The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (arXiv:1710.06416)
2. *Factorization homology of enriched ∞-categories* (arXiv:1710.06414)
3. *The geometry of the cyclotomic trace* (arXiv:1710.06409)

talk 1

talk 1

§1.1 traces in differential geometry

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 2

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 2

§2.1 stratified schemes and generalized recollements (interlude)

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 2

§2.1 stratified schemes and generalized recollements (interlude)

§2.2 the geometry of TC (conjectural refinement of §1.3)

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 2

§2.1 stratified schemes and generalized recollements (interlude)

§2.2 the geometry of TC (conjectural refinement of §1.3)

§2.3 the construction: factorization homology of enriched ∞-categories

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}(X)$, $\oplus$)

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory:

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X)$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\quad\rightsquigarrow\quad$ monoid $\left(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) \, , \ \oplus \right)$

topological K-theory: $K(X) :=$ group-completion of this monoid

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt}$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N}$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2}$ ...

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$[TS^2] + [\underline{\mathbb{R}^1}]$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$[TS^2] + [\underline{\mathbb{R}^1}] = [TS^2] + [NS^2]$$

$S^2 \xhookrightarrow{i} \mathbb{R}^3$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$[TS^2] + [\underline{\mathbb{R}^1}] = [TS^2] + [NS^2] = [TS^2 \oplus NS^2]$$

$S^2 \overset{i}{\hookrightarrow} \mathbb{R}^3$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid ($\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X)$, $\oplus$)

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$[TS^2] + [\underline{\mathbb{R}^1}] = [TS^2] + [NS^2] = [TS^2 \oplus NS^2]$$
$$= [i^* T\mathbb{R}^3]$$

$S^2 \xhookrightarrow{i} \mathbb{R}^3$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$[TS^2] + [\underline{\mathbb{R}^1}] = [TS^2] + [NS^2] = [TS^2 \oplus NS^2]$$
$$= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}]$$

$S^2 \xhookrightarrow{i} \mathbb{R}^3$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$[TS^2] + [\underline{\mathbb{R}^1}] = [TS^2] + [NS^2] = [TS^2 \oplus NS^2]$$

$S^2 \xhookrightarrow{i} \mathbb{R}^3$

$$= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}]$$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$\begin{aligned} [TS^2] + [\underline{\mathbb{R}^1}] &= [TS^2] + [NS^2] = [TS^2 \oplus NS^2] \\ &= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}] \end{aligned}$$

$S^2 \overset{i}{\hookrightarrow} \mathbb{R}^3$

$$\rightsquigarrow \quad [TS^2] = [\underline{\mathbb{R}^2}]$$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$\begin{aligned} [TS^2] + [\underline{\mathbb{R}^1}] &= [TS^2] + [NS^2] = [TS^2 \oplus NS^2] \\ &= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}] \end{aligned}$$

$S^2 \overset{i}{\hookrightarrow} \mathbb{R}^3$

$$\rightsquigarrow [TS^2] = [\underline{\mathbb{R}^2}] \quad (TS^2 \text{ is } \textit{stably trivial})$$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$[TS^2] + [\underline{\mathbb{R}^1}] = [TS^2] + [NS^2] = [TS^2 \oplus NS^2]$$

$S^2 \overset{i}{\hookrightarrow} \mathbb{R}^3$

$$= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}]$$

$$\rightsquigarrow [TS^2] = [\underline{\mathbb{R}^2}] \quad (TS^2 \text{ is } \textit{stably trivial})$$

Bott periodicity:

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$\begin{aligned} [TS^2] + [\underline{\mathbb{R}^1}] &= [TS^2] + [NS^2] = [TS^2 \oplus NS^2] \\ &= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}] \end{aligned}$$

$S^2 \overset{i}{\hookrightarrow} \mathbb{R}^3$

$$\rightsquigarrow [TS^2] = [\underline{\mathbb{R}^2}] \quad (TS^2 \text{ is } \textit{stably trivial})$$

Bott periodicity: $\widetilde{KU}(\Sigma^2 X) \cong \widetilde{KU}(X)$,

$\widetilde{K}(X) := \mathrm{coker}(K(\mathrm{pt}) \to K(X))$

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space $\rightsquigarrow$ monoid $(\mathrm{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X) , \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory $= KO$, complex K-theory $= KU$

elements: formal differences $[E_0, E_1] \approx$ "$E_0 - E_1$"

e.g. $X = \mathrm{pt} \rightsquigarrow \mathrm{VBdl}(\mathrm{pt}) \cong \mathbb{N} \rightsquigarrow K(\mathrm{pt}) \cong \mathbb{Z}$

e.g. $X = S^2 \rightsquigarrow TS^2 \ncong \underline{\mathbb{R}^2} \ldots$ but in $KO(S^2)$,

$$\begin{aligned} [TS^2] + [\underline{\mathbb{R}^1}] &= [TS^2] + [NS^2] = [TS^2 \oplus NS^2] \\ &= [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}] \end{aligned}$$

$S^2 \xhookrightarrow{i} \mathbb{R}^3$

$$\rightsquigarrow [TS^2] = [\underline{\mathbb{R}^2}] \quad (TS^2 \text{ is } \textit{stably trivial})$$

Bott periodicity: $\widetilde{KU}(\Sigma^2 X) \cong \widetilde{KU}(X)$, $\widetilde{KO}(\Sigma^8 X) \cong \widetilde{KO}(X)$

$\widetilde{K}(X) := \mathrm{coker}(K(\mathrm{pt}) \to K(X))$

in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,

in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,

$$[E_0, E_1] \cdot [F_0, F_1] = [(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (F_0 \otimes E_1)]$$

in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,

$$[E_0, E_1] \cdot [F_0, F_1] = [(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (F_0 \otimes E_1)]$$

the ***Chern character***: a ring homomorphism

$$KU(X) \xrightarrow{\text{ch}} H^*(X; \mathbb{Q}) \ .$$

in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,

$$[E_0, E_1] \cdot [F_0, F_1] = [(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (F_0 \otimes E_1)]$$

the ***Chern character***: a ring homomorphism

$$KU(X) \xrightarrow{\text{ch}} H^*(X; \mathbb{Q}) \ .$$

Thm. $KU(X) \longrightarrow H^{even}(X; \mathbb{Q})$

$$KU(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^{even}(X; \mathbb{Q})$$

in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,

$$[E_0, E_1] \cdot [F_0, F_1] = [(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (F_0 \otimes E_1)]$$

the ***Chern character***: a ring homomorphism

$$KU(X) \xrightarrow{\text{ch}} H^*(X; \mathbb{Q}) \ .$$

Thm. $KU(X) \longrightarrow H^{even}(X; \mathbb{Q})$

$$KU(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^{even}(X; \mathbb{Q})$$

idea: $H^{even}(X; \mathbb{Q})$ is an approximation to $KU(X)$ (loses torsion)

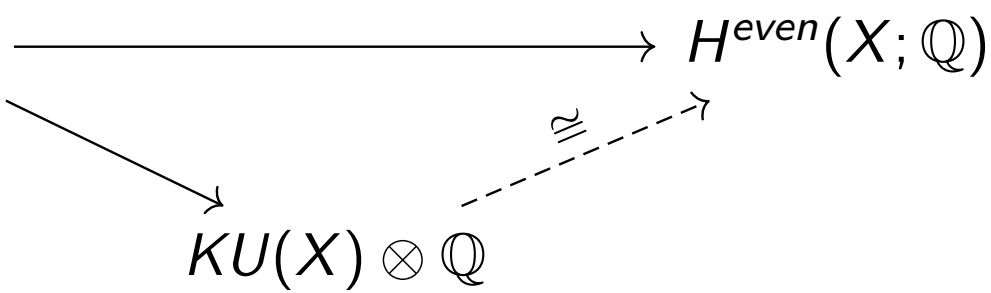
in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,

$$[E_0, E_1] \cdot [F_0, F_1] = [(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (F_0 \otimes E_1)]$$

the ***Chern character***: a ring homomorphism

$$KU(X) \xrightarrow{\text{ch}} H^*(X; \mathbb{Q}) \ .$$

Thm.



idea: $H^{even}(X; \mathbb{Q})$ is an approximation to $KU(X)$ (loses torsion)

chromatic homotopy theory: over $\mathbb{Q}$, have $\widehat{\mathbb{G}}_a \cong \widehat{\mathbb{G}}_m$ (via exp/log)

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇:

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$, $F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\mathrm{VBdl}^\nabla_{\mathbb{C}}(M) \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} \Omega^*_{dR}(M)$$

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{array}{ccc} \mathrm{VBdl}^\nabla_{\mathbb{C}}(M) & \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} & \Omega^*_{dR}(M) \\ & & \uparrow \\ & & Z^*_{dR}(M) \end{array}$$

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{array}{ccc} \mathrm{VBdl}^\nabla_{\mathbb{C}}(M) & \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} & \Omega^*_{dR}(M) \\ & \searrow & \uparrow \\ & & Z^*_{dR}(M) \end{array}$$

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{array}{ccc} \mathrm{VBdl}^\nabla_{\mathbb{C}}(M) & \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} & \Omega^*_{dR}(M) \\ \downarrow & \nearrow & \uparrow \\ \mathrm{VBdl}_{\mathbb{C}}(M) & & Z^*_{dR}(M) \end{array}$$

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{array}{ccc} \mathrm{VBdl}^\nabla_{\mathbb{C}}(M) & \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} & \Omega^*_{dR}(M) \\ \downarrow & \nearrow & \uparrow \\ \mathrm{VBdl}_{\mathbb{C}}(M) & & Z^*_{dR}(M) \\ & & \downarrow \\ & & H^*_{dR}(M) \end{array}$$

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{array}{ccc} \mathrm{VBdl}^\nabla_{\mathbb{C}}(M) & \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} & \Omega^*_{dR}(M) \\ \downarrow & \nearrow & \uparrow \\ \mathrm{VBdl}_{\mathbb{C}}(M) & & Z^*_{dR}(M) \\ & \searrow & \downarrow \\ & & H^*_{dR}(M) \end{array}$$

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{array}{ccc} \mathrm{VBdl}^\nabla_{\mathbb{C}}(M) & \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} & \Omega^*_{dR}(M) \\ \downarrow & \nearrow & \uparrow \\ \mathrm{VBdl}_{\mathbb{C}}(M) & & Z^*_{dR}(M) \\ \downarrow & \nearrow & \downarrow \\ KU(M) & & H^*_{dR}(M) \end{array}$$

for $X = M$ a smooth manifold, can get Chern character $KU(M) \to H^*_{dR}(M)$ via ***Chern–Weil theory:***

given $E \downarrow M$, choose a *connection* ∇: for $v \in T_pM$ and section s, $\nabla_v(s) \in E_p \approx$ "derivative of s in the v direction"

get *curvature*, an $\mathrm{End}(E)$-valued 2-form: for $v, w \in T_pM$,
$F^\nabla(v,w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v},\tilde{w}]}$ $\tilde{v}, \tilde{w}$ any extensions of v, w

$\rightsquigarrow \quad F^\nabla \approx$ "monodromy around infinitesimal parallelograms"

$$\begin{array}{ccc} \mathrm{VBdl}^\nabla_{\mathbb{C}}(M) & \xrightarrow{\mathrm{tr}(e^{iF/2\pi})} & \Omega^*_{dR}(M) \\ \downarrow & \nearrow & \uparrow \\ \mathrm{VBdl}_{\mathbb{C}}(M) & & Z^*_{dR}(M) \\ \downarrow & \nearrow & \downarrow \\ KU(M) & \xrightarrow[\mathrm{ch}]{} & H^*_{dR}(M) \end{array}$$

§1.2: TRACES IN ALGEBRAIC GEOMETRY

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\quad\rightsquigarrow\quad$ $\mathrm{Spec}(R)$ an *affine scheme*

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\quad\rightsquigarrow\quad$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\quad\approx$ Serre–Swan theorem

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}\cdot}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:= \mathrm{Proj}_R^{\mathbf{f.g.}\cdot}$

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:= \mathrm{Proj}_R^{\mathbf{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...): glued together from affines

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:= \mathrm{Proj}_R^{\mathbf{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...): glued together from affines

to define ***algebraic K-theory*** of X, have a choice: not all exact seq's split $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in $\mathrm{Mod}_{\mathbb{Z}}$: does this enforce $[\mathbb{Z}/4] = [\mathbb{Z}/2] + [\mathbb{Z}/2]$ in $\mathrm{K}(\mathbb{Z})$?

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:= \mathrm{Proj}_R^{\mathbf{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...): glued together from affines

to define ***algebraic K-theory*** of X, have a choice: not all exact seq's split $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in $\mathrm{Mod}_{\mathbb{Z}}$: does this enforce $[\mathbb{Z}/4] = [\mathbb{Z}/2] + [\mathbb{Z}/2]$ in $\mathrm{K}(\mathbb{Z})$?

make easier choice: $E_0 \rightarrowtail E_1 \twoheadrightarrow E_2 \rightsquigarrow [E_1] = [E_0] + [E_2]$

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:= \mathrm{Proj}_R^{\mathbf{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...): glued together from affines

to define ***algebraic K-theory*** of X, have a choice: not all exact seq's split $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in $\mathrm{Mod}_{\mathbb{Z}}$: does this enforce $[\mathbb{Z}/4] = [\mathbb{Z}/2] + [\mathbb{Z}/2]$ in $\mathrm{K}(\mathbb{Z})$?

make easier choice: $E_0 \rightarrowtail E_1 \twoheadrightarrow E_2 \rightsquigarrow [E_1] = [E_0] + [E_2]$

enforce relations *derivedly*: record relations, relations between relations, ... $\rightsquigarrow$ $\mathrm{K}(X)$ a *spectrum* $\approx$ chain complex

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:= \mathrm{Proj}_R^{\mathbf{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...): glued together from affines

to define ***algebraic K-theory*** of X, have a choice: not all exact seq's split $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in $\mathrm{Mod}_{\mathbb{Z}}$: does this enforce $[\mathbb{Z}/4] = [\mathbb{Z}/2] + [\mathbb{Z}/2]$ in $\mathrm{K}(\mathbb{Z})$?

make easier choice: $E_0 \rightarrowtail E_1 \twoheadrightarrow E_2 \rightsquigarrow [E_1] = [E_0] + [E_2]$

enforce relations *derivedly*: record relations, relations between relations, ... $\rightsquigarrow$ $\mathrm{K}(X)$ a *spectrum* $\approx$ chain complex

for *any* category $\mathcal{C}$ with "exact sequences", get spectrum $\mathrm{K}(\mathcal{C})$

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow$ $\mathrm{Spec}(R)$ an *affine scheme*

$\mathrm{VBdl}(\mathrm{Spec}(R)) \simeq \mathrm{Proj}_R^{\mathrm{f.g.}}$ $\approx$ Serre–Swan theorem

R an assoc. ring $\rightsquigarrow$ "$\mathrm{VBdl}(\mathrm{Spec}(R))$" $:= \mathrm{Proj}_R^{\mathbf{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...): glued together from affines

to define ***algebraic K-theory*** of X, have a choice: not all exact seq's split $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in $\mathrm{Mod}_{\mathbb{Z}}$: does this enforce $[\mathbb{Z}/4] = [\mathbb{Z}/2] + [\mathbb{Z}/2]$ in $\mathrm{K}(\mathbb{Z})$?

make easier choice: $E_0 \rightarrowtail E_1 \twoheadrightarrow E_2 \rightsquigarrow [E_1] = [E_0] + [E_2]$

enforce relations *derivedly*: record relations, relations between relations, ... $\rightsquigarrow$ $\mathrm{K}(X)$ a *spectrum* $\approx$ chain complex

for *any* category $\mathcal{C}$ with "exact sequences", get spectrum $\mathrm{K}(\mathcal{C})$

$\mathrm{K}(X) := \mathrm{K}(\mathrm{VBdl}(X)) \xrightarrow{\sim} \mathrm{K}(\mathrm{Perf}(X))$

the target of the trace from algebraic K-theory...

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category,

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal ∞-category, $\mathcal{C}$ a $\mathcal{V}$-enriched ∞-category,

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category,

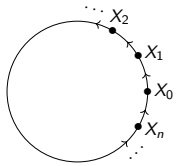
the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

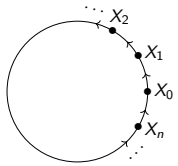
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$

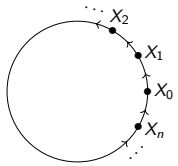


really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



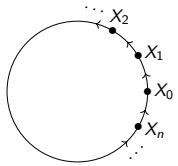
really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} =$ spectra $\quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



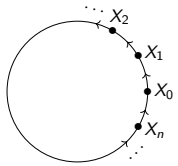
really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$.

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim}_{\text{  }} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$

really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

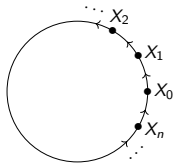
$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

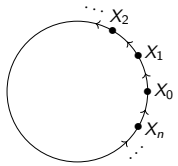
have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

§2.3

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

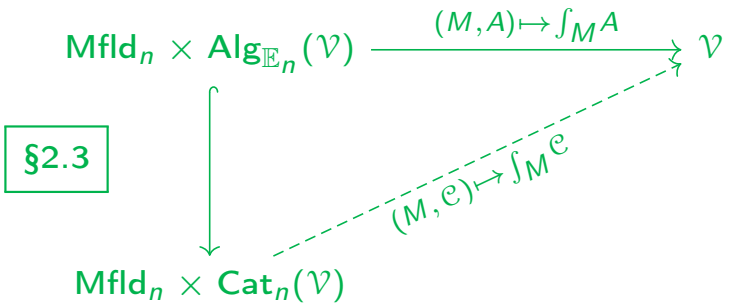
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

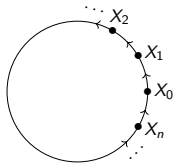
have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)



the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

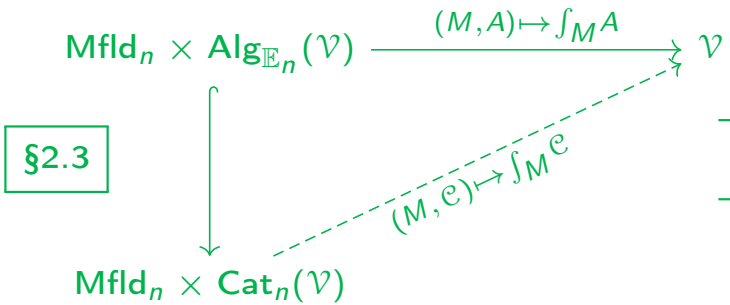
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

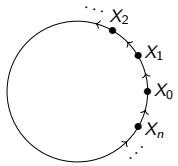


simplicial			
Δ^{op}			

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

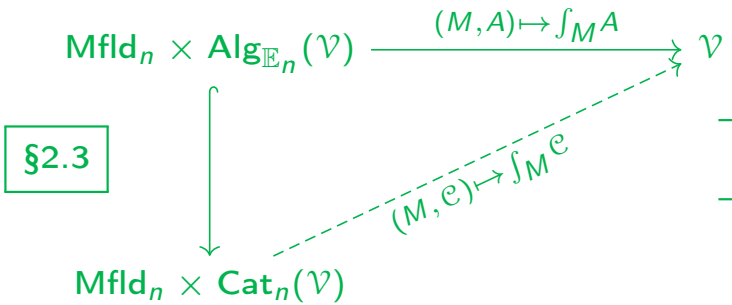
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

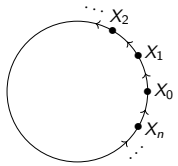


simplicial			
Δ^{op}			
S^1_*			

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

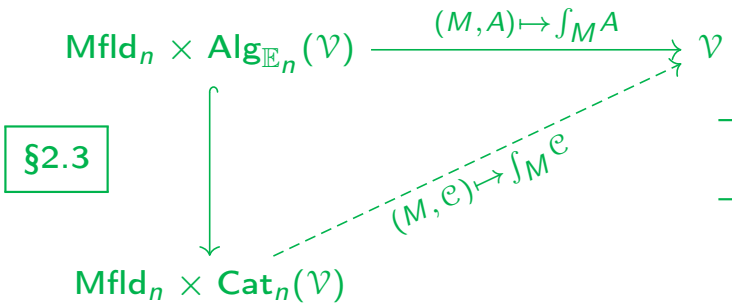
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

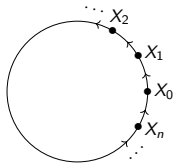


simplicial	paracyclic		
$\Delta^{\mathbf{op}}$	$\Delta^{\mathbf{op}}_{\circlearrowleft}$		
$S^{\mathbf{1}}_*$			

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

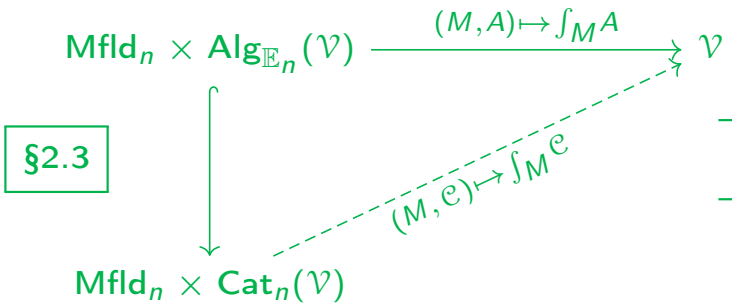
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

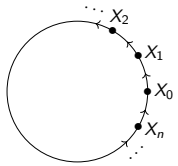


simplicial	paracyclic		
Δ^{op}	$\Delta^{\mathrm{op}}_{\circlearrowleft}$		
S^1_*	S^1		

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

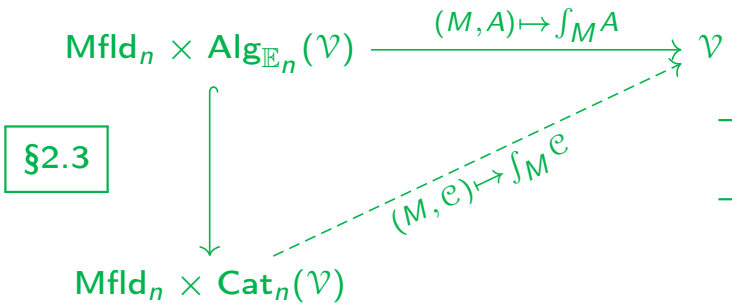
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

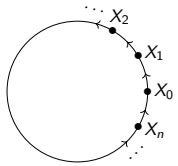


simplicial	paracyclic	cyclic	
$\Delta^{\mathbf{op}}$	$\Delta^{\mathbf{op}}_{\circlearrowleft}$	$\Lambda^{\mathbf{op}}$	
$S^{\mathbf{1}}_*$	$S^{\mathbf{1}}$		

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

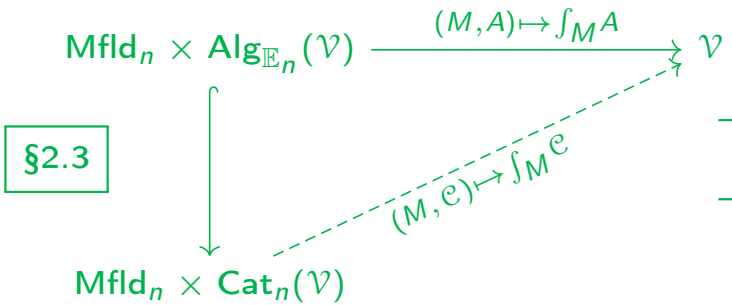
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

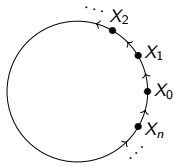


simplicial	paracyclic	cyclic	
Δ^{op}	$\Delta^{\mathrm{op}}_{\circlearrowleft}$	Λ^{op}	
S^1_*	S^1	S^1 & auto's	

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

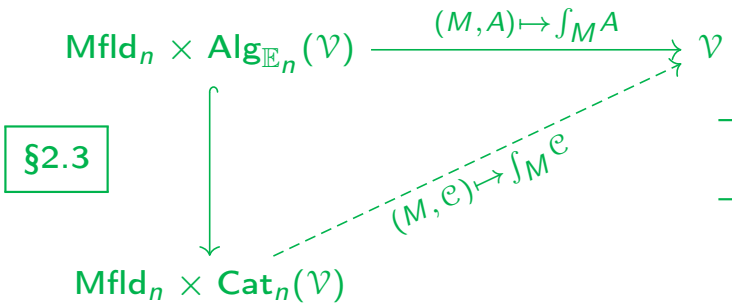
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim} \left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0) \right).$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)

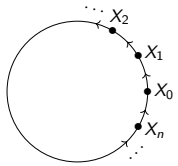


simplicial	paracyclic	cyclic	epicyclic
Δ^{op}	$\Delta^{\mathrm{op}}_{\circlearrowleft}$	Λ^{op}	$\widetilde{\Lambda}^{\mathrm{op}}$
S^1_*	S^1	S^1 & auto's	

the target of the trace from algebraic K-theory...

Def. $(\mathcal{V}, \boxtimes)$ a monoidal category, $\mathcal{C}$ a $\mathcal{V}$-enriched category, the ***Hochschild homology*** of $\mathcal{C}$ is its ***factorization homology*** over the circle:

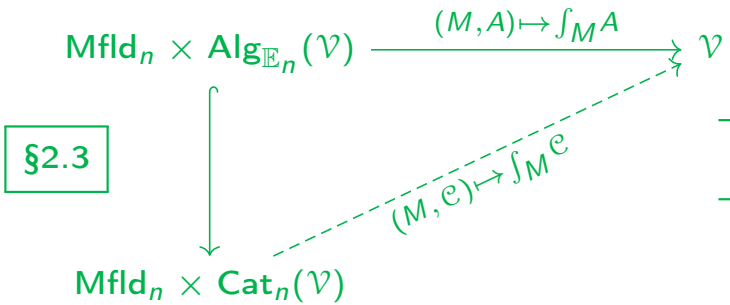
$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} :\approx \operatorname*{colim}_{\left(\underline{\hom}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\hom}_{\mathcal{C}}(X_n, X_0)\right)} .$$



really hocolim (or ∞-categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathscr{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathcal{S}\mathrm{p} = \text{spectra} \quad \rightsquigarrow \quad$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathcal{S}\mathrm{p} \rightleftarrows \mathscr{D}(R)$, just like $\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathcal{S}\mathrm{p} = \mathscr{D}(\mathcal{S}\mathrm{et})$!)



simplicial	paracyclic	cyclic	epicyclic
$\Delta^{\mathbf{op}}$	$\Delta^{\mathbf{op}}_{\circlearrowleft}$	$\Lambda^{\mathbf{op}}$	$\widetilde{\Lambda}^{\mathbf{op}}$
S^1_*	S^1	S^1 & auto's	S^1 & endo's

Thm (Hochschild–Kostant–Rosenberg '65).

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k,

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$\mathbb{T} \curvearrowright \mathrm{HH}(X)$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ;$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}}$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H^*_{dR}(X)$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H^*_{dR}(X)$$

if X is a *spectral* scheme (e.g. $\mathrm{Spec}(R)$ for R a ring spectrum), define ***topological negative cyclic homology:***

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H^*_{dR}(X)$$

if X is a *spectral* scheme (e.g. $\mathrm{Spec}(R)$ for R a ring spectrum), define ***topological negative cyclic homology:***

$$\mathrm{THC}^-(X) := \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}}$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H^*_{dR}(X)$$

if X is a *spectral* scheme (e.g. $\mathrm{Spec}(R)$ for R a ring spectrum), define ***topological negative cyclic homology:***

$$\mathrm{THC}^-(X) := \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{THC}^-(X)) \approx H^*_{dR}(X)$$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H^*_{dR}(X)$$

if X is a *spectral* scheme (e.g. $\mathrm{Spec}(R)$ for R a ring spectrum), define ***topological negative cyclic homology:***

$$\mathrm{THC}^-(X) := \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{THC}^-(X)) \approx H^*_{dR}(X)$$

$\star\star\star$ **from now on, *everything* is spectral** $\star\star\star$

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k, then

$$H_*(\mathrm{HH}(X)) \cong \Omega^*_{dR}(X) \ .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \ \cong \ k[\varepsilon]/\varepsilon^2 \ , \quad |\varepsilon| = 1 \ ; \quad \text{and} \quad \varepsilon \leftrightarrow \mathrm{d}_{dR} \ !$$

so, define ***negative cyclic homology:***

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H^*_{dR}(X)$$

if X is a *spectral* scheme (e.g. $\mathrm{Spec}(R)$ for R a ring spectrum), define ***topological negative cyclic homology:***

$$\mathrm{THC}^-(X) := \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{THC}^-(X)) \approx H^*_{dR}(X)$$

$\star\star\star$ **from now on, *everything* is spectral** $\star\star\star$

any variety or scheme can be considered as spectral, through the forgetful functor $\mathrm{Mod}_{\mathbb{Z}} \to \mathscr{D}(\mathbb{Z}) \to \mathcal{S}\mathrm{p}$

enter DAG...

enter DAG...

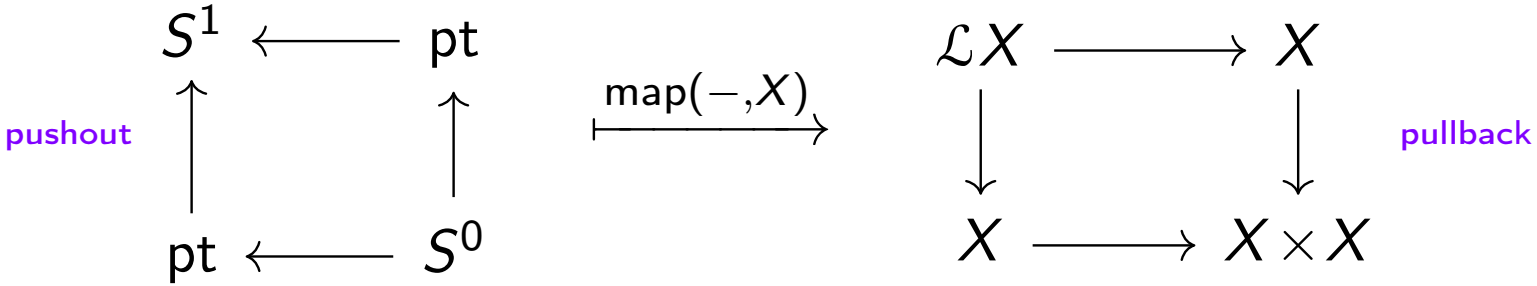
Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

$$\mathcal{L}X = \mathrm{map}(S^1, X) \ .$$

enter DAG...

Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

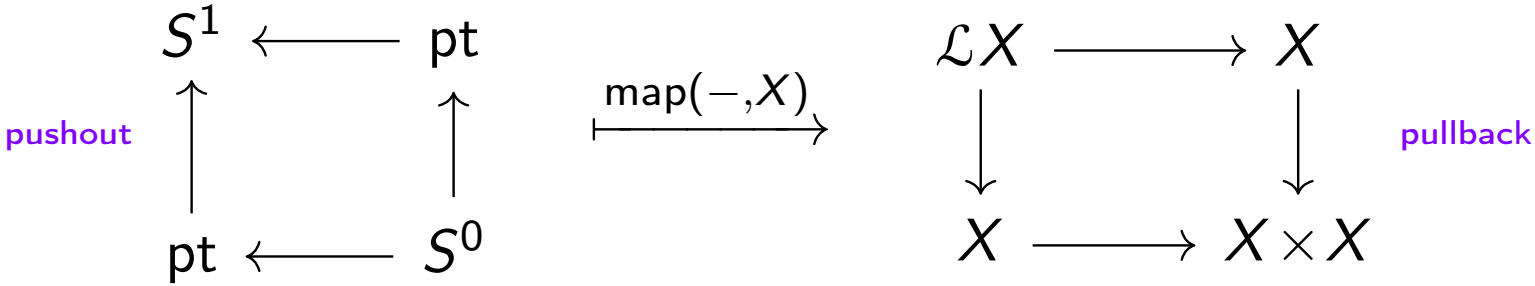
$$\mathcal{L}X = \mathrm{map}(S^1, X) \ .$$



enter DAG...

Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

$$\mathcal{L}X = \mathrm{map}(S^1, X) \ .$$



$$\rightsquigarrow \quad \mathscr{O}_{\mathcal{L}X} \simeq \mathscr{O}_X \underset{\mathscr{O}_{X\times X}}{\otimes} \mathscr{O}_X$$

enter DAG...

Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

$$\mathcal{L}X = \mathrm{map}(S^1, X) \ .$$

$$\begin{array}{ccc} S^1 & \longleftarrow & \mathrm{pt} \\ \uparrow & & \uparrow \\ \mathrm{pt} & \longleftarrow & S^0 \end{array} \quad \xmapsto{\mathrm{map}(-,X)} \quad \begin{array}{ccc} \mathcal{L}X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

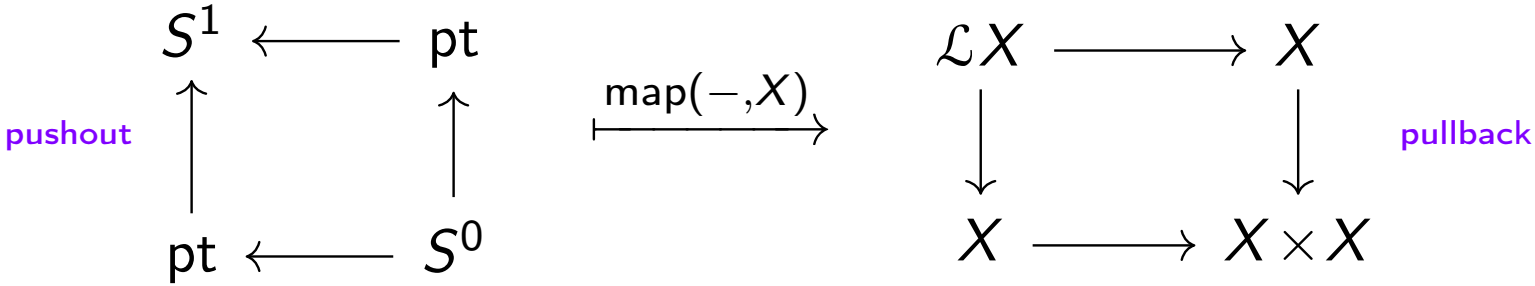
pushout (left square) — pullback (right square)

$$\rightsquigarrow \quad \mathscr{O}_{\mathcal{L}X} \simeq \mathscr{O}_X \overset{\mathbb{L}}{\underset{\mathscr{O}_{X\times X}}{\otimes}} \mathscr{O}_X$$

enter DAG...

Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

$$\mathcal{L}X = \mathrm{map}(S^1, X) \ .$$

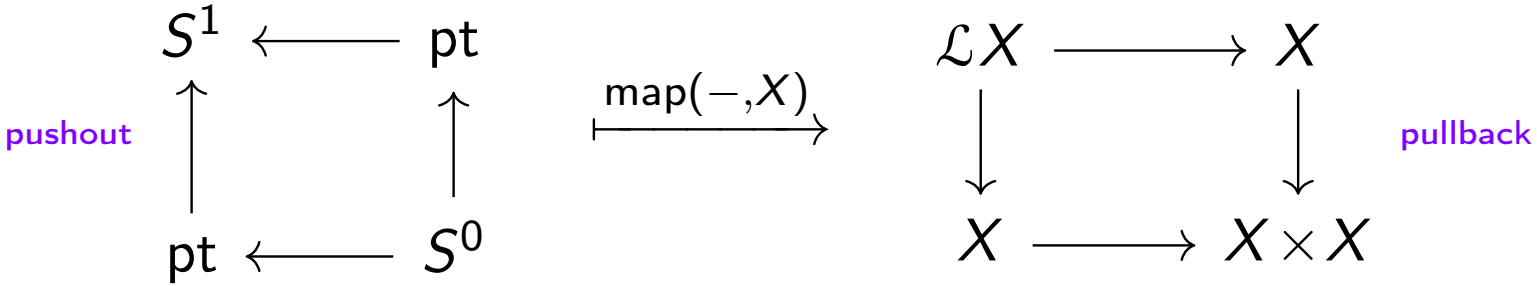


$$\rightsquigarrow \quad \mathscr{O}_{\mathcal{L}X} \simeq \mathscr{O}_X \underset{\mathscr{O}_{X\times X}}{\otimes} \mathscr{O}_X$$

enter DAG...

Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

$$\mathcal{L}X = \mathrm{map}(S^1, X) \ .$$



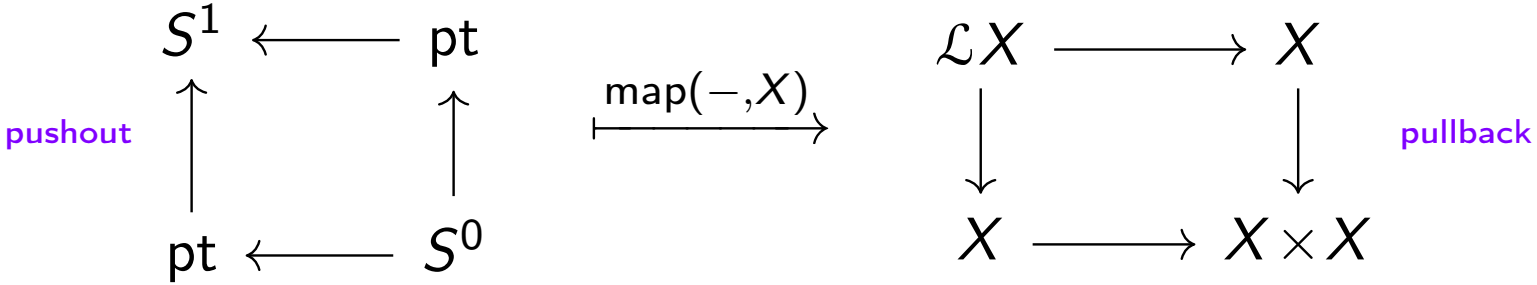
$$\rightsquigarrow \quad \mathscr{O}_{\mathcal{L}X} \simeq \mathscr{O}_X \underset{\mathscr{O}_{X\times X}}{\otimes} \mathscr{O}_X$$

$\rightsquigarrow$ $\mathscr{O}(\mathcal{L}X)$ is computed by the cyclic bar construction for THH!

enter DAG...

Def. for a(n ordinary or derived) scheme X, its ***free loopspace*** is the derived mapping stack

$$\mathcal{L}X = \mathrm{map}(S^1, X) \ .$$



$$\rightsquigarrow \quad \mathscr{O}_{\mathcal{L}X} \simeq \mathscr{O}_X \underset{\mathscr{O}_{X\times X}}{\otimes} \mathscr{O}_X$$

$\rightsquigarrow$ $\mathscr{O}(\mathcal{L}X)$ is computed by the cyclic bar construction for THH!

$\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X)$, $\mathbb{T}$-action $\leftrightarrow$ rotation of loops

traces approximate *rationalized* algebraic K-theory...

traces approximate *rationalized* algebraic K-theory...

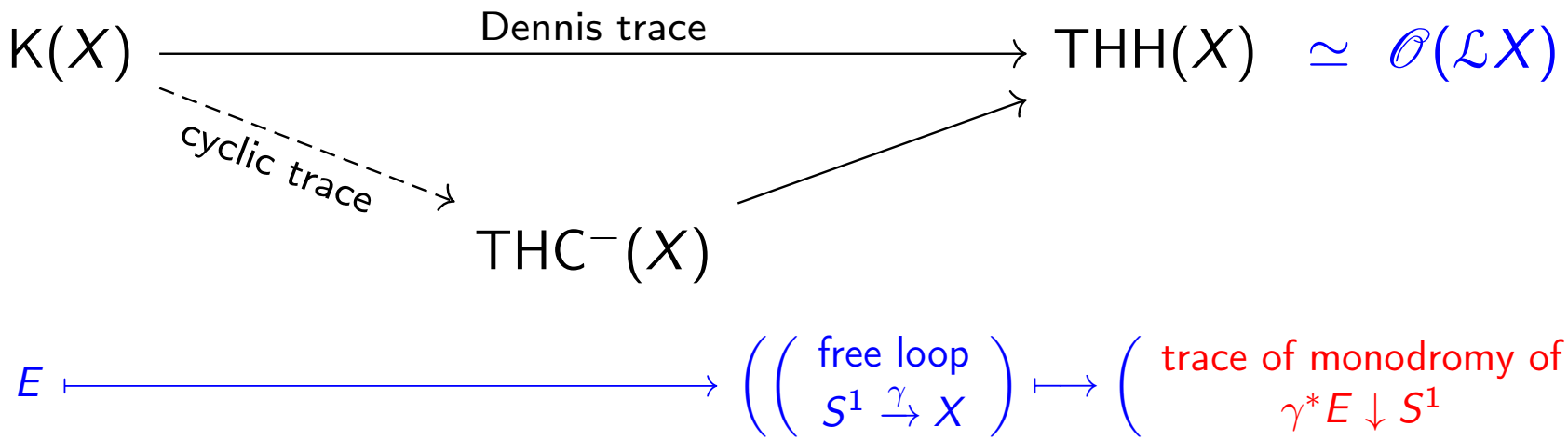
$$\mathrm{K}(X) \xrightarrow{\text{Dennis trace}} \mathrm{THH}(X)$$

traces approximate *rationalized* algebraic K-theory...

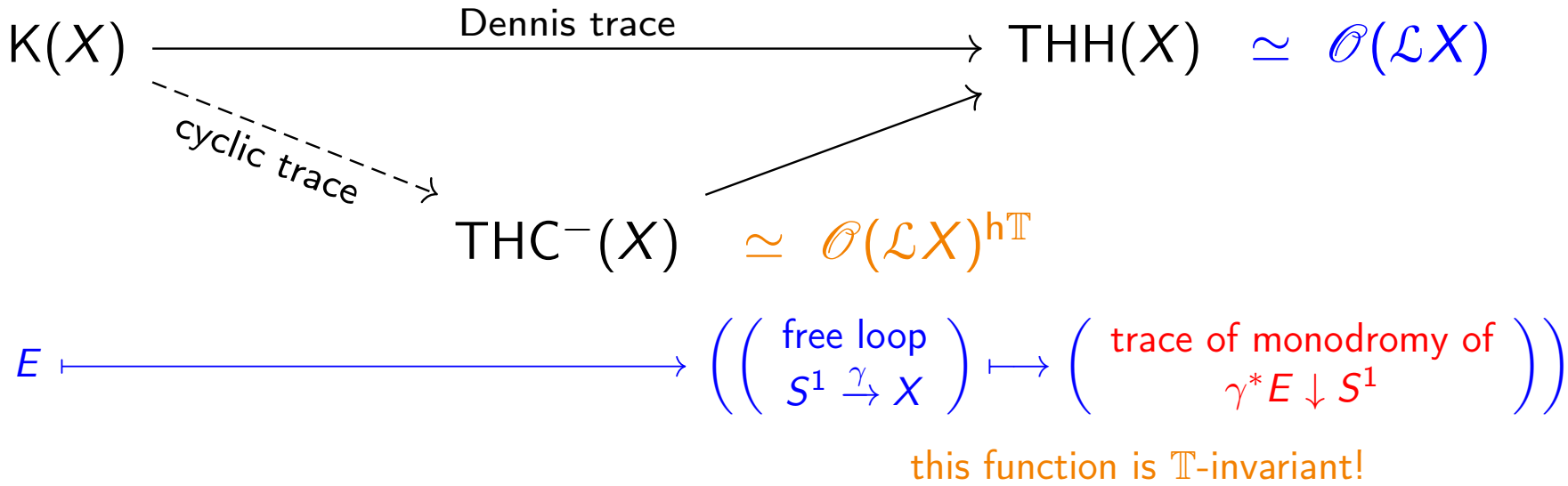
$$\mathrm{K}(X) \xrightarrow{\text{Dennis trace}} \mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X)$$

$$E \longmapsto \left(\begin{pmatrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{pmatrix} \longmapsto \begin{pmatrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{pmatrix} \right)$$

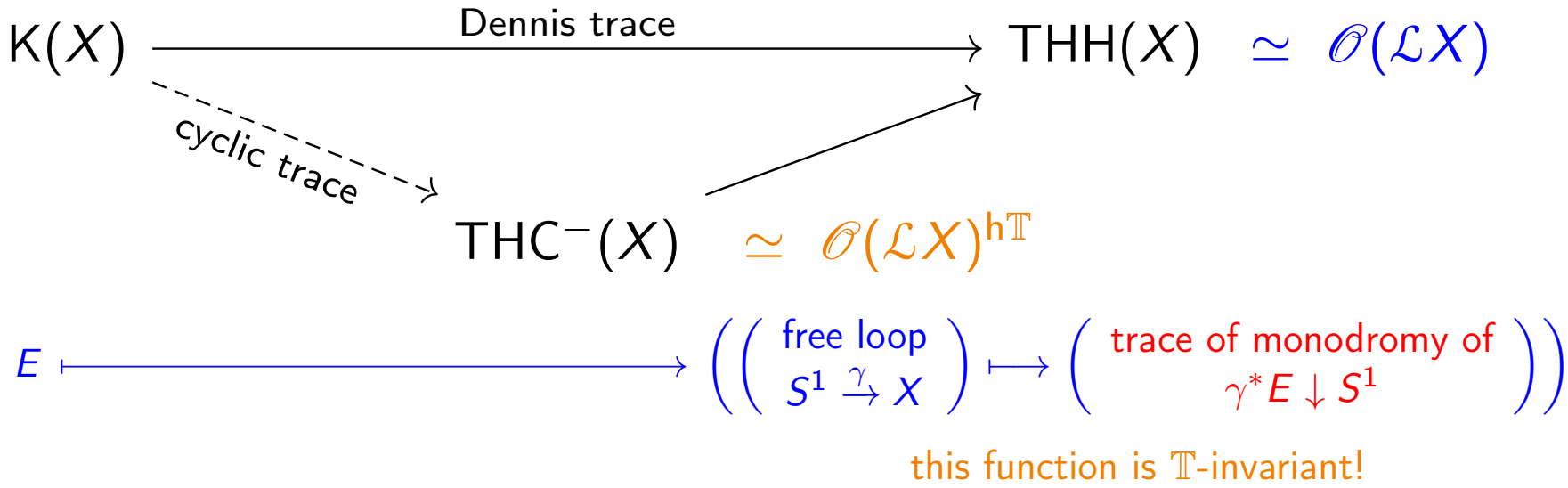
traces approximate *rationalized* algebraic K-theory...



traces approximate *rationalized* algebraic K-theory...

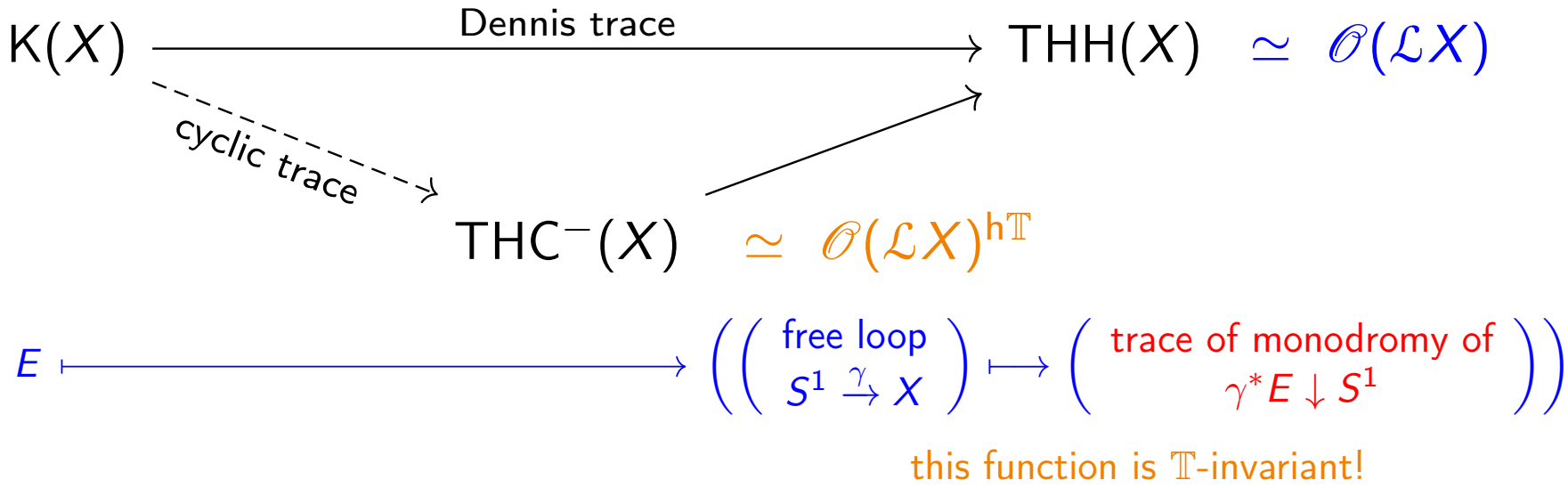


traces approximate *rationalized* algebraic K-theory...



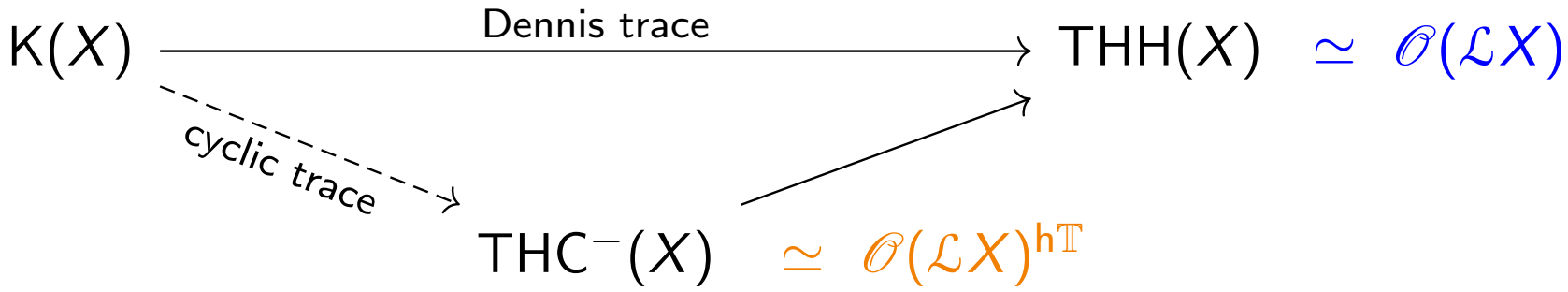
Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*:

traces approximate *rationalized* algebraic K-theory...



Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

traces approximate *rationalized* algebraic K-theory...



$$E \longmapsto \left(\begin{pmatrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{pmatrix} \longmapsto \begin{pmatrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{pmatrix} \right)$$

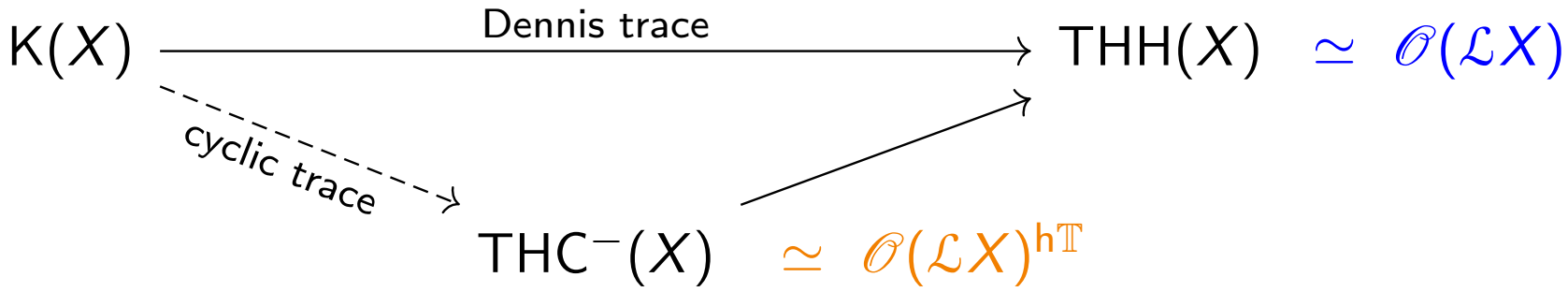
this function is $\mathbb{T}$-invariant!

Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{THC}^-(R) & \longrightarrow & \mathrm{THC}^-(R_0) \end{array}$$

pullback after $- \otimes \mathbb{Q}$.

traces approximate *rationalized* algebraic K-theory...



$$E \longmapsto \left(\begin{pmatrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{pmatrix} \longmapsto \begin{pmatrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{pmatrix} \right)$$

this function is $\mathbb{T}$-invariant!

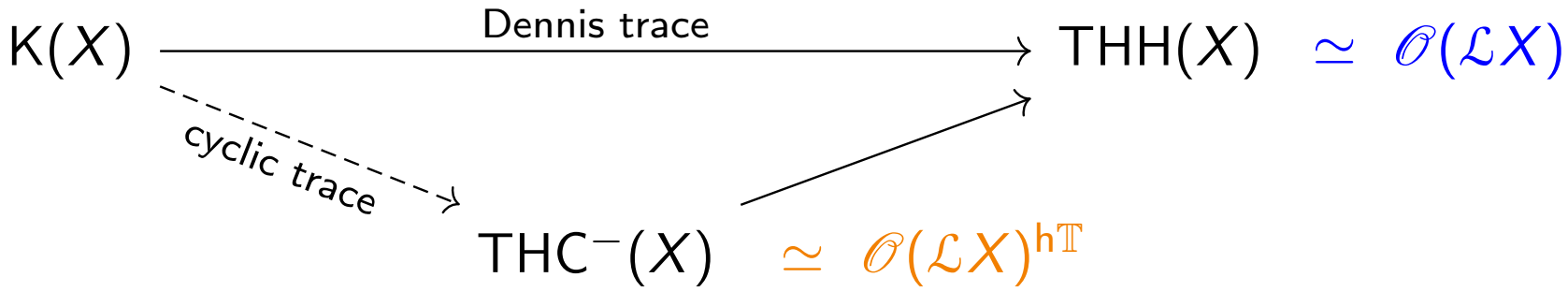
Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{THC}^-(R) & \longrightarrow & \mathrm{THC}^-(R_0) \end{array}$$

pullback after $- \otimes \mathbb{Q}$.

slogan:

traces approximate *rationalized* algebraic K-theory...



$E \longmapsto \left(\begin{pmatrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{pmatrix} \longmapsto \begin{pmatrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{pmatrix} \right)$

this function is $\mathbb{T}$-invariant!

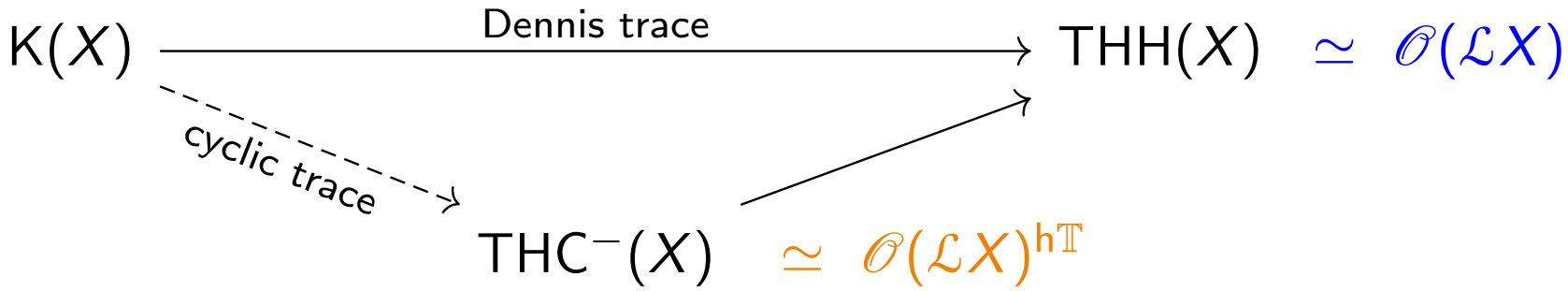
Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{THC}^-(R) & \longrightarrow & \mathrm{THC}^-(R_0) \end{array}$$

pullback after $- \otimes \mathbb{Q}$.

slogan: $\mathrm{vbdl}/\mathrm{Spec}(R) \overset{\mathbb{Q}}{\approx}$

traces approximate *rationalized* algebraic K-theory...



$E \longmapsto \left(\begin{pmatrix} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{pmatrix} \longmapsto \begin{pmatrix} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{pmatrix} \right)$

this function is $\mathbb{T}$-invariant!

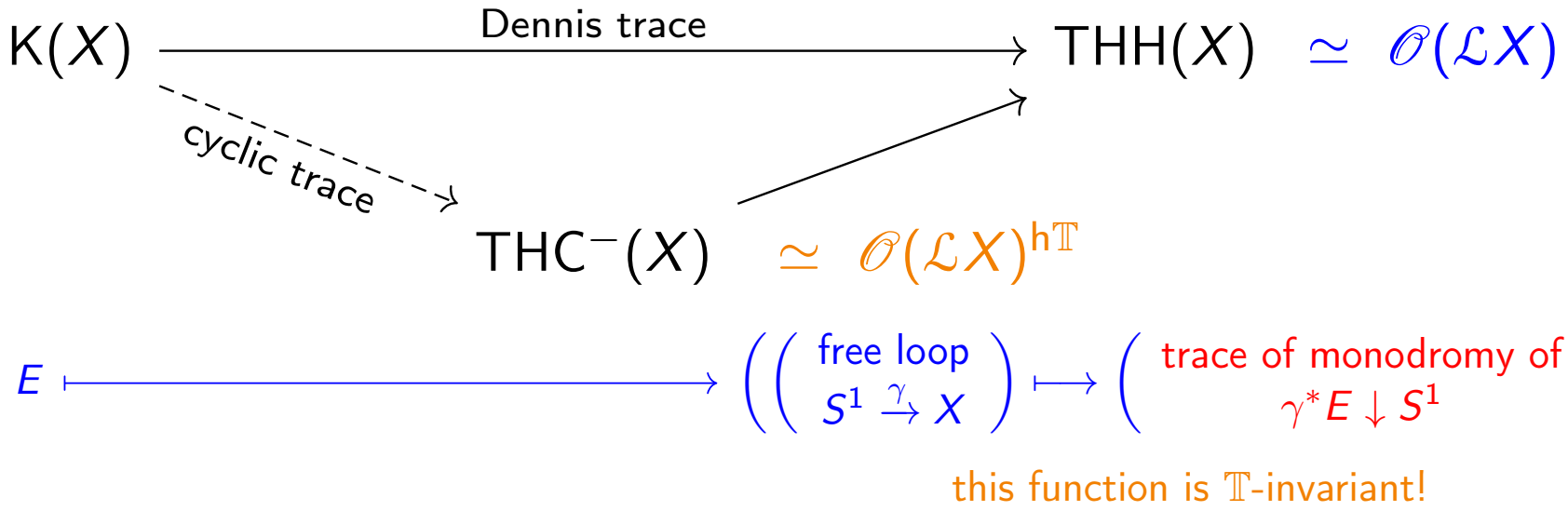
Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{THC}^-(R) & \longrightarrow & \mathrm{THC}^-(R_0) \end{array}$$

pullback after $- \otimes \mathbb{Q}$.

slogan: vbdl/Spec(R) $\overset{\mathbb{Q}}{\approx}$ restriction to Spec(R_0)

traces approximate *rationalized* algebraic K-theory...



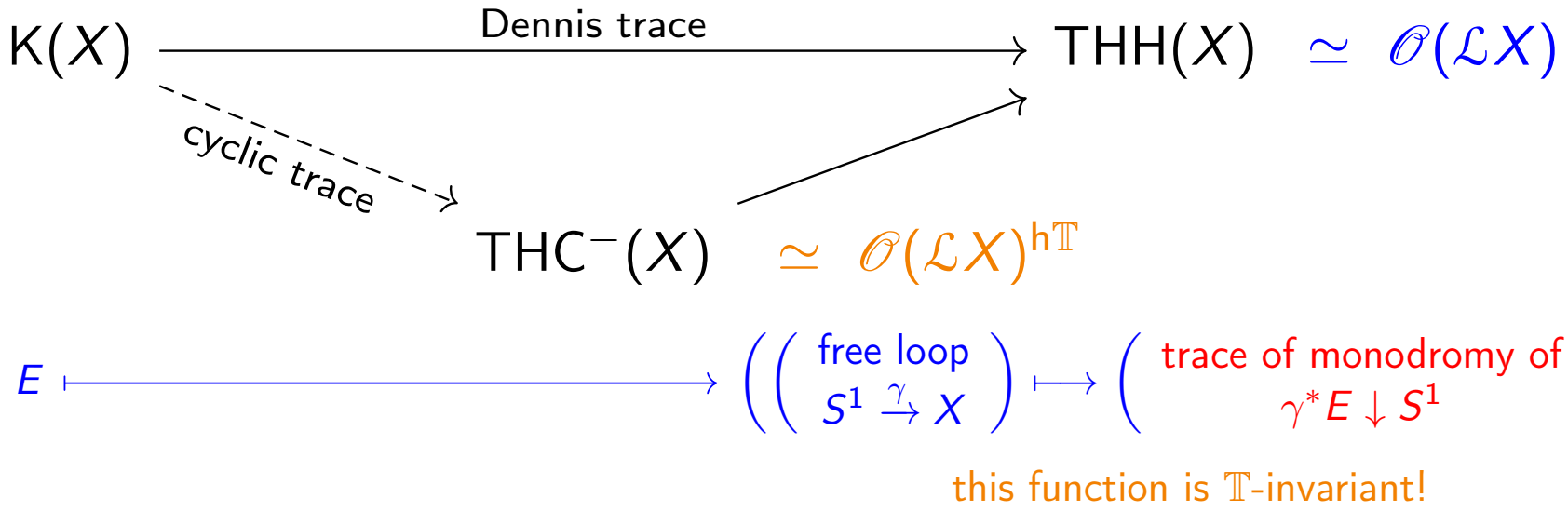
Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{THC}^-(R) & \longrightarrow & \mathrm{THC}^-(R_0) \end{array}$$

pullback after $- \otimes \mathbb{Q}$.

slogan: vbdl/Spec(R) $\overset{\mathbb{Q}}{\approx}$ restriction to Spec(R_0) + compatible trace-of-monodromy function

traces approximate *rationalized* algebraic K-theory...



Thm (Goodwillie '86). the cyclic trace is a *local* $\mathbb{Q}$*-equivalence*: for $R \to R_0$ a nilpotent extension of connective ring spectra,

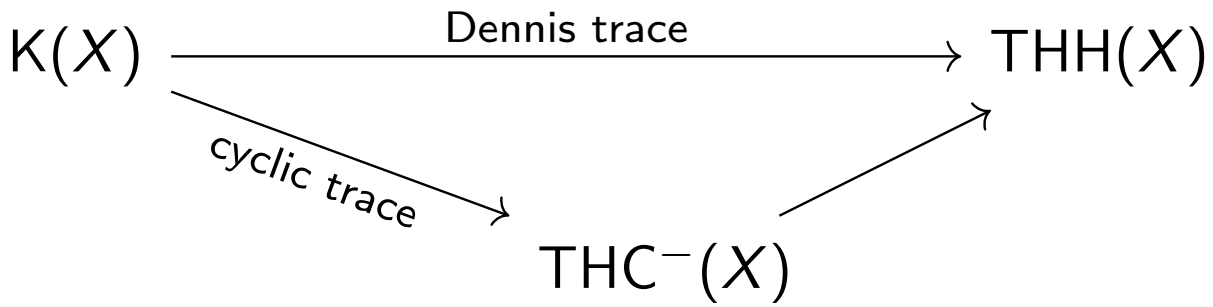
$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{THC}^-(R) & \longrightarrow & \mathrm{THC}^-(R_0) \end{array}$$

pullback after $- \otimes \mathbb{Q}$.

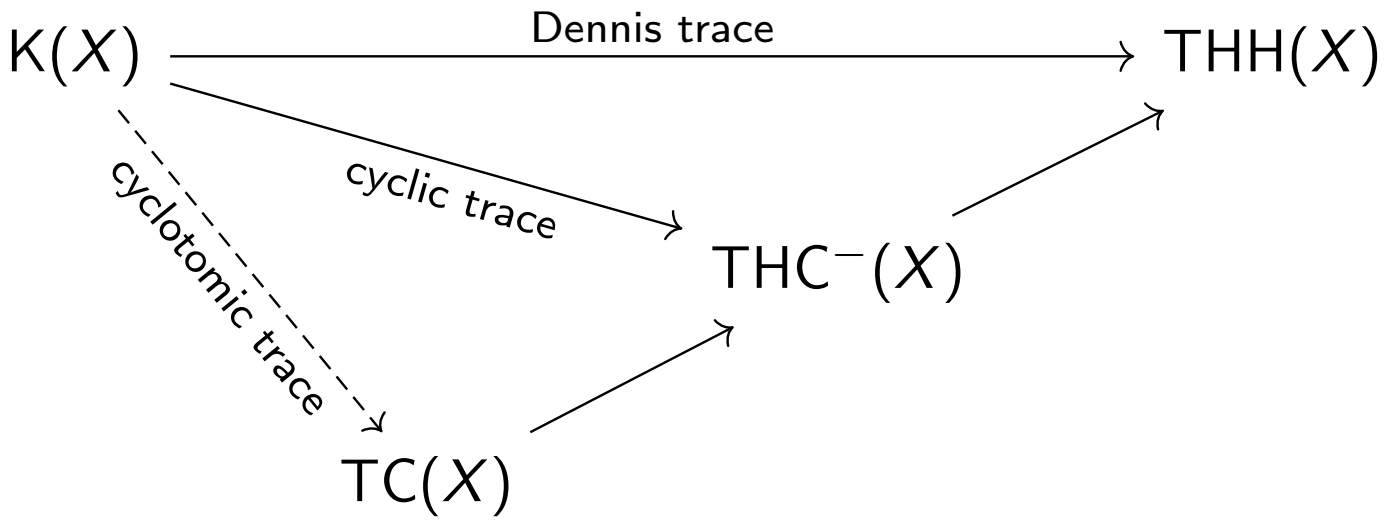
slogan: vbdl/Spec(R) $\overset{\mathbb{Q}}{\approx}$ restriction to Spec(R_0)
+ compatible trace-of-monodromy function
+ *data* of $\mathbb{T}$-invariance of this function

the integral story...

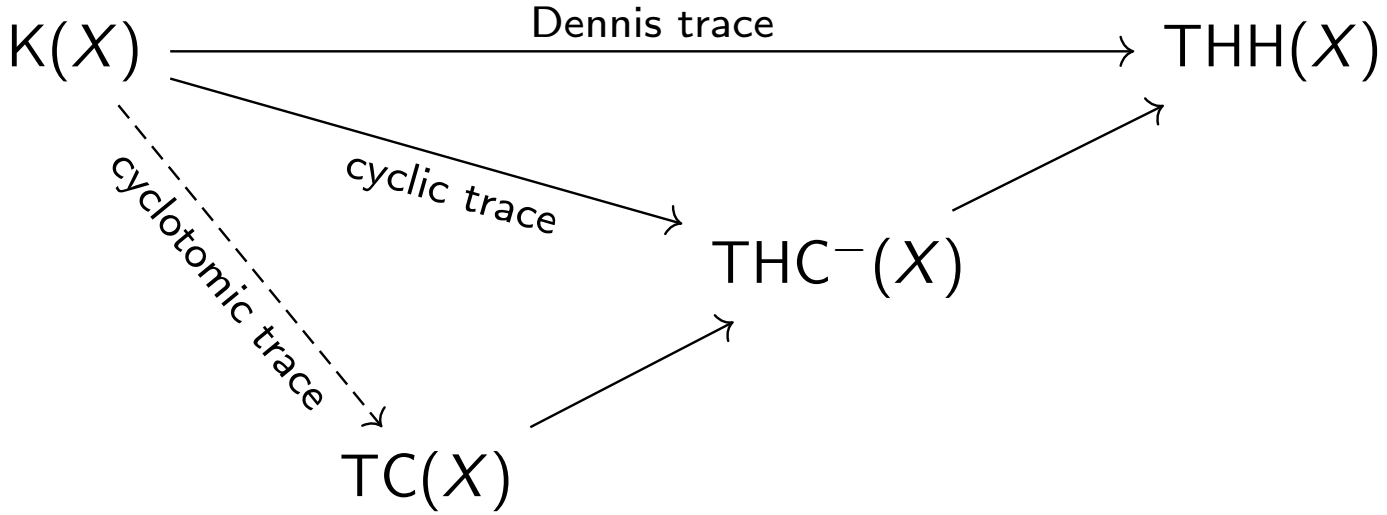
the integral story...



the integral story...

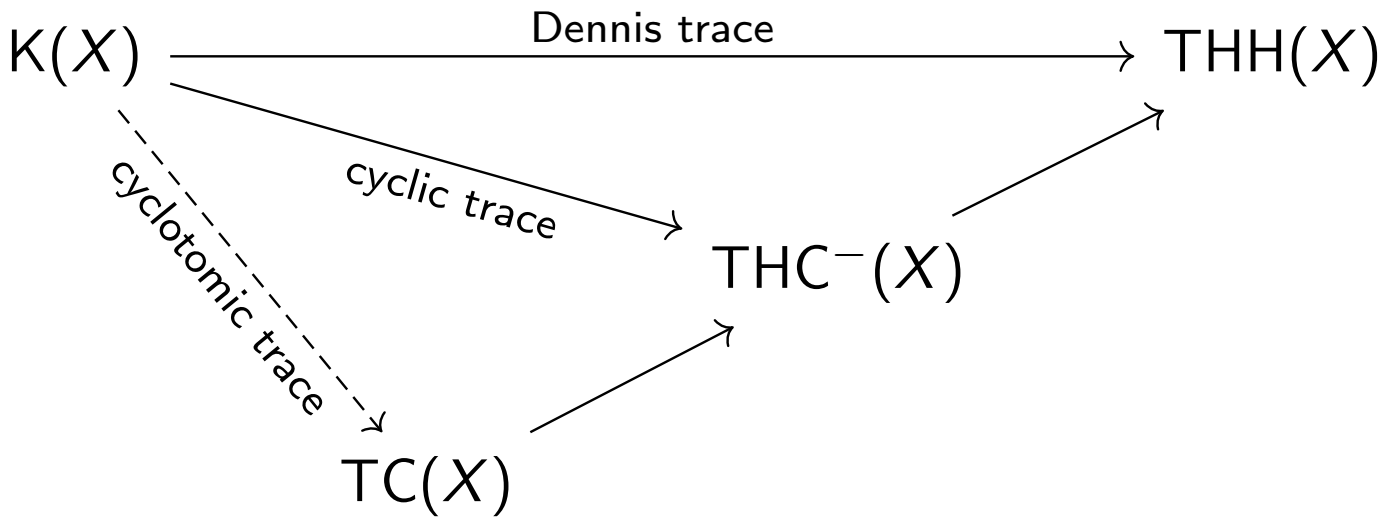


the integral story...



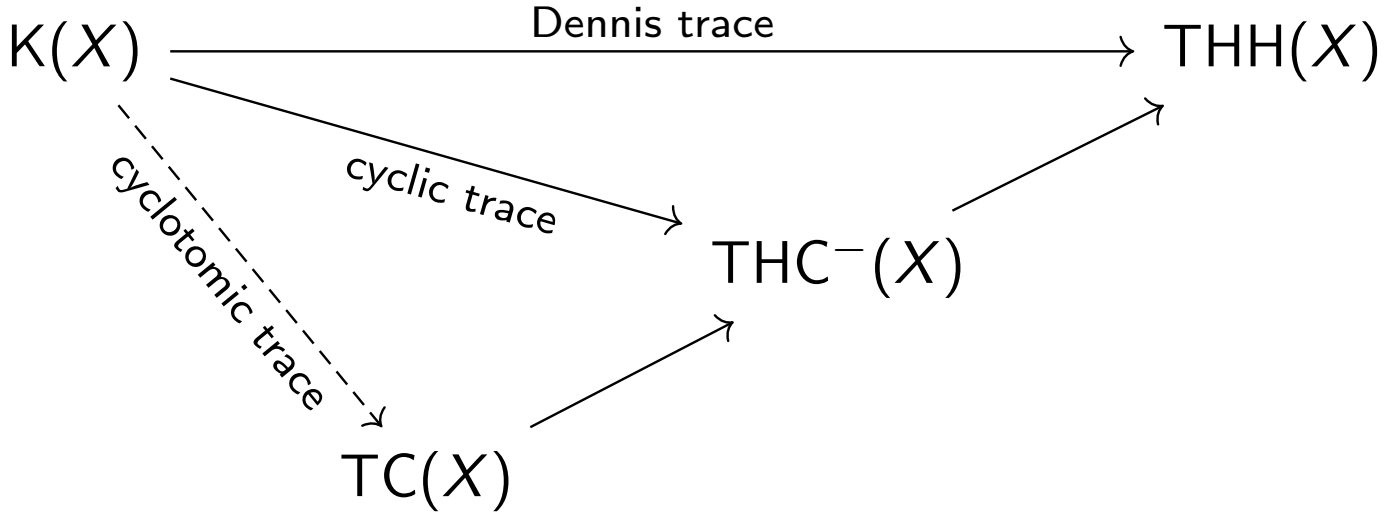
construction of the cyclotomic trace:

the integral story...



construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

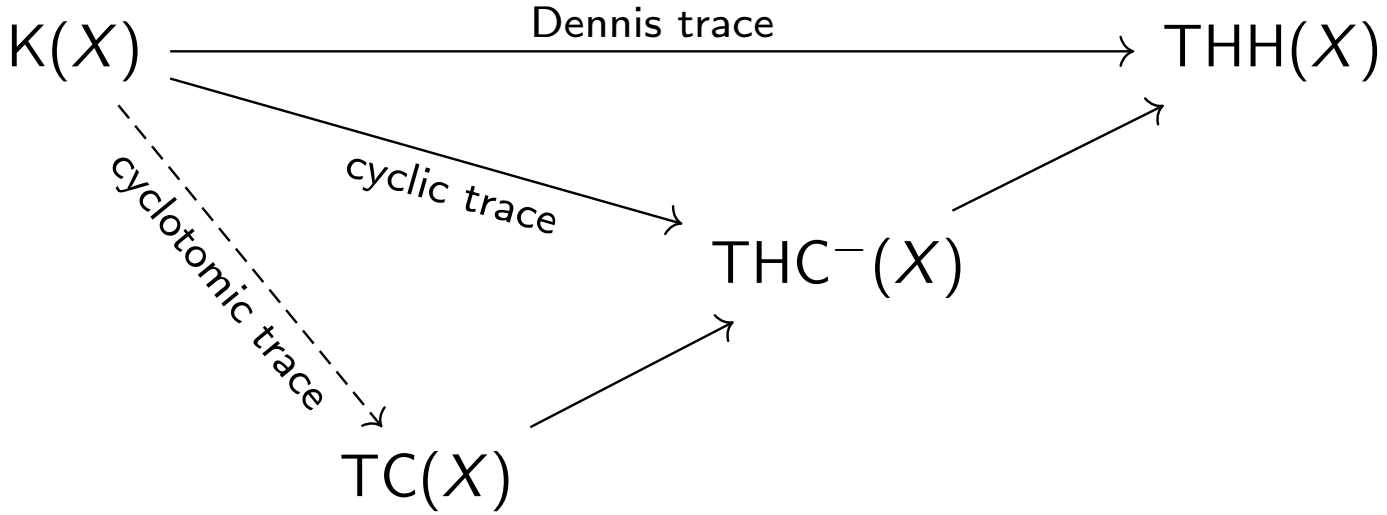
the integral story...



construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

Thm (Dundas–McCarthy '97). the cyclotomic trace is a *local equivalence* (without rationalization).

the integral story...



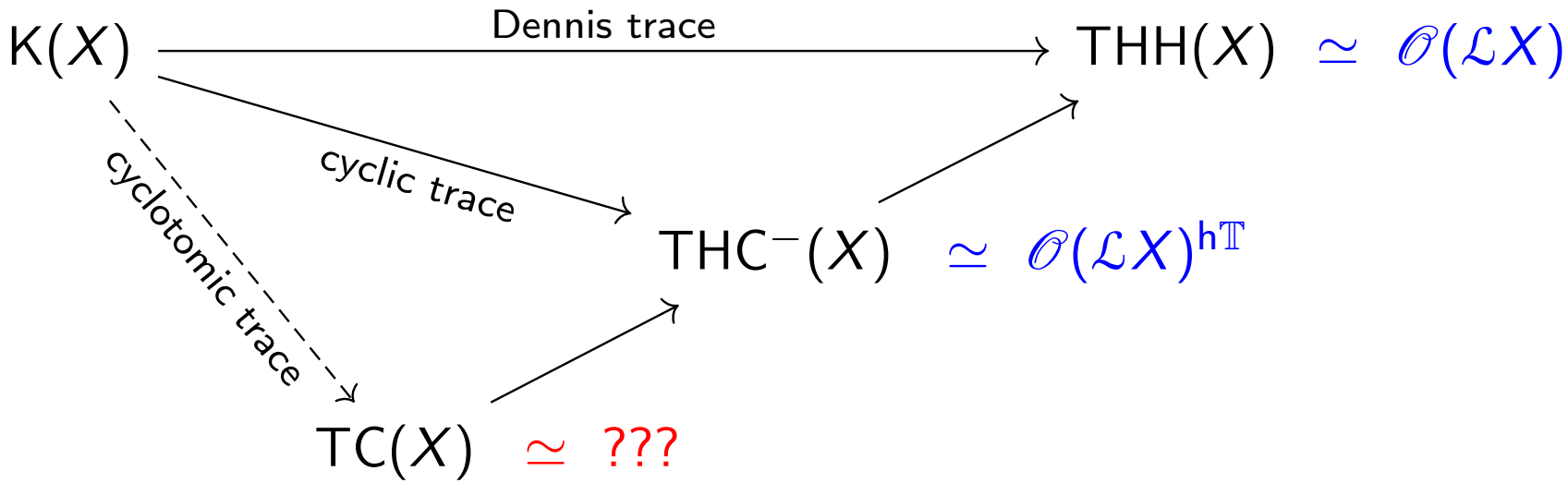
construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

Thm (Dundas–McCarthy '97). the cyclotomic trace is a *local equivalence* (without rationalization).

"This is how people other than Quillen compute algebraic K-theory."

$\sim$ *A. Blumberg*, algebraic K-theorist

the integral story...



construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

Thm (Dundas–McCarthy '97). the cyclotomic trace is a *local equivalence* (without rationalization).

"This is how people other than Quillen compute algebraic K-theory."
$\sim$ *A. Blumberg*, algebraic K-theorist

Main Question: In terms of DAG, what is $\mathrm{TC}(X)$?

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X)$$

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a ***cyclotomic spectrum***; TC is the *homotopy invariants* of its cyclotomic structure

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a *cyclotomic spectrum*; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\rightleftarrows}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longmapsto & THH(X) \end{array}$$

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a ***cyclotomic spectrum***; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\rightleftarrows}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longmapsto & THH(X) \end{array}$$

§2.2

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a ***cyclotomic spectrum***; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longleftarrow & THH(X) \end{array}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a ***cyclotomic spectrum***; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\rightleftarrows}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longleftarrow & THH(X) \end{array}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

recall: $THH(X) = $ functions on $\mathcal{L}X$

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a *cyclotomic spectrum*; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\rightleftarrows}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longleftarrow & THH(X) \end{array}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

recall: $THH(X) =$ functions on $\mathcal{L}X$

main idea:

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a *cyclotomic spectrum*; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\rightleftarrows}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longleftarrow & THH(X) \end{array}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

recall: $THH(X)$ = functions on $\mathcal{L}X$

main idea: $TC(X)$ = functions on $\mathcal{L}X$ that are:

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a *cyclotomic spectrum*; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longleftarrow & THH(X) \end{array}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

recall: $THH(X) = $ functions on $\mathcal{L}X$

main idea: $TC(X) = $ functions on $\mathcal{L}X$ that are:

- invariant under $\mathbb{T}$-action on $\mathcal{L}X$;

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a *cyclotomic spectrum*; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\rightleftarrows}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longleftarrow & THH(X) \end{array}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

recall: $THH(X) =$ functions on $\mathcal{L}X$

main idea: $TC(X) =$ functions on $\mathcal{L}X$ that are:

- invariant under $\mathbb{T}$-action on $\mathcal{L}X$;
- "sensitive" to relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$.

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a *cyclotomic spectrum*; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc} Sp & \underset{(-)^{hCyc}}{\overset{\text{triv}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & Cyc(Sp) \\ \cup & & \cup \\ TC(X) & \longleftarrow\!\shortmid & THH(X) \end{array}$$

§2.2

right adjoint (limit-type construction): *imposing conditions*

recall: $THH(X)$ = functions on $\mathcal{L}X$

main idea: $TC(X)$ = functions on $\mathcal{L}X$ that are:

- invariant under $\mathbb{T}$-action on $\mathcal{L}X$;
- "sensitive" to relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$. **Q.** What does "sensitive" mean?

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n \times n}(R)$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n \times n}(R)$

$$\mathrm{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j \quad ,$$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n \times n}(R)$

$$\mathrm{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j \quad , \quad \mathrm{tr}(M^2) = \sum_k m_k \cdot m_k$$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n \times n}(R)$

$$\mathrm{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \otimes m_j \quad , \quad \mathrm{tr}(M^{\otimes 2}) = \sum_k m_k \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \otimes m_j \quad , \quad \mathrm{tr}(M^{\otimes 2}) = \sum_{k} m_k \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is ***norms***:

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \otimes m_j \quad , \quad \mathrm{tr}(M^{\otimes 2}) = \sum_k m_k \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is ***norms***: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{array}{ccc} (R \otimes R)_{C_2} & \xrightarrow{\quad\mathrm{Nm}\quad} & (R \otimes R)^{C_2} \\ [x \otimes y] & \longmapsto & \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{array}$$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \otimes m_j \quad , \quad \mathrm{tr}(M^{\otimes 2}) = \sum_k m_k \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is ***norms***: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{array}{ccc} (R \otimes R)_{C_2} & \xrightarrow{\quad\mathrm{Nm}\quad} & (R \otimes R)^{C_2} \\ [x \otimes y] & \longmapsto & \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{array}$$

$\rightsquigarrow$ become equal in the ***Tate construction***, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{\mathrm{t}C_2}$$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \otimes m_j \quad , \quad \mathrm{tr}(M^{\otimes 2}) = \sum_k m_k \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is ***norms***: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{array}{ccc} (R \otimes R)_{C_2} & \xrightarrow{\quad\mathrm{Nm}\quad} & (R \otimes R)^{C_2} \\ [x \otimes y] & \longmapsto & \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{array}$$

$\rightsquigarrow$ become equal in the ***Tate construction***, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{\mathrm{t}C_2}$$

over $\mathbb{Q}$, norm an iso! $\rightsquigarrow$ $(R \otimes R)^{\mathrm{t}C_2} = 0$, assertion is vacuous

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

main point:

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

main point: for $\mathcal{C}$ a spectrally enriched ∞-category,

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

main point: for $\mathcal{C}$ a spectrally enriched ∞-category,
$S^1_b \xleftarrow{r} S^1_a$ a covering map of framed circles,

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

main point: for $\mathcal{C}$ a spectrally enriched ∞-category,
$S^1_b \xleftarrow{r} S^1_a$ a covering map of framed circles,
get ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$...

Q. for M an $n \times n$ matrix, difference between $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(R)$, r arbitrary

now, difference between

$$\mathrm{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \mathrm{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $\mathrm{tr}(M^r) \equiv \mathrm{tr}(M)^r$ in *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

main point: for $\mathcal{C}$ a spectrally enriched ∞-category,
$S^1_b \xleftarrow{r} S^1_a$ a covering map of framed circles,
get ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

doesn't exist over $\mathbb{Z}$! only have "Tate diagonal" in $\mathcal{S}\mathrm{p}$, not $\mathscr{D}(\mathbb{Z})$.

Thm (A–M-G–R).

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \qquad\qquad \mathrm{Sp}^{h\mathbb{T}} := \mathrm{Fun}(B\mathbb{T}, \mathrm{Sp})$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \qquad\qquad \mathrm{Sp}^{h\mathbb{T}}$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \left(\mathrm{Sp}^{h\mathbb{T}} \underset{\tau}{\curvearrowleft} \mathbb{N}^{\times}\right)$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\text{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times} \right)$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\text{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right)$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right)$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$,

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$,

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$,

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

§2.1

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

§2.1 stratification of a scheme/stack Y $\rightsquigarrow$ *generalized recollement:* $\mathrm{QC}(Y) \simeq \lim^{\mathbf{r.lax}}(\ldots \mathrm{l.lax} \ldots)$.

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

§2.1 stratification of a scheme/stack Y $\rightsquigarrow$ *generalized recollement:* $\mathrm{QC}(Y) \simeq \lim^{\mathrm{r.lax}}(\ldots \mathrm{l.lax} \ldots)$.

§2.2

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

§2.1 stratification of a scheme/stack Y $\rightsquigarrow$ *generalized recollement:* $\mathrm{QC}(Y) \simeq \lim^{\mathbf{r.lax}}(\ldots \mathrm{l.lax} \ldots)$.

§2.2 conjecture: there's a stratified derived stack $\mathscr{B}\mathbb{T}$ with $\mathbb{N}^{\times}$-action such that

$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^{\times}},$$

Thm (A–M-G–R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{h\mathbb{T}} \overset{\mathrm{l.lax}}{\underset{\tau}{\circlearrowleft}} \mathbb{N}^{\times} \right).$$

$\star$ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{h\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & \left(T^{\tau C_s}\right)^{\tau C_r} \end{array}$$

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

§2.1 stratification of a scheme/stack Y $\rightsquigarrow$ *generalized recollement:* $\mathrm{QC}(Y) \simeq \lim^{\mathbf{r.lax}}(\ldots \mathrm{l.lax} \ldots)$.

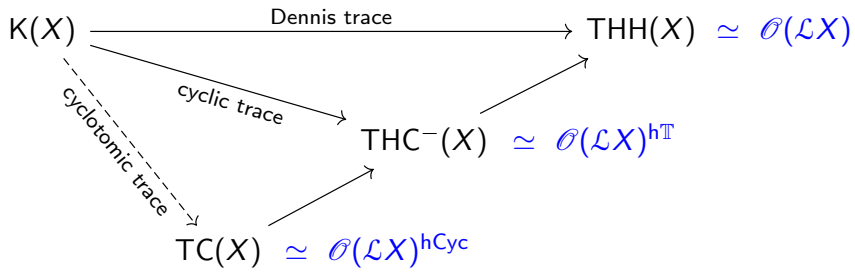
§2.2 conjecture: there's a stratified derived stack $\mathscr{B}\mathbb{T}$ with $\mathbb{N}^{\times}$-action such that

$$\mathrm{Cyc}(\mathbb{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathbf{h}\mathbb{N}^{\times}},$$

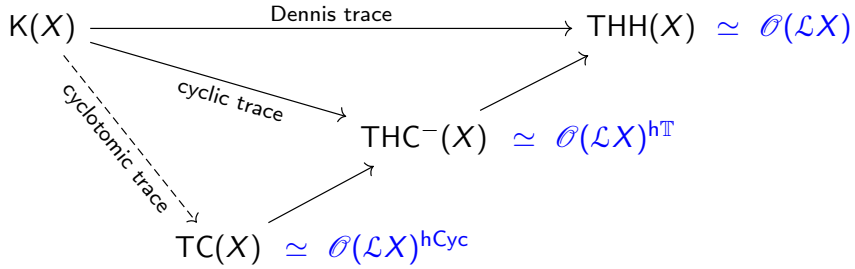
and $\mathscr{L}X$ enhances to $\mathbb{N}^{\times}$-equivariant $\mathscr{L}X \downarrow \mathscr{B}\mathbb{T}$ such that $\mathrm{THH}(X) \leftrightarrow \mathscr{O}_{\mathscr{L}X}$.

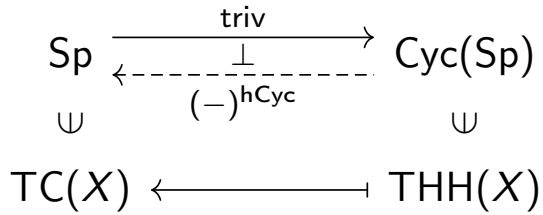
the geometry of the cyclotomic trace:

the geometry of the cyclotomic trace:

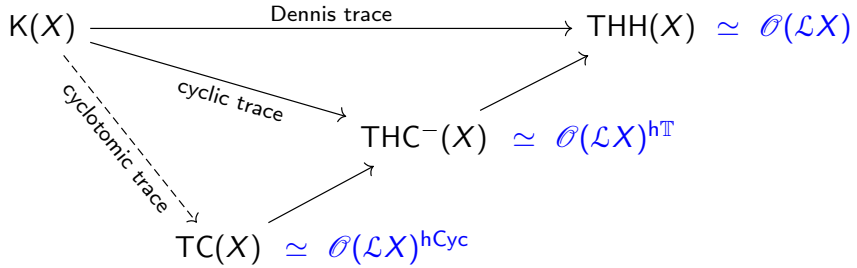


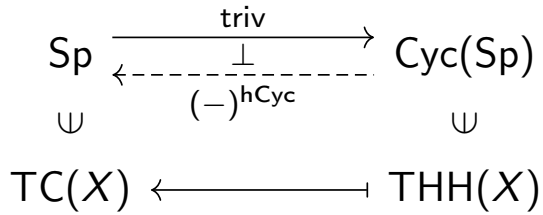
the geometry of the cyclotomic trace:





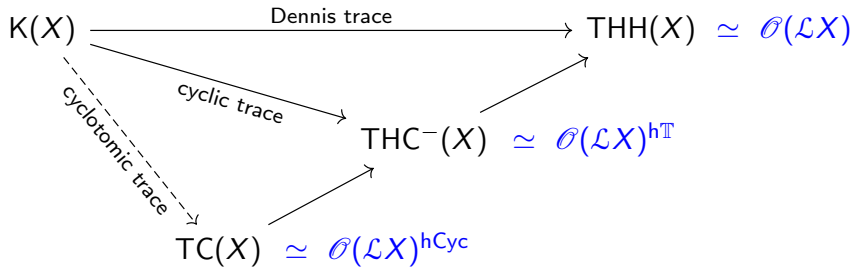
the geometry of the cyclotomic trace:

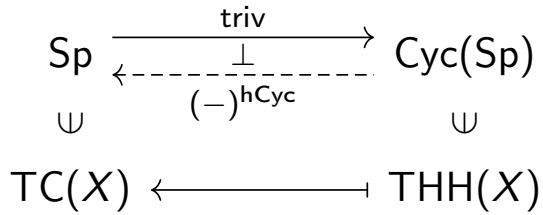




$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T\ ,\ \left\{ \begin{array}{c} T \\ \downarrow{\scriptstyle \sigma_r} \\ T^{\tau C_r} \end{array} \right\}_{r\in\mathbb{N}^\times}\ ,\ \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s\in\mathbb{N}^\times}\ ,\ \cdots \right) \right\}$$

the geometry of the cyclotomic trace:

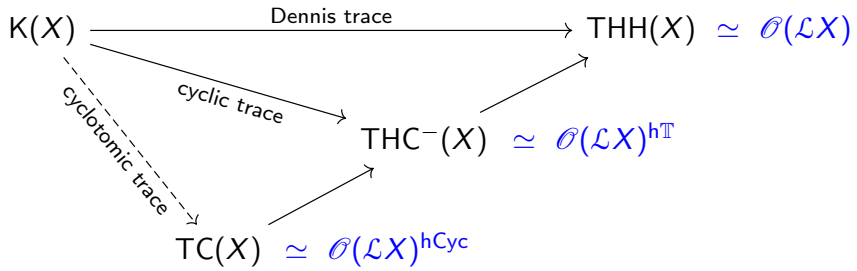


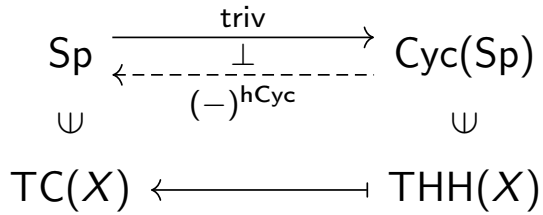


$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T\ ,\ \left\{ \begin{array}{c} T \\ \downarrow{\scriptstyle \sigma_r} \\ T^{\tau C_r} \end{array} \right\}_{r\in\mathbb{N}^\times}\ ,\ \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s\in\mathbb{N}^\times}\ ,\ \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

the geometry of the cyclotomic trace:



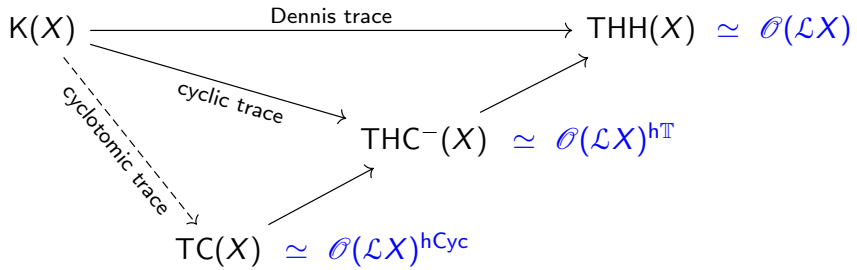


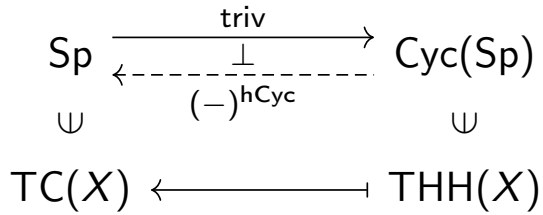
$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T\ ,\ \left\{ \begin{array}{c} T \\ \downarrow{\scriptstyle \sigma_r} \\ T^{\tau C_r} \end{array} \right\}_{r\in\mathbb{N}^\times}\ ,\ \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s\in\mathbb{N}^\times}\ ,\ \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;

the geometry of the cyclotomic trace:



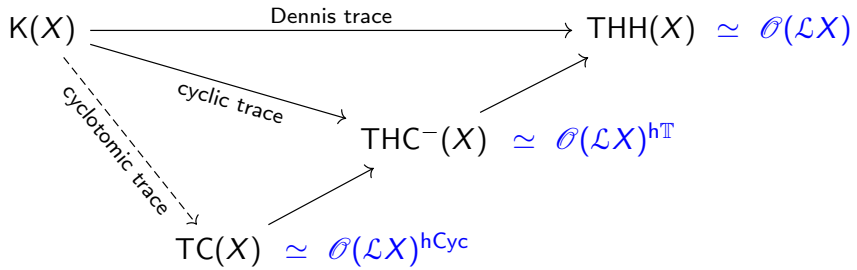


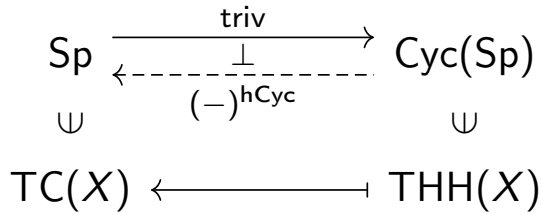
$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T\ ,\ \left\{ \begin{array}{c} T \\ \downarrow{\scriptstyle \sigma_r} \\ T^{\tau C_r} \end{array} \right\}_{r\in\mathbb{N}^\times}\ ,\ \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s\in\mathbb{N}^\times}\ ,\ \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible"

the geometry of the cyclotomic trace:



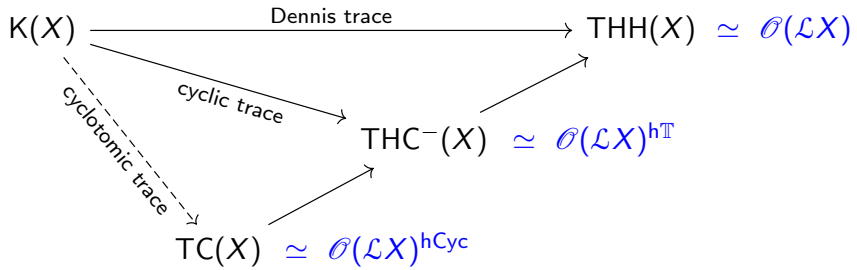


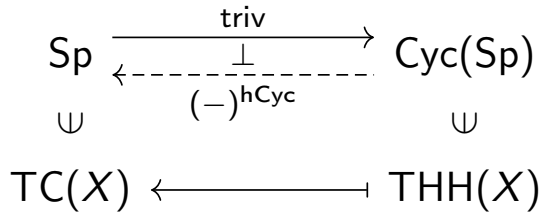
$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T\,,\ \left\{ \begin{array}{c} T \\ \downarrow{\scriptstyle \sigma_r} \\ T^{\tau C_r} \end{array} \right\}_{r\in\mathbb{N}^\times},\ \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s\in\mathbb{N}^\times},\ \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible" (+ higher coherences)...

the geometry of the cyclotomic trace:





$$\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T\ ,\ \left\{ \begin{array}{c} T \\ \downarrow{\scriptstyle \sigma_r} \\ T^{\tau C_r} \end{array} \right\}_{r\in\mathbb{N}^\times}\ ,\ \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \xrightarrow[\text{can.}]{} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s\in\mathbb{N}^\times}\ ,\ \cdots \right) \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are $\mathbb{T}$-invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible" (+ higher coherences)...

...which is precisely the structure present on trace-of-monodromy functions of vector bundles! $\mathbf{tr}(M)^r \equiv \mathbf{tr}(M^r)$, ...

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 2

§2.1 stratified schemes and generalized recollements (interlude)

§2.2 the geometry of TC (conjectural refinement of §1.3)

§2.3 the construction: factorization homology of enriched ∞-categories

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

illustrate general theorem through examples.

§2.1: STRATIFIED SCHEMES AND GEN^{IZED} RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U$$

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \xhookrightarrow[\text{closed}]{i} X \xhookleftarrow[\text{open}]{j} U$$

$$\begin{array}{ccccc} \mathbb{A}^0 & \hookrightarrow & \mathbb{A}^1 & \hookleftarrow & \mathbb{G}_m \\ & & \| & & \\ \mathrm{Spec}(\, k & \xleftarrow{0 \hookleftarrow x} & k[x] & \hookrightarrow & k[x^{\pm}] \,) \end{array}$$

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U$$

$$\begin{array}{ccccc} \mathbb{A}^0 & \hookrightarrow & \mathbb{A}^1 & \hookleftarrow & \mathbb{G}_m \\ & & \| & & \\ \mathrm{Spec}(\, k & \xleftarrow{0 \hookleftarrow x} & k[x] & \hookrightarrow & k[x^{\pm}] \,) \end{array}$$

Def. qcoh over X ***supported on*** Z***:*** $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.

§2.1: STRATIFIED SCHEMES AND GEN^{IZED} RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U \qquad \mathbb{A}^0 \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{G}_m$$

$$\mathbb{A}^1 = \operatorname{Spec}(k[x]) \qquad \operatorname{Spec}(k) \xleftarrow{0 \leftarrow\!\shortmid x} k[x] \hookrightarrow k[x^{\pm}]$$

Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.

$$\mathrm{QC}_Z(X) \hookrightarrow \mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U) \qquad \mathrm{Mod}_{k[x]}^{(x)\text{-tors}} \hookrightarrow \mathrm{Mod}_{k[x]} \xrightarrow{-\otimes_{k[x]} k[x^{\pm}]} \mathrm{Mod}_{k[x^{\pm}]}$$

$\mathrm{Mod}_{k[x]}^{(x)\text{-tors}} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq 1}$

§2.1: STRATIFIED SCHEMES AND GEN^{IZED} RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U \qquad \mathbb{A}^0 \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{G}_m$$

$$\mathbb{A}^1 = \operatorname{Spec}(k[x]), \quad \mathbb{G}_m = \operatorname{Spec}(k[x^{\pm}]), \quad \mathbb{A}^0 = \operatorname{Spec}(k) \text{ via } 0 \mapsfrom x$$

Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.

$$\mathrm{QC}_Z(X) \hookrightarrow \mathrm{QC}(X) \underset{\perp}{\overset{j^*}{\underset{j_*}{\leftrightarrows}}} \mathrm{QC}(U) \qquad \mathrm{Mod}_{k[x]}^{(x)\text{-tors}} \hookrightarrow \mathrm{Mod}_{k[x]} \underset{\perp}{\overset{-\otimes_{k[x]} k[x^{\pm}]}{\underset{\mathrm{fgt}}{\leftrightarrows}}} \mathrm{Mod}_{k[x^{\pm}]}$$

$\mathrm{Mod}_{k[x]}^{(x)\text{-tors}} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq 1}$

§2.1: STRATIFIED SCHEMES AND GEN^{IZED} RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U \qquad \mathbb{A}^0 \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{G}_m$$

$$\mathbb{A}^1 = \operatorname{Spec}(k[x]) \qquad \operatorname{Spec}(k \xleftarrow{0 \leftarrow\!\shortmid x} k[x] \hookrightarrow k[x^{\pm}])$$

Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.

$$\mathrm{QC}_Z(X) \underset{\ker(\mathrm{id} \to j_* j^*)}{\overset{\hookrightarrow}{\underset{\leftarrow}{\overset{\perp}{}}}} \mathrm{QC}(X) \underset{\perp}{\overset{j^*}{\leftarrow j_* \rightharpoonup}} \mathrm{QC}(U) \qquad \mathrm{Mod}_{k[x]}^{(x)\text{-tors}} \underset{\leftarrow}{\overset{\hookrightarrow}{\overset{\perp}{}}} \mathrm{Mod}_{k[x]} \underset{\perp}{\overset{-\otimes_{k[x]} k[x^{\pm}]}{\leftarrow \mathrm{fgt} \rightharpoonup}} \mathrm{Mod}_{k[x^{\pm}]}$$

$\mathrm{Mod}_{k[x]}^{(x)\text{-tors}} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq 1}$

§2.1: STRATIFIED SCHEMES AND GEN^{IZED} RECOLLEMENTS

illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U \qquad Z \xrightarrow{\hat{i}} X^{\wedge}_{Z}$$

$$\mathbb{A}^0 \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{G}_m \qquad \mathbb{A}^1 = \operatorname{Spec}(k[x]) \qquad \operatorname{Spec}(k \xleftarrow{0 \leftarrow\!\shortmid x} k[x] \hookrightarrow k[x^{\pm}])$$

Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.

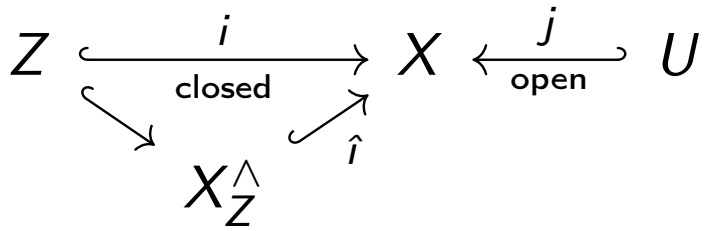
$$\mathrm{QC}_Z(X) \underset{\ker(\mathrm{id} \to j_* j^*)}{\overset{\hookrightarrow}{\underset{\leftarrow}{\perp}}} \mathrm{QC}(X) \overset{j^*}{\underset{\perp}{\leftarrow j_* \rightarrow}} \mathrm{QC}(U) \qquad \mathrm{Mod}^{(x)\text{-tors}}_{k[x]} \underset{\leftarrow}{\overset{\hookrightarrow}{\perp}} \mathrm{Mod}_{k[x]} \overset{-\otimes_{k[x]} k[x^{\pm}]}{\underset{\perp}{\leftarrow \mathrm{fgt} \rightarrow}} \mathrm{Mod}_{k[x^{\pm}]}$$

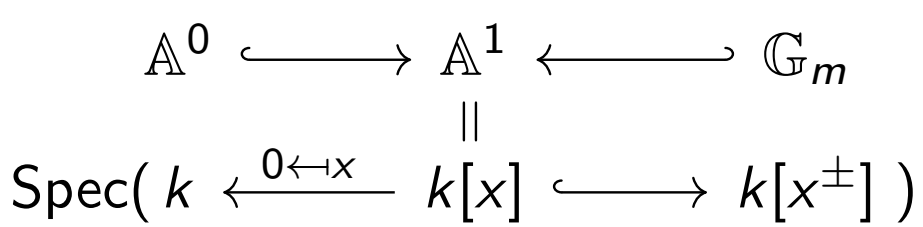
$\mathrm{Mod}^{(x)\textbf{-tors}}_{k[x]} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq \mathbf{1}}$

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

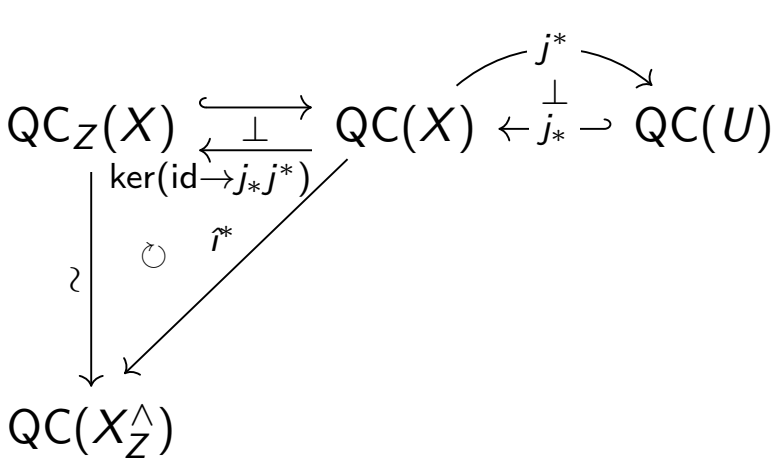
illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

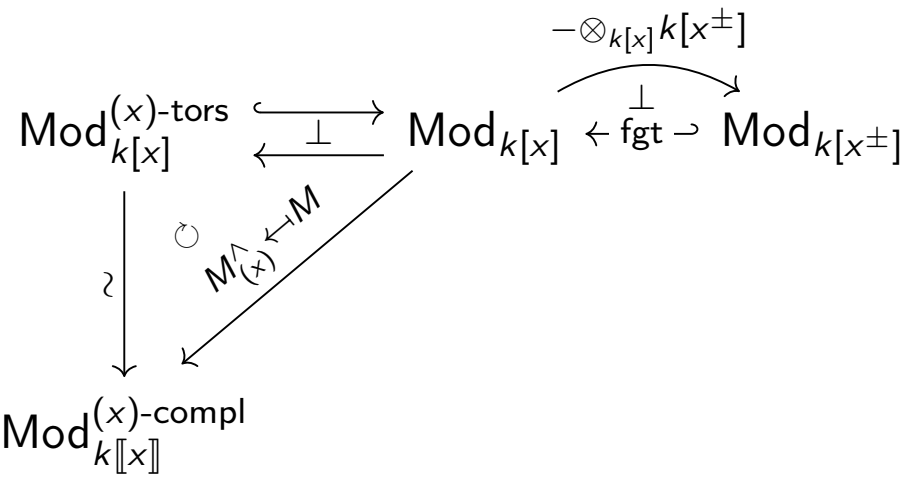
Ex. 1





Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.





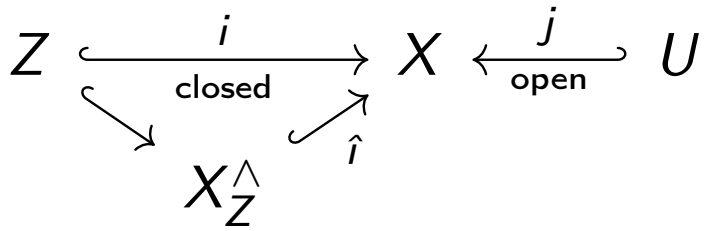
$\mathrm{Mod}^{(x)\textbf{-tors}}_{k[x]} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq 1}$

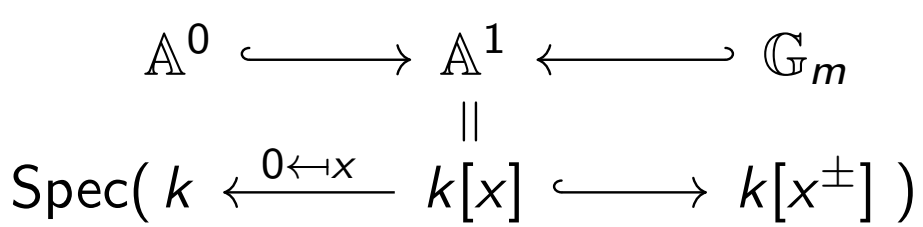
$\mathrm{Mod}^{(x)\textbf{-compl}}_{k[\![x]\!]} \simeq \lim(\cdots \to \mathrm{Mod}_{k[x]/x^2} \to \mathrm{Mod}_{k[x]/x}) \quad \rightsquigarrow \quad \{- \otimes_{k[x]} k[x]/x^n\}_{n \geq 1}$

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

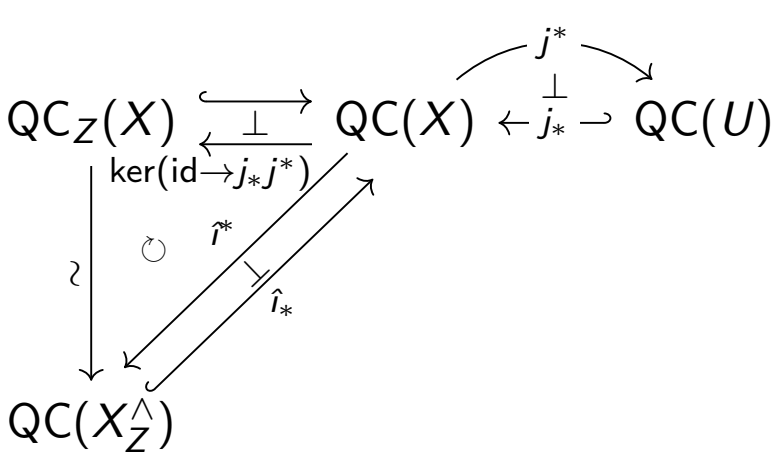
illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

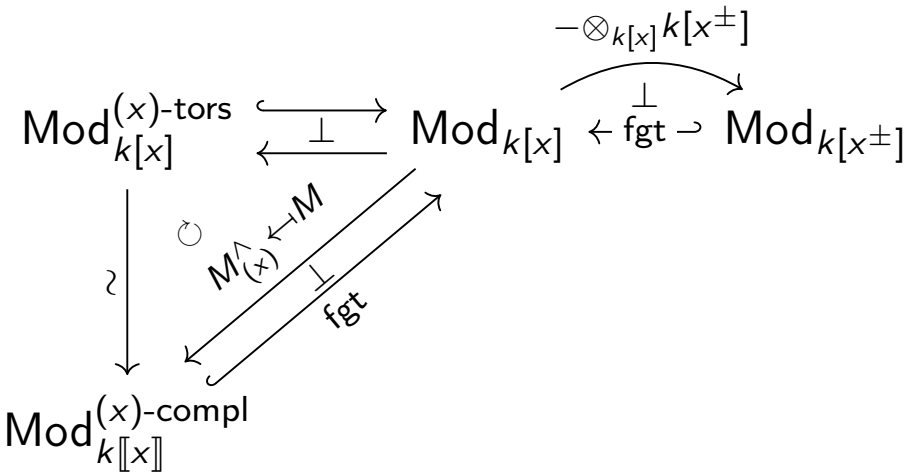
Ex. 1





Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.





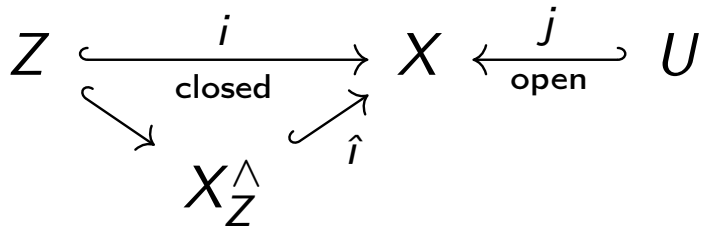
$\mathrm{Mod}^{(x)\textbf{-tors}}_{k[x]} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq 1}$

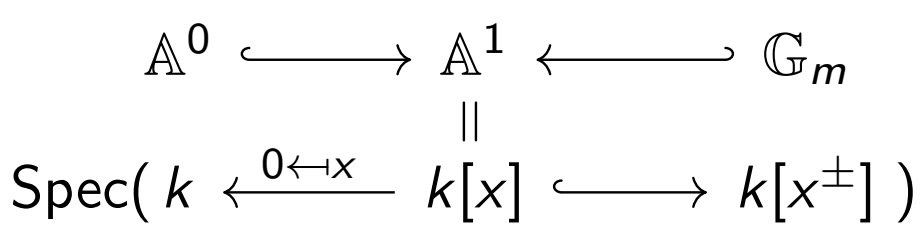
$\mathrm{Mod}^{(x)\textbf{-compl}}_{k[\![x]\!]} \simeq \lim(\cdots \to \mathrm{Mod}_{k[x]/x^2} \to \mathrm{Mod}_{k[x]/x})$ $\rightsquigarrow$ $\{- \otimes_{k[x]} k[x]/x^n\}_{n \geq 1} \dashv$ "lim"

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

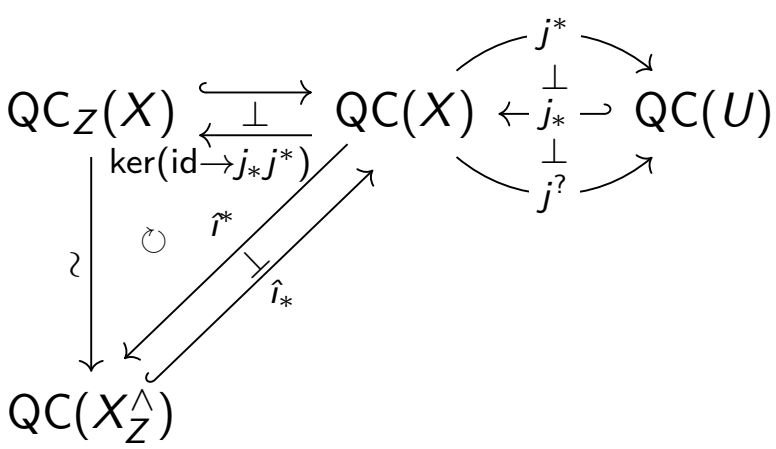
illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

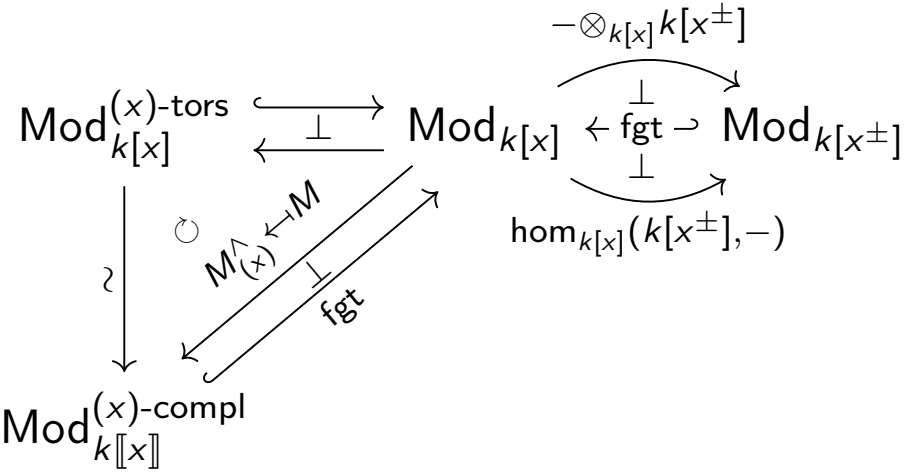
Ex. 1





Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.





$\mathrm{Mod}^{(x)\text{-tors}}_{k[x]} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq 1}$

$\mathrm{Mod}^{(x)\text{-compl}}_{k[\![x]\!]} \simeq \lim(\cdots \to \mathrm{Mod}_{k[x]/x^2} \to \mathrm{Mod}_{k[x]/x}) \quad \rightsquigarrow \quad \{- \otimes_{k[x]} k[x]/x^n\}_{n \geq 1} \dashv$ "lim"

§2.1: STRATIFIED SCHEMES AND GEN$^{\text{IZED}}$ RECOLLEMENTS

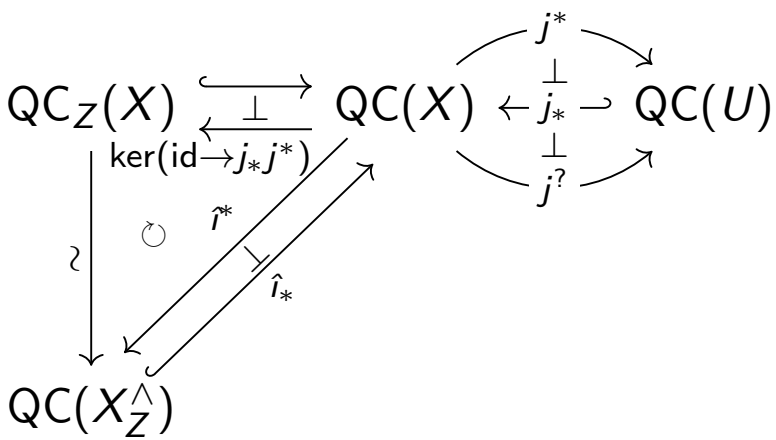
illustrate general theorem through examples.
truly need stable ∞-categories (not triangulated categories).
should feel a lot like *categorified group theory* (extension sequences, etc.).

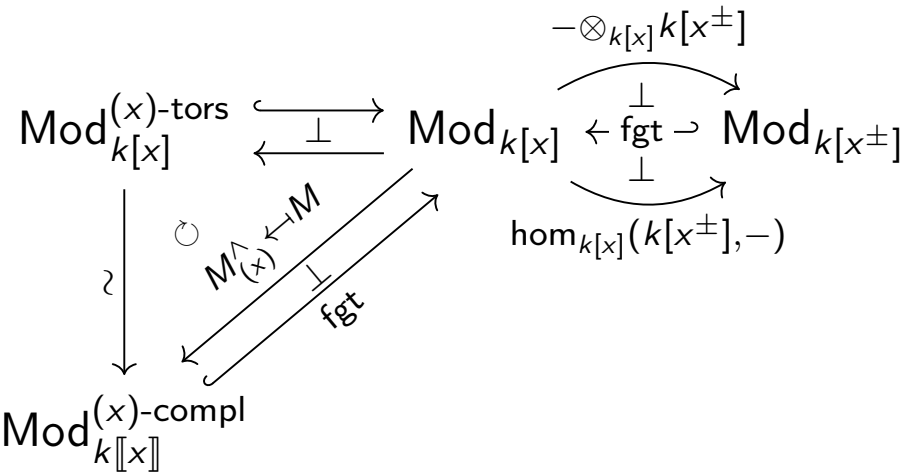
Ex. 1

$$Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U \qquad X \xleftarrow{\hat{\imath}} X^{\wedge}_{Z}$$

$$\mathbb{A}^0 \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{G}_m \qquad \mathbb{A}^1 = \operatorname{Spec}(k[x]) \qquad \mathbb{G}_m = \operatorname{Spec}(k[x^{\pm}])$$

Def. qcoh over X ***supported on*** Z: $\mathrm{QC}_Z(X) := \ker(\mathrm{QC}(X) \xrightarrow{j^*} \mathrm{QC}(U))$.

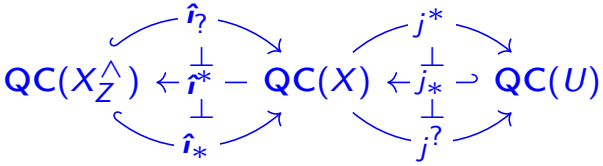


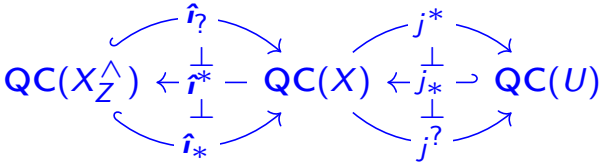


$\mathrm{Mod}^{(x)\text{-tors}}_{k[x]} \subset \mathrm{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n\geq 1}$

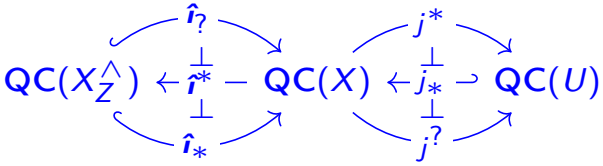
$\mathrm{Mod}^{(x)\text{-compl}}_{k[\![x]\!]} \simeq \lim(\cdots \to \mathrm{Mod}_{k[x]/x^2} \to \mathrm{Mod}_{k[x]/x}) \quad \rightsquigarrow \quad \{- \otimes_{k[x]} k[x]/x^n\}_{n\geq 1} \dashv$ "lim"

add pix here!!!

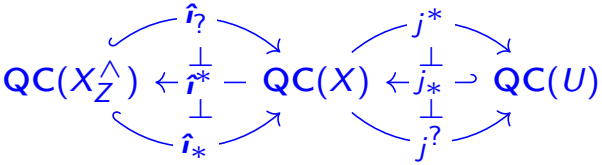
so, get a ***recollement*** of $\mathrm{QC}(X)$: 

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

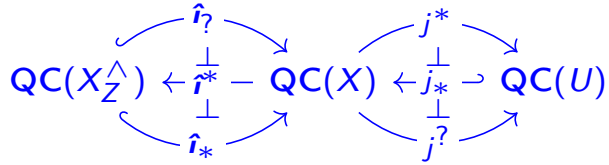
Microcosm thm for recollements:

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$,

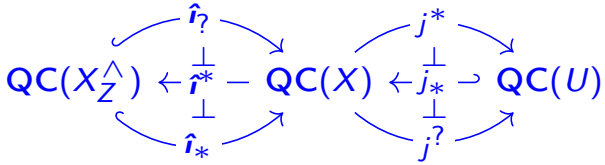
so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ {\scriptstyle \text{pullback}} \quad {\scriptstyle \eta_{\hat{\imath}}(F)} \downarrow & & \downarrow {\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F))} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} .$$

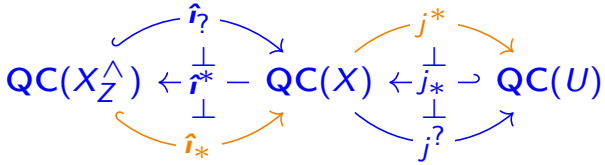
so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ {\scriptstyle \text{pullback}} \quad {\scriptstyle \eta_{\hat{\imath}}(F)} \downarrow & & \downarrow {\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F))} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements:

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

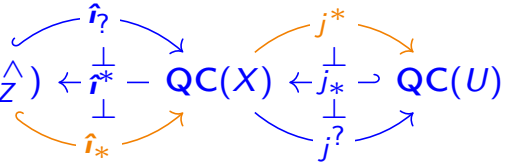
Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ \text{pullback} \quad {\scriptstyle \eta_{\hat{\imath}}(F)}\downarrow & & \downarrow{\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F)) \,=\, j_*(\mu)} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} \quad .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements: $\mu =$ ***monodromy*** $=$ "gluing data"

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim{}^{\mathrm{r.lax}} \left(\mathrm{QC}(X^{\wedge}_Z) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right) := \left\{ \left(\begin{array}{c} F_0 \in \mathrm{QC}(X^{\wedge}_Z) \,, \; F_1 \in \mathrm{QC}(U) \,, \quad F_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \mu \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad j^* \hat{\imath}_* F_0 \end{array} \right) \right\} .$$

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

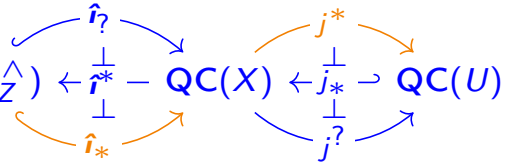
$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ \text{pullback} \quad {\scriptstyle \eta_{\hat{\imath}}(F)}\downarrow & & \downarrow{\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F)) \,=\, j_*(\mu)} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} \quad .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements: $\mu =$ ***monodromy*** $=$ "gluing data"

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim{}^{\mathrm{r.lax}} \left(\mathrm{QC}(X^\wedge_Z) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right) := \left\{ \left(\begin{array}{c} F_0 \in \mathrm{QC}(X^\wedge_Z) \,, \; F_1 \in \mathrm{QC}(U) \,, \quad F_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \mu \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad j^* \hat{\imath}_* F_0 \end{array} \right) \right\} .$$

given $\mathcal{I} \xrightarrow{\mathcal{C}^\bullet}$ Cat, an obj of $\left\{ \begin{array}{c} \text{strict} \\ \text{left-lax} \\ \text{right-lax} \end{array} \right.$ limit is:

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

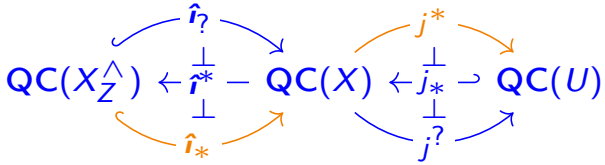
$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ {\scriptstyle \text{pullback}} \quad {\scriptstyle \eta_{\hat{\imath}}(F)} \downarrow & & \downarrow {\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F)) \,=\, j_*(\mu)} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} \quad .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements: $\mu =$ ***monodromy*** $=$ "gluing data"

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim{}^{\mathrm{r.lax}} \left(\mathrm{QC}(X^\wedge_Z) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right) := \left\{ \left(\begin{array}{c} F_0 \in \mathrm{QC}(X^\wedge_Z) \,, \; F_1 \in \mathrm{QC}(U) \,, \\ \end{array} \; \begin{array}{c} F_1 \\ {\scriptstyle \mu} \downarrow \\ j^* \hat{\imath}_* F_0 \end{array} \right) \right\} .$$

given $\mathcal{I} \xrightarrow{\mathcal{C}^\bullet}$ Cat, an obj of $\left\{ \begin{array}{l} \text{strict} \\ \text{left-lax} \\ \text{right-lax} \end{array} \right.$ limit is: $\{x_i \in \mathcal{C}_i\}_{i \in \mathcal{I}}$

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

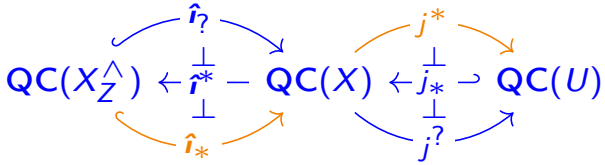
$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ \text{pullback} \quad {\scriptstyle \eta_{\hat{\imath}}(F)}\downarrow & & \downarrow{\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F)) \,=\, j_*(\mu)} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} \quad .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements: $\mu =$ ***monodromy*** $=$ "gluing data"

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\mathrm{QC}(X_Z^\wedge) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right) := \left\{ \left(\begin{array}{c} F_0 \in \mathrm{QC}(X_Z^\wedge) \,,\; F_1 \in \mathrm{QC}(U) \,, \\ \end{array} \begin{array}{c} F_1 \\ {\scriptstyle \mu}\downarrow \\ j^* \hat{\imath}_* F_0 \end{array} \right) \right\} .$$

given $\mathcal{I} \xrightarrow{\mathcal{C}^\bullet}$ Cat, an obj of $\left\{ \begin{array}{l} \text{strict} \\ \text{left-lax} \\ \text{right-lax} \end{array} \right.$ limit is: $\{x_i \in \mathcal{C}_i\}_{i \in \mathcal{I}}$ + $\forall\ i \xrightarrow{\varphi} j$ in $\mathcal{I}$, $\left\{ \begin{array}{l} \mathcal{C}_\varphi(x_i) \simeq x_j \\ \mathcal{C}_\varphi(x_i) \to x_j \\ \mathcal{C}_\varphi(x_i) \leftarrow x_j \end{array} \right.$

so, get a ***recollement*** of $\mathrm{QC}(X)$: 

Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{\imath}^* F$ and $F_1 := j^* F$,

$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ \text{pullback} \quad {\scriptstyle \eta_{\hat{\imath}}(F)}\downarrow & & \downarrow{\scriptstyle j_* j^*(\eta_{\hat{\imath}}(F)) \,=\, j_*(\mu)} \\ \hat{\imath}_* F_0 & \xrightarrow[\eta_j(\hat{\imath}_* F_0)]{} & j_* j^* \hat{\imath}_* F_0 \end{array} \quad .$$

...and this pullback square is *unique*, in the sense of the

Macrocosm thm for recollements: $\mu =$ ***monodromy*** $=$ "gluing data"

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim{}^{\mathrm{r.lax}} \left(\mathrm{QC}(X_Z^\wedge) \xrightarrow{j^* \hat{\imath}_*} \mathrm{QC}(U) \right) := \left\{ \left(\begin{array}{c} F_0 \in \mathrm{QC}(X_Z^\wedge) \,, \; F_1 \in \mathrm{QC}(U) \,, \quad {\scriptstyle \mu}\downarrow \\ j^* \hat{\imath}_* F_0 \end{array} \right) \right\} .$$

given $\mathcal{I} \xrightarrow{\mathcal{C}^\bullet}$ Cat, an obj of $\begin{cases} \text{strict} \\ \text{left-lax} \\ \text{right-lax} \end{cases}$ limit is: $\{x_i \in \mathcal{C}_i\}_{i \in \mathcal{I}}$ $+$ $\forall\, i \xrightarrow{\varphi} j$ in $\mathcal{I}$, $\begin{cases} \mathcal{C}_\varphi(x_i) \simeq x_j \\ \mathcal{C}_\varphi(x_i) \to x_j \\ \mathcal{C}_\varphi(x_i) \leftarrow x_j \end{cases}$

$+$ higher coherence data

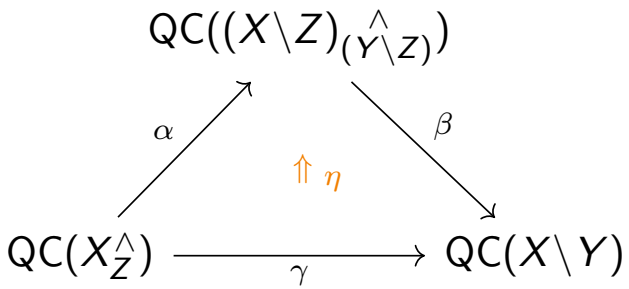
Ex. 2

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X$$

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$



add pix here!!!

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)^{\wedge}_{(Y\backslash Z)}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\wedge}_{Z}) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

add pix here!!!

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)^{\wedge}_{(Y\backslash Z)}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\wedge}_{Z}) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

F_1 F_0 F_2

add pix here!!!

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)_{(\hat{Y}\backslash Z)}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\wedge}_{Z}) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

$$\begin{array}{ccc} F_1 & & \\ \downarrow{\scriptstyle\mu_{01}} & & \\ \alpha F_0 & & \end{array} \qquad \begin{array}{ccc} F_0 & & \end{array} \qquad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ {\scriptstyle\mu_{02}}\downarrow & & \\ \gamma F_0 & & \end{array}$$

add pix here!!!

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)_{(\hat{Y}\backslash Z)}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\wedge}_{Z}) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

$$\begin{array}{ccc} F_1 & & \\ \downarrow{\scriptstyle\mu_{01}} & & \\ \alpha F_0 & & \end{array} \qquad F_0 \qquad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ \downarrow{\scriptstyle\mu_{02}} & & \downarrow{\scriptstyle\beta(\mu_{01})} \\ \gamma F_0 & & \beta\alpha F_0 \end{array}$$

add pix here!!!

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)_{(\widehat{Y\backslash Z})}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\widehat{}}_Z) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

$$\begin{array}{ccc} F_1 & & \\ \downarrow{\scriptstyle\mu_{01}} & & \\ \alpha F_0 & & \end{array} \qquad F_0 \qquad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ {\scriptstyle\mu_{02}}\downarrow & & \downarrow{\scriptstyle\beta(\mu_{01})} \\ \gamma F_0 & \xrightarrow[\eta(F_0)]{} & \beta\alpha F_0 \end{array}$$

add pix here!!!

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)_{(\widehat{Y\backslash Z})}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\widehat{}}_Z) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

$$\begin{array}{ccc} F_1 & & \\ \downarrow{\scriptstyle\mu_{01}} & & \\ \alpha F_0 & & \end{array} \qquad F_0 \qquad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ {\scriptstyle\mu_{02}}\downarrow & \circlearrowleft\ \mu_{012} & \downarrow{\scriptstyle\beta(\mu_{01})} \\ \gamma F_0 & \xrightarrow[\eta(F_0)]{} & \beta\alpha F_0 \end{array}$$

add pix here!!!

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)_{(\widehat{Y\backslash Z})}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\widehat{}}_Z) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

$$\begin{array}{ccc} F_1 & & \\ \downarrow{\scriptstyle\mu_{01}} & & \\ \alpha F_0 & & \end{array} \qquad F_0 \qquad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ {\scriptstyle\mu_{02}}\downarrow & \circlearrowleft\ \mu_{012} & \downarrow{\scriptstyle\beta(\mu_{01})} \\ \gamma F_0 & \xrightarrow[\eta(F_0)]{} & \beta\alpha F_0 \end{array}$$

add pix here!!!

simpler: $\lim^{\text{l.lax}}$ $\left(\begin{array}{ccc} & \mathcal{C}_1 & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathcal{C}_0 & \xrightarrow[\gamma]{} & \mathcal{C}_2 \end{array} \right) \simeq \left\{ \left(\begin{array}{ccc} & x_1 & \\ & \uparrow{\scriptstyle\mu_{01}} & \\ & \alpha x_0 & \\ x_0 & & x_2 \xleftarrow{\mu_{12}} \beta x_1 \end{array} \right) \right\}$ because now $\begin{array}{ccc} x_2 & \xleftarrow{\mu_{12}} & \beta x_1 \\ \exists!\ {\scriptstyle\mu_{02}}\uparrow & & \uparrow{\scriptstyle\beta(\mu_{01})} \\ \gamma x_0 & \xrightarrow[\eta(x_0)]{} & \beta\alpha x_0 \end{array}$

Ex. 2

$$Z \xhookrightarrow[\text{closed}]{} Y \xhookrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad [2] \xrightarrow{\text{l.lax}} \text{Cat}$$

Thm (A–M-G–R).

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \mathrm{QC}((X\backslash Z)_{(\widehat{Y\backslash Z})}) & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathrm{QC}(X^{\widehat{}}_Z) & \xrightarrow[\gamma]{} & \mathrm{QC}(X\backslash Y) \end{array} \right).$$

$$\begin{array}{ccc} F_1 & & \\ \downarrow{\scriptstyle\mu_{01}} & & \\ \alpha F_0 & & \end{array} \qquad F_0 \qquad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ {\scriptstyle\mu_{02}}\downarrow & \circlearrowleft\ \mu_{012} & \downarrow{\scriptstyle\beta(\mu_{01})} \\ \gamma F_0 & \xrightarrow[\eta(F_0)]{} & \beta\alpha F_0 \end{array}$$

add pix here!!!

simpler: $\lim^{\text{l.lax}}$ $\left(\begin{array}{ccc} & \mathcal{C}_1 & \\ {\scriptstyle\alpha}\nearrow & \Uparrow \eta & \searrow{\scriptstyle\beta} \\ \mathcal{C}_0 & \xrightarrow[\gamma]{} & \mathcal{C}_2 \end{array} \right)$ $\simeq$ $\left\{ \left(\begin{array}{ccc} & x_1 & \\ & \uparrow{\scriptstyle\mu_{01}} & \\ & \alpha x_0 & \\ x_0 & & x_2 \xleftarrow{\mu_{12}} \beta x_1 \end{array} \right) \right\}$ because now $\begin{array}{ccc} x_2 & \xleftarrow{\mu_{12}} & \beta x_1 \\ \exists!\ {\scriptstyle\mu_{02}}\uparrow & & \uparrow{\scriptstyle\beta(\mu_{01})} \\ \gamma x_0 & \xrightarrow[\eta(x_0)]{} & \beta\alpha x_0 \end{array}$

$\star$ above, need μ_{012} because laxness of diagram $\neq$ laxness of limit

the general picture:

the general picture:

fix any scheme X and any set of closed subschemes including X.

the general picture:

fix any scheme X and any set of closed subschemes including X.

order by inclusion $\rightsquigarrow$ determines a poset P

the general picture:

fix any scheme X and any set of closed subschemes including X.

order by inclusion $\rightsquigarrow$ determines a poset P

$\rightsquigarrow$ a ***stratification*** of X over P: a map of posets

$$\mathrm{P} \xrightarrow{X_\bullet} \{\text{closed subschemes of } X\} .$$

the general picture:

fix any scheme X and any set of closed subschemes including X.

order by inclusion $\rightsquigarrow$ determines a poset P

$\rightsquigarrow$ a ***stratification*** of X over P: a map of posets

$$\mathrm{P} \xrightarrow{X_\bullet} \{\text{closed subschemes of } X\} .$$

notation: $X_{<p} := \bigcup_{q<p} X_q$.

the general picture:

fix any scheme X and any set of closed subschemes including X.

order by inclusion $\rightsquigarrow$ determines a poset P

$\rightsquigarrow$ a ***stratification*** of X over P: a map of posets

$$\mathrm{P} \xrightarrow{X_\bullet} \{\text{closed subschemes of } X\} .$$

notation: $X_{<p} := \bigcup_{q<p} X_q$.

assumption: P satisfies *descending chain condition*.

the general picture:

fix any scheme X and any set of closed subschemes including X.

order by inclusion $\rightsquigarrow$ determines a poset P

$\rightsquigarrow$ a ***stratification*** of X over P: a map of posets

$$\mathrm{P} \xrightarrow{X_\bullet} \{\text{closed subschemes of } X\} .$$

notation: $X_{<p} := \bigcup_{q<p} X_q$.

assumption: P satisfies *descending chain condition*.

Thm (A–M-G–R). $\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} \mathrm{P} & \longrightarrow^{\text{l.lax}} & \mathrm{Cat} \\ p & \longmapsto & \mathrm{QC}((X \backslash X_{<p})_{(X_p \backslash \hat{X}_{<p})}) \end{array} \right)$.

§2.2: THE GEOMETRY OF TC

§2.2: THE GEOMETRY OF TC

recall: $\mathsf{THH} \in \mathrm{Cyc}(\mathcal{S}\mathrm{p})$ a *cyclotomic spectrum*, $\mathsf{TC} := \mathsf{THH}^{\mathsf{hCyc}}$.

§2.2: THE GEOMETRY OF TC

recall: THH $\in$ Cyc($\mathcal{S}$p) a *cyclotomic spectrum*, TC $:=$ THH$^{\mathbf{hCyc}}$.

cyclotomic spectra defined in terms of

cyclonic spectra $:=$ genuine-proper $\mathbb{T}$-spectra $:= \mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}}$.

remember "genuine" fixedpoints for proper subgroups of $\mathbb{T}$. ***genuine***: e.g. $E\mathcal{F} \xrightarrow{\sim}$ pt.

§2.2: THE GEOMETRY OF TC

recall: $\mathrm{THH} \in \mathrm{Cyc}(\mathcal{S}\mathrm{p})$ a *cyclotomic spectrum*, $\mathrm{TC} := \mathrm{THH}^{\mathrm{hCyc}}$.

cyclotomic spectra defined in terms of

cyclonic spectra $:=$ genuine-proper $\mathbb{T}$-spectra $:= \mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}}$.

remember "genuine" fixedpoints for proper subgroups of $\mathbb{T}$. ***genuine***: e.g. $EG \xrightarrow{\nsim} \mathrm{pt}$.

Thm (A–M-G–R). $\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}} \xrightarrow{\sim} \lim^{\mathrm{r.lax}}$

$$\begin{pmatrix} \mathbb{N}^{\mathrm{div}} & \xrightarrow{\text{l.lax}} & \mathrm{Cat} \\ \cup & & \cup \\ i & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_i)} \xleftarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \\ \downarrow j & \longmapsto & \downarrow (-)^{\tau C_j} \\ ij & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_{ij})} \xrightarrow[\sim]{} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \end{pmatrix} .$$

$\mathbb{N}^{\mathbf{div}} \cong$ poset of proper closed subgroups of $\mathbb{T}$

$i \leftrightarrow C_i$

§2.2: THE GEOMETRY OF TC

recall: $\mathrm{THH} \in \mathrm{Cyc}(\mathcal{S}\mathrm{p})$ a *cyclotomic spectrum*, $\mathrm{TC} := \mathrm{THH}^{\mathrm{hCyc}}$.

cyclotomic spectra defined in terms of

cyclonic spectra $:=$ genuine-proper $\mathbb{T}$-spectra $:= \mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}}$.

remember "genuine" fixedpoints for proper subgroups of $\mathbb{T}$. ***genuine***: e.g. $EG \xrightarrow{\nsim} \mathrm{pt}$.

Thm (A–M-G–R). $\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}} \xrightarrow{\sim} \lim^{\mathrm{r.lax}}$

$$\begin{pmatrix} \mathbb{N}^{\mathrm{div}} & \xrightarrow{\text{l.lax}} & \mathrm{Cat} \\ \cup & & \cup \\ i & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_i)} \xleftarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \\ \downarrow{\scriptstyle j} & \mapsto & \downarrow{\scriptstyle (-)^{\tau C_j}} \\ ij & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_{ij})} \xrightarrow[\sim]{} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \end{pmatrix} .$$

$\mathbb{N}^{\mathbf{div}} \cong$ poset of proper closed subgroups of $\mathbb{T}$

$i \leftrightarrow C_i$

Larger thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}G}$ is stratified over poset of closed subgroups of G up to subconjugacy.

§2.2: THE GEOMETRY OF TC

recall: $\mathrm{THH} \in \mathrm{Cyc}(\mathcal{S}\mathrm{p})$ a *cyclotomic spectrum*, $\mathrm{TC} := \mathrm{THH}^{\mathrm{hCyc}}$.

cyclotomic spectra defined in terms of

cyclonic spectra $:=$ genuine-proper $\mathbb{T}$-spectra $:= \mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}}$.

remember "genuine" fixedpoints for proper subgroups of $\mathbb{T}$. ***genuine***: e.g. $EG \xrightarrow{\nsim}$ pt.

Thm (A–M-G–R). $\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}} \xrightarrow{\sim} \lim^{\mathrm{r.lax}}$

$$\begin{pmatrix} \mathbb{N}^{\mathrm{div}} & \xrightarrow{\text{l.lax}} & \mathrm{Cat} \\ \cup & & \cup \\ i & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_i)} \xleftarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \\ \downarrow j & \longmapsto & \downarrow (-)^{\tau C_j} \\ ij & & \mathcal{S}\mathrm{p}^{\mathrm{h}(\mathbb{T}/C_{ij})} \xrightarrow[\sim]{} \mathcal{S}\mathrm{p}^{\mathrm{h}\mathbb{T}} \end{pmatrix} .$$

$\mathbb{N}^{\mathrm{div}} \cong$ poset of proper closed subgroups of $\mathbb{T}$

$i \leftrightarrow C_i$

Larger thm. $\mathcal{S}\mathrm{p}^{\mathrm{g}G}$ is stratified over poset of closed subgroups of G up to subconjugacy.

Conj. there's an $\mathbb{N}^{\mathrm{div}}$-stratified stack $\mathscr{B}\mathbb{T}$, a "mapping telescope" of copies of $B\mathbb{T}$, such that $\mathcal{S}\mathrm{p}^{\mathrm{g}<\mathbb{T}} \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})$ compatibly with the above generalized recollement.

Thm. $\mathbb{S}\mathrm{p}^{\mathbf{g}<\mathbb{T}} \xrightarrow{\sim} \lim^{\mathbf{r.lax}}(\ \mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ )$. ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}\ \mathbf{strat}/\mathbb{N}^{\mathbf{div}}\ \mathbf{s.t.}\ \mathbb{S}\mathrm{p}^{\mathbf{g}<\mathbb{T}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$.

Thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<}\mathbb{T}} \xrightarrow{\sim} \lim^{\mathbf{r.lax}}(\ \mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ )$. ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}$ **strat**$/\mathbb{N}^{\mathbf{div}}$ **s.t.** $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<}\mathbb{T}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$.

"geometric fixedpoints" $\Phi^{C_n} : \mathcal{S}\mathrm{p}^{\mathrm{g}^{<}\mathbb{T}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{\mathrm{g}^{<}(\mathbb{T}/C_n)} \xrightarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{g}^{<}\mathbb{T}}$ $\rightsquigarrow$ $\mathbb{N}^{\times} \curvearrowright \mathcal{S}\mathrm{p}^{\mathrm{g}^{<}\mathbb{T}}$

Thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \xrightarrow{\sim} \lim^{\mathbf{r.lax}}(\ \mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ )$. ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}$ **strat**$/\mathbb{N}^{\mathbf{div}}$ **s.t.** $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$.

"geometric fixedpoints" $\Phi^{C_n} : \mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{\mathrm{g}^{<(\mathbb{T}/C_n)}} \xrightarrow{\sim} \mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}} \quad \rightsquigarrow \quad \mathbb{N}^{\times} \curvearrowright \mathcal{S}\mathrm{p}^{\mathrm{g}^{<\mathbb{T}}}$

$\leftrightarrow$ pullback of $\lim^{\mathrm{r.lax}}$ along dilation action $\mathbb{N}^{\times} \curvearrowright \mathbb{N}^{\mathrm{div}}$ $\quad n \mapsto (i \mapsto ni)$

Thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}<\mathbb{T}} \xrightarrow{\sim} \lim^{\mathbf{r.lax}}(\ \mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ )$. ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}$ **strat**$/\mathbb{N}^{\mathbf{div}}$ **s.t.** $\mathcal{S}\mathrm{p}^{\mathbf{g}<\mathbb{T}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$.

"geometric fixedpoints" $\Phi^{C_n} : \mathcal{S}\mathrm{p}^{g<\mathbb{T}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{g<(\mathbb{T}/C_n)} \xrightarrow{\sim} \mathcal{S}\mathrm{p}^{g<\mathbb{T}} \quad \rightsquigarrow \quad \mathbb{N}^\times \curvearrowright \mathcal{S}\mathrm{p}^{g<\mathbb{T}}$

$\leftrightarrow$ pullback of $\lim^{r.lax}$ along dilation action $\mathbb{N}^\times \curvearrowright \mathbb{N}^{div}$ $\quad n \mapsto (i \mapsto ni)$

cyclotomic spectra $:= \mathrm{Cyc}(\mathcal{S}\mathrm{p}) := (\mathcal{S}\mathrm{p}^{g<\mathbb{T}})^{h\mathbb{N}^\times}$

Thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \xrightarrow{\sim} \lim^{\mathbf{r.lax}}(\ \mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ )$. ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}$ **strat**$/\mathbb{N}^{\mathbf{div}}$ **s.t.** $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$.

"geometric fixedpoints" $\Phi^{C_n} : \mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{g^{<(\mathbb{T}/C_n)}} \xrightarrow{\sim} \mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}} \quad \rightsquigarrow \quad \mathbb{N}^\times \curvearrowright \mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}}$

$\leftrightarrow$ pullback of $\lim^{r.lax}$ along dilation action $\mathbb{N}^\times \curvearrowright \mathbb{N}^{div}$ $\quad n \mapsto (i \mapsto ni)$

cyclotomic spectra $:= \mathrm{Cyc}(\mathcal{S}\mathrm{p}) := (\mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}})^{h\mathbb{N}^\times} \overset{\textbf{Cor.}}{\simeq} \lim^{r.lax}(\ B\mathbb{N}^\times \xrightarrow{l.lax} \mathrm{Cat}\ )$.

§1.3

note: $B\mathbb{N}^\times \simeq (\mathbb{N}^{\mathbf{div}})_{\mathbf{h}\mathbb{N}^\times}$

Thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \xrightarrow{\sim} \lim^{\mathbf{r.lax}}(\ \mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ )$. ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}$ **strat**$/\mathbb{N}^{\mathbf{div}}$ **s.t.** $\mathcal{S}\mathrm{p}^{\mathbf{g}^{<\mathbb{T}}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$...

"geometric fixedpoints" $\Phi^{C_n} : \mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{g^{<(\mathbb{T}/C_n)}} \xrightarrow{\sim} \mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}} \quad \rightsquigarrow \quad \mathbb{N}^\times \curvearrowright \mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}}$

$\leftrightarrow$ pullback of $\lim^{r.lax}$ along dilation action $\mathbb{N}^\times \curvearrowright \mathbb{N}^{div}$ $\quad n \mapsto (i \mapsto ni)$

cyclotomic spectra $:= \mathrm{Cyc}(\mathcal{S}\mathrm{p}) := (\mathcal{S}\mathrm{p}^{g^{<\mathbb{T}}})^{h\mathbb{N}^\times} \underset{\simeq}{\overset{\textbf{Cor.}}{}}$ §1.3 $\lim^{r.lax}(\ B\mathbb{N}^\times \xrightarrow{l.lax} \mathrm{Cat}\ )$.

note: $B\mathbb{N}^\times \simeq (\mathbb{N}^{\mathbf{div}})_{\mathbf{h}\mathbb{N}^\times}$

...and stratification is $\mathbb{N}^\times$-equivariant, so that $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times}$.

Thm. $\mathcal{S}\mathrm{p}^{\mathbf{g}<\mathbb{T}} \xrightarrow{\simeq} \lim^{\mathbf{r.lax}}(\ \mathbb{N}^{\mathbf{div}} \xrightarrow{\mathbf{l.lax}} \mathbf{Cat}\ )$. ***Conj.*** $\exists\ \mathscr{B}\mathbb{T}$ **strat/**$\mathbb{N}^{\mathbf{div}}$ **s.t.** $\mathcal{S}\mathrm{p}^{\mathbf{g}<\mathbb{T}} \simeq \mathbf{QC}(\mathscr{B}\mathbb{T})$...

"geometric fixedpoints" $\Phi^{C_n} : \mathcal{S}\mathrm{p}^{g<\mathbb{T}} \xrightarrow{\Phi^{C_n}} \mathcal{S}\mathrm{p}^{g<(\mathbb{T}/C_n)} \xrightarrow{\simeq} \mathcal{S}\mathrm{p}^{g<\mathbb{T}} \quad \rightsquigarrow \quad \mathbb{N}^\times \curvearrowright \mathcal{S}\mathrm{p}^{g<\mathbb{T}}$

$\leftrightarrow$ pullback of $\lim^{r.lax}$ along dilation action $\mathbb{N}^\times \curvearrowright \mathbb{N}^{div}$ $\quad n \mapsto (i \mapsto ni)$

cyclotomic spectra $:= \mathrm{Cyc}(\mathcal{S}\mathrm{p}) := (\mathcal{S}\mathrm{p}^{g<\mathbb{T}})^{h\mathbb{N}^\times} \overset{\textbf{Cor.}}{\simeq} \lim^{r.lax}(\ B\mathbb{N}^\times \xrightarrow{l.lax} \mathrm{Cat}\ )$.

§1.3

note: $B\mathbb{N}^\times \simeq (\mathbb{N}^{\mathbf{div}})_{\mathbf{h}\mathbb{N}^\times}$

...and stratification is $\mathbb{N}^\times$-equivariant, so that $\mathrm{Cyc}(\mathcal{S}\mathrm{p}) \simeq \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times}$. Moreover,

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} & & \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & & : & \\ & \mathcal{S}\mathrm{p} & & & \end{array}$$

homotopy invariants of cyclotomic structures correspond to $\mathbb{N}^\times$-equivariant global sections.

the geometry of TC:

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} \\ & {\scriptstyle (-)^{\mathrm{hCyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{\mathrm{h}\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} & & \\ & {\scriptstyle (-)^{\mathrm{hCyc}}}\searrow & & \swarrow{\scriptstyle \Gamma^{\mathrm{h}\mathbb{N}^\times}} & \\ & & \mathcal{S}\mathrm{p} & & \end{array}$$

the $\mathbb{T}$-action on THH:

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times} & & \\ & {\scriptstyle (-)^{\mathrm{hCyc}}}\searrow & & \swarrow{\scriptstyle \Gamma^{\mathrm{h}\mathbb{N}^\times}} & \\ & & \mathcal{S}\mathrm{p} & & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{c} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) \\ \downarrow \\ B\mathbb{T} \end{array}$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} & & \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & & & \\ & & \mathcal{S}\mathrm{p} & & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & & \\ B\mathbb{T} & \longleftarrow & K \end{array}$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} & & \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & & & \\ & & \mathcal{S}\mathrm{p} & & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

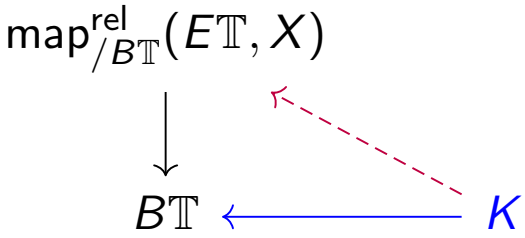
$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array}$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}} \searrow \quad \swarrow {\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,



the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}} \searrow \quad \swarrow {\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nearrow & \\ B\mathbb{T} & \longleftarrow & K \end{array}$$

$$\begin{array}{ccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K \\ \downarrow & & \downarrow \\ B\mathbb{T} & \longleftarrow & K \end{array}$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}} \searrow \quad \swarrow {\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array}$$

$$\begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nearrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \qquad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T},X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1,X)} = \mathscr{O}_{\mathcal{L}X}$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} & & \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow & \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & & \\ & & \mathcal{S}\mathrm{p} & & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nearrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \qquad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T},X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1,X)} = \mathscr{O}_{\mathcal{L}X}$

the cyclotomic structure on THH:

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} & & \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow & & \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & & \mathcal{S}\mathrm{p} & & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \qquad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T},X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1,X)} = \mathscr{O}_{\mathcal{L}X}$

the cyclotomic structure on THH: using universal circle bundle $\mathscr{E}\mathbb{T} \downarrow \mathscr{B}\mathbb{T}$, get

$$\mathscr{L}X := \mathrm{map}^{\mathrm{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \xrightarrow{\mathbb{N}^\times\text{-eq}} \mathscr{B}\mathbb{T}$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}} \searrow \quad \swarrow {\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \qquad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T},X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1,X)} = \mathscr{O}_{\mathcal{L}X}$

the cyclotomic structure on THH: using universal circle bundle $\mathscr{E}\mathbb{T} \downarrow \mathscr{B}\mathbb{T}$, get

$$\mathscr{L}X := \mathrm{map}^{\mathrm{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \xrightarrow{\mathbb{N}^\times\text{-eq}} \mathscr{B}\mathbb{T} \qquad \rightsquigarrow \qquad \begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ \cup & & \cup \\ \mathrm{THH}(X) & \longleftrightarrow & \mathscr{O}_{\mathscr{L}X} \end{array}.$$

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \qquad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T},X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1,X)} = \mathscr{O}_{\mathcal{L}X}$

the cyclotomic structure on THH: using universal circle bundle $\mathscr{E}\mathbb{T} \downarrow \mathscr{B}\mathbb{T}$, get

$$\mathscr{L}X := \mathrm{map}^{\mathrm{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \xrightarrow{\mathbb{N}^\times\text{-eq}} \mathscr{B}\mathbb{T} \qquad \rightsquigarrow \qquad \begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ \cup & & \cup \\ \mathrm{THH}(X) & \longleftrightarrow & \mathscr{O}_{\mathscr{L}X} \end{array}.$$

the $\mathbb{N}^{\mathrm{div}}$-stratified stack $\mathscr{L}X \downarrow \mathscr{B}X$ records the symmetries of $\mathcal{L}X$ coming from rotations and covering maps of circles, and $\mathrm{TC}(X)$ is its $\mathbb{T}$- and $\mathbb{N}^\times$-equivariant global functions.

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \qquad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T},X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1,X)} = \mathscr{O}_{\mathcal{L}X}$

the cyclotomic structure on THH: using universal circle bundle $\mathscr{E}\mathbb{T} \downarrow \mathscr{B}\mathbb{T}$, get

$$\mathscr{L}X := \mathrm{map}^{\mathrm{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \xrightarrow{\mathbb{N}^\times\text{-eq}} \mathscr{B}\mathbb{T} \qquad \rightsquigarrow \qquad \begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ \cup & & \cup \\ \mathrm{THH}(X) & \longleftrightarrow & \mathscr{O}_{\mathscr{L}X} \end{array}.$$

the $\mathbb{N}^{\mathrm{div}}$-stratified stack $\mathscr{L}X \downarrow \mathscr{B}X$ records the symmetries of $\mathcal{L}X$ coming from rotations and covering maps of circles, and $\mathrm{TC}(X)$ is its $\mathbb{T}$- and $\mathbb{N}^\times$-equivariant global functions.

recall §1.3: $\mathrm{TC}(X)$ = functions on $\mathcal{L}X$ that are $\mathbb{T}$-invariant and "sensitive" to covering maps of circles.

the geometry of TC:

recall: $\mathrm{THH}(X) \simeq \mathscr{O}(\mathcal{L}X) = \mathscr{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ & {\scriptstyle (-)^{h\mathrm{Cyc}}}\searrow \quad \swarrow{\scriptstyle \Gamma^{h\mathbb{N}^\times}} & \\ & \mathcal{S}\mathrm{p} & \end{array}$$

the $\mathbb{T}$-action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T}, X) & & \\ \downarrow & \nwarrow & \\ B\mathbb{T} & \longleftarrow & K \end{array} \qquad \begin{array}{ccccc} E\mathbb{T} & \longleftarrow & E\mathbb{T} \times_{B\mathbb{T}} K & \dashrightarrow & X \\ \downarrow & & \downarrow & & \\ B\mathbb{T} & \longleftarrow & K & & \end{array}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathrm{map}^{\mathrm{rel}}_{/B\mathbb{T}}(E\mathbb{T},X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits $\mathbb{T}$-action on fiber $= \mathscr{O}_{\mathrm{map}(S^1,X)} = \mathscr{O}_{\mathcal{L}X}$

the cyclotomic structure on THH: using universal circle bundle $\mathscr{E}\mathbb{T} \downarrow \mathscr{B}\mathbb{T}$, get

$$\mathscr{L}X := \mathrm{map}^{\mathrm{rel}}_{/\mathscr{B}\mathbb{T}}(\mathscr{E}\mathbb{T}, X) \xrightarrow{\mathbb{N}^\times\text{-eq}} \mathscr{B}\mathbb{T} \qquad \rightsquigarrow \qquad \begin{array}{ccc} \mathrm{Cyc}(\mathcal{S}\mathrm{p}) & \simeq & \mathrm{QC}(\mathscr{B}\mathbb{T})^{h\mathbb{N}^\times} \\ \cup & & \cup \\ \mathrm{THH}(X) & \longleftrightarrow & \mathscr{O}_{\mathscr{L}X} \end{array}.$$

the $\mathbb{N}^{\mathrm{div}}$-stratified stack $\mathscr{L}X \downarrow \mathscr{B}X$ records the symmetries of $\mathcal{L}X$ coming from rotations and covering maps of circles, and $\mathrm{TC}(X)$ is its $\mathbb{T}$- and $\mathbb{N}^\times$-equivariant global functions.

recall §1.3: $\mathrm{TC}(X)$ = functions on $\mathcal{L}X$ that are $\mathbb{T}$-invariant and "sensitive" to covering maps of circles.

§2.3 the ***Witt monoid*** $:= \mathbb{W} := \mathbb{T} \rtimes \mathbb{N}^\times$ (captures "Frobenius" and "Verschiebung" on THH)

§2.3: factorization homology of enriched ∞-categories

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \to TC$ come from?

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \to TC$ come from?

mechanism 1: factorization homology.

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \to TC$ come from?

mechanism 1: factorization homology.

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathbb{S}_{\mathbf{P}}} \mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over S^1.

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \to TC$ come from?

mechanism 1: factorization homology.

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathbb{S}} \mathbf{P}\mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over S^1.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace K $\to$ TC come from?

mechanism 1: factorization homology.

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathcal{S}^{\mathbf{P}}} \mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over S^1.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \to M\}$ a covering, $\int_M \mathcal{C} \in \mathscr{D}(\mathrm{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \to TC$ come from?

mechanism 1: factorization homology.

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathcal{S}^{\mathbf{P}}} \mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over S^1.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \to M\}$ a covering, $\int_M \mathcal{C} \in \mathscr{D}(\mathrm{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

mechanism 2: linearization.

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \to TC$ come from?

mechanism 1: factorization homology.

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathcal{S}\mathbf{p}} \mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over S^1.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \to M\}$ a covering, $\int_M \mathcal{C} \in \mathscr{D}(\mathrm{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

mechanism 2: linearization.

differential calculus: $(M, p) \rightsquigarrow T_p M$ $\quad\leftrightsquigarrow\quad$ Goodwillie calculus: $(\mathcal{S}, X) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{p}}(X)$

$(\mathcal{S}, \mathrm{pt}) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{p}}(\mathrm{pt}) \simeq \mathcal{S}\mathrm{p}$

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace K $\to$ TC come from?

mechanism 1: factorization homology...

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathcal{S}^{\mathbf{p}}} \mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over $S^{\mathbf{1}}$.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \to M\}$ a covering, $\int_M \mathcal{C} \in \mathscr{D}(\mathrm{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

mechanism 2: linearization...

differential calculus: $(M, p) \rightsquigarrow T_p M \quad \nleftrightarrow \quad$ Goodwillie calculus: $(\mathcal{S}, X) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}^{\mathbf{p}}}(X)$

$(\mathcal{S}, \mathrm{pt}) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}^{\mathbf{p}}}(\mathrm{pt}) \simeq \mathcal{S}\mathrm{p}$

...as applied to *fiber bundles* among compact framed manifolds:

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \to TC$ come from?

mechanism 1: factorization homology...

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathcal{S}\mathbf{p}} \mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over S^1.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \to M\}$ a covering, $\int_M \mathcal{C} \in \mathscr{D}(\mathrm{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

mechanism 2: linearization...

differential calculus: $(M, p) \rightsquigarrow T_p M$ $\quad\leftrightsquigarrow\quad$ Goodwillie calculus: $(\mathcal{S}, X) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{p}}(X)$

$(\mathcal{S}, \mathrm{pt}) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{p}}(\mathrm{pt}) \simeq \mathcal{S}\mathrm{p}$

...as applied to *fiber bundles* among compact framed manifolds:

A1: covering maps $S^1 \xrightarrow{r} S^1$ (for $r \in \mathbb{N}^\times$).

§2.3: factorization homology of enriched ∞-categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace K $\to$ TC come from?

mechanism 1: factorization homology...

recall: THH$(\mathcal{C}) := \int_{S^1}^{\mathcal{S}_{\mathbf{p}}} \mathcal{C} :=$ spectrally-enriched factorization homology of $\mathcal{C}$ over S^1.

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology $\leftrightarrow$ TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \to M\}$ a covering, $\int_M \mathcal{C} \in \mathscr{D}(\text{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

mechanism 2: linearization...

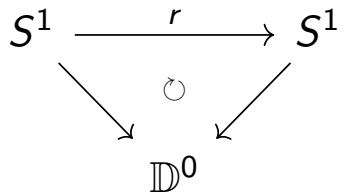
differential calculus: $(M, p) \rightsquigarrow T_p M$ $\quad\leftrightsquigarrow\quad$ Goodwillie calculus: $(\mathcal{S}, X) \rightsquigarrow \text{Shv}_{\mathcal{S}_{\mathbf{p}}}(X)$

$(\mathcal{S}, \text{pt}) \rightsquigarrow \text{Shv}_{\mathcal{S}_{\mathbf{p}}}(\text{pt}) \simeq \mathcal{S}\text{p}$

...as applied to *fiber bundles* among compact framed manifolds:

A1: covering maps $S^1 \xrightarrow{r} S^1$ (for $r \in \mathbb{N}^\times$).

A2: the fiber bundle $S^1 \to \mathbb{D}^0$, and its invariance:



.

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

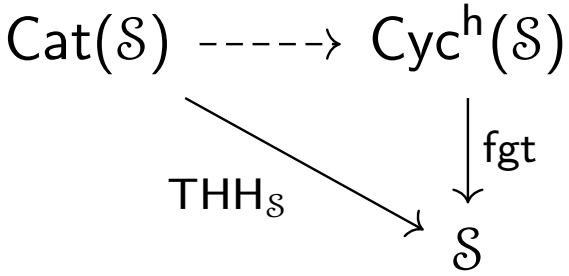
1. *diagonal package* for spaces

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

1. ***diagonal package*** for spaces $\rightsquigarrow$

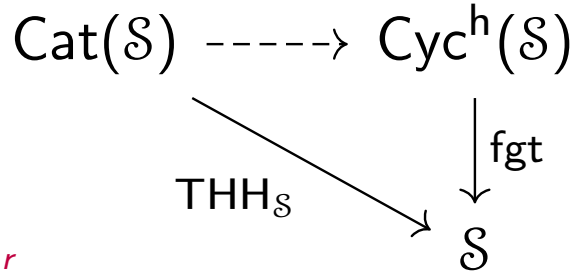


Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

1. ***diagonal package*** for spaces $\rightsquigarrow$



packages *diagonal maps* for spaces: $X \to (X^{\times r})^{hC_r}$

$\mathrm{Cyc}^h(\mathbb{S}) := \mathrm{Fun}(B\mathbb{W}, \mathbb{S}) :=$ "unstable cyclotomic spaces"

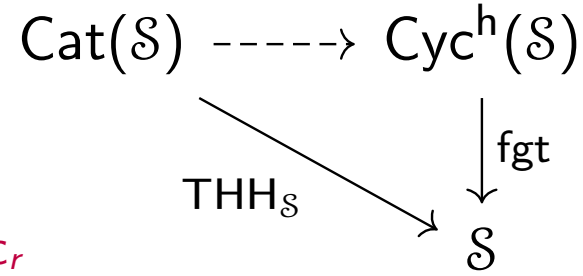
$B\mathbb{W} :=$ (objects: framed S^1's , morphisms: covering maps)

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

1. ***diagonal package*** for spaces $\rightsquigarrow$



packages *diagonal maps* for spaces: $X \to (X^{\times r})^{hC_r}$

$\mathrm{Cyc}^h(\mathcal{S}) := \mathrm{Fun}(B\mathbb{W}, \mathcal{S}) :=$ "unstable cyclotomic spaces"

$B\mathbb{W} :=$ (objects: framed S^1's , morphisms: covering maps)

2. ***diagonal package*** for spaces

linearization $\rightsquigarrow$

Tate package for spectra

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

1. ***diagonal package*** for spaces $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathcal{S}) & \dashrightarrow & \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) \\ & \searrow^{\mathrm{THH}_{\mathcal{S}}} & \downarrow \mathrm{fgt} \\ & & \mathcal{S} \end{array}$$

packages *diagonal maps* for spaces: $X \to (X^{\times r})^{\mathrm{h}C_r}$

$\mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) := \mathrm{Fun}(B\mathbb{W}, \mathcal{S}) :=$ "unstable cyclotomic spaces"

$B\mathbb{W} :=$ (objects: framed S^1's , morphisms: covering maps)

2. ***diagonal package*** for spaces

linearization $\rightsquigarrow$

Tate package for spectra $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathrm{Sp}) & \dashrightarrow & \mathrm{Cyc}(\mathrm{Sp}) \\ & \searrow^{\mathrm{THH}} & \downarrow \mathrm{fgt} \\ & & \mathrm{Sp} \end{array}$$

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A–M-G–R).

1. ***diagonal package*** for spaces $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathcal{S}) & \dashrightarrow & \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) \\ & \searrow^{\mathrm{THH}_{\mathcal{S}}} & \downarrow \mathrm{fgt} \\ & & \mathcal{S} \end{array}$$

packages *diagonal maps* for spaces: $X \to (X^{\times r})^{\mathrm{h}C_r}$

$\mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) := \mathrm{Fun}(B\mathbb{W}, \mathcal{S}) :=$ "unstable cyclotomic spaces"

$B\mathbb{W} :=$ (objects: framed S^1's , morphisms: covering maps)

2. ***diagonal package*** for spaces

linearization $\rightsquigarrow$

Tate package for spectra $\rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Cat}(\mathrm{Sp}) & \dashrightarrow & \mathrm{Cyc}(\mathrm{Sp}) \\ & \searrow^{\mathrm{THH}} & \downarrow \mathrm{fgt} \\ & & \mathrm{Sp} \end{array}$$

packages *Tate diagonal maps* for spectra: $E \to (E^{\otimes r})^{\tau C_r}$

$\mathrm{Cyc}(\mathrm{Sp}) :=$ cyclotomic spectra

Q2: where does the cyclotomic trace $K \to TC$ come from?

A2: factorization homology + linearization + the fiber bundle $S^1 \downarrow \mathbb{D}^0$.

Q2: where does the cyclotomic trace $K \to TC$ come from?

A2: factorization homology + linearization + the fiber bundle $S^1 \downarrow \mathbb{D}^0$.

Thm (A–M-G–R).

Q2: where does the cyclotomic trace $K \to TC$ come from?

A2: factorization homology + linearization + the fiber bundle $S^1 \downarrow \mathbb{D}^0$.

Thm (A–M-G–R).

the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

$$\max^{\ell} \text{ subgpd of } \mathcal{C} := \iota\mathcal{C} \simeq \int_{\mathbb{D}^0} \mathcal{C} \longrightarrow \left(\int_{S^1} \mathcal{C}\right)^{h\mathbb{W}} =: \mathrm{THH}_{\mathcal{S}}(\mathcal{C})^{h\mathbb{W}} =: \mathrm{TC}^{h}_{\mathcal{S}}(\mathcal{C})$$

input: the fiber bundle $S^1 \downarrow \mathbb{D}^0$ is invariant for the $\mathbb{W}$-action on S^1

Q2: where does the cyclotomic trace $K \to TC$ come from?

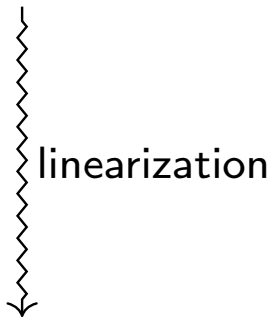
A2: factorization homology + linearization + the fiber bundle $S^1 \downarrow \mathbb{D}^0$.

Thm (A–M-G–R).

the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

$$\max^{\ell} \text{subgpd of } \mathcal{C} := \iota\mathcal{C} \simeq \textstyle\int_{\mathbb{D}^0} \mathcal{C} \longrightarrow \left(\int_{S^1} \mathcal{C}\right)^{h\mathbb{W}} =: \mathrm{THH}_{\mathcal{S}}(\mathcal{C})^{h\mathbb{W}} =: \mathrm{TC}^{h}_{\mathcal{S}}(\mathcal{C})$$

input: the fiber bundle $S^1 \downarrow \mathbb{D}^0$ is invariant for the $\mathbb{W}$-action on S^1



the ***cyclotomic trace***: for $\mathcal{C}$ a stable ∞-category,

$$K(\mathcal{C}) \longrightarrow \mathrm{THH}(\mathcal{C})^{h\mathrm{Cyc}} =: \mathrm{TC}(\mathcal{C})$$

Q2: where does the cyclotomic trace $K \to TC$ come from?

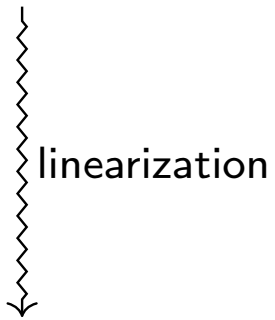
A2: factorization homology + linearization + the fiber bundle $S^1 \downarrow \mathbb{D}^0$.

Thm (A–M-G–R).

the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

$$\max^{\ell} \text{subgpd of } \mathcal{C} := \iota\mathcal{C} \simeq \textstyle\int_{\mathbb{D}^0} \mathcal{C} \longrightarrow \left(\int_{S^1} \mathcal{C}\right)^{h\mathbb{W}} =: \mathrm{THH}_{\mathcal{S}}(\mathcal{C})^{h\mathbb{W}} =: \mathrm{TC}^{h}_{\mathcal{S}}(\mathcal{C})$$

input: the fiber bundle $S^1 \downarrow \mathbb{D}^0$ is invariant for the $\mathbb{W}$-action on S^1



the ***cyclotomic trace***: for $\mathcal{C}$ a stable ∞-category,

$$K(\mathcal{C}) \longrightarrow \mathrm{THH}(\mathcal{C})^{h\mathrm{Cyc}} =: \mathrm{TC}(\mathcal{C})$$

input: K-theory is the ***universal additive invariant*** of stable ∞-categories (Blumberg–Gepner–Tabuada)

enriched factorization homology:

enriched factorization homology:

given:

- $M \in \mathrm{Mfld}_1$ a *1-manifold*
- $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ a *$\mathcal{V}$-enriched ∞-category*

want to define: $\int_M \mathcal{C} \in \mathcal{V}$.

enriched factorization homology:

given:

- $M \in \mathrm{Mfld}_1$ a *1-manifold*
- $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ a *$\mathcal{V}$-enriched ∞-category*

want to define: $\int_M \mathcal{C} \in \mathcal{V}$. first define both of these terms...

enriched factorization homology:

given:

- $M \in \mathrm{Mfld}_1$ a *1-manifold*
- $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ a *$\mathcal{V}$-enriched ∞-category*

want to define: $\int_M \mathcal{C} \in \mathcal{V}$. first define both of these terms...

idea: this will be a colimit in $\mathcal{V}$

- over *disk-refinements* R of M
- of compatible labelings λ
  - of the vertices $R^{(0)}$ by objects of $\mathcal{C}$ and
  - of the edges $R^{(1)}$ by morphisms in $\mathcal{C}$.

$$\int_M \mathcal{C} := \operatorname*{colim}_{(R \to M) \in \mathcal{D}(M)} \left(\operatorname*{colim}_{R^{(0)} \xrightarrow{\lambda} \iota\mathcal{C}} \left(\bigboxtimes_{e \in R^{(1)}} \underline{\hom}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right) .$$

enriched factorization homology:

given:

- $M \in \mathrm{Mfld}_1$ a *1-manifold*
- $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ a *$\mathcal{V}$-enriched ∞-category*

want to define: $\int_M \mathcal{C} \in \mathcal{V}$. first define both of these terms...

idea: this will be a colimit in $\mathcal{V}$

- over *disk-refinements* R of M
- of compatible labelings λ
  - of the vertices $R^{(0)}$ by objects of $\mathcal{C}$ and
  - of the edges $R^{(1)}$ by morphisms in $\mathcal{C}$.

$$\int_M \mathcal{C} := \operatorname*{colim}_{(R \to M) \in \mathcal{D}(M)} \left(\operatorname*{colim}_{R^{(0)} \xrightarrow{\lambda} \iota\mathcal{C}} \left(\bigboxtimes_{e \in R^{(1)}} \underline{\hom}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right) .$$

slogan: *enriched* factorization homology via *categorified* factorization homology.

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

- obj: $\coprod_{\text{finite}}$ (finite directed graphs) $\coprod$ (framed S^1's) ,

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

- obj: $\coprod_{\text{finite}}$ (finite directed graphs) $\coprod$ (framed S^1's) ,
- mor: TBD (to be described).

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

- obj: $\coprod_{\text{finite}}$ (finite directed graphs) $\coprod$ (framed S^1's) ,
- mor: TBD (to be described).

given $M \in \mathcal{M}$, its category $\mathcal{D}(M)$ of ***disk-refinements*** has:

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

- obj: $\coprod_{\text{finite}}$ (finite directed graphs) $\coprod$ (framed S^1's) ,
- mor: TBD (to be described).

given $M \in \mathcal{M}$, its category $\mathcal{D}(M)$ of ***disk-refinements*** has:

- obj: config$^{\text{ns}}$ of dots in 1-dim$^{\ell}$ strata of M s.t. $\nexists$ smooth S^1's;

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

- obj: $\coprod_{\text{finite}}$ (finite directed graphs) $\coprod$ (framed S^1's) ,
- mor: TBD (to be described).

given $M \in \mathcal{M}$, its category $\mathcal{D}(M)$ of ***disk-refinements*** has:

- obj: config$^{\text{ns}}$ of dots in 1-dim$^{\ell}$ strata of M s.t. $\nexists$ smooth S^1's;
- mor: disappearances, anticollisions, and isotopies.

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

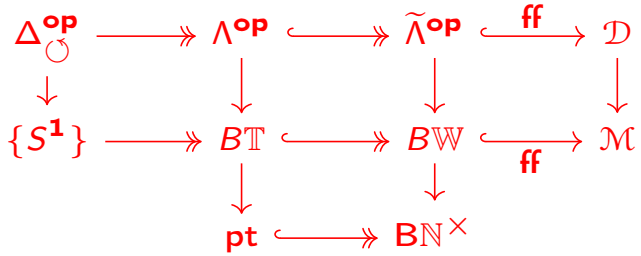
Defs (Ayala–Francis–Rozenblyum). the ∞-category $\mathcal{M} := \mathrm{Mfld}_1$ of ***compact variframed 1-manifolds*** has

- obj: $\coprod_{\text{finite}}$ (finite directed graphs) $\coprod$ (framed S^1's) ,
- mor: TBD (to be described).

given $M \in \mathcal{M}$, its category $\mathcal{D}(M)$ of ***disk-refinements*** has:

- obj: config$^{\text{ns}}$ of dots in 1-dim$^{\ell}$ strata of M s.t. $\nexists$ smooth S^1's;
- mor: disappearances, anticollisions, and isotopies.

add pix here!!! $\mathcal{D}(S^1) = \Delta^{\mathbf{op}}_{\circlearrowleft}$, full diagram



$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\quad \rightsquigarrow \quad$ $\int_M \mathcal{C} \in \mathcal{V}$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathsf{Alg}(\mathsf{Cat})$ $\leftrightsquigarrow$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathsf{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathsf{Alg}(\mathsf{Cat}) \quad \leftrightsquigarrow \quad$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat:
for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

$M \in \mathcal{M}$, $\mathcal{C} \in \mathrm{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat}) \quad \leftrightsquigarrow \quad$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat:
for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;

$M \in \mathcal{M}$, $\mathcal{C} \in \mathrm{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat}) \quad \leftrightsquigarrow \quad$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat:
for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;
- ∞-category of morphisms: $\underline{\hom}_{\mathfrak{B}\mathcal{V}}(\mathrm{pt}, \mathrm{pt}) \simeq \mathcal{V}$;

$M \in \mathcal{M}$, $\mathcal{C} \in \mathrm{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat}) \quad \leftrightsquigarrow \quad$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat:
for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;
- ∞-category of morphisms: $\underline{\hom}_{\mathfrak{B}\mathcal{V}}(\mathrm{pt}, \mathrm{pt}) \simeq \mathcal{V}$;
- composition: $\mathcal{V} \times \mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$.

$M \in \mathcal{M}$, $\mathcal{C} \in \mathrm{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat}) \quad \leftrightsquigarrow \quad$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat: for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;
- ∞-category of morphisms: $\underline{\hom}_{\mathfrak{B}\mathcal{V}}(\mathrm{pt}, \mathrm{pt}) \simeq \mathcal{V}$;
- composition: $\mathcal{V} \times \mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$.

another category object in Cat (in fact in $\mathcal{S}$): for any space X, the ***codiscrete*** category object $\mathrm{cd}(X) : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}$.
$[n] \longmapsto X^{\times (n+\mathbf{1})}$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathrm{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat}) \quad \leftrightsquigarrow \quad$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat: for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;
- ∞-category of morphisms: $\underline{\hom}_{\mathfrak{B}\mathcal{V}}(\mathrm{pt}, \mathrm{pt}) \simeq \mathcal{V}$;
- composition: $\mathcal{V} \times \mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$.

another category object in Cat (in fact in $\mathcal{S}$): for any space X, the ***codiscrete*** category object $\mathrm{cd}(X) : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}$.
$[n] \longmapsto X^{\times (n+\mathbf{1})}$

- space of objects: $\mathrm{cd}(X)_0 = X^{\times 1} = X$;

$M \in \mathcal{M}$, $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat})$ $\leftrightsquigarrow$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat:
for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;
- ∞-category of morphisms: $\underline{\hom}_{\mathfrak{B}\mathcal{V}}(\mathrm{pt}, \mathrm{pt}) \simeq \mathcal{V}$;
- composition: $\mathcal{V} \times \mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$.

another category object in Cat (in fact in $\mathcal{S}$): for any space X, the ***codiscrete*** category object $\mathrm{cd}(X) : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}$.
$[n] \longmapsto X^{\times (n+\mathbf{1})}$

- space of objects: $\mathrm{cd}(X)_0 = X^{\times 1} = X$;
- space of morphisms: $\underline{\hom}_{\mathrm{cd}(X)}(x, y) = \mathrm{pt}$; hence "codiscrete"

$M \in \mathcal{M}$, $\mathcal{C} \in \mathrm{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$(\mathcal{V}, \boxtimes) \in \mathrm{Alg}(\mathrm{Cat}) \quad \leftrightsquigarrow \quad$ bar construction $\mathfrak{B}\mathcal{V} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, $[n] \longmapsto \mathcal{V}^{\times n}$

$\mathfrak{B}\mathcal{V}$ is a ***category object*** (a.k.a. *Segal object*) in Cat: for all $n \geq 2$, $\mathfrak{B}\mathcal{V}_n \xrightarrow{\sim} \mathfrak{B}\mathcal{V}_1 \times_{\mathfrak{B}\mathcal{V}_0} \cdots \times_{\mathfrak{B}\mathcal{V}_0} \mathfrak{B}\mathcal{V}_1$.

- ∞-category of objects: $\mathfrak{B}\mathcal{V}_0 = \mathcal{V}^{\times 0} \simeq \mathrm{pt}$;
- ∞-category of morphisms: $\underline{\hom}_{\mathfrak{B}\mathcal{V}}(\mathrm{pt}, \mathrm{pt}) \simeq \mathcal{V}$;
- composition: $\mathcal{V} \times \mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$.

another category object in Cat (in fact in $\mathcal{S}$): for any space X, the ***codiscrete*** category object $\mathrm{cd}(X) : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}$.
$[n] \longmapsto X^{\times (n+1)}$

- space of objects: $\mathrm{cd}(X)_0 = X^{\times 1} = X$;
- space of morphisms: $\underline{\hom}_{\mathrm{cd}(X)}(x, y) = \mathrm{pt}$; hence "codiscrete"
- composition: $\mathrm{pt} \times \mathrm{pt} \xrightarrow{\sim} \mathrm{pt}$.

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\quad \rightsquigarrow \quad$ $\int_M \mathcal{C} \in \mathcal{V}$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\mathrm{cd}(\iota\mathcal{C}) \xrightarrow{\underline{\hom}_{\mathcal{C}}} \mathfrak{B}\mathcal{V}$$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{array}{ccc} \mathrm{cd}(\iota\mathcal{C}) & \xrightarrow{\underline{\hom}_{\mathcal{C}}} & \mathfrak{B}\mathcal{V} \qquad \Delta^{\mathrm{op}} \\ \iota\mathcal{C} & \xrightarrow{\quad ! \quad} & \mathrm{pt} \qquad [0] \end{array}$$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{array}{ccccc}
\mathrm{cd}(\iota\mathcal{C}) & \xrightarrow{\underline{\hom}_{\mathcal{C}}} & \mathfrak{B}\mathcal{V} & & \Delta^{\mathrm{op}} \\
\iota\mathcal{C} & \xrightarrow{!} & \mathrm{pt} & & [0] \\
(\iota\mathcal{C})^{\times 2} & \xrightarrow{(x,y)\mapsto \underline{\hom}_{\mathcal{C}}(x,y)} & \mathcal{V} & & [1]
\end{array}$$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{array}{lll}
\mathrm{cd}(\iota\mathcal{C}) & \xrightarrow{\underline{\hom}_{\mathcal{C}}} & \mathfrak{B}\mathcal{V} \qquad \Delta^{\mathrm{op}} \\
\iota\mathcal{C} & \xrightarrow{\quad ! \quad} & \mathrm{pt} \qquad [0] \\
(\iota\mathcal{C})^{\times 2} & \xrightarrow{(x,y)\mapsto\underline{\hom}_{\mathcal{C}}(x,y)} & \mathcal{V} \qquad [1] \\
(\iota\mathcal{C})^{\times 3} & \xrightarrow{(x,y,z)\mapsto(\underline{\hom}_{\mathcal{C}}(x,y),\underline{\hom}_{\mathcal{C}}(y,z))} & \mathcal{V}^{\times 2} \qquad [2]
\end{array}$$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{array}{ccccc}
\mathrm{cd}(\iota\mathcal{C}) & & \xrightarrow{\quad \underline{\hom}_{\mathcal{C}} \quad} & & \mathfrak{B}\mathcal{V} \qquad \Delta^{\mathrm{op}} \\
\iota\mathcal{C} & & \xrightarrow{\quad ! \quad} & & \mathrm{pt} \qquad [0] \\
{\scriptstyle x \mapsto (x,x)}\downarrow & & & & \downarrow{\scriptstyle \mathbb{1}_{\mathcal{V}}} \qquad \downarrow{\scriptstyle \sigma_0} \\
(\iota\mathcal{C})^{\times 2} & & \xrightarrow{\quad (x,y) \mapsto \underline{\hom}_{\mathcal{C}}(x,y) \quad} & & \mathcal{V} \qquad [1] \\
\\
(\iota\mathcal{C})^{\times 3} & & \xrightarrow{\quad (x,y,z) \mapsto (\underline{\hom}_{\mathcal{C}}(x,y), \underline{\hom}_{\mathcal{C}}(y,z)) \quad} & & \mathcal{V}^{\times 2} \qquad [2]
\end{array}$$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{array}{ccccc}
\mathrm{cd}(\iota\mathcal{C}) & \xrightarrow{\quad\underline{\hom}_{\mathcal{C}}\quad} & \mathfrak{B}\mathcal{V} & & \Delta^{\mathrm{op}} \\
\iota\mathcal{C} & \xrightarrow{\quad!\quad} & \mathrm{pt} & & [0] \\
{\scriptstyle x\mapsto(x,x)}\downarrow & \swarrow & \downarrow{\scriptstyle \mathbb{1}_{\mathcal{V}}} & & \downarrow{\scriptstyle \sigma_0} \\
(\iota\mathcal{C})^{\times 2} & \xrightarrow{\quad(x,y)\mapsto\underline{\hom}_{\mathcal{C}}(x,y)\quad} & \mathcal{V} & & [1] \\
\\
(\iota\mathcal{C})^{\times 3} & \xrightarrow{\quad(x,y,z)\mapsto(\underline{\hom}_{\mathcal{C}}(x,y),\underline{\hom}_{\mathcal{C}}(y,z))\quad} & \mathcal{V}^{\times 2} & & [2]
\end{array}$$

$\mathbb{1}_{\mathcal{V}} \to \underline{\hom}_{\mathcal{C}}(x,x)$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{array}{ccccc}
\mathrm{cd}(\iota\mathcal{C}) & & \xrightarrow{\quad \underline{\hom}_{\mathcal{C}} \quad} & & \mathfrak{B}\mathcal{V} \qquad \Delta^{\mathrm{op}} \\
\iota\mathcal{C} & & \xrightarrow{\quad ! \quad} & & \mathrm{pt} \qquad [0] \\
{\scriptstyle x\mapsto(x,x)}\downarrow & \swarrow & & & \downarrow{\scriptstyle \mathbb{1}_{\mathcal{V}}} \qquad \downarrow{\scriptstyle \sigma_0} \\
(\iota\mathcal{C})^{\times 2} & & \xrightarrow{\quad (x,y)\mapsto\underline{\hom}_{\mathcal{C}}(x,y) \quad} & & \mathcal{V} \qquad [1] \\
{\scriptstyle (x,y,z)\mapsto(x,z)}\uparrow & & & & \uparrow{\scriptstyle \boxtimes} \qquad \uparrow{\scriptstyle \delta_1} \\
(\iota\mathcal{C})^{\times 3} & & \xrightarrow{\quad (x,y,z)\mapsto(\underline{\hom}_{\mathcal{C}}(x,y),\underline{\hom}_{\mathcal{C}}(y,z)) \quad} & & \mathcal{V}^{\times 2} \qquad [2]
\end{array}$$

$\mathbb{1}_{\mathcal{V}} \to \underline{\hom}_{\mathcal{C}}(x,x)$

Def (Gepner–Haugseng). a $\mathcal{V}$***-enriched*** ∞***-category*** $\mathcal{C} \in \mathrm{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\begin{array}{ccccc}
\mathrm{cd}(\iota\mathcal{C}) & & \xrightarrow{\quad \underline{\hom}_{\mathcal{C}} \quad} & & \mathfrak{B}\mathcal{V} \qquad \Delta^{\mathrm{op}} \\
\iota\mathcal{C} & & \xrightarrow{\quad ! \quad} & & \mathrm{pt} \qquad [0] \\
{\scriptstyle x\mapsto(x,x)}\downarrow & \swarrow & & & \downarrow{\scriptstyle \mathbb{1}_{\mathcal{V}}} \qquad \downarrow{\scriptstyle \sigma_0} \\
(\iota\mathcal{C})^{\times 2} & & \xrightarrow{\quad (x,y)\mapsto\underline{\hom}_{\mathcal{C}}(x,y) \quad} & & \mathcal{V} \qquad [1] \\
{\scriptstyle (x,y,z)\mapsto(x,z)}\uparrow & \nwarrow & & & \uparrow{\scriptstyle \boxtimes} \qquad \uparrow{\scriptstyle \delta_1} \\
(\iota\mathcal{C})^{\times 3} & & \xrightarrow{\quad (x,y,z)\mapsto(\underline{\hom}_{\mathcal{C}}(x,y),\underline{\hom}_{\mathcal{C}}(y,z)) \quad} & & \mathcal{V}^{\times 2} \qquad [2]
\end{array}$$

$$\mathbb{1}_{\mathcal{V}} \to \underline{\hom}_{\mathcal{C}}(x,x) \qquad \underline{\hom}_{\mathcal{C}}(x,y) \boxtimes \underline{\hom}_{\mathcal{C}}(y,z) \to \underline{\hom}_{\mathcal{C}}(x,z)$$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

recall slogan: *enriched* factorization homology via *categorified* factorization homology.

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

recall slogan: *enriched* factorization homology via *categorified* factorization homology.

the ***Grothendieck construction***: for a category object $\mathcal{Y}$ in Cat,

$$\mathcal{M} \xrightarrow{\int_{(-)} \mathcal{Y}} \mathbf{Cat} \qquad \leftrightsquigarrow \qquad \begin{array}{c} \int_{|\mathcal{M}} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \end{array}$$

"covariant bundle of ∞-categories"

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V}) \quad \rightsquigarrow \quad \int_M \mathcal{C} \in \mathcal{V}$

recall slogan: *enriched* factorization homology via *categorified* factorization homology.

the ***Grothendieck construction***: for a category object $\mathcal{Y}$ in Cat,

$$\mathcal{M} \xrightarrow{\int_{(-)} \mathcal{Y}} \mathrm{Cat} \qquad \leftrightsquigarrow \qquad \begin{array}{c} \int_{|\mathcal{M}} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \end{array}$$

"covariant bundle of ∞-categories"

$$\int_M \mathcal{C} := \mathrm{colim} \left(\int_{|\mathcal{D}(M)} \mathrm{cd}(\iota \mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\mathrm{hom}}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right)$$

in general, requires $\mathcal{V}$ to be *symmetric* monoidal
(or e.g. only *cyclically* monoidal if $M = S^1$)

add pix here!!!

$M \in \mathcal{M}$, $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ $\rightsquigarrow$ $\int_M \mathcal{C} \in \mathcal{V}$

recall slogan: *enriched* factorization homology via *categorified* factorization homology.

the ***Grothendieck construction***: for a category object $\mathcal{Y}$ in Cat,

$$\mathcal{M} \xrightarrow{\int_{(-)} \mathcal{Y}} \mathrm{Cat} \qquad \leftrightsquigarrow \qquad \begin{array}{c} \int_{|\mathcal{M}} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \end{array}$$

"covariant bundle of ∞-categories"

$$\int_M \mathcal{C} := \mathrm{colim}\left(\int_{|\mathcal{D}(M)} \mathrm{cd}(\iota \mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\hom}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right)$$

in general, requires $\mathcal{V}$ to be *symmetric* monoidal (or e.g. only *cyclically* monoidal if $M = S^1$)

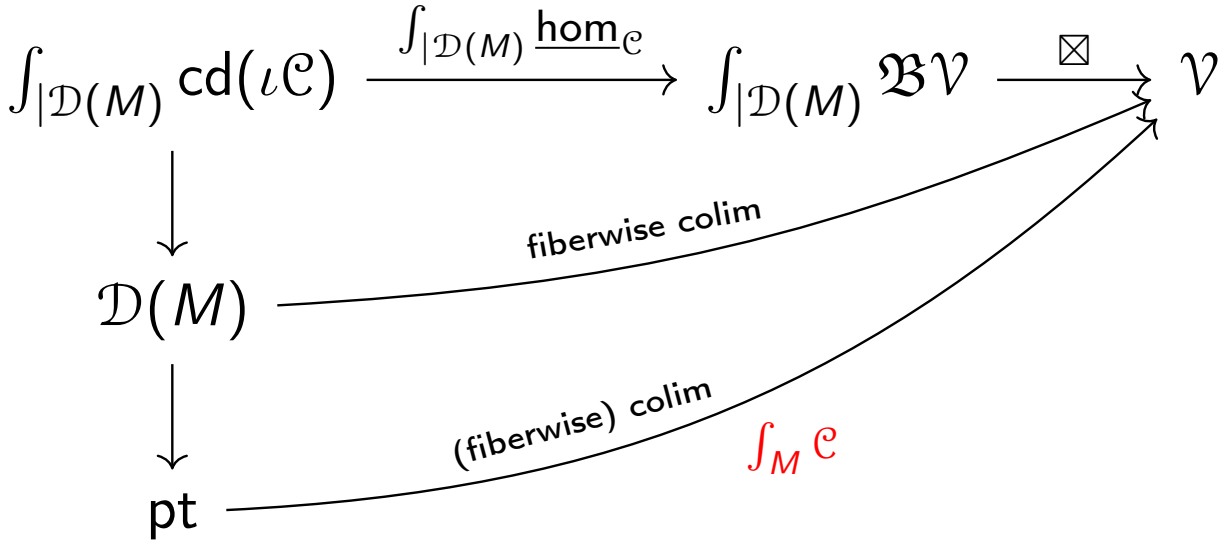
add pix here!!!

rigorizes heuristic formula $\int_M \mathcal{C} := \underset{(R \to M) \in \mathcal{D}(M)}{\mathrm{colim}} \left(\underset{R(\mathbf{0}) \xrightarrow{\lambda} \iota \mathcal{C}}{\mathrm{colim}} \left(\boxtimes_{e \in R(\mathbf{1})} \underline{\hom}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right)$

$$\int_M \mathcal{C} := \mathbf{colim}\left(\int_{\mid \mathcal{D}(M)} \mathbf{cd}(\iota\mathcal{C}) \xrightarrow{\int_{\mid \mathcal{D}(M)} \underline{\mathbf{hom}}\mathcal{C}} \int_{\mid \mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}\right)$$

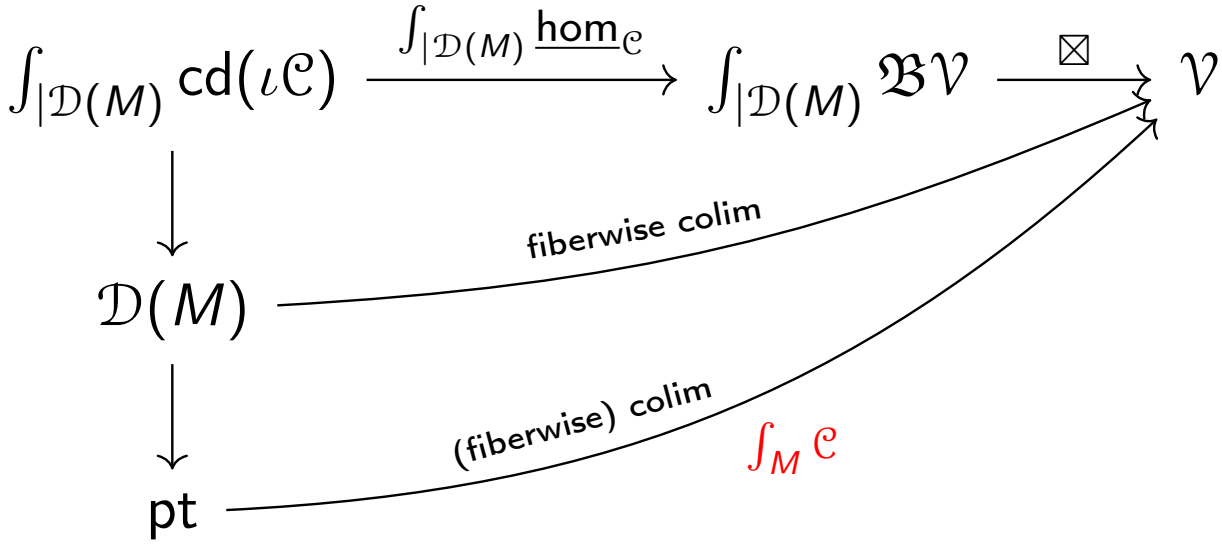
$$\int_M \mathcal{C} := \mathbf{colim}\left(\int_{|\mathcal{D}(M)} \mathbf{cd}(\iota\mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\mathbf{hom}}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}\right)$$

equivalently,



$$\int_M \mathcal{C} := \mathbf{colim}\left(\int_{|\mathcal{D}(M)} \mathbf{cd}(\iota\mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\mathbf{hom}}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}\right)$$

equivalently,

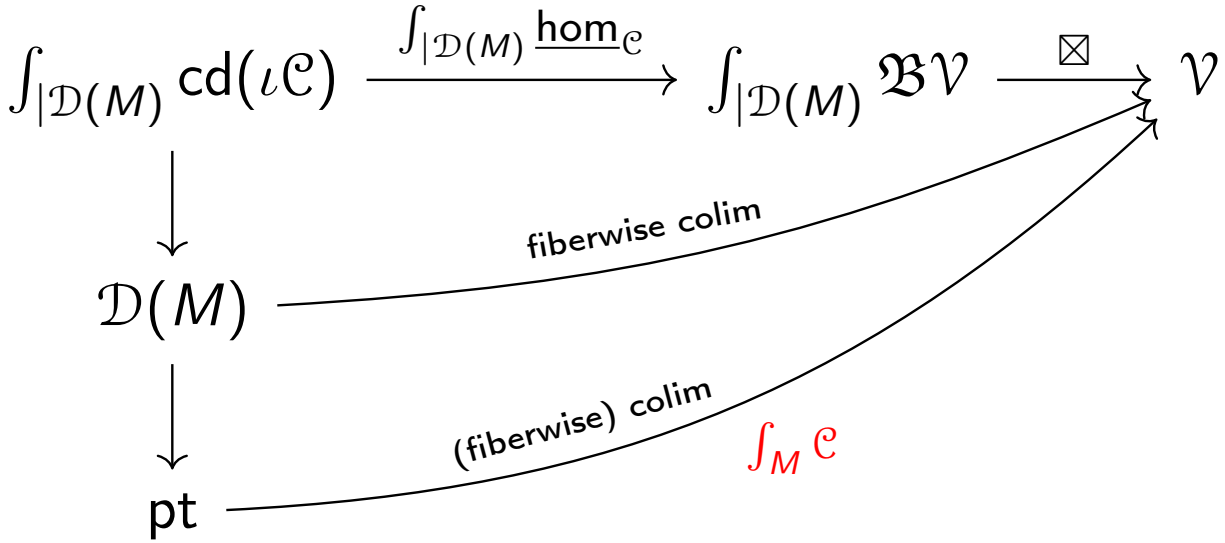


e.g. $\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} \simeq \mathrm{colim}(\Delta^{\mathrm{op}}_{\circlearrowleft} \to \mathcal{V})$

recall: $\Delta^{\mathbf{op}}_{\circlearrowleft} := \mathcal{D}(S^1)$

$$\int_M \mathcal{C} := \mathbf{colim}\left(\int_{|\mathcal{D}(M)} \mathbf{cd}(\iota\mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\mathbf{hom}}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}\right)$$

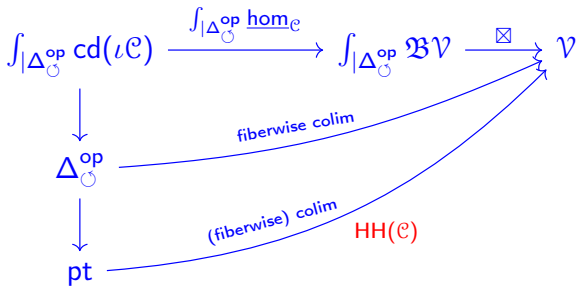
equivalently,

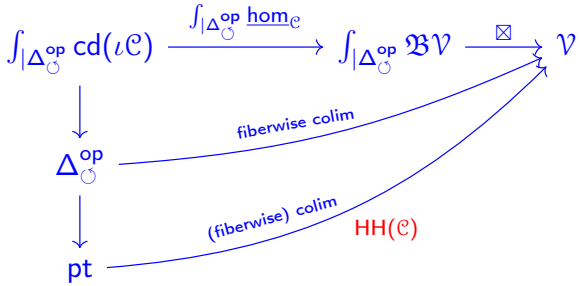


e.g. $\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} \simeq \mathrm{colim}(\Delta^{\mathrm{op}}_{\circlearrowleft} \to \mathcal{V}) \simeq \mathrm{colim}(B^{\mathrm{cyc}}\mathcal{C} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}_{\circlearrowleft} \to \mathcal{V})$

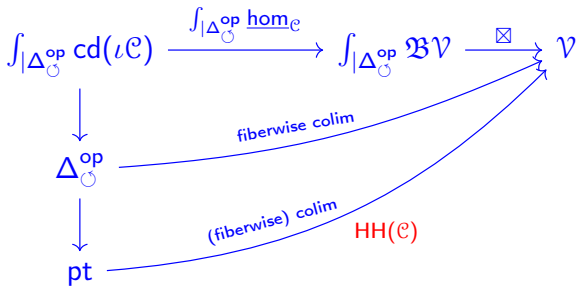
recall: $\Delta^{\mathbf{op}}_{\circlearrowleft} := \mathcal{D}(S^1)$, $\Delta^{\mathbf{op}} \simeq \mathcal{D}(S^1_*)$ *cyclic bar construction* (classical def$^{\mathbf{n}}$ of HH)

fact: $\Delta^{\mathbf{op}} := \mathcal{D}(S^1_*) \to \mathcal{D}(S^1) =: \Delta^{\mathbf{op}}_{\circlearrowleft}$ **is** *final*

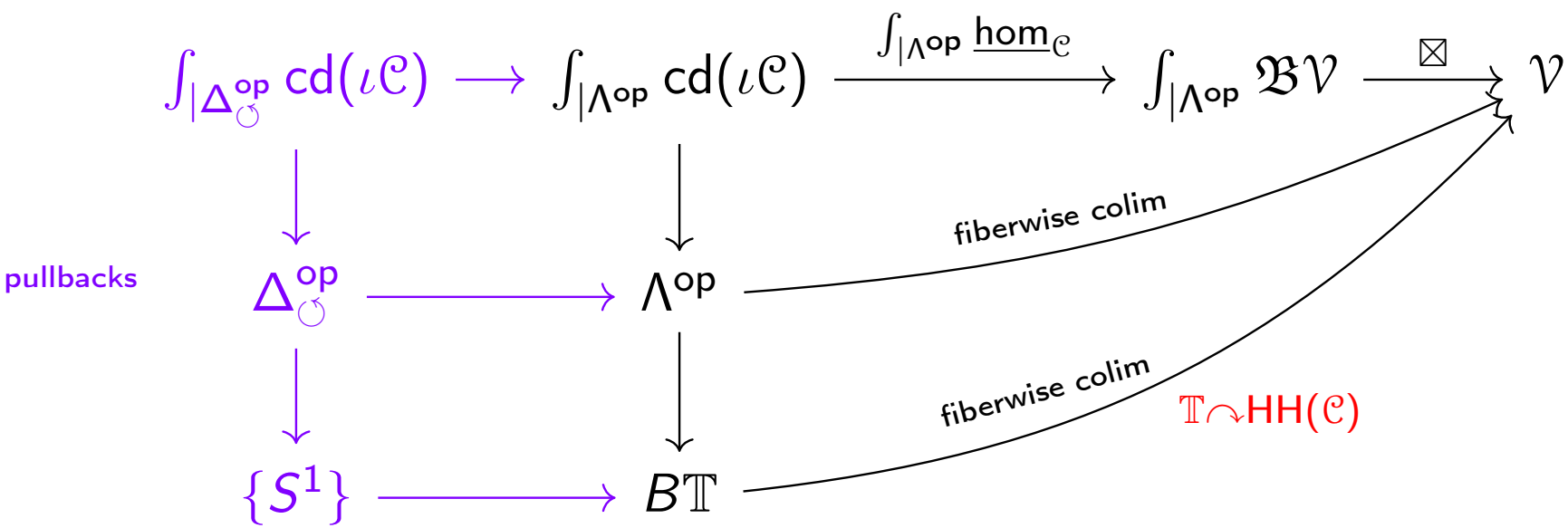

$\int_{|\Delta^{\mathrm{op}}_{\circlearrowleft}} \mathsf{cd}(\iota\mathcal{C})$
$\int_{|\Delta^{\mathrm{op}}_{\circlearrowleft}} \underline{\mathsf{hom}}_{\mathcal{C}}$
$\int_{|\Delta^{\mathrm{op}}_{\circlearrowleft}} \mathfrak{B}\mathcal{V}$
$\boxtimes$
$\mathcal{V}$
$\Delta^{\mathrm{op}}_{\circlearrowleft}$
fiberwise colim
(fiberwise) colim
$\mathsf{HH}(\mathcal{C})$
pt

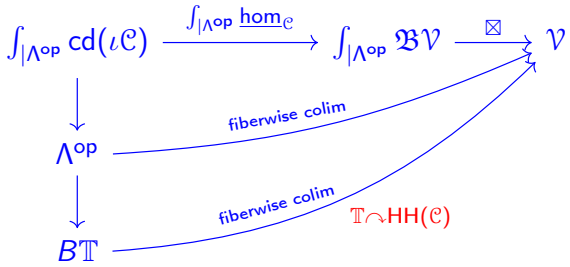


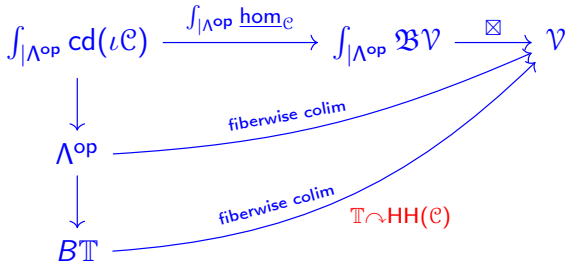
build in functoriality for rotations of S^1:



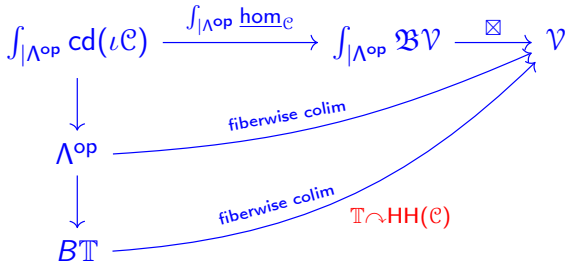
build in functoriality for rotations of S^1:



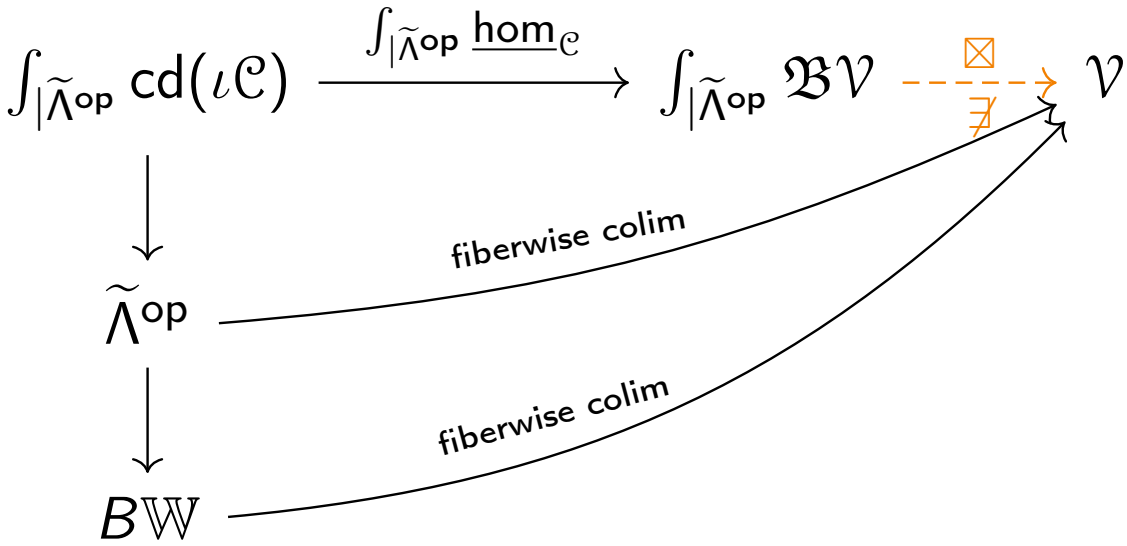

$\int_{|\Lambda^{op}} \mathsf{cd}(\iota\mathcal{C})$
$\int_{|\Lambda^{op}} \underline{\mathsf{hom}}_{\mathcal{C}}$
$\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}$
$\boxtimes$
$\mathcal{V}$
fiberwise colim
Λ^{op}
fiberwise colim
$\mathbb{T}\curvearrowright\mathsf{HH}(\mathcal{C})$
$B\mathbb{T}$

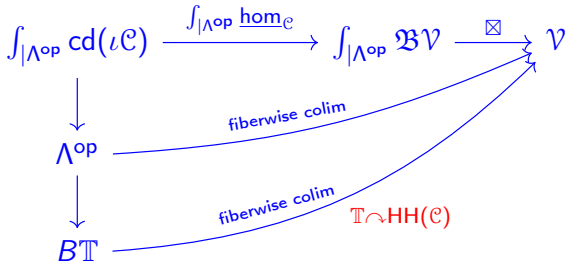


attempt to build in functoriality for self-coverings of S^1:

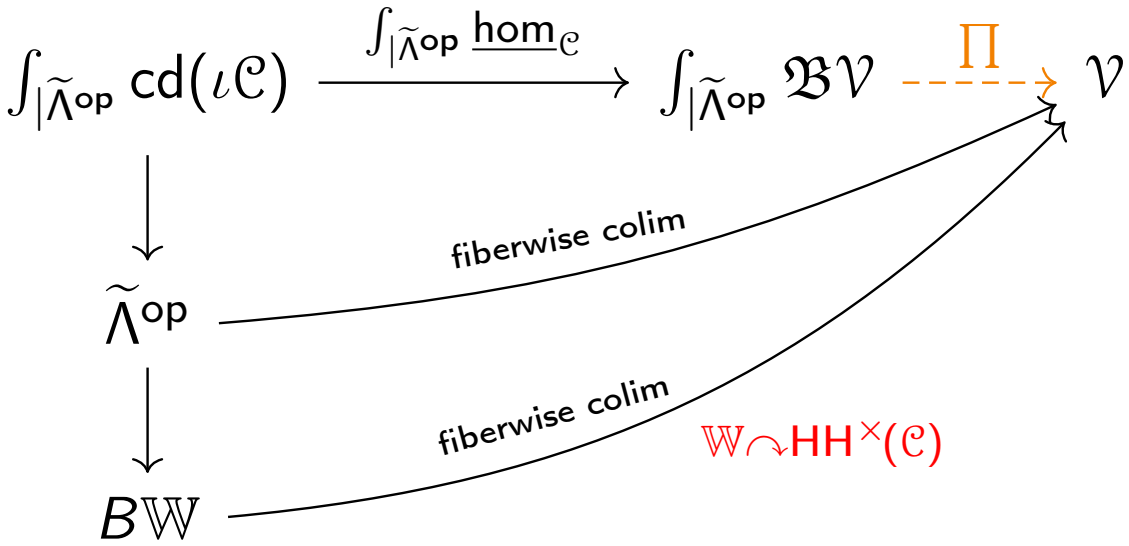


attempt to build in functoriality for self-coverings of S^1:





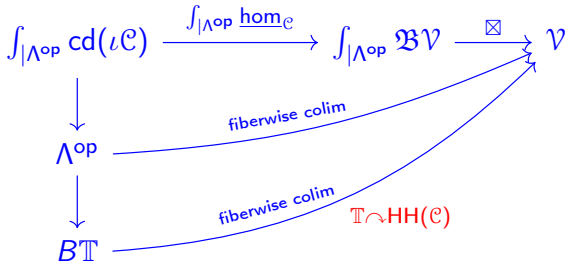
attempt to build in functoriality for self-coverings of S^1:



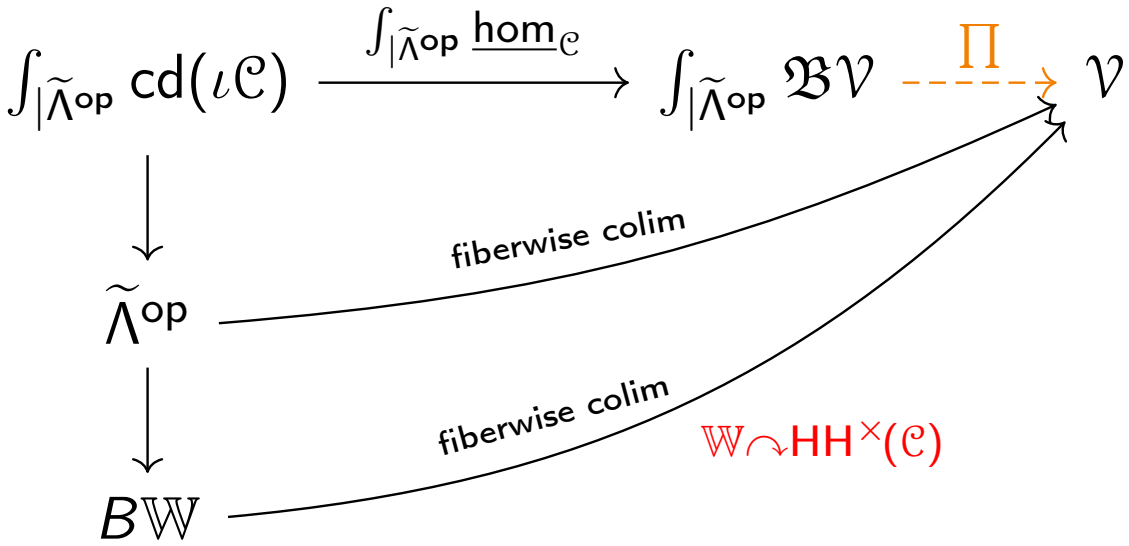
only exists when $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal! notation: $\mathrm{HH}^{\times}$

e.g. asks for maps $V \to V^{\boxtimes r}$, or more generally $\bigboxtimes_i V_i \to \bigboxtimes_i (V_i)^{\boxtimes r}$

add pix here!!!



attempt to build in functoriality for self-coverings of S^1:



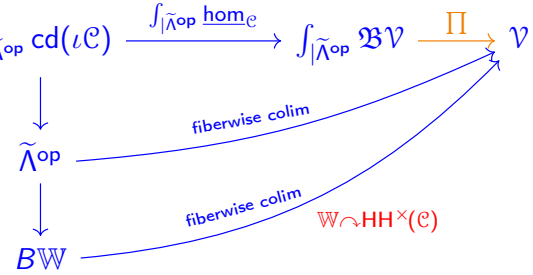
only exists when $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal! notation: $\mathrm{HH}^{\times}$

e.g. asks for maps $V \to V^{\boxtimes r}$, or more generally $\bigboxtimes_i V_i \to \bigboxtimes_i (V_i)^{\boxtimes r}$

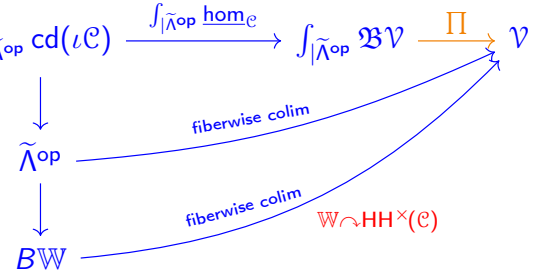
add pix here!!!

this functor $\prod$ is the ***diagonal package***, and the $\mathbb{W}$-action on $\mathrm{HH}^{\times}(\mathcal{C})$ is its ***unstable cyclotomic structure***. $\mathrm{Cyc}^{\mathbf{h}}(\mathcal{V}) := \mathrm{Fun}(B\mathbb{W}, \mathcal{V})$

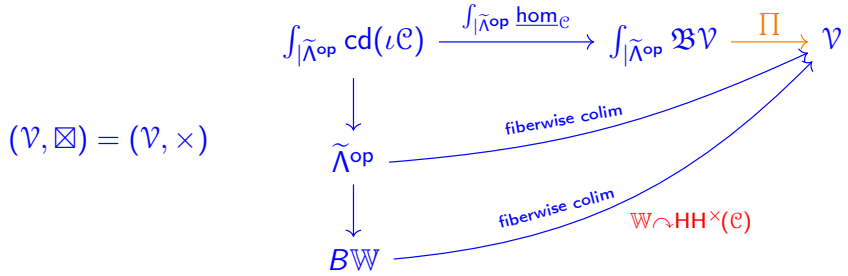
$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$



$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$

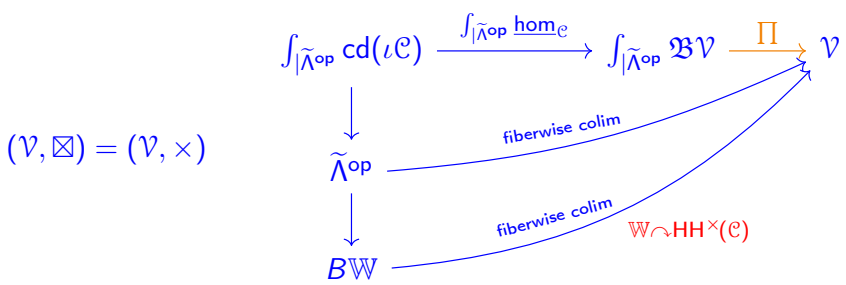


to understand the diagonal package, work "locally":



to understand the diagonal package, work "locally": equivalently,

$$\begin{array}{ccc} \int_{|\widetilde{\Lambda}^{op}} \mathfrak{B}\mathcal{V} & \xrightarrow{\Pi} & \underline{\mathcal{V}} := \mathcal{V} \times \mathrm{B}\mathbb{N}^{\times} \\ \downarrow & & \downarrow \\ \widetilde{\Lambda}^{op} & & \\ \downarrow & & \\ B\mathbb{W} & \longrightarrow & \mathrm{B}\mathbb{N}^{\times} \end{array}$$



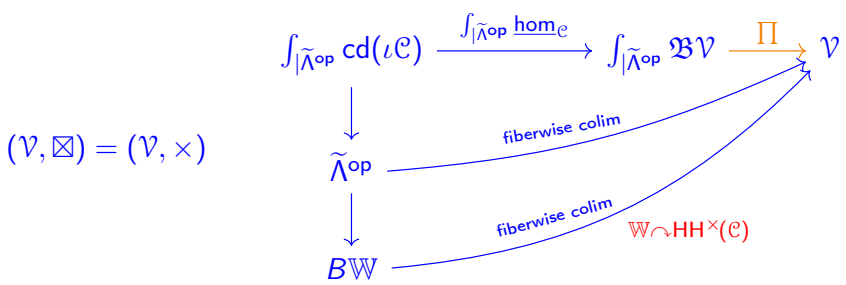
to understand the diagonal package, work "locally": equivalently,

$$\begin{array}{ccc} \int_{|\widetilde{\Lambda}^{op}} \mathfrak{B}\mathcal{V} & \xrightarrow{\Pi} & \underline{\mathcal{V}} := \mathcal{V} \times B\mathbb{N}^{\times} \\ \downarrow & & \downarrow \\ \widetilde{\Lambda}^{op} & & \\ \downarrow & & \\ B\mathbb{W} & \longrightarrow & B\mathbb{N}^{\times} \end{array}$$

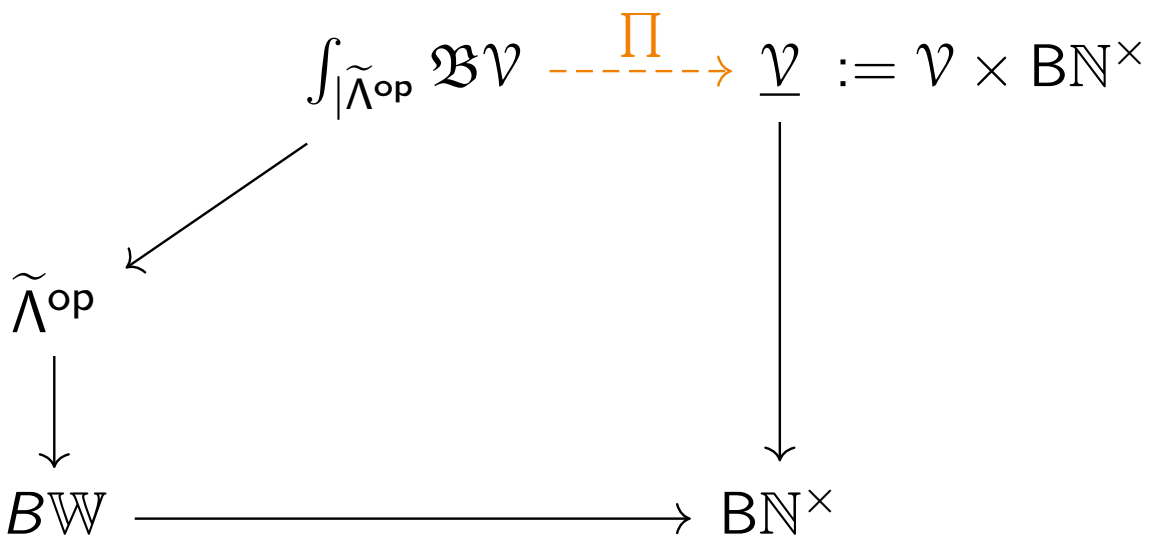
$$\mathrm{Fun}^{\mathrm{rel}}_{/B\mathbb{N}^{\times}}\left(\int_{|\widetilde{\Lambda}^{op}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}\right)$$

$$\Pi \downarrow$$

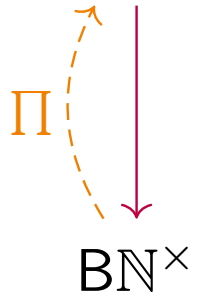
$$B\mathbb{N}^{\times}$$



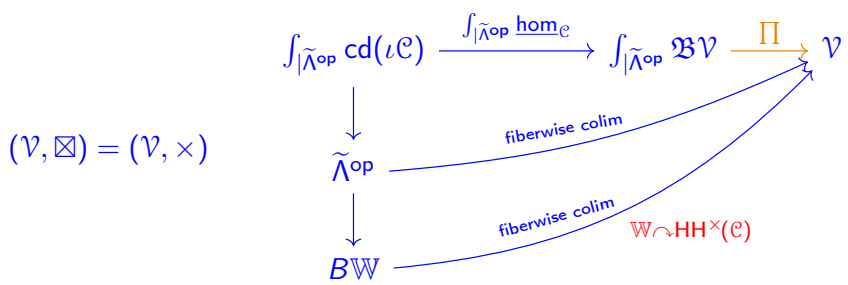
to understand the diagonal package, work "locally": equivalently,



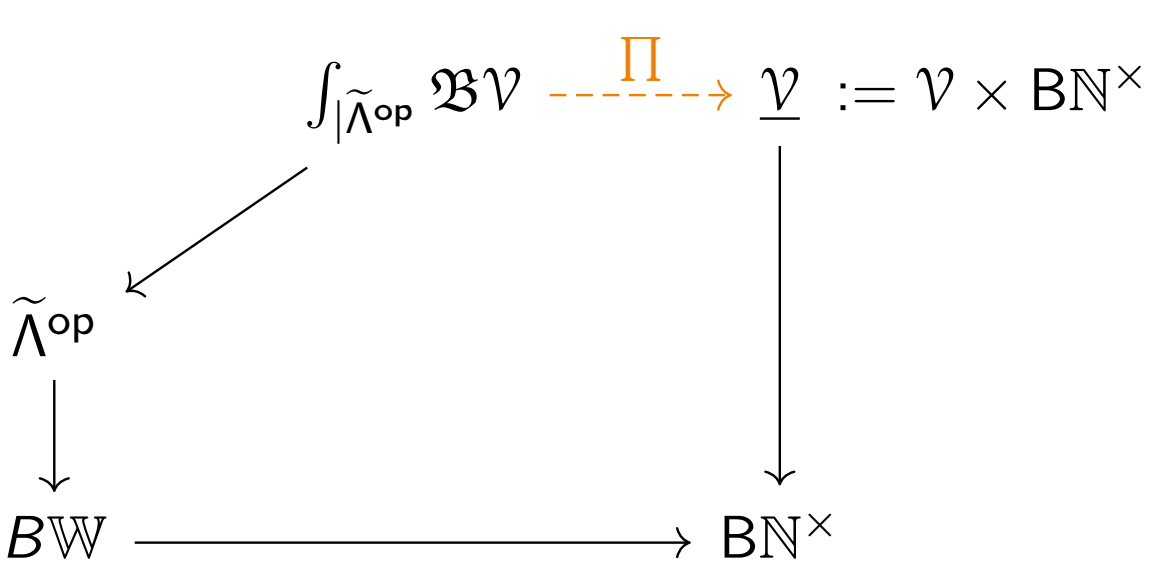
$\mathrm{Fun}^{\mathrm{rel}}_{/B\mathbb{N}^{\times}}\left(\int_{|\widetilde{\Lambda}^{op}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}\right)$

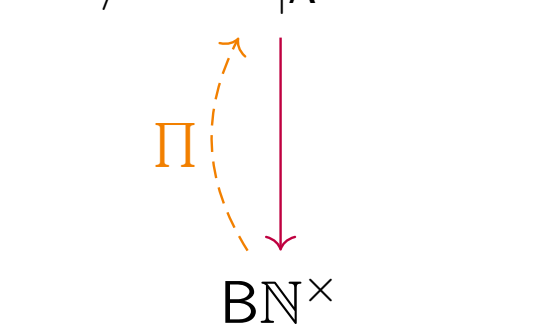


contravariant bundle of ∞-categories:



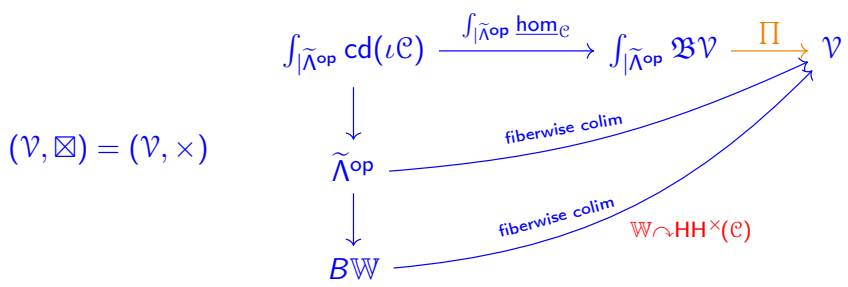
to understand the diagonal package, work "locally": equivalently,



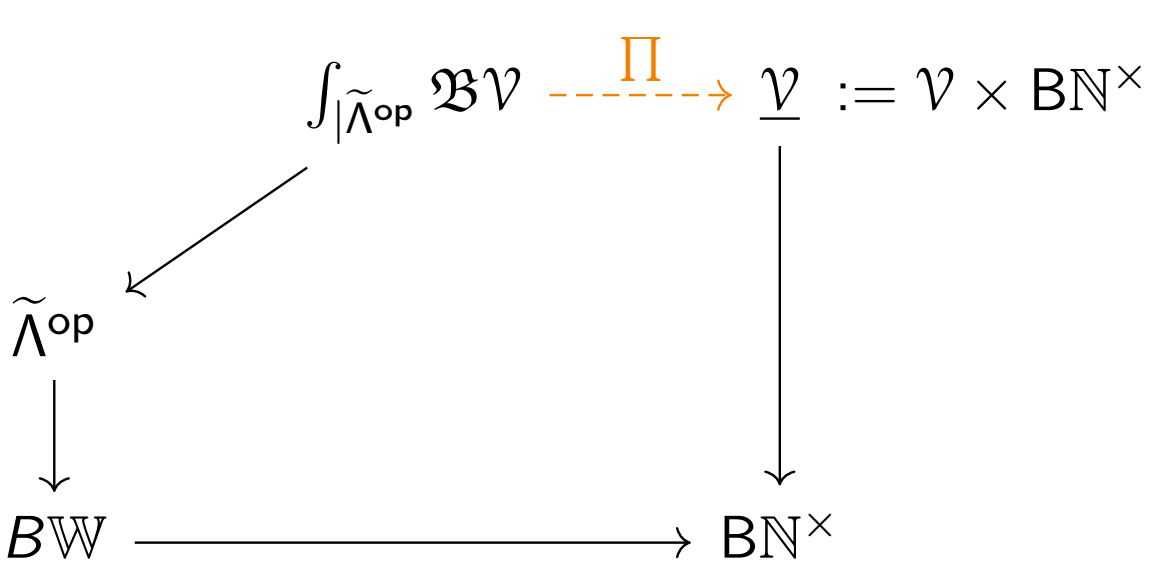


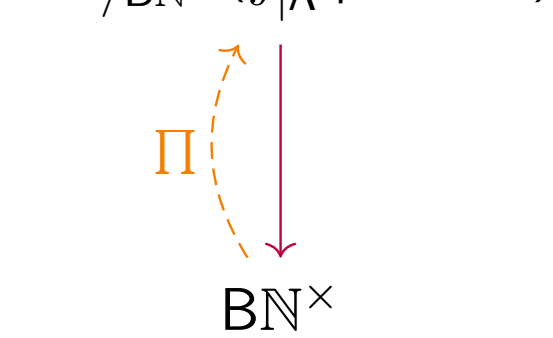
contravariant bundle of ∞-categories: defines right $\mathbb{N}^{\times}$-action on fiber $\mathrm{Fun}(\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}, \mathcal{V})$, where $r \in \mathbb{N}^{\times}$ acts as $F \mapsto (F(\pi_r^*(-)))^{hC_r}$

$\int_{|\Delta^{op}_{\circlearrowleft}} \mathfrak{B}\mathcal{V} \xrightarrow{\pi_r^*} \int_{|\Delta^{op}_{\circlearrowleft}} \mathfrak{B}\mathcal{V}$ induced by $S^1 \xleftarrow{r} S^1$ add pix here!!!



to understand the diagonal package, work "locally": equivalently,





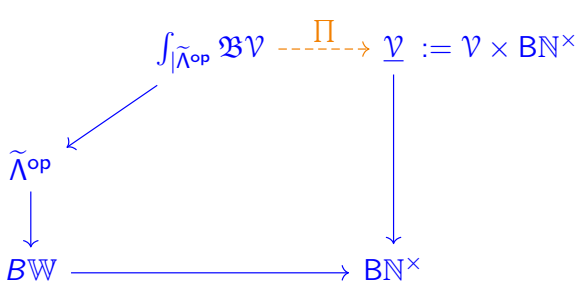
contravariant bundle of ∞-categories: defines right $\mathbb{N}^{\times}$-action on fiber $\mathrm{Fun}(\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}, \mathcal{V})$, where $r \in \mathbb{N}^{\times}$ acts as $F \mapsto (F(\pi_r^*(-)))^{hC_r}$

$\int_{|\Delta^{op}_{\circlearrowleft}} \mathfrak{B}\mathcal{V} \xrightarrow{\pi_r^*} \int_{|\Delta^{op}_{\circlearrowleft}} \mathfrak{B}\mathcal{V}$ induced by $S^1 \xleftarrow{r} S^1$ add pix here!!!

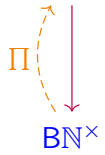
action by *homotopy* fixedpoints, so denote by $\mathrm{Fun}(\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}, \mathcal{V}) \underset{h}{\curvearrowright} \mathbb{N}^{\times}$;

$(\prod, h) := \prod$ an object of its (in fact strict!) limit strict because $V \xrightarrow{\sim} (V^{\times r})^{hC_r}$

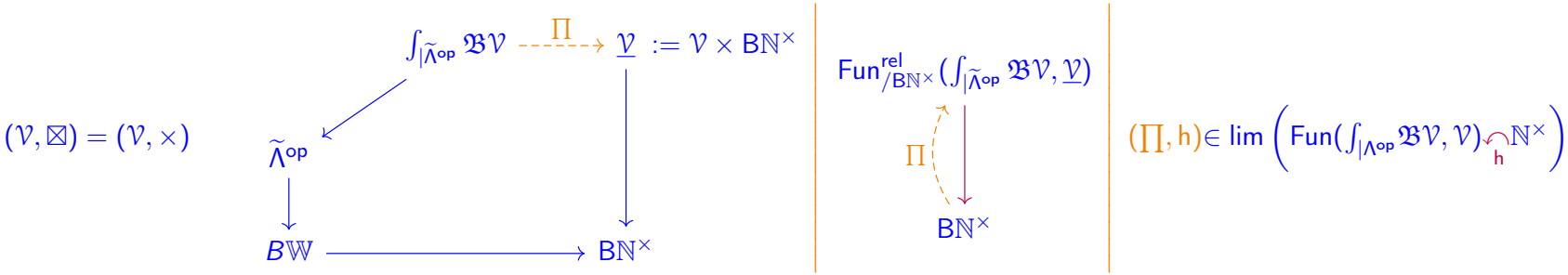
$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$



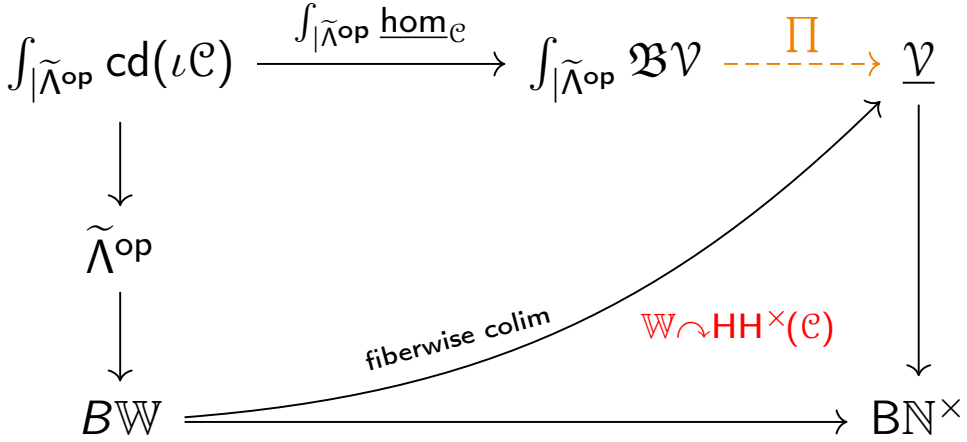
$\mathrm{Fun}^{rel}_{/B\mathbb{N}^{\times}}(\int_{\widetilde{\Lambda}^{op}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}})$

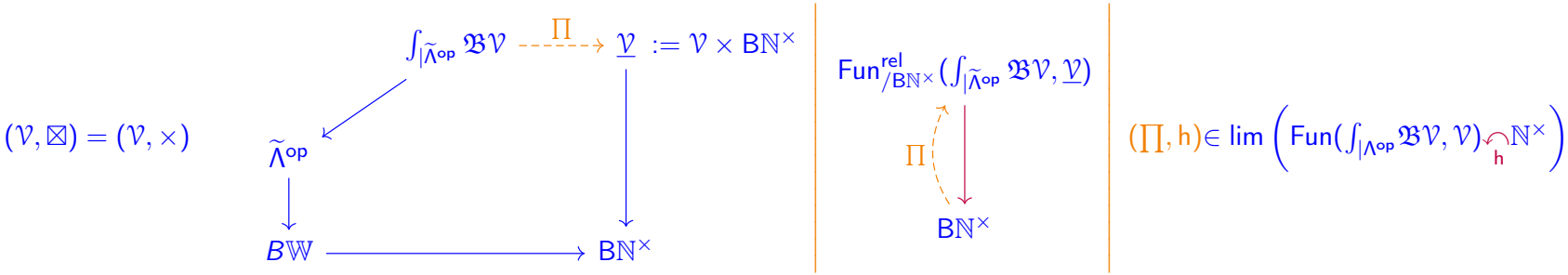


$(\prod, h) \in \lim \left(\mathrm{Fun}(\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}, \mathcal{V}) \underset{h}{\curvearrowright} \mathbb{N}^{\times} \right)$

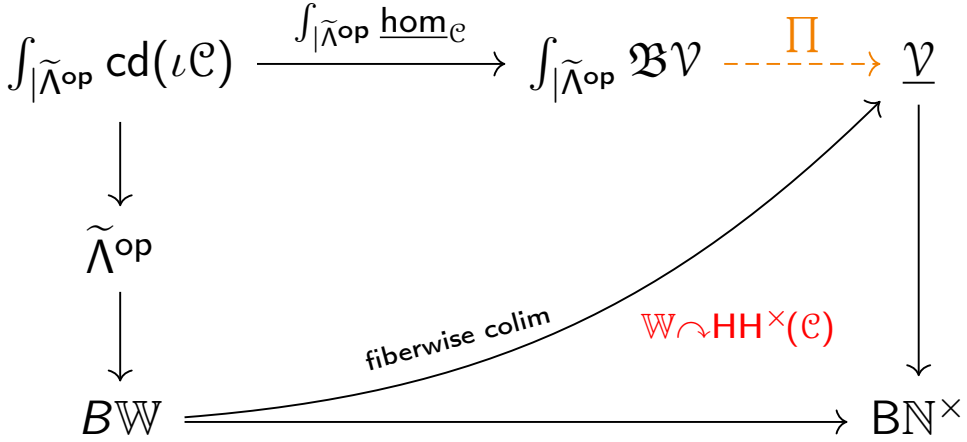


recall construction of unstable cyclotomic structure $\mathbb{W} \curvearrowright \mathrm{HH}^{\times}$:





recall construction of unstable cyclotomic structure $\mathbb{W}\curvearrowright \mathrm{HH}^{\times}$:



rephrase in language of $\mathbb{N}^{\times}$-actions:

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{V},\mathcal{V})\underset{h}{\curvearrowleft}\mathbb{N}^{\times})\xrightarrow{\hom^{*}_{\mathcal{C}}}(\mathrm{Fun}(\textstyle\int_{|\Lambda^{op}}\mathrm{cd}(\iota\mathcal{C}),\mathcal{V})\underset{h}{\curvearrowleft}\mathbb{N}^{\times})\underset{\perp}{\overset{r.lax}{\rightleftarrows}}(\mathrm{Fun}(B\mathbb{T},\mathcal{V})\underset{h}{\curvearrowleft}\mathbb{N}^{\times})$$

$$\lim(\bullet)\qquad\qquad\lim{}^{r.lax}(\bullet)=:\mathrm{Cyc}^{h}(\mathcal{V})$$

$$\cup\qquad\qquad\cup$$

$$(\textstyle\prod,h)\longmapsto(\mathbb{W}\curvearrowright\mathrm{HH}^{\times}(\mathcal{C}))$$

$$(\mathsf{Fun}(\textstyle\int_{|\Lambda^{\mathsf{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times}) \xrightarrow{\mathsf{hom}^*_{\mathcal{C}}} (\mathsf{Fun}(\textstyle\int_{|\Lambda^{\mathsf{op}}} \mathsf{cd}(\iota\mathcal{C}), \mathcal{S}) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times}) \underset{\perp}{\overset{\mathsf{r.lax}}{\dashleftarrow\!\dashrightarrow}} (\mathsf{Fun}(B\mathbb{T}, \mathcal{S}) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times})$$

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) = (\mathcal{S}, \times)$

$$\begin{array}{ccc} \mathsf{lim}(\bullet) & & \mathsf{lim}^{\,\mathsf{r.lax}}(\bullet) =: \mathsf{Cyc}^{\mathsf{h}}(\mathcal{S}) \\ \cup & & \cup \\ (\prod, \mathsf{h}) & \longmapsto & (\mathbb{W} \curvearrowright \mathsf{HH}^{\times}(\mathcal{C})) \end{array}$$

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times}) \xrightarrow{\hom^*_{\mathcal{C}}} (\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathsf{cd}(\iota\mathcal{C}), \mathcal{S}) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times}) \underset{\perp}{\overset{\mathsf{r.lax}}{\dashleftarrow\!\dashrightarrow}} (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times})$$

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) = (\mathcal{S}, \times)$

$$\begin{array}{ccc} \lim(\bullet) & & \lim{}^{\mathsf{r.lax}}(\bullet) =: \mathrm{Cyc}^{\mathsf{h}}(\mathcal{S}) \\ \cup & & \cup \\ (\prod, \mathsf{h}) & \longmapsto & (\mathbb{W} \curvearrowright \mathrm{HH}^{\times}(\mathcal{C})) \end{array}$$

so, diagonal package for spaces induces unstable cyclotomic structure.

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathfrak{BS}, \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^\times) \xrightarrow{\underline{\hom}^*_{\mathcal{C}}} (\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^\times) \underset{\perp}{\overset{\mathrm{r.lax}}{\rightleftarrows}} (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^\times)$$

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) = (\mathcal{S}, \times)$

$\lim(\bullet)$ $\qquad\qquad$ $\lim^{\mathrm{r.lax}}(\bullet) =: \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S})$

$\cup$ $\qquad\qquad$ $\cup$

$(\prod, \mathrm{h}) \longmapsto (\mathbb{W} \curvearrowright \mathrm{HH}^\times(\mathcal{C}))$

so, diagonal package for spaces induces unstable cyclotomic structure.

completely analogously, Tate package for spectra induces cyclotomic structure:

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathfrak{BS}\mathrm{p}, \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\mathrm{l.lax}}{\curvearrowleft}} \mathbb{N}^\times) \xrightarrow{\underline{\hom}^*_{\mathcal{C}}} (\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\mathrm{l.lax}}{\curvearrowleft}} \mathbb{N}^\times) \underset{\perp}{\overset{\mathrm{r.lax}}{\rightleftarrows}} (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\mathrm{l.lax}}{\curvearrowleft}} \mathbb{N}^\times)$$

$\lim^{\mathrm{r.lax}}(\bullet)$ $\qquad\qquad$ $\lim^{\mathrm{r.lax}}(\bullet) \;=: \mathrm{Cyc}(\mathcal{S}\mathrm{p})$

$\cup$ $\qquad\qquad$ $\cup$

$(\bigotimes, \tau) \longmapsto (\mathrm{Cyc} \curvearrowright \mathrm{THH}(\mathcal{C}))$

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \xrightarrow{\underline{\hom}^*_{\mathcal{C}}} (\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \underset{\perp}{\overset{\mathrm{r.lax}}{\dashrightarrow}} (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times})$$

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) = (\mathcal{S}, \times)$

$\lim(\bullet)$ $\qquad\qquad$ $\lim^{\mathrm{r.lax}}(\bullet) =: \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S})$

$\cup \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup$

$(\prod, \mathrm{h}) \longmapsto (\mathbb{W} \curvearrowright \mathrm{HH}^{\times}(\mathcal{C}))$

so, diagonal package for spaces induces unstable cyclotomic structure.

completely analogously, Tate package for spectra induces cyclotomic structure:

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\mathrm{l.lax}}{\curvearrowleft}} \mathbb{N}^{\times}) \xrightarrow{\underline{\hom}^*_{\mathcal{C}}} (\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\mathrm{l.lax}}{\curvearrowleft}} \mathbb{N}^{\times}) \underset{\perp}{\overset{\mathrm{r.lax}}{\dashrightarrow}} (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\mathrm{l.lax}}{\curvearrowleft}} \mathbb{N}^{\times})$$

$\lim^{\mathrm{r.lax}}(\bullet)$ $\qquad\qquad$ $\lim^{\mathrm{r.lax}}(\bullet) \;=: \mathrm{Cyc}(\mathcal{S}\mathrm{p})$

$\cup \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup$

$(\bigotimes, \tau) \longmapsto (\mathrm{Cyc} \curvearrowright \mathrm{THH}(\mathcal{C}))$

Q.: where does the Tate package come from?

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}) \underset{h}{\curvearrowleft} \mathbb{N}^\times) \xrightarrow{\hom^*_{\mathcal{C}}} (\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}) \underset{h}{\curvearrowleft} \mathbb{N}^\times) \underset{\perp}{\overset{\text{r.lax}}{\dashrightarrow}} (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}) \underset{h}{\curvearrowleft} \mathbb{N}^\times)$$

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) = (\mathcal{S}, \times)$

$\lim(\bullet)$ $\qquad$ $\lim^{\text{r.lax}}(\bullet) =: \mathrm{Cyc}^h(\mathcal{S})$

$\cup$ $\qquad$ $\cup$

$(\prod, h) \longmapsto (\mathbb{W} \curvearrowright \mathrm{HH}^\times(\mathcal{C}))$

so, diagonal package for spaces induces unstable cyclotomic structure.

completely analogously, Tate package for spectra induces cyclotomic structure:

$$(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\text{l.lax}}{\curvearrowleft}} \mathbb{N}^\times) \xrightarrow{\hom^*_{\mathcal{C}}} (\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\text{l.lax}}{\curvearrowleft}} \mathbb{N}^\times) \underset{\perp}{\overset{\text{r.lax}}{\dashrightarrow}} (\mathrm{Fun}(B\mathbb{T}, \mathcal{S}\mathrm{p}) \underset{\tau}{\overset{\text{l.lax}}{\curvearrowleft}} \mathbb{N}^\times)$$

$\lim^{\text{r.lax}}(\bullet)$ $\qquad$ $\lim^{\text{r.lax}}(\bullet) =: \mathrm{Cyc}(\mathcal{S}\mathrm{p})$

$\cup$ $\qquad$ $\cup$

$(\bigotimes, \tau) \longmapsto (\mathrm{Cyc} \curvearrowright \mathrm{THH}(\mathcal{C}))$

Q.: where does the Tate package come from?

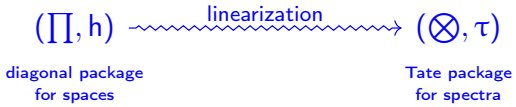
recall **A:**

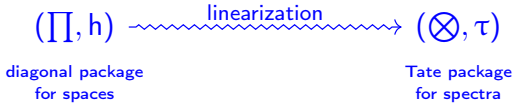
$$(\prod, h) \rightsquigarrow (\bigotimes, \tau)$$

linearization

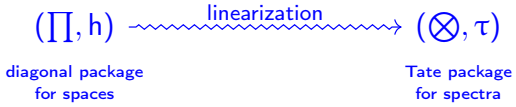
diagonal package for spaces

Tate package for spectra


$(\prod, h)$
linearization
$(\otimes, \tau)$
diagonal package for spaces
Tate package for spectra



three ingredients:



three ingredients:

(1) an action $\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^{\times}$

$(\prod, h) \rightsquigarrow^{\text{linearization}} (\otimes, \tau)$

diagonal package for spaces — Tate package for spectra

three ingredients:

(1) an action $\mathrm{Fun}(-, \mathcal{S}p) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times$

(2) a functor $(\mathrm{Fun}(-, \mathcal{S}p) \underset{h}{\curvearrowright} \mathbb{N}^\times) \xrightarrow{\text{r.lax}} (\mathrm{Fun}(-, \mathcal{S}p) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times)$

$(\prod, h) \rightsquigarrow (\otimes, \tau)$ (linearization)

diagonal package for spaces — Tate package for spectra

three ingredients:

(1) an action $\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^{\times}$

(2) a functor $(\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \underset{h}{\curvearrowright} \mathbb{N}^{\times}) \xrightarrow{\mathrm{r.lax}} (\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^{\times})$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors***

$(\prod, h) \rightsquigarrow (\otimes, \tau)$ (linearization)

diagonal package for spaces — Tate package for spectra

three ingredients:

(1) an action $\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times$

(2) a functor $(\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \underset{h}{\curvearrowright} \mathbb{N}^\times) \xrightarrow{\text{r.lax}} (\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times)$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors***

$$\begin{array}{ccccc} \mathcal{V}^{\times R^{(1)}} & \longrightarrow & \int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V} & \longrightarrow & \mathcal{S}\mathrm{p} \\ \downarrow & & \downarrow & & \\ \{R \to S^1\} & \longrightarrow & \Lambda^{\mathrm{op}} & & \end{array}$$

multilinear: preserves finite colimits separately in each variable*

$(\prod, h) \overset{\text{linearization}}{\rightsquigarrow} (\otimes, \tau)$

diagonal package for spaces — Tate package for spectra

three ingredients:

(1) an action $\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times$

(2) a functor $(\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \underset{h}{\curvearrowright} \mathbb{N}^\times) \xrightarrow{\text{r.lax}} (\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times)$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors***

$$\begin{array}{ccccc} \mathcal{V}^{\times R^{(1)}} & \longrightarrow & \int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V} & \longrightarrow & \mathcal{S}\mathrm{p} \\ \downarrow & & \downarrow & & \\ \{R \to S^1\} & \longrightarrow & \Lambda^{\mathrm{op}} & & \end{array}$$

multilinear: preserves finite colimits separately in each variable*

which is *stable* under action $\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times$

$(\prod, h) \xrightarrow{\text{linearization}} (\otimes, \tau)$

diagonal package for spaces — Tate package for spectra

three ingredients:

(1) an action $\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times$

(2) a functor $(\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \underset{h}{\curvearrowright} \mathbb{N}^\times) \xrightarrow{\text{r.lax}} (\mathrm{Fun}(-, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times)$

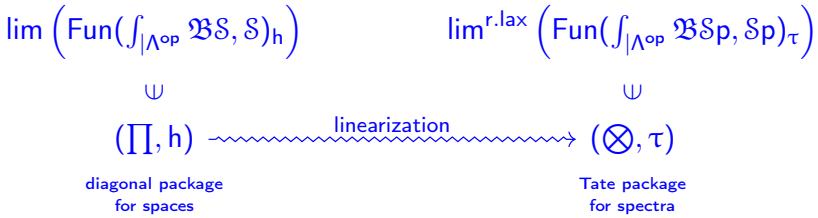
(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors***

$$\begin{array}{ccccc} \mathcal{V}^{\times R^{(1)}} & \longrightarrow & \int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V} & \longrightarrow & \mathcal{S}\mathrm{p} \\ \downarrow & & \downarrow & & \\ \{R \to S^1\} & \longrightarrow & \Lambda^{\mathrm{op}} & & \end{array}$$

multilinear: preserves finite colimits separately in each variable*

which is *stable* under action $\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}\mathrm{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^\times$

* sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathbf{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathbf{fin}}$)



(1) an action $\mathrm{Fun}(-,\mathcal{S}p)\overset{l.lax}{\underset{\tau}{\curvearrowleft}}\mathbb{N}^\times$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}p)\underset{h}{\curvearrowleft}\mathbb{N}^\times)\xrightarrow{r.lax}(\mathrm{Fun}(-,\mathcal{S}p)\overset{l.lax}{\underset{\tau}{\curvearrowleft}}\mathbb{N}^\times)$

(3) for $\mathcal{V}=\mathcal{S}$ or $\mathcal{S}p$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{V},\mathcal{S}p)\subset\mathrm{Fun}(\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{V},\mathcal{S}p)$ of ***fiberwise multilinear functors*** which is stable under $\overset{l.lax}{\underset{\tau}{\curvearrowleft}}\mathbb{N}^\times$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{fin}$, $\mathcal{S}p$ vs. $\mathcal{S}p^{fin}$)

$$\lim\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S})_{\mathsf{h}}\right) \qquad \lim^{\mathrm{r.lax}}\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p})_{\tau}\right)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$(\textstyle\prod,\mathsf{h}) \rightsquigarrow\rightsquigarrow\rightsquigarrow \text{linearization} \rightsquigarrow\rightsquigarrow\rightsquigarrow (\bigotimes,\tau)$$

diagonal package for spaces — Tate package for spectra

$$\left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\right)\curvearrowleft_{\mathsf{h}}\mathbb{N}^{\times}\right)$$

$$(\textstyle\prod,\mathsf{h})$$

(1) an action $\mathrm{Fun}(-,\mathcal{S}\mathrm{p})\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}}\mathbb{N}^{\times}$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}\mathrm{p})\underset{\mathsf{h}}{\curvearrowleft}\mathbb{N}^{\times})\xrightarrow{\mathrm{r.lax}}(\mathrm{Fun}(-,\mathcal{S}\mathrm{p})\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}}\mathbb{N}^{\times})$

(3) for $\mathcal{V}=\mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p})\subset\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors*** which is stable under $\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}}\mathbb{N}^{\times}$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathrm{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$)

$$\lim\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S})_{\mathsf{h}}\right) \qquad \lim^{\mathsf{r.lax}}\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p})_{\tau}\right)$$

$$\cup \qquad\qquad\qquad \cup$$

$$(\textstyle\prod,\mathsf{h}) \rightsquigarrow \text{linearization} \rightsquigarrow (\bigotimes,\tau)$$

diagonal package for spaces — Tate package for spectra

(1) an action $\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^{\times}) \xrightarrow{\mathsf{r.lax}} (\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times})$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors*** which is stable under $\overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathsf{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathsf{fin}}$)

$$\left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\right) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) \xrightarrow[\mathsf{r.lax}]{\Sigma^{\infty}_{+}\circ -} \left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^{\times}\right)$$

$$(\textstyle\prod,\mathsf{h}) \longmapsto \left(\bigotimes \circ \int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+},\mathsf{h}\right)$$

$\mathcal{S} \xrightarrow{\Sigma^{\infty}_{+}} \mathcal{S}\mathrm{p}$ **is symmetric monoidal** $\quad\leftrightsquigarrow\quad$

$$\begin{array}{ccc} \int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S} & \xrightarrow{\prod} & \mathcal{S} \\ {\scriptstyle \int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\Sigma^{\infty}}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Sigma^{\infty}_{+}} \\ \int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S}\mathrm{p} & \xrightarrow[\bigotimes]{} & \mathcal{S}\mathrm{p} \end{array}$$

$$\lim\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S})_h\right) \qquad \lim^{r.lax}\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S}p,\mathcal{S}p)_\tau\right)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$(\textstyle\prod, h) \rightsquigarrow^{\text{linearization}} (\bigotimes, \tau)$$

diagonal package for spaces — Tate package for spectra

(1) an action $\mathrm{Fun}(-,\mathcal{S}p) \overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}p) \underset{h}{\curvearrowleft} \mathbb{N}^\times) \xrightarrow{r.lax} (\mathrm{Fun}(-,\mathcal{S}p) \overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times)$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}p$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{V},\mathcal{S}p) \subset \mathrm{Fun}(\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{V},\mathcal{S}p)$ of ***fiberwise multilinear functors*** which is stable under $\overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{fin}$, $\mathcal{S}p$ vs. $\mathcal{S}p^{fin}$)

$$\left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S}\right) \underset{h}{\curvearrowleft} \mathbb{N}^\times\right) \xrightarrow[r.lax]{\Sigma^\infty_+\circ -} \left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S}p\right) \underset{h}{\curvearrowleft} \mathbb{N}^\times\right) \xrightarrow[r.lax]{(2)} \left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S}p\right) \overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times\right)$$

$$(\textstyle\prod, h) \longmapsto \left(\bigotimes \circ \int_{|\Lambda^{op}}\mathfrak{B}\Sigma^\infty_+, h\right) \longmapsto \left(\bigotimes \circ \int_{|\Lambda^{op}}\mathfrak{B}\Sigma^\infty_+, \tau\right)$$

$\mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathcal{S}p$ **is symmetric monoidal** $\quad\leftrightsquigarrow\quad$

$$\begin{array}{ccc} \int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S} & \xrightarrow{\prod} & \mathcal{S} \\ {\scriptstyle \int_{|\Lambda^{op}}\mathfrak{B}\Sigma^\infty}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Sigma^\infty_+} \\ \int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S}p & \xrightarrow[\bigotimes]{} & \mathcal{S}p \end{array}$$

$$\lim\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S})_h\right) \qquad \lim^{r.lax}\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S}p,\mathcal{S}p)_\tau\right)$$

$$\cup \qquad\qquad\qquad \cup$$

$$(\textstyle\prod,h) \rightsquigarrow^{\text{linearization}} (\bigotimes,\tau)$$

diagonal package for spaces — Tate package for spectra

(1) an action $\mathrm{Fun}(-,\mathcal{S}p) \overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}p) \underset{h}{\curvearrowleft} \mathbb{N}^\times) \xrightarrow{r.lax} (\mathrm{Fun}(-,\mathcal{S}p) \overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times)$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}p$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{V},\mathcal{S}p) \subset \mathrm{Fun}(\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{V},\mathcal{S}p)$ of ***fiberwise multilinear functors*** which is stable under $\overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{fin}$, $\mathcal{S}p$ vs. $\mathcal{S}p^{fin}$)

$$\left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S}\right) \underset{h}{\curvearrowleft} \mathbb{N}^\times\right) \xrightarrow[r.lax]{\Sigma^\infty_+\circ -} \left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S}p\right) \underset{h}{\curvearrowleft} \mathbb{N}^\times\right) \xrightarrow[r.lax]{(2)} \left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S},\mathcal{S}p\right) \overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times\right) \xleftarrow{-\circ\left(\int_{|\Lambda^{op}}\mathfrak{B}\Sigma^\infty_+\right)} \left(\mathrm{Fun}\left(\textstyle\int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S}p,\mathcal{S}p\right) \overset{l.lax}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times\right)$$

$$(\textstyle\prod,h) \longmapsto \left(\bigotimes\circ\int_{|\Lambda^{op}}\mathfrak{B}\Sigma^\infty_+,h\right) \longmapsto \left(\bigotimes\circ\int_{|\Lambda^{op}}\mathfrak{B}\Sigma^\infty_+,\tau\right)$$

$\mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathcal{S}p$ **is symmetric monoidal** $\quad\leftrightsquigarrow\quad$

$$\begin{array}{ccc} \int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S} & \xrightarrow{\prod} & \mathcal{S} \\ {\scriptstyle \int_{|\Lambda^{op}}\mathfrak{B}\Sigma^\infty}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Sigma^\infty_+} \\ \int_{|\Lambda^{op}}\mathfrak{B}\mathcal{S}p & \xrightarrow[\bigotimes]{} & \mathcal{S}p \end{array}$$

$$\lim\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S})_{\mathrm{h}}\right) \qquad \lim^{\mathrm{r.lax}}\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p})_{\tau}\right)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$(\textstyle\prod,\mathrm{h}) \overset{\text{linearization}}{\rightsquigarrow} (\otimes,\tau)$$

diagonal package for spaces — Tate package for spectra

(1) an action $\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \xrightarrow{\mathrm{r.lax}} (\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times})$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors*** which is stable under $\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathrm{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$)

$$\begin{array}{ccccccc}
\left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) & \xrightarrow[\mathrm{r.lax}]{\Sigma^{\infty}_{+}\circ -} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) & \xrightarrow[\mathrm{r.lax}]{(2)} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) & \xleftarrow{-\circ\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+}\right)} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \\
 & & & & \uparrow & & \uparrow \\
 & & & & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) & \longleftarrow & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right)
\end{array}$$

$$(\textstyle\prod,\mathrm{h}) \longmapsto \left(\otimes\circ\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+},\mathrm{h}\right) \longmapsto \left(\otimes\circ\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+},\tau\right)$$

$\mathcal{S} \xrightarrow{\Sigma^{\infty}_{+}} \mathcal{S}\mathrm{p}$ **is symmetric monoidal** $\quad\Longleftrightarrow\quad$

$$\begin{array}{ccc}
\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S} & \xrightarrow{\prod} & \mathcal{S} \\
{\scriptstyle\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\Sigma^{\infty}}\downarrow & \circlearrowright & \downarrow{\scriptstyle\Sigma^{\infty}_{+}} \\
\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S}\mathrm{p} & \xrightarrow[\otimes]{} & \mathcal{S}\mathrm{p}
\end{array}$$

$$\lim\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S})_{\mathrm{h}}\right) \qquad \lim^{\mathrm{r.lax}}\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p})_{\tau}\right)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$(\textstyle\prod,\mathrm{h}) \overset{\text{linearization}}{\rightsquigarrow} (\otimes,\tau)$$

diagonal package for spaces — Tate package for spectra

(1) an action $\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \xrightarrow{\mathrm{r.lax}} (\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times})$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors*** which is stable under $\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathrm{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$)

$$\begin{array}{ccccccc}
\left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) & \xrightarrow[\mathrm{r.lax}]{\Sigma^{\infty}_{+}\circ -} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) & \xrightarrow[\mathrm{r.lax}]{(2)} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) & \xleftarrow{-\circ\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+}\right)} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \\
 & & & & \uparrow & & \uparrow \\
 & & & & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) & \xleftarrow{\sim} & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right)
\end{array}$$

$$(\textstyle\prod,\mathrm{h}) \longmapsto \left(\otimes\circ\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+},\mathrm{h}\right) \longmapsto \left(\otimes\circ\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+},\tau\right)$$

$\mathcal{S} \xrightarrow{\Sigma^{\infty}_{+}} \mathcal{S}\mathrm{p}$ **is symmetric monoidal** $\quad\leftrightsquigarrow\quad$

$$\begin{array}{ccc}
\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S} & \xrightarrow{\prod} & \mathcal{S} \\
{\scriptstyle\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\Sigma^{\infty}}\downarrow & \circlearrowright & \downarrow{\scriptstyle\Sigma^{\infty}_{+}} \\
\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S}\mathrm{p} & \xrightarrow[\otimes]{} & \mathcal{S}\mathrm{p}
\end{array}$$

universal property: $\mathcal{S}\mathrm{p}$ **is the *stabilization* of** $\mathcal{S}$ $\quad\leftrightsquigarrow\quad$ $\mathrm{Lin}(\mathcal{S},\mathcal{S}\mathrm{p}) \xleftarrow{\sim} \mathrm{Lin}(\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p})$

$$\lim\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S})_{\mathrm{h}}\right) \qquad \lim^{\mathrm{r.lax}}\left(\mathrm{Fun}(\textstyle\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p})_{\tau}\right)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$(\textstyle\prod,\mathrm{h}) \overset{\text{linearization}}{\rightsquigarrow} (\otimes,\tau)$$

diagonal package for spaces — Tate package for spectra

(1) an action $\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

(2) a functor $(\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}) \xrightarrow{\mathrm{r.lax}} (\mathrm{Fun}(-,\mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times})$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}\mathrm{p}$, a subcat $\mathrm{Lin}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p}) \subset \mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{V},\mathcal{S}\mathrm{p})$ of ***fiberwise multilinear functors*** which is stable under $\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}$

sweeping minor issues under the rug ($\mathcal{S}$ vs. $\mathcal{S}^{\mathrm{fin}}$, $\mathcal{S}\mathrm{p}$ vs. $\mathcal{S}\mathrm{p}^{\mathrm{fin}}$)

$$\begin{array}{ccccccc}
\left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) & \xrightarrow[\mathrm{r.lax}]{\Sigma^{\infty}_{+}\circ -} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \underset{\mathrm{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) & \xrightarrow[\mathrm{r.lax}]{(2)} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) & \xleftarrow{-\circ\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+}\right)} & \left(\mathrm{Fun}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \\
 & & & & \uparrow & & \uparrow \\
 & & & & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) & \xleftarrow{\sim} & \left(\mathrm{Lin}\left(\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right)
\end{array}$$

$$(\textstyle\prod,\mathrm{h}) \longmapsto \left(\otimes\circ\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+},\mathrm{h}\right) \longmapsto \left(\otimes\circ\int_{|\Lambda^{\mathrm{op}}}\mathfrak{B}\Sigma^{\infty}_{+},\tau\right) \longmapsto (\otimes,\tau)$$

$\mathcal{S} \xrightarrow{\Sigma^{\infty}_{+}} \mathcal{S}\mathrm{p}$ **is symmetric monoidal** $\quad\leftrightsquigarrow\quad$

$$\begin{array}{ccc}
\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S} & \xrightarrow{\prod} & \mathcal{S} \\
{\scriptstyle \int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\Sigma^{\infty}}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Sigma^{\infty}_{+}} \\
\int_{|\Lambda^{\mathbf{op}}}\mathfrak{B}\mathcal{S}\mathrm{p} & \xrightarrow[\otimes]{} & \mathcal{S}\mathrm{p}
\end{array}$$

universal property: $\mathcal{S}\mathrm{p}$ **is the *stabilization* of** $\mathcal{S}$ $\quad\leftrightsquigarrow\quad$ $\mathrm{Lin}(\mathcal{S},\mathcal{S}\mathrm{p}) \xleftarrow{\sim} \mathrm{Lin}(\mathcal{S}\mathrm{p},\mathcal{S}\mathrm{p})$

summary

summary

talk 1

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\textbf{mdrmy}} H^*_{dR}$

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\textbf{mdrmy}} H^*_{dR}$

§1.2 traces in algebraic geometry

$\mathrm{K}(X) \xrightarrow{\textbf{mdrmy}} \mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$, $\mathrm{K} \to \mathrm{THC}^-$ a local $\mathbb{Q}$-equiv$^{\textbf{ce}}$, $\mathrm{K} \to \mathrm{TC}$ a local equiv$^{\textbf{ce}}$

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\text{mdrmy}} H^*_{dR}$

§1.2 traces in algebraic geometry

$\mathrm{K}(X) \xrightarrow{\text{mdrmy}} \mathrm{THH}(X) = \mathcal{O}(\mathcal{L}X)$, $\mathrm{K} \to \mathrm{THC}^-$ a local $\mathbb{Q}$-equiv$^{\text{ce}}$, $\mathrm{K} \to \mathrm{TC}$ a local equiv$^{\text{ce}}$

§1.3 the geometry of the cyclotomic trace

$\mathrm{TC}(X) = \mathbb{T}$-inv$^{\text{t}}$ functions on $\mathcal{L}X$ that are also "sensitive" to $\gamma \mapsto r^*\gamma$ (e.g. $\mathrm{tr}(M^{\otimes r}) \equiv \mathrm{tr}(M)^{\otimes r}$)

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\text{mdrmy}} H^*_{dR}$

§1.2 traces in algebraic geometry

$\mathrm{K}(X) \xrightarrow{\text{mdrmy}} \mathrm{THH}(X) = \mathcal{O}(\mathcal{L}X)$, $\mathrm{K} \to \mathrm{THC}^-$ a local $\mathbb{Q}$-equiv$^{\mathbf{ce}}$, $\mathrm{K} \to \mathrm{TC}$ a local equiv$^{\mathbf{ce}}$

§1.3 the geometry of the cyclotomic trace

$\mathrm{TC}(X) = \mathbb{T}$-inv$^{\mathbf{t}}$ functions on $\mathcal{L}X$ that are also "sensitive" to $\gamma \mapsto r^*\gamma$ (e.g. $\mathrm{tr}(M^{\otimes r}) \equiv \mathrm{tr}(M)^{\otimes r}$)

talk 2

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\textbf{mdrmy}} H^*_{dR}$

§1.2 traces in algebraic geometry

$K(X) \xrightarrow{\textbf{mdrmy}} THH(X) = \mathcal{O}(\mathcal{L}X)$, $K \to THC^-$ a local $\mathbb{Q}$-equiv$^{\textbf{ce}}$, $K \to TC$ a local equiv$^{\textbf{ce}}$

§1.3 the geometry of the cyclotomic trace

$TC(X) = \mathbb{T}$-inv$^{\textbf{t}}$ functions on $\mathcal{L}X$ that are also "sensitive" to $\gamma \mapsto r^*\gamma$ (e.g. $\mathrm{tr}(M^{\otimes r}) \equiv \mathrm{tr}(M)^{\otimes r}$)

talk 2

§2.1 stratified schemes and generalized recollements

for X a stratified scheme, $QC(X) \xrightarrow{\sim} \lim^{\textbf{r.lax}}(\ldots\text{l.lax}\ldots)$

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\textbf{mdrmy}} H^*_{dR}$

§1.2 traces in algebraic geometry

$\mathrm{K}(X) \xrightarrow{\textbf{mdrmy}} \mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$, $\mathrm{K} \to \mathrm{THC}^-$ a local $\mathbb{Q}$-equiv$^{\textbf{ce}}$, $\mathrm{K} \to \mathrm{TC}$ a local equiv$^{\textbf{ce}}$

§1.3 the geometry of the cyclotomic trace

$\mathrm{TC}(X) = \mathbb{T}$-inv$^{\textbf{t}}$ functions on $\mathcal{L}X$ that are also "sensitive" to $\gamma \mapsto r^*\gamma$ (e.g. $\mathrm{tr}(M^{\otimes r}) \equiv \mathrm{tr}(M)^{\otimes r}$)

talk 2

§2.1 stratified schemes and generalized recollements

for X a stratified scheme, $\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\textbf{r.lax}}(\ldots\text{l.lax}\ldots)$

§2.2 the geometry of TC

$\mathbb{N}^\times$-eq $\mathbb{N}^{\textbf{div}}$-strat$^{\textbf{d}}$ $\mathscr{L}X \downarrow \mathscr{B}\mathbb{T}$ such that $\mathrm{THH}(X) \leftrightarrow \mathscr{O}_{\mathscr{L}X}$ and $\mathrm{TC}(X) = \mathbb{W}$-eq fxns on $\mathscr{L}X$

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\textbf{mdrmy}} H^*_{dR}$

§1.2 traces in algebraic geometry

$\mathrm{K}(X) \xrightarrow{\textbf{mdrmy}} \mathrm{THH}(X) = \mathscr{O}(\mathcal{L}X)$, $\mathrm{K} \to \mathrm{THC}^-$ a local $\mathbb{Q}$-equiv$^{\textbf{ce}}$, $\mathrm{K} \to \mathrm{TC}$ a local equiv$^{\textbf{ce}}$

§1.3 the geometry of the cyclotomic trace

$\mathrm{TC}(X) = \mathbb{T}$-inv$^{\textbf{t}}$ functions on $\mathcal{L}X$ that are also "sensitive" to $\gamma \mapsto r^*\gamma$ (e.g. $\mathrm{tr}(M^{\otimes r}) \equiv \mathrm{tr}(M)^{\otimes r}$)

talk 2

§2.1 stratified schemes and generalized recollements

for X a stratified scheme, $\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\textbf{r.lax}}(\ldots\text{l.lax}\ldots)$

§2.2 the geometry of TC

$\mathbb{N}^\times$-eq $\mathbb{N}^{\textbf{div}}$-strat$^{\textbf{d}}$ $\mathscr{L}X \downarrow \mathscr{B}\mathbb{T}$ such that $\mathrm{THH}(X) \leftrightarrow \mathscr{O}_{\mathscr{L}X}$ and $\mathrm{TC}(X) = \mathbb{W}$-eq fxns on $\mathscr{L}X$

§2.3 factorization homology of enriched ∞-categories

via fact hlgy and linearizi$^{\textbf{n}}$: $S^{\textbf{1}} \downarrow S^{\textbf{1}} \rightsquigarrow$ cyclo str on THH and $S^{\textbf{1}} \downarrow \mathbb{D}^{\textbf{0}} \rightsquigarrow$ cyclo trace