{T}$-action on $\mathcal{L}X$;
- "sensitive" to relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$. **Q.** What does "sensitive" mean?

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \cdot \otimes m_j \quad , \quad \mathrm{tr}(M^{\otimes 2}) = \sum_k m_k \cdot \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is ***norms***: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{tikzcd}
(R \otimes R)_{C_2} \arrow[r, "\mathrm{Nm}"] & (R \otimes R)^{C_2} \\
{[x \otimes y]} \arrow[r, mapsto] & \sum_{\sigma \in C_2} \sigma(x \otimes y)
\end{tikzcd}$$

$\rightsquigarrow$ become equal in the ***Tate construction***, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{\mathrm{t}C_2}$$

over $\mathbb{Q}$, norm an iso! $\rightsquigarrow$ $(R \otimes R)^{\mathrm{t}C_2} = 0$, assertion is vacuous

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

main point: for $\mathcal{C}$ a spectrally enriched ∞-category,
$S^1_b \xleftarrow{r} S^1_a$ a covering map of framed circles,
get ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

doesn't exist over $\mathbb{Z}$! only have "Tate diagonal" in $\mathcal{S}p$, not $\mathscr{D}(\mathbb{Z})$.

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{\mathrm{h}\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right)$$

$$:= \mathrm{Fun}(\mathrm{B}\mathbb{T}, \mathrm{Sp})$$

⋆ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{\mathrm{h}\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{tikzcd}
T \arrow[r, "\sigma_r"] \arrow[d, "\sigma_{rs}"'] & T^{\tau C_r} \arrow[d, "(\sigma_s)^{\tau C_r}"] \\
T^{\tau C_{rs}} \arrow[r, "\mathrm{can.}"'] & (T^{\tau C_s})^{\tau C_r}
\end{tikzcd}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

§2.1 stratification of a scheme/stack Y ⇝ *generalized recollement:* $\mathrm{QC}(Y) \simeq \lim^{\mathbf{r.lax}}(\ldots\mathrm{l.lax}\ldots)$.

§2.2 conjecture: there's a stratified derived stack $\mathscr{B}\mathbb{T}$ with $\mathbb{N}^{\times}$-action such that

$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathbf{h}\mathbb{N}^{\times}} ,$$

and $\mathcal{L}X$ enhances to $\mathbb{N}^{\times}$-equivariant $\mathscr{L}X \downarrow \mathscr{B}\mathbb{T}$ such that $\mathrm{THH}(X) \leftrightarrow \mathscr{O}_{\mathscr{L}X}$.

the geometry of the cyclotomic trace:

$$\begin{tikzcd}
\mathrm{K}(X) \arrow[rrr, "\text{Dennis trace}"] \arrow[drr, "\text{cyclic trace}"] \arrow[ddr, dashed, "\text{cyclotomic trace}"'] & & & \mathrm{THH}(X) & \simeq \mathscr{O}(\mathcal{L}X) \\
& & \mathrm{THC}^{-}(X) \arrow[ur] & \simeq \mathscr{O}(\mathcal{L}X)^{\mathrm{h}\mathbb{T}} & \\
& \mathrm{TC}(X) \arrow[ur] & \simeq \mathscr{O}(\mathcal{L}X)^{\mathrm{hCyc}} & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, shift left, "\mathrm{triv}"] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathrm{Cyc}(\mathrm{Sp}) \arrow[l, shift left, dashed, "(-)^{\mathrm{hCyc}}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\mathrm{TC}(X) & \mathrm{THH}(X) \arrow[l, mapsto]
\end{tikzcd}$$

$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T \,,\, \left\{ \begin{tikzcd} T \arrow[d, "\sigma_r"] \\ T^{\tau C_r} \end{tikzcd} \right\}_{r\in\mathbb{N}^\times} , \left\{ \begin{tikzcd} T \arrow[r, "\sigma_r"] \arrow[d, "\sigma_{rs}"'] & T^{\tau C_r} \arrow[d, "(\sigma_s)^{\tau C_r}"] \\ T^{\tau C_{rs}} \arrow[r, "\mathrm{can.}"'] & (T^{\tau C_s})^{\tau C_r} \end{tikzcd} \right\}_{r,s\in\mathbb{N}^\times} , \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible" (+ higher coherences)...

...which is precisely the structure present on trace-of-monodromy functions of vector bundles! $\mathbf{tr}(M)^r \equiv \mathbf{tr}(M^r)$, ...

talk 1

- §1.1 traces in differential geometry
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talk 2

- §2.1 stratified schemes and generalized recollements (interlude)
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- §2.3 the construction: factorization homology of enriched ∞-categories

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$\begin{tikzcd}
Z \arrow[rr, hook, "i", "\text{closed}"'] \arrow[dr, hook] & & X & U \arrow[l, hook', "j"', "\text{open}"] \\
& X^{\wedge}_{Z} \arrow[ur, "\hat{\imath}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}^0 \arrow[r, hook] & \mathbb{A}^1 \arrow[d, equal] & \mathbb{G}_m \arrow[l, hook'] \\
\mathrm{Spec}(\, k & k[x] \arrow[l, "0 \mapsfrom x"'] \arrow[r, hook] & k[x^{\pm}]\,)
\end{tikzcd}$$

Def. qcoh over X ***supported on*** Z**:** $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{QC}_Z(X) \arrow[r, hook, shift left=2] \arrow[r, phantom, "\perp"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{QC}(X) \arrow[l, shift left=2, "\ker(\mathrm{id}\to j_* j^*)"] \arrow[r, bend left=50, "j^*"] \arrow[r, bend right=50, "j^?"'] \arrow[dl, shift right=2, "\hat{\imath}^*"'] \arrow[dl, phantom, "\perp"] & \mathrm{QC}(U) \arrow[l, hook', "j_*" description] \arrow[l, phantom, shift left=4, "\perp"] \arrow[l, phantom, shift right=4, "\perp"] \\
\mathrm{QC}(X^{\wedge}_{Z}) \arrow[ur, hook, shift right=2, "\hat{\imath}_*"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Mod}^{(x)\text{-tors}}_{k[x]} \arrow[r, hook, shift left=2] \arrow[r, phantom, "\perp"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Mod}_{k[x]} \arrow[l, shift left=2] \arrow[r, bend left=50, "-\otimes_{k[x]} k[x^{\pm}]"] \arrow[r, bend right=50, "\hom_{k[x]}(k[x^{\pm}]{,}-)"'] \arrow[dl, shift right=2, "M^{\wedge}_{(x)} \mapsfrom M"'] \arrow[dl, phantom, "\perp"] & \mathrm{Mod}_{k[x^{\pm}]} \arrow[l, hook', "\mathrm{fgt}" description] \arrow[l, phantom, shift left=4, "\perp"] \arrow[l, phantom, shift right=4, "\perp"] \\
\mathrm{Mod}^{(x)\text{-compl}}_{k[\![x]\!]} \arrow[ur, hook, shift right=2, "\mathrm{fgt}"'] & &
\end{tikzcd}$$

$\mathrm{Mod}^{(x)\textbf{-tors}}_{k[x]} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n\geq\mathbf{1}}$

$\mathrm{Mod}^{(x)\textbf{-compl}}_{k[\![x]\!]} \simeq \lim(\cdots \to \mathrm{Mod}_{k[x]/x^{\mathbf{2}}} \to \mathrm{Mod}_{k[x]/x}) \quad \rightsquigarrow \quad \{- \otimes_{k[x]} k[x]/x^n\}_{n\geq\mathbf{1}} \dashv$ "lim"

add pix here!!!

so, get a ***recollement*** of $\mathrm{QC}(X)$: $\begin{tikzcd} \mathrm{QC}(X^{\wedge}_{Z}) \arrow[r, hook, bend left=40, "\hat{\imath}_?"] \arrow[r, hook, bend right=40, "\hat{\imath}_*"'] \arrow[r, phantom, bend left=20, "\perp"] \arrow[r, phantom, bend right=20, "\perp"] & \mathrm{QC}(X) \arrow[l, "\hat{\imath}^*" description] \arrow[r, bend left=40, "j^*"] \arrow[r, bend right=40, "j^?"'] \arrow[r, phantom, bend left=20, "\perp"] \arrow[r, phantom, bend right=20, "\perp"] & \mathrm{QC}(U) \arrow[l, hook', "j_*" description] \end{tikzcd}$

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

$$\text{pullback} \qquad \begin{tikzcd} F \arrow[r, "\eta_j(F)"] \arrow[d, "\eta_{\hat{\imath}}(F)"'] & j_* F_1 \arrow[d, "j_* j^* (\eta_{\hat{\imath}}(F)) = j_*(\mu)"] \\ \hat{\imath}_* F_0 \arrow[r, "\eta_j(\hat{\imath}_* F_0)"'] & j_* j^* \hat{\imath}_* F_0 \end{tikzcd} .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements: μ = ***monodromy*** = "gluing data"

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\mathrm{QC}(X^{\wedge}_{Z}) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right) := \left\{ \left(F_0 \in \mathrm{QC}(X^{\wedge}_{Z}) \ , \ F_1 \in \mathrm{QC}(U) \ , \ \begin{tikzcd} F_1 \arrow[d, "\mu"'] \\ j^* \hat{\imath}_* F_0 \end{tikzcd} \right) \right\} .$$

given $\mathcal{I} \xrightarrow{\mathcal{C}_\bullet}$ Cat, an obj of $\begin{cases} \text{strict} \\ \text{left-lax} \\ \text{right-lax} \end{cases}$ limit is: $\{x_i \in \mathcal{C}_i\}_{i \in \mathcal{I}}$ $+$ $\forall\ i \xrightarrow{\varphi} j$ in $\mathcal{I}$, $\begin{cases} \mathcal{C}_\varphi(x_i) \simeq x_j \\ \mathcal{C}_\varphi(x_i) \rightarrow x_j \\ \mathcal{C}_\varphi(x_i) \leftarrow x_j \end{cases}$

$+$ higher coherence data

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \mathrm{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{tikzcd} & \mathrm{QC}((X\backslash Z)^{\wedge}_{(Y\backslash Z)}) \arrow[dr, "\beta"] & \\ \mathrm{QC}(X^{\wedge}_{Z}) \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & \arrow[u, phantom, "\Uparrow \eta"] & \mathrm{QC}(X\backslash Y) \end{tikzcd} \right) .$$

$$\begin{tikzcd} & F_1 \arrow[d, "\mu_{01}"] & & \\ & \alpha F_0 & & \\ F_0 & & F_2 \arrow[r, "\mu_{12}"] \arrow[d, "\mu_{02}"'] \arrow[dr, phantom, "\circlearrowleft \ \mu_{012}"] & \beta F_1 \arrow[d, "\beta(\mu_{01})"] \\ & & \gamma F_0 \arrow[r, "\eta(F_0)"'] & \beta\alpha F_0 \end{tikzcd}$$

add pix here!!!

simpler: $\lim^{\text{l.lax}} \left(\begin{tikzcd} & \mathcal{C}_1 \arrow[dr, "\beta"] & \\ \mathcal{C}_0 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & \arrow[u, phantom, "\Uparrow \eta"] & \mathcal{C}_2 \end{tikzcd} \right) \simeq \left\{ \left(\begin{tikzcd} & x_1 & \\ & \alpha x_0 \arrow[u, "\mu_{01}"'] & \\ x_0 & x_2 & \beta x_1 \arrow[l, "\mu_{12}"'] \end{tikzcd} \right) \right\}$ because now $\begin{tikzcd} x_2 & \beta x_1 \arrow[l, "\mu_{12}"'] \\ \gamma x_0 \arrow[u, dashed, "\exists! \ \mu_{02}"] \arrow[r, "\eta(x_0)"'] & \beta\alpha x_0 \arrow[u, "\beta(\mu_{01})"'] \end{tikzcd}$

⋆ above, need μ_{012} because laxness of diagram $\neq$ laxness of limit

the general picture:

fix any scheme X and any set of closed subschemes including X.

order by inclusion $\rightsquigarrow$ determines a poset P

$\rightsquigarrow$ a ***stratification*** of X over P: a map of posets

$$\mathrm{P} \xrightarrow{X_\bullet} \{\text{closed subschemes of } X\} .$$

notation: $X_{<p} := \bigcup_{q<p} X_q$.

assumption: P satisfies *descending chain condition*.

Thm (A–M-G–R). $\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\begin{array}{ccc} \mathrm{P} & \xrightarrow{\quad \mathrm{l.lax} \quad} & \mathrm{Cat} \\ p & \longmapsto & \mathrm{QC}((X \backslash X_{<p})^{\wedge}_{(X_p \backslash X_{<p})}) \end{array} \right)$.

§2.2: THE GEOMETRY OF TC

recall: THH $\in \mathrm{Cyc}(\mathcal{S}\mathrm{p})$ a *cyclotomic spectrum*, TC $:= \mathrm{THH}^{\mathbf{hCyc}}$.

cyclotomic spectra defined in terms of

cyclonic spectra := genuine-proper $\mathbb{T}$-spectra $:= \mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}}$.

remember "genuine" fixedpoints for proper subgroups of $\mathbb{T}$. ***genuine***: e.g. $EG \xrightarrow{\not\sim} \mathrm{pt}$.

Thm (A–M-G–R). $$\mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}} \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\begin{tikzcd} \mathbb{N}^{\mathrm{div}} \arrow[rr, "\mathrm{l.lax}"] & & \mathrm{Cat} & \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ i \arrow[d, "j"'] & {} \arrow[d, phantom, "\longmapsto"] & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/\mathrm{C}_i)} \arrow[d, "(-)^{\tau \mathrm{C}_j}"] & \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \arrow[l, hook', "\sim"'] \\ ij & {} & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/\mathrm{C}_{ij})} \arrow[r, "\sim"'] & \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \end{tikzcd} \right) .$$

$\mathbb{N}^{\mathbf{div}} \cong$ poset of proper closed subgroups of $\mathbb{T}$

$i \leftrightarrow \mathrm{C}_i$

Larger thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}G}$ is stratified over poset of closed subgroups of G up to subconjugacy.

Conj. there's an $\mathbb{N}^{\mathrm{div}}$-stratified stack $\mathscr{B}\mathbb{T}$, a "mapping telescope" of copies of $B\mathbb{T}$, such that $\mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}} \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})$ compatibly with the above generalized recollement.

Thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \xrightarrow{\sim} \lim^{\mathbf{r.lax}}\big(\ \mathbb{N}^{\mathbf{div}}\mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ \big)$.   ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}$ **strat/**$\mathbb{N}^{\mathbf{div}}$ **s.t.** $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$**...**

"geometric fixedpoints" $\quad \Phi^{C_n} : \mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{\mathrm{g}^{<(\mathbb{T}/C_n)}} \xrightarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}} \quad \rightsquigarrow \quad \mathbb{N}^{\times} \curvearrowright \mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}}$

$\leftrightarrow$ pullback of $\lim^{\mathrm{r.lax}}$ along dilation action $\mathbb{N}^{\times} \curvearrowright \mathbb{N}^{\mathrm{div}}$ $n \mapsto (i \mapsto ni)$

cyclotomic spectra $:= \mathrm{Cyc}(\mathcal{S}\mathrm{p}) := (\mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}})^{\mathrm{h}\mathbb{N}^{\times}} \underset{\boxed{\S 1.3}}{\overset{\textbf{\textit{Cor.}}}{\simeq}} \lim^{\mathrm{r.lax}}\big(\ \mathrm{B}\mathbb{N}^{\times} \xrightarrow{\mathrm{l.lax}} \mathrm{Cat}\ \big)$.

note: $\mathbf{B}\mathbb{N}^{\times} \simeq (\mathbb{N}^{\mathbf{div}})_{\mathbf{h}\mathbb{N}^{\times}}$

...and stratification is $\mathbb{N}^{\times}$-equivariant, so that $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^{\times}}$. Moreover,

$$\begin{tikzcd}
\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \arrow[r, phantom, "\simeq"] \arrow[dr, "(-)^{\mathrm{hCyc}}"'] & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^{\times}} \arrow[d, "\Gamma^{\mathrm{h}\mathbb{N}^{\times}}"] \\
& \mathcal{S}\mathrm{p}
\end{tikzcd}$$

:

homotopy invariants of cyclotomic structures correspond to $\mathbb{N}^{\times}$-equivariant global sections.

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{tikzcd}
\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \arrow[rr, phantom, "\simeq"] \arrow[dr, "(-)^{\mathrm{hCyc}}"'] & & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} \arrow[dl, "\Gamma^{\mathrm{h}\mathbb{N}^\times}"] \\
& \mathcal{S}\mathrm{p} &
\end{tikzcd}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{tikzcd}
\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) \arrow[d] & \\
B\mathbb{T} & K \arrow[l] \arrow[ul, dashed]
\end{tikzcd}
\qquad
\begin{tikzcd}
E\mathbb{T} \arrow[d] & E\mathbb{T} \times_{B\mathbb{T}} K \arrow[l] \arrow[d] \arrow[r, dashed] & X \\
B\mathbb{T} & K \arrow[l] &
\end{tikzcd}$$

$\rightsquigarrow \quad \mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1, X)} = \mathscr{O}_{\mathcal{L}X}$

the cyclotomic structure on THH: using universal circle bundle $\mathscr{E}\mathbb{T} \downarrow \mathscr{B}\mathbb{T}$, get

$$\mathscr{L}X := \mathrm{map}^{\mathrm{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \xrightarrow{\mathbb{N}^\times\text{-eq}} \mathscr{B}\mathbb{T} \qquad \rightsquigarrow \qquad
\begin{tikzcd}
\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \arrow[r, phantom, "\simeq"] & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\mathrm{THH}(X) \arrow[r, leftrightarrow] & \mathscr{O}_{\mathscr{L}X}
\end{tikzcd} .$$

the $\mathbb{N}^{\mathrm{div}}$-stratified stack $\mathscr{L}X \downarrow \mathscr{B}X$ records the symmetries of $\mathcal{L}X$ coming from rotations and covering maps of circles, and $\mathrm{TC}(X)$ is its $\mathbb{T}$- and $\mathbb{N}^\times$-equivariant global functions.

recall §1.3: $\mathrm{TC}(X)$ = functions on $\mathcal{L}X$ that are $\mathbb{T}$-invariant and "sensitive" to covering maps of circles.

§2.3 the ***Witt monoid*** $:= \mathbb{W} := \mathbb{T} \rtimes \mathbb{N}^\times$ (captures "Frobenius" and "Verschiebung" on THH)

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $\mathrm{K} \to \mathrm{TC}$ come from?

mechanism 1: factorization homology....

recall: $\mathbf{THH}(\mathcal{C}) := \int_{S^{\mathbf{1}}}^{\mathcal{S}\mathbf{p}} \mathcal{C}$:= spectrally-enriched factorization homology of $\mathcal{C}$ over $S^{\mathbf{1}}$.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \to M\}$ a covering, $\int_M \mathcal{C} \in \mathscr{D}(\mathrm{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

mechanism 2: linearization....

differential calculus: $(M, p) \rightsquigarrow T_p M$ $\quad\leftrightsquigarrow\quad$ Goodwillie calculus: $(\mathcal{S}, X) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{p}}(X)$

$(\mathcal{S}, \mathrm{pt}) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{p}}(\mathrm{pt}) \simeq \mathcal{S}\mathbf{p}$

...as applied to *fiber bundles* among compact framed manifolds:

A1: covering maps $S^1 \xrightarrow{r} S^1$ (for $r \in \mathbb{N}^{\times}$).

A2: the fiber bundle $S^1 \to \mathbb{D}^0$, and its invariance:

$$\begin{tikzcd} S^1 \arrow[rr, "r"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & S^1 \arrow[dl] \\ & \mathbb{D}^0 & \end{tikzcd} .$$

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

❶

$$\begin{tikzcd} & \mathrm{Cat}(\mathcal{S}) \arrow[r, dashed] \arrow[dr, "\mathrm{THH}_{\mathcal{S}}"'] & \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) \arrow[d, "\mathrm{fgt}"] \\ \textbf{\textit{diagonal package}} \text{ for spaces} \arrow[ur, squiggly] & & \mathcal{S} \end{tikzcd}$$

packages *diagonal maps* for spaces: $X \rightarrow (X^{\times r})^{\mathrm{h}\mathrm{C}_r}$

$\mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) := \mathrm{Fun}(\mathrm{B}\mathbb{W}, \mathcal{S})$:= “unstable cyclotomic spaces”

$B\mathbb{W}$:= (objects: framed S^1's , morphisms: covering maps)

❷

$$\begin{tikzcd} \textbf{\textit{diagonal package}} \text{ for spaces} \arrow[d, squiggly, "\text{linearization}"'] & & \\ & \mathrm{Cat}(\mathrm{Sp}) \arrow[r, dashed] \arrow[dr, "\mathrm{THH}"'] & \mathrm{Cyc}(\mathrm{Sp}) \arrow[d, "\mathrm{fgt}"] \\ \textbf{\textit{Tate package}} \text{ for spectra} \arrow[ur, squiggly] & & \mathrm{Sp} \end{tikzcd}$$

packages *Tate diagonal maps* for spectra: $E \rightarrow (E^{\otimes r})^{\tau \mathrm{C}_r}$

$\mathrm{Cyc}(\mathrm{Sp})$:= cyclotomic spectra

Q2: where does the cyclotomic trace $\mathrm{K} \to \mathrm{TC}$ come from?

A2: factorization homology + linearization + the fiber bundle $S^{\mathbf{1}} \downarrow \mathbb{D}^{\mathbf{0}}$.

Thm (A–M-G–R).

the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

$$\max^{\ell} \text{ subgpd of } \mathcal{C} := \iota\mathcal{C} \simeq \int_{\mathbb{D}^{\mathbf{0}}} \mathcal{C} \longrightarrow \left(\int_{S^{\mathbf{1}}} \mathcal{C} \right)^{\mathsf{h}\mathbb{W}} =: \mathsf{THH}_{\mathcal{S}}(\mathcal{C})^{\mathsf{h}\mathbb{W}} =: \mathsf{TC}^{\mathsf{h}}_{\mathcal{S}}(\mathcal{C})$$

input: the fiber bundle $S^{\mathbf{1}} \downarrow \mathbb{D}^{\mathbf{0}}$ is invariant for the $\mathbb{W}$-action on $S^{\mathbf{1}}$

$$\begin{tikzcd} {} \arrow[d, squiggly, "\text{linearization}"] \\ {} \end{tikzcd}$$

the ***cyclotomic trace***: for $\mathcal{C}$ a stable ∞-category,

$$\mathsf{K}(\mathcal{C}) \longrightarrow \mathsf{THH}(\mathcal{C})^{\mathsf{hCyc}} =: \mathsf{TC}(\mathcal{C})$$

input: K-theory is the ***universal additive invariant*** of stable ∞-categories (Blumberg–Gepner–Tabuada)

enriched factorization homology:

given:

- $M \in \mathrm{Mfld}_1 \mathrm{Mfld}_1$ a *1-manifold*
- $\mathcal{C} \in \mathrm{Cat}(\mathcal{V}) \mathrm{Cat}(\mathcal{V})$ a *$\mathcal{V}$-enriched ∞-category*

want to define: $\int_M \mathcal{C} \in \mathcal{V}$. first define both of these terms...

idea: this will be a colimit in $\mathcal{V}$

- over *disk-refinements* R of M
- of compatible labelings λ
  - of the vertices $R^{(0)}$ by objects of $\mathcal{C}$ and
  - of the edges $R^{(1)}$ by morphisms in $\mathcal{C}$.

$$\int_M \mathcal{C} := \operatorname*{colim}_{(R \to M) \in \mathcal{D}(M)} \left(\operatorname*{colim}_{R^{(0)} \xrightarrow{\lambda} \iota\mathcal{C}} \left(\boxtimes_{e \in R^{(1)}} \underline{\hom}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right) .$$

slogan: *enriched* factorization homology via *categorified* factorization homology.

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

- obj: $\displaystyle\coprod_{\text{finite}}$ (finite directed graphs) $\coprod$ (framed S^1's) ,
- mor: TBD (to be described).

given $M \in \mathcal{M}$, its category $\mathcal{D}(M)$ of ***disk-refinements*** has:

- obj: config$^{\text{ns}}$ of dots in 1-dim$^{\ell}$ strata of M s.t. $\nexists$ smooth S^1's;
- mor: disappearances, anticollisions, and isotopies.

add pix here!!! $\mathcal{D}(S^{\mathbf{1}}) = \Delta^{\mathbf{op}}_{\circlearrowleft}$, full diagram

$$\begin{tikzcd}
\Delta^{\mathbf{op}}_{\circlearrowleft} \arrow[r, two heads] \arrow[d] & \Lambda^{\mathbf{op}} \arrow[r, hook, two heads] \arrow[d] & \widetilde{\Lambda}^{\mathbf{op}} \arrow[r, hook, "\mathbf{ff}"] \arrow[d] & \mathcal{D} \arrow[d] \\
\{S^{\mathbf{1}}\} \arrow[r, two heads] & B\mathbb{T} \arrow[r, hook, two heads] \arrow[d] & B\mathbb{W} \arrow[r, hook, "\mathbf{ff}"'] \arrow[d] & \mathcal{M} \\
 & \mathbf{pt} \arrow[r, hook, two heads] & \mathrm{B}\mathbb{N}^{\times} &
\end{tikzcd}$$

$M \in \mathcal{M}$, $\mathcal{C} \in$ **Cat**$(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat})$ $\leftrightsquigarrow$ bar construction $\mathfrak{B}\mathcal{V} : \underset{[n] \longmapsto \mathcal{V}^{\times n}}{\Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat:
for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;
- ∞-category of morphisms: $\underline{\hom}_{\mathfrak{B}\mathcal{V}}(\mathrm{pt}, \mathrm{pt}) \simeq \mathcal{V}$;
- composition: $\mathcal{V} \times \mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$.

another category object in Cat (in fact in $\mathcal{S}$): for any space X, the ***codiscrete*** category object $\mathrm{cd}(X) : \underset{[n] \longmapsto X^{\times(n+1)}}{\Delta^{\mathrm{op}} \longrightarrow \mathcal{S}}$.

- space of objects: $\mathrm{cd}(X)_0 = X^{\times 1} = X$;
- space of morphisms: $\underline{\hom}_{\mathrm{cd}(X)}(x, y) = \mathrm{pt}$; hence "codiscrete"
- composition: $\mathrm{pt} \times \mathrm{pt} \xrightarrow{\sim} \mathrm{pt}$.

Def (Gepner–Haugseng). a ***$\mathcal{V}$-enriched ∞-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{tikzcd}
\mathrm{cd}(\iota\mathcal{C}) \arrow[rr, "\underline{\mathrm{hom}}_{\mathcal{C}}"] & & \mathfrak{B}\mathcal{V} & \Delta^{\mathrm{op}} \\
\iota\mathcal{C} \arrow[rr, "!"] \arrow[dd, "x\mapsto(x,x)"'] & & \mathrm{pt} \arrow[dd, "\mathbb{1}_{\mathcal{V}}"] & {[0]} \arrow[dd, "\sigma_0"] \\
 & \Swarrow & & \\
(\iota\mathcal{C})^{\times 2} \arrow[rr, "(x,y)\mapsto\underline{\mathrm{hom}}_{\mathcal{C}}(x,y)"] & & \mathcal{V} & {[1]} \\
 & \Nwarrow & & \\
(\iota\mathcal{C})^{\times 3} \arrow[rr, "(x,y,z)\mapsto(\underline{\mathrm{hom}}_{\mathcal{C}}(x,y)\text{,}\underline{\mathrm{hom}}_{\mathcal{C}}(y,z))"] \arrow[uu, "(x,y,z)\mapsto(x,z)"] & & \mathcal{V}^{\times 2} \arrow[uu, "\boxtimes"'] & {[2]} \arrow[uu, "\delta_1"']
\end{tikzcd}$$

$\mathbb{1}_{\mathcal{V}} \to \underline{\mathrm{hom}}_{\mathcal{C}}(x,x)$ $\underline{\mathrm{hom}}_{\mathcal{C}}(x,y) \boxtimes \underline{\mathrm{hom}}_{\mathcal{C}}(y,z) \to \underline{\mathrm{hom}}_{\mathcal{C}}(x,z)$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

recall slogan: *enriched* factorization homology via *categorified* factorization homology.

the ***Grothendieck construction***: for a category object $\mathcal{Y}$ in Cat,

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\int_{(-)}\mathcal{Y}"] & \mathrm{Cat} & \leftrightsquigarrow & \int_{|\mathcal{M}}\mathcal{Y} \arrow[d] \\ & & & \mathcal{M} \end{tikzcd}$$

“covariant bundle
of ∞-categories”

$$\int_M \mathcal{C} := \operatorname{colim}\left(\int_{|\mathcal{D}(M)} \mathrm{cd}(\iota\mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\mathrm{hom}}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right)$$

in general, requires $\mathcal{V}$ to be *symmetric* monoidal
(or e.g. only *cyclically* monoidal if $M = S^{\mathbf{1}}$)

add pix here!!!

rigorizes heuristic formula $\int_M \mathcal{C} := \operatorname*{colim}_{(R \to M) \in \mathcal{D}(M)} \left(\operatorname*{colim}_{R^{(\mathbf{0})} \xrightarrow{\lambda} \iota\mathcal{C}} \left(\mathop{\boxtimes}\limits_{e \in R^{(\mathbf{1})}} \underline{\mathrm{hom}}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right)$

$$\int_M \mathcal{C} := \mathbf{colim}\left(\int_{|\mathcal{D}(M)} \mathbf{cd}(\iota\mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\mathbf{hom}}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right)$$

equivalently,

$$\begin{tikzcd}
\int_{|\mathcal{D}(M)} \mathrm{cd}(\iota\mathcal{C}) \arrow[r, "\int_{|\mathcal{D}(M)} \underline{\mathrm{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \arrow[r, "\boxtimes"] & \mathcal{V} \\
\mathcal{D}(M) \arrow[d] \arrow[rru, bend right=15, "\text{fiberwise colim}"] & & \\
\mathrm{pt} \arrow[rruu, bend right=20, "\text{(fiberwise) colim}", "\int_M \mathcal{C}"'] & &
\end{tikzcd}$$

e.g. $\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} \simeq \mathrm{colim}(\Delta^{\mathrm{op}}_{\circlearrowleft} \to \mathcal{V}) \simeq \mathrm{colim}(B^{\mathrm{cyc}}\mathcal{C} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}_{\circlearrowleft} \to \mathcal{V})$

recall: $\Delta^{\mathbf{op}}_{\circlearrowleft} := \mathcal{D}(S^{\mathbf{1}})$, $\Delta^{\mathbf{op}} \simeq \mathcal{D}(S^{\mathbf{1}}_*)$

cyclic bar construction (classical def$^{\mathbf{n}}$ of HH)

fact: $\Delta^{\mathbf{op}} := \mathcal{D}(S^{\mathbf{1}}_*) \to \mathcal{D}(S^{\mathbf{1}}) =: \Delta^{\mathbf{op}}_{\circlearrowleft}$ is *final*

$$\begin{tikzcd}
\int_{|\Delta^{op}_{\circlearrowleft}} \mathrm{cd}(\iota\mathcal{C}) \arrow[r, "\int_{|\Delta^{op}_{\circlearrowleft}} \underline{\mathrm{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\Delta^{op}_{\circlearrowleft}} \mathfrak{B}\mathcal{V} \arrow[r, "\boxtimes"] & \mathcal{V} \\
\Delta^{op}_{\circlearrowleft} \arrow[d] \arrow[urr, bend right, "\text{fiberwise colim}"] & & \\
\mathrm{pt} \arrow[uurr, bend right, "\text{(fiberwise) colim}", "\mathrm{HH}(\mathcal{C})"'] & &
\end{tikzcd}$$

build in functoriality for rotations of S^1:

$$\begin{tikzcd}
 & \int_{|\Delta^{op}_{\circlearrowleft}} \mathrm{cd}(\iota\mathcal{C}) \arrow[r] \arrow[d] & \int_{|\Lambda^{op}} \mathrm{cd}(\iota\mathcal{C}) \arrow[r, "\int_{|\Lambda^{op}} \underline{\mathrm{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V} \arrow[r, "\boxtimes"] & \mathcal{V} \\
\text{pullbacks} & \Delta^{op}_{\circlearrowleft} \arrow[r] \arrow[d] & \Lambda^{op} \arrow[d] \arrow[urr, bend right, "\text{fiberwise colim}"] & & \\
 & \{S^1\} \arrow[r] & B\mathbb{T} \arrow[uurr, bend right, "\text{fiberwise colim}", "\mathbb{T}\curvearrowright \mathrm{HH}(\mathcal{C})"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}) \arrow[r, "\int_{|\Lambda^{\mathrm{op}}} \underline{\hom}_{\mathcal{C}}"] \arrow[d] & \int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \arrow[r, "\boxtimes"] & \mathcal{V} \\
\Lambda^{\mathrm{op}} \arrow[rru, bend right=10, "\text{fiberwise colim}"] \arrow[d] & & \\
B\mathbb{T} \arrow[rruu, bend right=20, "\text{fiberwise colim}", "\mathbb{T}\curvearrowright \mathrm{HH}(\mathcal{C})"'] & &
\end{tikzcd}$$

attempt to build in functoriality for self-coverings of S^1:

$$\begin{tikzcd}
\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}) \arrow[r, "\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \underline{\hom}_{\mathcal{C}}"] \arrow[d] & \int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \arrow[r, dashed, "\boxtimes", "\nexists"'] & \mathcal{V} \\
\widetilde{\Lambda}^{\mathrm{op}} \arrow[rru, bend right=10, "\text{fiberwise colim}"] \arrow[d] & & \\
B\mathbb{W} \arrow[rruu, bend right=20, "\text{fiberwise colim}", "\mathbb{W}\curvearrowright \mathrm{HH}^{\times}(\mathcal{C})"'] & &
\end{tikzcd}$$

only exists when $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal! **notation: $\mathrm{HH}^{\times}$**

e.g. asks for maps $V \to V^{\boxtimes r}$, or more generally $\bigboxtimes_i V_i \to \bigboxtimes_i (V_i)^{\boxtimes r}$

add pix here!!!

this functor $\prod$ is the ***diagonal package***, and the $\mathbb{W}$-action on $\mathrm{HH}^{\times}(\mathcal{C})$ is its ***unstable cyclotomic structure***. $\mathrm{Cyc}^{\mathbf{h}}(\mathcal{V}) := \mathrm{Fun}(B\mathbb{W}, \mathcal{V})$

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$

$$\begin{tikzcd}
\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}) \arrow[r, "\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \underline{\mathrm{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \arrow[r, "\prod"] & \mathcal{V} \\
\widetilde{\Lambda}^{\mathrm{op}} \arrow[urr, bend right=10, "\text{fiberwise colim}"] \arrow[d] & & \\
B\mathbb{W} \arrow[uurr, bend right=20, "\text{fiberwise colim}", "\mathbb{W} \curvearrowright \mathrm{HH}^{\times}(\mathcal{C})"'] & &
\end{tikzcd}$$

to understand the diagonal package, work “locally”: equivalently,

$$\begin{tikzcd}
{} & \int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \arrow[r, dashed, "\prod"] \arrow[dl] & \underline{\mathcal{V}} := \mathcal{V} \times \mathrm{B}\mathbb{N}^{\times} \arrow[dd] \\
\widetilde{\Lambda}^{\mathrm{op}} \arrow[d] & & \\
B\mathbb{W} \arrow[rr] & & \mathrm{B}\mathbb{N}^{\times}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{rel}}_{/\mathrm{B}\mathbb{N}^{\times}}\left(\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}\right) \arrow[d] \\
\mathrm{B}\mathbb{N}^{\times} \arrow[u, dashed, bend left, "\prod"]
\end{tikzcd}$$

contravariant bundle of ∞-categories: defines right $\mathbb{N}^{\times}$-action on fiber $\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V})$, where $r \in \mathbb{N}^{\times}$ acts as $F \mapsto (F(\pi_r^*(-)))^{\mathrm{hC}_r}$

$\int_{|\Delta^{\mathbf{op}}_{\circlearrowleft}} \mathfrak{B}\mathcal{V} \xrightarrow{\pi_r^*} \int_{|\Delta^{\mathbf{op}}_{\circlearrowleft}} \mathfrak{B}\mathcal{V}$ **induced by $S^{\mathbf{1}} \xleftarrow{r} S^{\mathbf{1}}$** **add pix here!!!**

action by *homotopy* fixedpoints, so denote by $\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}$;

$(\prod, \mathrm{h}) := \prod$ an object of its (in fact strict!) limit **strict because $V \xrightarrow{\sim} (V^{\times r})^{\mathbf{hC}_r}$**

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$

$$\begin{tikzcd}
& \int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \arrow[r, dashed, "\Pi"] \arrow[dl] & \underline{\mathcal{V}} := \mathcal{V} \times \mathrm{B}\mathbb{N}^{\times} \arrow[dd] \\
\widetilde{\Lambda}^{\mathrm{op}} \arrow[d] & & \\
B\mathbb{W} \arrow[rr] & & \mathrm{B}\mathbb{N}^{\times}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{rel}}_{/\mathrm{B}\mathbb{N}^{\times}}\left(\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}\right) \arrow[d] \\
\mathrm{B}\mathbb{N}^{\times} \arrow[u, dashed, bend left, "\Pi"]
\end{tikzcd}$$

$(\prod, \mathrm{h}) \in \lim \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times} \right)$

recall construction of unstable cyclotomic structure $\mathbb{W} \curvearrowright \mathrm{HH}^{\times}$:

$$\begin{tikzcd}
\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}) \arrow[r, "\int_{|\widetilde{\Lambda}^{\mathrm{op}}} \underline{\mathrm{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\widetilde{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \arrow[r, dashed, "\prod"] & \underline{\mathcal{V}} \arrow[dd] \\
\widetilde{\Lambda}^{\mathrm{op}} \arrow[d] & \arrow[r, phantom, "\mathbb{W} \curvearrowright \mathrm{HH}^{\times}(\mathcal{C})"] & \\
B\mathbb{W} \arrow[rr] \arrow[uurr, bend right, "\text{fiberwise colim}"] & & \mathrm{B}\mathbb{N}^{\times}
\end{tikzcd}$$

rephrase in language of $\mathbb{N}^{\times}$-actions:

$$\begin{tikzcd}
\left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) \arrow[r, "\underline{\mathrm{hom}}_{\mathcal{C}}^{*}"] & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{V}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) \arrow[r, dashed, shift left, "\mathrm{r.lax}"] \arrow[r, phantom, "\perp"] & \left(\mathrm{Fun}(B\mathbb{T}, \mathcal{V}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) \arrow[l, shift left] \\
\lim(\bullet) & & \lim^{\mathrm{r.lax}}(\bullet) =: \mathrm{Cyc}^{\mathrm{h}}(\mathcal{V}) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
(\prod, \mathrm{h}) \arrow[rr, mapsto] & & (\mathbb{W} \curvearrowright \mathrm{HH}^{\times}(\mathcal{C}))
\end{tikzcd}$$

$$\begin{tikzcd}
& (\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \arrow[r, "\mathrm{hom}^{*}_{\mathcal{C}}"] & (\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \arrow[r, dashed, shift left, "\mathrm{r.lax}"] \arrow[r, phantom, "\perp"] & (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \arrow[l, shift left] \\
(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) = (\mathcal{S}, \times) & \lim(\bullet) & & \lim^{\mathrm{r.lax}}(\bullet) =: \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) \\
& \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
& (\prod, \mathrm{h}) \arrow[rr, mapsto] & & (\mathbb{W} \curvearrowright \mathrm{HH}^{\times}(\mathcal{C}))
\end{tikzcd}$$

so, diagonal package for spaces induces unstable cyclotomic structure.

completely analogously, Tate package for spectra induces cyclotomic structure:

$$\begin{tikzcd}
(\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}) \arrow[r, "\mathrm{hom}^{*}_{\mathcal{C}}"] & (\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}) \arrow[r, dashed, shift left, "\mathrm{r.lax}"] \arrow[r, phantom, "\perp"] & (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}) \arrow[l, shift left] \\
\lim^{\mathrm{r.lax}}(\bullet) & & \lim^{\mathrm{r.lax}}(\bullet) =: \mathrm{Cyc}(\mathcal{S}\mathrm{p}) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
(\bigotimes, \tau) \arrow[rr, mapsto] & & (\mathrm{Cyc} \curvearrowright \mathrm{THH}(\mathcal{C}))
\end{tikzcd}$$

Q.: where does the Tate package come from?

recall **A:**

$$\begin{tikzcd}
(\prod, \mathrm{h}) \arrow[r, squiggly, "\text{linearization}"] & (\bigotimes, \tau) \\
\text{diagonal package for spaces} & \text{Tate package for spectra}
\end{tikzcd}$$

$$\begin{tikzcd}(\prod, h) \arrow[r, squiggly, "\text{linearization}"] & (\bigotimes, \tau)\end{tikzcd}$$

diagonal package
for spaces

Tate package
for spectra

three ingredients:

(1) an action $\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

(2) a functor $(\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \xrightarrow{\mathrm{r.lax}} (\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times})$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors***

$$\begin{tikzcd}
\mathcal{V}^{\times R^{(1)}} \arrow[r] \arrow[d] \arrow[rr, dashed, bend left] & \int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \arrow[r] \arrow[d] & \mathcal{S}\mathrm{p} \\
\{R \to S^1\} \arrow[r] & \Lambda^{\mathrm{op}} &
\end{tikzcd}$$

multilinear: preserves finite colimits separately in each variable*

which is *stable* under action $\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

* sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathrm{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$)

$$\begin{tikzcd}[column sep=huge]
\lim \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}\right)_{\mathrm{h}} \right) & \lim^{\mathrm{r.lax}} \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}\right)_{\tau} \right) \\
\left(\prod, \mathrm{h}\right) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, rightsquigarrow, "\text{linearization}"] & \left(\bigotimes, \tau\right) \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

diagonal package
for spaces

Tate package
for spectra

(1) an action $\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

(2) a functor $\left(\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) \xrightarrow{\mathrm{r.lax}} \left(\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right)$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors*** which is stable under $\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathrm{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$)

$$\begin{tikzcd}
\left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) \arrow[r, "\Sigma^{\infty}_{+} \circ -", "\mathrm{r.lax}"'] & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}\mathrm{p}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) \arrow[r, "(2)", "\mathrm{r.lax}"'] & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \arrow[l, "-\circ\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\Sigma^{\infty}_{+}\right)"'] \\
 & & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \arrow[u, hook] & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \arrow[u, hook] \arrow[l, "\sim"'] \\
\left(\prod, \mathrm{h}\right) \arrow[r, mapsto] & \left(\bigotimes \circ \int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\Sigma^{\infty}_{+}, \mathrm{h}\right) \arrow[dr, mapsto] & & \\
 & & \left(\bigotimes \circ \int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\Sigma^{\infty}_{+}, \tau\right) \arrow[r, mapsto] & \left(\bigotimes, \tau\right)
\end{tikzcd}$$

$\mathcal{S} \xrightarrow{\Sigma^{\infty}_{+}} \mathcal{S}\mathrm{p}$ **is symmetric monoidal** $\rightsquigarrow$

$$\begin{tikzcd}
\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S} \arrow[r, "\prod"] \arrow[d, "\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\Sigma^{\infty}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{S} \arrow[d, "\Sigma^{\infty}_{+}"] \\
\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathrm{p} \arrow[r, "\bigotimes"'] & \mathcal{S}\mathrm{p}
\end{tikzcd}$$

universal property: $\mathcal{S}\mathrm{p}$ **is the *stabilization* of** $\mathcal{S}$ $\rightsquigarrow$ $\mathrm{Lin}(\mathcal{S}, \mathcal{S}\mathrm{p}) \xleftarrow{\sim} \mathrm{Lin}(\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p})$

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\textbf{mdrmy}} H^*_{dR}$

§1.2 traces in algebraic geometry

$\mathrm{K}(X) \xrightarrow{\textbf{mdrmy}} \mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$, $\mathrm{K} \to \mathrm{THC}^-$ a local $\mathbb{Q}$-equiv$^{\mathbf{ce}}$, $\mathrm{K} \to \mathrm{TC}$ a local equiv$^{\mathbf{ce}}$

§1.3 the geometry of the cyclotomic trace

$\mathrm{TC}(X) = \mathbb{T}$-inv$^{\mathbf{t}}$ functions on $\mathcal{L}X$ that are also "sensitive" to $\gamma \mapsto r^*\gamma$ (e.g. $\mathrm{tr}(M^{\otimes r}) \equiv \mathrm{tr}(M)^{\otimes r}$)

talk 2

§2.1 stratified schemes and generalized recollements

for X a stratified scheme, $\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\mathbf{r.lax}}(\ldots\mathrm{l.lax}\ldots)$

§2.2 the geometry of TC

$\mathbb{N}^\times$-eq $\mathbb{N}^{\mathbf{div}}$-strat$^{\mathbf{d}}$ $\mathscr{L}X \downarrow \mathscr{B}\mathbb{T}$ such that $\mathrm{THH}(X) \leftrightarrow \mathscr{O}_{\mathscr{L}X}$ and $\mathrm{TC}(X) = \mathbb{W}$-eq fxns on $\mathscr{L}X$

§2.3 factorization homology of enriched ∞-categories

via fact hlgy and lineariz$^{\mathbf{n}}$: $S^{\mathbf{1}} \downarrow S^{\mathbf{1}} \rightsquigarrow$ cyclo str on THH and $S^{\mathbf{1}} \downarrow \mathbb{D}^{\mathbf{0}} \rightsquigarrow$ cyclo trace