The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (arXiv:1710.06416)
2. *Factorization homology of enriched ∞-categories* (arXiv:1710.06414)
3. *The geometry of the cyclotomic trace* (arXiv:1710.06409)

X a scheme (derived)

$\mathrm{K}(X) =$ ***algebraic K-theory*** of X
$:= \mathrm{K}(\mathrm{Perf}_X)$
$\approx$ group-completion of $(\mathrm{VBdl(X)/iso.}, \oplus)$
[hard to compute!]

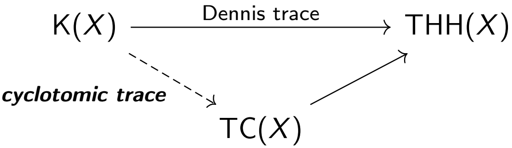
$\mathrm{THH}(X) =$ ***topological Hochschild homology*** of X
$:= \mathrm{THH}(\mathrm{Perf}_X) := \int_{S^1} \mathrm{Perf}_X$
$\approx \mathscr{O}(\mathscr{L}X)$, functions on the free loopspace of X
[easier to compute]

the ***Dennis trace*** map

$$\mathrm{K}(X) \longrightarrow \mathrm{THH}(X)$$

$$(E \downarrow X) \longmapsto \left((S^1 \xrightarrow{\gamma} X) \longmapsto \begin{array}{l} \text{trace of} \\ \text{monodromy} \\ \text{of } E|_\gamma \end{array} \right)$$

a refinement:



$\mathsf{TC}(X)$ = ***topological cyclic homology*** of X

$\approx$ .....???!!!?

[computationally accessible, but conceptually mysterious]

why we care about TC

Thm (Goodwillie/$\mathbb{Q}$ '86, McCarthy/$\mathbb{Z}$ '97, Dundas/$\mathbb{S}$ '97). The cyclotomic trace is "locally constant": for $\tilde{A} \to A$ a nilpotent extension of associative rings (or of connective ring spectra),

$$\begin{array}{ccc} \mathrm{K}(\tilde{A}) & \longrightarrow & \mathrm{K}(A) \\ \downarrow & & \downarrow \\ \mathrm{TC}(\tilde{A}) & \longrightarrow & \mathrm{TC}(A) \end{array}$$

is a pullback.

"This is how people other than Quillen compute algebraic K-theory."

$\sim$ *A. Blumberg*, algebraic K-theorist

...but what *is* $\mathrm{TC}(X)$, really?

intermediate factorization through *negative cyclic homology*:

$$\mathrm{K}(X) \longrightarrow \mathrm{TC}(X) \longrightarrow \mathrm{THC}^{-}(X) \longrightarrow \mathrm{THH}(X)$$

	differential algebra	derived algebraic geometry
$\mathrm{THH}(X)$	$\Omega^*_{\mathrm{dR}}(X)$	functions on $\mathscr{L}X$
$\mathrm{THC}^{-}(X)$ $:= \mathrm{THH}(X)^{\mathrm{h}\mathbb{T}}$	$\mathrm{H}^*_{\mathrm{dR}}(X)$	$\mathbb{T}$-invariant functions on $\mathscr{L}X$
$\mathrm{TC}(X)$ $:= \mathrm{THH}(X)^{\mathrm{hCyc}}$	$???^*_{\mathrm{dR}}(X)$	**TODAY**

constructions of TC

original definition (Bökstedt–Hsiang–Madsen '93):

- uses *genuine-equivariant* stable homotopy theory
  - useful (e.g. equivariant Poincaré duality)...
  - but not conceptual (no DAG interpretation known)
- used opaque point-set manipulations
- based on vague analogy with free loopspaces

firmer categorical footing (Blumberg–Mandell '13):

- define homotopy theory of "cyclotomic spectra"

more recent definition (Nikolaus–Scholze '17?):

- removes genuine-equivariance
- restricts to *connective* ring spectra

this talk, inspired by Nikolaus–Scholze:

- applies to any spectrally-enriched ∞-category
- uses *factorization homology* to keep track of symmetries
- admits direct interpretation in DAG via $\mathscr{L}X$
- suggests higher-dim generalizations ($\rightsquigarrow$ "higher K-theory")

overview

$$
\begin{array}{ccc}
\mathrm{Sp} & \underset{(-)^{\mathrm{hCyc}}}{\overset{\mathrm{triv}}{\underset{\dashleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathrm{Cyc}(\mathrm{Sp}) \\
\rotatebox[origin=c]{-90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\
\mathrm{TC}(X) & \longleftarrow\!\shortmid & \mathrm{THH}(X)
\end{array}
$$

$\mathrm{TC}(X) :=$ fixedpoints of ***cyclotomic structure*** on $\mathrm{THH}(X)$
$\rightsquigarrow$ built by "imposing conditions" on functions on $\mathscr{L}X$

main idea: $\mathrm{TC}(X) \approx$ functions on $\mathscr{L}X$ that are...

- invariant under the $\mathbb{T}$-action on $\mathscr{L}X$;
- "sensitive" to the relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$.

$\mathbb{T}$-invariance is easy, but what does "sensitive" mean?

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: M an $n \times n$ matrix, compare $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \mathrm{diag}(m_1, \ldots, m_n) \in \mathrm{M}_{n\times n}(R)$

$$\mathrm{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j \quad , \quad \mathrm{tr}(M^2) = \sum_k m_k \cdot m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is *norms*: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{aligned} (R \otimes R)_{C_2} &\xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \\ [x \otimes y] &\longmapsto \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{aligned}$$

$\rightsquigarrow$ become equal in the ***Tate construction***, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\mathrm{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{\mathrm{t}C_2}$$

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: M an $n \times n$ matrix, compare $\mathrm{tr}(M)^r$ and $\mathrm{tr}(M^r)$?

Ex. 2: $M = \mathrm{diag}(m_1, m_2)$, $r \in \mathbb{N}^\times$ arbitrary

now, difference between

$$\mathrm{tr}(M)^r = (m_1 + m_2)^r \quad , \quad \mathrm{tr}(M^r) = ((m_1)^r + (m_2)^r)$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

fact: these are never coprime to r

$\rightsquigarrow$ quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

$\rightsquigarrow$ $tr(M^r) \equiv \mathrm{tr}(M)^r$ in the *generalized* Tate construction $(R^{\otimes r})^{\tau C_r}$

$\star$ for $\mathcal{C}$ a spectrally enriched ∞-category, a covering map

$$S^1_b \xleftarrow{r} S^1_a$$

of oriented circles induces a ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

Theorem 1 (A & M-G & R)

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{\mathrm{h}\mathbb{T}} \underset{\tau}{\overset{\mathrm{l.lax}}{\circlearrowleft}} \mathbb{N}^{\times} \right) \quad := \mathrm{Fun}(\mathrm{B}\mathbb{T}, \mathrm{Sp})$$

⋆ an object of $\lim^{\mathrm{r.lax}}$ is given by $T \in \mathrm{Sp}^{\mathrm{h}\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^{\times}$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ {\scriptstyle \sigma_{rs}}\downarrow & & \downarrow {\scriptstyle (\sigma_s)^{\tau C_r}} \\ T^{\tau C_{rs}} & \underset{\text{can.}}{\xrightarrow{\sim}} & (T^{\tau C_s})^{\tau C_r} \end{array}$$

Thm. [Nikolaus–Scholze]
for T connective and $r = s = p$ prime
Cor.: suff to specify just σ_p
(since $\sigma_{p^n} = (\sigma_p)^{\circ n}$, and n-cubes
canonically commute $\forall\ n \geq 2$)

- for each $r_1, \ldots, r_n \in \mathbb{N}^{\times}$, the *data* of a commutative n-cube...

⋆ **slogan:** $\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) \approx \mathscr{O}(\mathscr{L}X)$ by selecting just those functions:

- that are $\mathbb{T}$-invariant;
- whose values on $S^1 \xrightarrow{\gamma} X$ determine their values on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ "to the greatest extent possible", subject to all possible coherences between these determinations.

Theorem 1 (A & M-G & R)

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{Sp}^{\mathrm{h}\mathbb{T}} \underset{\tau}{\overset{\mathrm{l.lax}}{\curvearrowleft}} \mathbb{N}^{\times} \right).$$

main input (inspired by Glasman & many others)...

notation: G a compact Lie group, P_G its poset of closed subgroups under subconjugacy.

Theorem 2 (A & M-G & R)

There's a canonical left-lax left P_G*-module* $\mathbf{Sp}^{\mathrm{g}G}$*, whose value on* $H \in \mathrm{P}_G$ *is* $\mathrm{Sp}^{\mathrm{hW}(H)}$*, with*

$$\mathrm{Sp}^{\mathrm{g}G} \simeq \lim^{\mathrm{r.lax}} \left(\mathrm{P}_G \overset{\mathrm{l.lax}}{\curvearrowright} \mathbf{Sp}^{\mathrm{g}G} \right).$$

⋆ over $H \in \mathrm{P}_G$, functor is $\mathrm{Sp}^{\mathrm{g}G} \xrightarrow{\Phi^H} \mathrm{Sp}^{\mathrm{gW}(H)} \xrightarrow{\mathrm{fgt}} \mathrm{Sp}^{\mathrm{hW}(H)}$

⋆ a ***generalized recollement*** over P_G; classical is over poset [1]

⋆ hints at a DAG description of genuine G-spectra...

Q.: Where does the cyclotomic structure on THH come from?

Theorem 3 (A & M-G & R)

1. ***diagonal package*** for spaces $\rightsquigarrow$
$$\begin{array}{ccc} \mathrm{Cat}(\mathcal{S}) & \dashrightarrow & \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) \\ & \searrow^{\mathrm{THH}_{\mathcal{S}}} & \downarrow \mathrm{fgt} \\ & & \mathcal{S} \end{array}$$
$\mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) := \mathrm{Fun}(\mathrm{B}\mathbb{W}, \mathcal{S}) :=$ "unstable cyclotomic spaces"
$\mathbb{W} \simeq \mathbb{T} \rtimes \mathbb{N}^{\times}$ the "Witt monoid"

2. ***diagonal package*** for spaces
$\Big\downarrow$ linearization (à la Goodwillie calculus)
Tate package for spectra $\rightsquigarrow$
$$\begin{array}{ccc} \mathrm{Cat}(\mathrm{Sp}) & \dashrightarrow & \mathrm{Cyc}(\mathrm{Sp}) \\ & \searrow^{\mathrm{THH}} & \downarrow \mathrm{fgt} \\ & & \mathrm{Sp} \end{array}$$
$\mathrm{Cyc}(\mathrm{Sp}) :=$ cyclotomic spectra

Q.: Where does the cyclotomic trace come from?

Theorem 4 (A & M-G & R)

the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

$$\iota\mathcal{C} \longrightarrow \mathsf{TC}^{\mathsf{h}}_{\mathcal{S}}(\mathcal{C}) := \mathsf{THH}_{\mathcal{S}}(\mathcal{C})^{\mathsf{h}\mathbb{W}}$$

$$\Big\downarrow \text{ linearization (à la Goodwillie calculus)}$$

the ***cyclotomic trace***: for $\mathcal{C}$ a stable ∞-category,

$$\mathsf{K}(\mathcal{C}) \longrightarrow \mathsf{TC}(\mathcal{C}) := \mathsf{THH}(\mathcal{C})^{\mathsf{hCyc}}$$