The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

2. "Factorization homology of enriched $\infty$-categories" (arXiv:1710.06414)
3. "The geometry of the cyclotomic trace" (arXiv:1710.06409)
$X$ a scheme (derived)

$$K(X) = \text{algebraic K-theory of } X$$

$$:= K(\text{Perf}_X)$$

$$\approx \text{group-completion of } (\text{VBdl}(X)/\text{iso.}, \oplus)$$

[hard to compute!]

$$\text{THH}(X) = \text{topological Hochschild homology of } X$$

$$:= \text{THH}(\text{Perf}_X) := \int_{S^1} \text{Perf}_X$$

$$\approx \mathcal{O}(\mathcal{L}X), \text{ functions on the free loop space of } X$$

[easier to compute]

the $\text{Dennis trace}$ map

$$K(X) \longrightarrow \text{THH}(X)$$

$$(E \downarrow X) \longmapsto \left( \left( S^1 \xrightarrow{\gamma} X \right) \longmapsto \text{trace of monodromy of } E|_{\gamma} \right)$$
a refinement:

\[ K(X) \xrightarrow{\text{Dennis trace}} \text{THH}(X) \]
\[ \text{cycloptic trace} \]
\[ \xrightarrow{\text{}} \text{TC}(X) \]

\[ \text{TC}(X) = \textit{topological cyclic homology} \text{ of } X \]
\[ \approx \ldots ???!!!? \]

[computationally accessible, but conceptually mysterious]
why we care about $TC$

**Thm (Goodwillie/\mathbb{Q} '86, McCarthy/\mathbb{Z} '97, Dundas/\mathbb{S} '97).** The cyclotomic trace is “locally constant”: for $\tilde{A} \to A$ a nilpotent extension of associative rings (or of connective ring spectra),

\[
\begin{array}{ccc}
K(\tilde{A}) & \longrightarrow & K(A) \\
\downarrow & & \downarrow \\
TC(\tilde{A}) & \longrightarrow & TC(A)
\end{array}
\]

is a pullback.

“This is how people other than Quillen compute algebraic $K$-theory.”

~ A. Blumberg, algebraic $K$-theorist
...but what is $TC(X)$, really?

Intermediate factorization through *negative cyclic homology*:

$$K(X) \longrightarrow TC(X) \longrightarrow THC^-(X) \longrightarrow THH(X)$$

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<th>differential algebra</th>
<th>derived algebraic geometry</th>
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<td>$THH(X)$</td>
<td>$\Omega^* ! _{dR}(X)$</td>
<td>functions on $\mathcal{L}X$</td>
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<td>$THC^-(X)$</td>
<td>$H^* ! _{dR}(X)$</td>
<td>$\mathbb{T}$-invariant functions on $\mathcal{L}X$</td>
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<tr>
<td>$TC(X)$</td>
<td>$???^* ! _{dR}(X)$</td>
<td><strong>TODAY</strong></td>
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<td>$:= THH(X)^{h\mathbb{T}}$</td>
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<td>$:= THH(X)^{hCyc}$</td>
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constructions of TC

original definition (Bökstedt–Hsiang–Madsen ’93):
- uses genuine-equivariant stable homotopy theory
  - useful (e.g. equivariant Poincaré duality)...
  - but not conceptual (no DAG interpretation known)
- used opaque point-set manipulations
- based on vague analogy with free loopspaces

firmer categorical footing (Blumberg–Mandell ’13):
- define homotopy theory of “cyclotomic spectra”

more recent definition (Nikolaus–Scholze ’17?):
- removes genuine-equivariance
- restricts to connective ring spectra

this talk, inspired by Nikolaus–Scholze:
- applies to any spectrally-enriched ∞-category
- uses factorization homology to keep track of symmetries
- admits direct interpretation in DAG via $\mathcal{L}X$
- suggests higher-dim generalizations ($\rightsquigarrow$ “higher K-theory”)

Aaron Mazel-Gee
The geometry of the cyclotomic trace
overview

\[
\begin{array}{c}
\text{Sp} \quad \xrightarrow{\text{triv}} \quad \text{Cyc(Sp)} \\
\cup \\
\cup \\
\text{TC}(X) \quad \xleftarrow{(-)^{hCyc}} \quad \text{THH}(X)
\end{array}
\]

TC(X) := fixedpoints of \textit{cyclotomic structure} on THH(X)

\( \rightsquigarrow \) built by “imposing conditions” on functions on \( L X \)

\textbf{main idea:} TC(X) \( \approx \) functions on \( L X \) that are...

- invariant under the \( \mathbb{T} \)-action on \( L X \);
- “sensitive” to the relationship between \( S^1 \xrightarrow{\gamma} X \) and \( S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X \).

\( \mathbb{T} \)-invariance is easy, but what does “sensitive” mean?
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \text{diag}(m_1, \ldots, m_n) \in M_{n \times n}(R)$

$$\text{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j, \quad \text{tr}(M^2) = \sum_k m_k \cdot m_k$$

- both cyclically invariant, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is norms: image of $\sum_{i < j} [m_i \otimes m_j]$ under

$$\begin{align*}
(R \otimes R)^{C_2} &\xrightarrow{Nm} (R \otimes R)^{C_2} \\
[x \otimes y] &\mapsto \sum_{\sigma \in C_2} \sigma(x \otimes y)
\end{align*}$$

$\sim$ become equal in the *Tate construction*, the cofiber

$$(R \otimes R)^{C_2} \xrightarrow{Nm} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{tC_2}$$
Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 2: $M = \text{diag}(m_1, m_2)$, $r \in \mathbb{N}^\times$ arbitrary

now, difference between

\[
\text{tr}(M)^r = (m_1 + m_2)^r, \quad \text{tr}(M^r) = ((m_1)^r + (m_2)^r)
\]

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

fact: these are never coprime to $r$

$\leadsto$ quotient $(R \otimes r)^{C_r}$ by norms from all proper subgroups of $C_r$

$\leadsto$ $\text{tr}(M^r) \equiv \text{tr}(M)^r$ in the generalized Tate construction $(R \otimes r)^{\tau C_r}$

$\star$ for $\mathcal{C}$ a spectrally enriched $\infty$-category, a covering map

\[
S^1_b \xleftarrow{r} S^1_a
\]

of oriented circles induces a **cyclotomic structure map**

\[
\text{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left( \int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \text{THH}(\mathcal{C})^{\tau C_r}
\]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim_{r}^{lax} \left( \text{Sp}^{hT} \overset{\lax}{\longrightarrow} \tau \text{Fin}(B_{T}, \text{Sp}) \right) \]

\( \star \) an object of \( \lim_{r}^{lax} \) is given by \( T \in \text{Sp}^{hT} \) equipped with:

- for each \( r \in \mathbb{N}^{\times} \), a cyclotomic structure map \( T \overset{\sigma_{r}}{\longrightarrow} T \tau C_{r} \);
- for each \( r, s \in \mathbb{N}^{\times} \), the data of a commutative square

\[
\begin{array}{ccc}
T & \overset{\sigma_{r}}{\longrightarrow} & T \tau C_{r} \\
\sigma_{rs} \downarrow & & \downarrow (\sigma_{s}) \tau C_{r} \\
T \tau C_{rs} & \overset{\sim \text{can.}}{\longrightarrow} & (T \tau C_{s}) \tau C_{r}
\end{array}
\]

- for each \( r_1, \ldots, r_n \in \mathbb{N}^{\times} \), the data of a commutative \( n \)-cube...

Thm. [Nikolaus–Scholze]
for \( T \) connective and \( r = s = p \) prime
Cor.: suff to specify just \( \sigma_{p} \)
(since \( \sigma_{pn} = (\sigma_{p})^{o_n} \), and \( n \)-cubes canonically commute \( \forall n \geq 2 \))

\( \star \) slogan: \( TC(X) \) is built from \( \text{THH}(X) \approx \mathcal{O}(\mathcal{L}X) \) by selecting just those functions:

- that are \( \mathbb{T} \)-invariant;
- whose values on \( S^{1} \overset{\gamma}{\longrightarrow} X \) determine their values on \( S^{1} \overset{r}{\longrightarrow} S^{1} \overset{\gamma}{\longrightarrow} X \) “to the greatest extent possible”, subject to all possible coherences between these determinations.
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \cong \lim^r \text{lax} \left( \text{Sp}^h \text{lax}_{\tau} \mathbb{N}^\\times \right). \]

main input (inspired by Glasman & many others)...

notation: \( G \) a compact Lie group, \( P_G \) its poset of closed subgroups under subconjugacy.

Theorem 2 (A & M-G & R)

There’s a canonical left-lax left \( P_G \)-module \( \text{Sp}^g G \), whose value on \( H \in P_G \) is \( \text{Sp}^h W(H) \), with

\[ \text{Sp}^g G \cong \lim^r \text{lax} \left( P_G \text{lax}_{\tau} \text{Sp}^g G \right). \]

⋆ over \( H \in P_G \), functor is \( \text{Sp}^g G \xrightarrow{\Phi^H} \text{Sp}^g W(H) \xrightarrow{\text{fgt}} \text{Sp}^h W(H) \)

⋆ a \textit{generalized recollement} over \( P_G \); classical is over poset [1]

⋆ hints at a DAG description of genuine \( G \)-spectra...
Q.: Where does the cyclotomic structure on THH come from?

Theorem 3 (A & M-G & R)

1. **diagonal package** for spaces

\[
\text{Cat}(\mathcal{S}) \longrightarrow \text{Cyc}^h(\mathcal{S})
\]

\[
\text{THH}_S \longrightarrow S
\]

\[
\text{Cyc}^h(\mathcal{S}) := \text{Fun}(\mathbb{B}W, \mathcal{S}) := \text{"unstable cyclotomic spaces"}
\]

\[
W \cong T \rtimes N \times \text{the "Witt monoid"}
\]

2. **diagonal package** for spaces

\[
\text{Cat}(\text{Sp}) \longrightarrow \text{Cyc}(\text{Sp})
\]

\[
\text{THH} \longrightarrow \text{Sp}
\]

**Tate package** for spectra

\[
\text{Cyc}(\text{Sp}) := \text{cyclotomic spectra}
\]
Q.: Where does the cyclotomic trace come from?

Theorem 4 (A & M-G & R)

the unstable cyclotomic trace: for $\mathcal{C}$ a $S$-enriched $\infty$-category,

$$\iota \mathcal{C} \longrightarrow \text{TC}_S^h(\mathcal{C}) \coloneqq \text{THH}_S(\mathcal{C})^{hW}$$

linearization (à la Goodwillie calculus)

the cyclotomic trace: for $\mathcal{C}$ a stable $\infty$-category,

$$K(\mathcal{C}) \longrightarrow \text{TC}(\mathcal{C}) \coloneqq \text{THH}(\mathcal{C})^{h\text{Cyc}}$$