

The geometry of the cyclotomic trace

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with David Ayala and Nick Rozenblyum

- 1 *A naive approach to genuine G -spectra and cyclotomic spectra* (arXiv:1710.06416)
- 2 *Factorization homology of enriched ∞ -categories* (arXiv:1710.06414)
- 3 *The geometry of the cyclotomic trace* (arXiv:1710.06409)

talk 1

§1.1 traces in differential geometry

§1.2 traces in algebraic geometry

§1.3 the geometry of the cyclotomic trace

talk 2

§2.1 stratified schemes and generalized recollements (interlude)

§2.2 the geometry of TC (conjectural refinement of §1.3)

§2.3 the construction: factorization homology of enriched ∞ -categories

§1.1: TRACES IN DIFFERENTIAL GEOMETRY

X a topological space \rightsquigarrow monoid $(\text{VBdl}_{\mathbb{R} \text{ or } \mathbb{C}}(X), \oplus)$

topological K-theory: $K(X) :=$ group-completion of this monoid

real K-theory = KO , complex K-theory = KU

elements: formal differences $[E_0, E_1] \approx "E_0 - E_1"$

e.g. $X = \text{pt}$ \rightsquigarrow $\text{VBdl}(\text{pt}) \cong \mathbb{N}$ \rightsquigarrow $K(\text{pt}) \cong \mathbb{Z}$

e.g. $X = S^2$ \rightsquigarrow $TS^2 \not\cong \underline{\mathbb{R}^2} \dots$ but in $KO(S^2)$,

$$\begin{aligned} [TS^2] + [\underline{\mathbb{R}^1}] &= [TS^2] + [NS^2] = [TS^2 \oplus NS^2] \\ s^2 \hookrightarrow \mathbb{R}^3 & \qquad \qquad \qquad = [i^* T\mathbb{R}^3] = [\underline{\mathbb{R}^3}] = [\underline{\mathbb{R}^2}] + [\underline{\mathbb{R}^1}] \\ & \rightsquigarrow [TS^2] = [\underline{\mathbb{R}^2}] \quad (TS^2 \text{ is stably trivial}) \end{aligned}$$

Bott periodicity: $\widetilde{KU}(\Sigma^2 X) \cong \widetilde{KU}(X)$, $\widetilde{KO}(\Sigma^8 X) \cong \widetilde{KO}(X)$

$\widetilde{K}(X) := \text{coker}(K(\text{pt}) \rightarrow K(X))$

in fact, $K(X)$ is a *commutative ring*: $[E] \cdot [F] = [E \otimes F]$,
 $[E_0, E_1] \cdot [F_0, F_1] = [(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (F_0 \otimes E_1)]$

the ***Chern character***: a ring homomorphism

$$KU(X) \xrightarrow{\text{ch}} H^*(X; \mathbb{Q}) .$$

Thm.

$$\begin{array}{ccc}
 KU(X) & \xrightarrow{\quad\quad\quad} & H^{\text{even}}(X; \mathbb{Q}) \\
 & \searrow & \nearrow \cong \\
 & & KU(X) \otimes \mathbb{Q}
 \end{array}$$

idea: $H^{\text{even}}(X; \mathbb{Q})$ is an approximation to $KU(X)$ (loses torsion)

chromatic homotopy theory: over \mathbb{Q} , have $\widehat{G}_a \cong \widehat{G}_m$ (via exp/log)

for $X = M$ a smooth manifold, can get Chern character
 $KU(M) \rightarrow H_{dR}^*(M)$ via ***Chern–Weil theory***:

given $E \downarrow M$, choose a *connection* ∇ : for $v \in T_p M$ and section s ,
 $\nabla_v(s) \in E_p \approx$ “derivative of s in the v direction”

get *curvature*, an $\text{End}(E)$ -valued 2-form: for $v, w \in T_p M$,
 $F^\nabla(v, w) = \nabla_{\tilde{v}}\nabla_{\tilde{w}} - \nabla_{\tilde{w}}\nabla_{\tilde{v}} - \nabla_{[\tilde{v}, \tilde{w}]}$ \tilde{v}, \tilde{w} any extensions of v, w

$\rightsquigarrow F^\nabla \approx$ “**monodromy around infinitesimal parallelograms**”

$$\begin{array}{ccc}
 \text{VBdl}_{\mathbb{C}}^\nabla(M) & \xrightarrow{\text{tr}(e^{iF/2\pi})} & \Omega_{dR}^*(M) \\
 \downarrow & \searrow & \uparrow \\
 \text{VBdl}_{\mathbb{C}}(M) & & Z_{dR}^*(M) \\
 \downarrow & \searrow & \downarrow \\
 KU(M) & \xrightarrow{\text{ch}} & H_{dR}^*(M)
 \end{array}$$

§1.2: TRACES IN ALGEBRAIC GEOMETRY

R a comm. ring $\rightsquigarrow \text{Spec}(R)$ an *affine scheme*

$\text{VBdl}(\text{Spec}(R)) \simeq \text{Proj}_R^{\text{f.g.}} \approx \text{Serre-Swan theorem}$

R an assoc. ring \rightsquigarrow “ $\text{VBdl}(\text{Spec}(R))$ ” := $\text{Proj}_R^{\text{f.g.}}$

more generally, X a *nonaffine* scheme (stack, derived stack, ...):
glued together from affines

to define *algebraic K-theory* of X , have a choice: not all exact
seq's split $\mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in $\text{Mod}_{\mathbb{Z}}$: does this enforce $[\mathbb{Z}/4] = [\mathbb{Z}/2] + [\mathbb{Z}/2]$ in $K(\mathbb{Z})$?

make easier choice: $E_0 \twoheadrightarrow E_1 \twoheadrightarrow E_2 \rightsquigarrow [E_1] = [E_0] + [E_2]$


enforce relations *derivedly*: record relations, relations between
relations, ... $\rightsquigarrow K(X)$ a *spectrum* \approx chain complex

for *any* category \mathcal{C} with “exact sequences”, get spectrum $K(\mathcal{C})$

$K(X) := K(\text{VBdl}(X)) \xrightarrow{\sim} K(\text{Perf}(X))$

the target of the trace from algebraic K-theory...

Def. (\mathcal{V}, \boxtimes) a monoidal ∞ -category, \mathcal{C} a \mathcal{V} -enriched ∞ -category, the **Hochschild homology** of \mathcal{C} is its **factorization homology** over the circle:

$$\mathrm{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} \simeq \operatorname{colim} \left(\underline{\operatorname{hom}}_{\mathcal{C}}(X_0, X_1) \boxtimes \cdots \boxtimes \underline{\operatorname{hom}}_{\mathcal{C}}(X_n, X_0) \right).$$


really **hocolim** (or ∞ -categorical colim); $\mathrm{HH}(\mathcal{C}) \in \mathcal{D}(\mathcal{V})$ (or just $\in \mathcal{V}$)

$\mathcal{V} = \mathrm{Sp} = \text{spectra} \rightsquigarrow$ *topological* Hochschild homology (THH)

have free/forget adjunction $\mathrm{Sp} \rightleftarrows \mathcal{D}(R)$, just like $\mathrm{Set} \rightleftarrows \mathrm{Mod}_R$. (in fact, $\mathrm{Sp} = \mathcal{D}(\mathrm{Set})!$)

§2.3

$$\begin{array}{ccc} \mathrm{Mfld}_n \times \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{V}) & \xrightarrow{(M, A) \mapsto \int_M A} & \mathcal{V} \\ \downarrow & \nearrow (M, \mathcal{C}) \mapsto \int_M \mathcal{C} & \\ \mathrm{Mfld}_n \times \mathrm{Cat}_n(\mathcal{V}) & & \end{array}$$

simplicial	paracyclic	cyclic	epicyclic
Δ^{op}	$\Delta^{\mathrm{op}}_{\circlearrowleft}$	Λ^{op}	$\tilde{\Lambda}^{\mathrm{op}}$
S^1_*	S^1	S^1 & auto's	S^1 & endo's

Thm (Hochschild–Kostant–Rosenberg '65). k a field, X a smooth variety or scheme over k , then

$$H_*(\mathrm{HH}(X)) \cong \Omega_{dR}^*(X) .$$

even better, this is *equivariant*:

$$\mathbb{T} \curvearrowright \mathrm{HH}(X) \quad \rightsquigarrow \quad H_*(\mathbb{T}) \curvearrowright H_*(\mathrm{HH}(X))$$

$$H_*(\mathbb{T}) \cong k[\varepsilon]/\varepsilon^2, \quad |\varepsilon| = 1; \quad \text{and} \quad \varepsilon \leftrightarrow d_{dR} !$$

so, define **negative cyclic homology**:

$$\mathrm{HC}^-(X) := \mathrm{HH}(X)^{h\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{HC}^-(X)) \approx H_{dR}^*(X)$$

if X is a *spectral* scheme (e.g. $\mathrm{Spec}(R)$ for R a ring spectrum), define **topological negative cyclic homology**:

$$\mathrm{THC}^-(X) := \mathrm{THH}(X)^{h\mathbb{T}} \quad \rightsquigarrow \quad H_*(\mathrm{THC}^-(X)) \approx H_{dR}^*(X)$$

*** **from now on, everything is spectral** ***

any variety or scheme can be considered as spectral, through the forgetful functor $\mathrm{Mod}_{\mathbb{Z}} \rightarrow \mathcal{D}(\mathbb{Z}) \rightarrow \mathrm{Sp}$

enter DAG...

Def. for a(n ordinary or derived) scheme X , its *free loop space* is the derived mapping stack

$$\mathcal{L}X = \text{map}(S^1, X) .$$

$$\begin{array}{ccc}
 S^1 & \longleftarrow & \text{pt} \\
 \uparrow & & \uparrow \\
 \text{pt} & \longleftarrow & S^0
 \end{array}
 \xrightarrow{\text{map}(-, X)}
 \begin{array}{ccc}
 \mathcal{L}X & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & X \times X
 \end{array}$$

pushout pullback

$$\rightsquigarrow \mathcal{O}_{\mathcal{L}X} \simeq \mathcal{O}_X \underset{\mathcal{O}_{X \times X}}{\overset{\mathbb{L}}{\otimes}} \mathcal{O}_X$$

$\rightsquigarrow \mathcal{O}(\mathcal{L}X)$ is computed by the cyclic bar construction for THH!

$\text{THH}(X) \simeq \mathcal{O}(\mathcal{L}X)$, T-action \leftrightarrow rotation of loops

traces approximate *rationalized* **rationalized** algebraic K-theory...

$$\begin{array}{ccc}
 K(X) & \xrightarrow{\text{Dennis trace}} & THH(X) \simeq \mathcal{O}(\mathcal{L}X) \\
 & \searrow \text{cyclic trace} & \nearrow \\
 & & THC^-(X) \simeq \mathcal{O}(\mathcal{L}X)^{h\mathbb{T}} \\
 E & \xrightarrow{\quad\quad\quad} & \left(\left(\begin{array}{c} \text{free loop} \\ S^1 \xrightarrow{\gamma} X \end{array} \right) \mapsto \left(\begin{array}{c} \text{trace of monodromy of} \\ \gamma^* E \downarrow S^1 \end{array} \right) \right)
 \end{array}$$

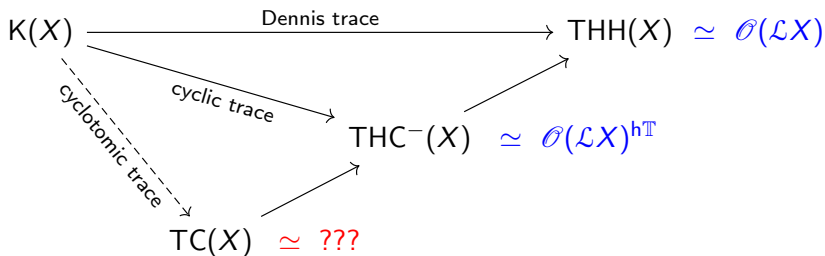
this function is \mathbb{T} -invariant!

Thm (Goodwillie '86). the cyclic trace is a *local* \mathbb{Q} -equivalence:
for $R \rightarrow R_0$ a nilpotent extension of connective ring spectra,

$$\begin{array}{ccc}
 K(R) & \longrightarrow & K(R_0) \\
 \downarrow & & \downarrow \\
 THC^-(R) & \longrightarrow & THC^-(R_0)
 \end{array}
 \quad \text{pullback after } - \otimes \mathbb{Q} .$$

slogan: $v\text{bdI}/\text{Spec}(R) \overset{\mathbb{Q}}{\approx} \text{restriction to } \text{Spec}(R_0)$
 + compatible trace-of-monodromy function
 + *data* of \mathbb{T} -invariance of this function

the integral story...



construction of the cyclotomic trace: Bökstedt–Hsiang–Madsen '92

Thm (Dundas–McCarthy '97). the cyclotomic trace is a *local equivalence* (without rationalization).

"This is how people other than Quillen compute algebraic K-theory."

~ A. Blumberg, algebraic K-theorist

Main Question: In terms of DAG, what is $TC(X)$?

§1.3: THE GEOMETRY OF THE CYCLOTOMIC TRACE

$$K(X) \xrightarrow{\text{cyclotomic trace}} TC(X) := THH(X)^{hCyc}$$

THH is a *cyclotomic spectrum*; TC is the *homotopy invariants* of its cyclotomic structure

$$\begin{array}{ccc}
 Sp & \xrightarrow{\text{triv}} & Cyc(Sp) \\
 \leftarrow \text{---} \perp \text{---} \leftarrow & & \\
 & (-)^{hCyc} & \\
 \Psi & & \Psi
 \end{array}
 \quad \boxed{\S 2.2}$$

$$TC(X) \longleftarrow THH(X)$$

right adjoint (limit-type construction): *imposing conditions*

recall: $THH(X) =$ functions on $\mathcal{L}X$

main idea: $TC(X) =$ functions on $\mathcal{L}X$ that are:

- invariant under \mathbb{T} -action on $\mathcal{L}X$;
- “sensitive” to relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$. **Q. What does “sensitive” mean?**

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X \dots$

Q. for M an $n \times n$ matrix, difference between $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \in M_{n \times n}(R)$

$$\text{tr}(M)^{\otimes 2} = \sum_{i,j} m_i \cdot \otimes m_j \quad , \quad \text{tr}(M^{\otimes 2}) = \sum_k m_k \cdot \otimes m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is **norms**: image of $\sum_{i < j} [m_i \otimes m_j]$ under

$$\begin{array}{ccc} (R \otimes R)_{C_2} & \xrightarrow{\text{Nm}} & (R \otimes R)^{C_2} \\ [x \otimes y] & \longmapsto & \sum_{\sigma \in C_2} \sigma(x \otimes y) \end{array}$$

\rightsquigarrow become equal in the **Tate construction**, the cofiber

$$(R \otimes R)_{C_2} \xrightarrow{\text{Nm}} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{tC_2}$$

over \mathbb{Q} , norm an iso! $\rightsquigarrow (R \otimes R)^{tC_2} = 0$, assertion is vacuous

relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X \dots$

Q. for M an $n \times n$ matrix, difference between $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 2: $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in M_{2 \times 2}(R)$, r arbitrary

now, difference between

$$\text{tr}(M)^{\otimes r} = (m_1 + m_2)^{\otimes r} \quad , \quad \text{tr}(M^{\otimes r}) = (m_1)^{\otimes r} + (m_2)^{\otimes r}$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

key fact: these are never coprime to r

\rightsquigarrow quotient $(R^{\otimes r})^{C_r}$ by norms from *all* proper subgroups of C_r

\rightsquigarrow $\text{tr}(M^r) \equiv \text{tr}(M)^r$ in *generalized Tate construction* $(R^{\otimes r})^{\tau C_r}$

main point: for \mathcal{C} a spectrally enriched ∞ -category,

$S_b^1 \xleftarrow{r} S_a^1$ a covering map of framed circles,

get *cyclotomic structure map*

$$\text{THH}(\mathcal{C}) := \int_{S_b^1} \mathcal{C} \longrightarrow \left(\int_{S_a^1} \mathcal{C} \right)^{\tau C_r} =: \text{THH}(\mathcal{C})^{\tau C_r}$$

doesn't exist over \mathbb{Z} ! only have "Tate diagonal" in Sp , not $\mathcal{D}(\mathbb{Z})$.

Thm (A-M-G-R).

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \lim^{r.\mathrm{lax}} \left(\mathrm{Sp}^{\mathrm{h}\mathbb{T}} \begin{array}{c} \xrightarrow{\mathrm{lax}} \\ \xrightarrow{\tau} \mathrm{Fun}(\mathbb{B}\mathbb{T}, \mathrm{Sp}) \end{array} \right)$$

★ an object of $\lim^{r.\mathrm{lax}}$ is given by $T \in \mathrm{Sp}^{\mathrm{h}\mathbb{T}}$ equipped with:

- for each $r \in \mathbb{N}^\times$, a cyclotomic structure map $T \xrightarrow{\sigma_r} T^{\tau C_r}$;
- for each $r, s \in \mathbb{N}^\times$, the *data* of a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ \sigma_{rs} \downarrow & & \downarrow (\sigma_s)^{\tau C_r} \\ T^{\tau C_{rs}} & \xrightarrow{\mathrm{can.}} & (T^{\tau C_s})^{\tau C_r} \end{array}$$

- for each $r_1, \dots, r_n \in \mathbb{N}^\times$, the *data* of a commutative n -cube...

§2.1 stratification of a scheme/stack $Y \rightsquigarrow$ *generalized recollement*: $\mathrm{QC}(Y) \simeq \lim^{r.\mathrm{lax}}(\dots \mathrm{lax} \dots)$.

§2.2 conjecture: there's a stratified derived stack $\mathcal{B}\mathbb{T}$ with \mathbb{N}^\times -action such that

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \mathrm{QC}(\mathcal{B}\mathbb{T})^{\mathrm{h}\mathbb{N}^\times},$$

and $\mathcal{L}\mathcal{X}$ enhances to \mathbb{N}^\times -equivariant $\mathcal{L}\mathcal{X} \downarrow \mathcal{B}\mathbb{T}$ such that $\mathrm{THH}(X) \leftrightarrow \mathcal{O}_{\mathcal{L}\mathcal{X}}$.

the geometry of the cyclotomic trace:

$$\begin{array}{ccc}
 K(X) & \xrightarrow{\text{Dennis trace}} & \mathrm{THH}(X) \simeq \mathcal{O}(\mathcal{L}X) \\
 & \searrow^{\text{cyclic trace}} & \uparrow \\
 & & \mathrm{THC}^-(X) \simeq \mathcal{O}(\mathcal{L}X)^{\mathrm{h}\mathbb{T}} \\
 & \swarrow_{\text{cyclotomic trace}} & \uparrow \\
 & & \mathrm{TC}(X) \simeq \mathcal{O}(\mathcal{L}X)^{\mathrm{h}\mathrm{Cyc}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathrm{Sp} & \xrightarrow{\text{triv}} & \mathrm{Cyc}(\mathrm{Sp}) \\
 \downarrow \cup & \dashleftarrow^{\perp} & \downarrow \cup \\
 & & (-)^{\mathrm{h}\mathrm{Cyc}} \\
 \mathrm{TC}(X) & \longleftarrow & \mathrm{THH}(X)
 \end{array}$$

$$\mathrm{Cyc}(\mathrm{Sp}) \simeq \left\{ \left(\mathbb{T} \curvearrowright T, \left\{ \begin{array}{c} T \\ \downarrow \sigma_r \\ T^{\tau C_r} \end{array} \right\}_{r \in \mathbb{N}^\times}, \left\{ \begin{array}{ccc} T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\ \sigma_{rs} \downarrow & & \downarrow (\sigma_s)^{\tau C_r} \\ T^{\tau C_{rs}} & \xrightarrow{\text{can.}} & (T^{\tau C_s})^{\tau C_r} \end{array} \right\}_{r,s \in \mathbb{N}^\times} \right\}$$

$\mathrm{TC}(X)$ is built from $\mathrm{THH}(X) = \mathcal{O}(\mathcal{L}X)$ by selecting just those functions on $\mathcal{L}X$ such that:

- values are \mathbb{T} -invariant;
- value on $S^1 \xrightarrow{\gamma} X$ determines value on $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$ “to the greatest extent possible” (+ higher coherences)...

...which is precisely the structure present on trace-of-monodromy functions of vector bundles! $\mathrm{tr}(M)^r \equiv \mathrm{tr}(M^r), \dots$

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§2.1 stratified schemes and generalized recollements (interlude)

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§2.3 the construction: factorization homology of enriched ∞ -categories

§2.1: STRATIFIED SCHEMES AND GEN^{IZED} RECOLLEMENTS

illustrate general theorem through examples.

truly need stable ∞ -categories (not triangulated categories).

should feel a lot like *categorified group theory* (extension sequences, etc.).

Ex. 1

$$Z \begin{array}{c} \xleftarrow{i} \\ \searrow \\ \xrightarrow{\text{closed}} \end{array} X \begin{array}{c} \xleftarrow{j} \\ \swarrow \\ \xrightarrow{\text{open}} \end{array} U$$

X_Z^\wedge

$$\mathbb{A}^0 \begin{array}{c} \xrightarrow{\quad} \\ \parallel \\ \xrightarrow{\quad} \end{array} \mathbb{A}^1 \begin{array}{c} \xleftarrow{\quad} \\ \parallel \\ \xrightarrow{\quad} \end{array} \mathbb{G}_m$$

$$\text{Spec}(k \xleftarrow{0 \leftarrow x} k[x] \xrightarrow{\quad} k[x^\pm])$$

Def. qcoh over X *supported on* Z : $\text{QC}_Z(X) := \ker(\text{QC}(X) \xrightarrow{j^*} \text{QC}(U))$.

$$\begin{array}{ccc} \text{QC}_Z(X) & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \text{QC}(X) \begin{array}{c} \xrightarrow{j^*} \\ \perp \\ \xleftarrow{j_*} \end{array} \text{QC}(U) \\ \downarrow \wr & \text{ker}(\text{id} \rightarrow j_* j^*) & \downarrow \wr \\ \text{QC}(X_Z^\wedge) & \begin{array}{c} \xrightarrow{\hat{i}_*} \\ \perp \\ \xleftarrow{\hat{i}^*} \end{array} & \text{QC}(X) \end{array}$$

$$\begin{array}{ccc} \text{Mod}_{k[x]}^{(x)\text{-tors}} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \text{Mod}_{k[x]} \begin{array}{c} \xrightarrow{- \otimes_{k[x]} k[x^\pm]} \\ \perp \\ \xleftarrow{\text{fgt}} \end{array} \text{Mod}_{k[x^\pm]} \\ \downarrow \wr & \text{M}_{k[x]}^{(x)} \xleftarrow{\quad} \text{M} & \downarrow \wr \\ \text{Mod}_{k[[x]]}^{(x)\text{-compl}} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\text{fgt}} \end{array} & \text{Mod}_{k[x]} \end{array}$$

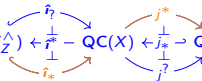
$\text{hom}_{k[x]}(k[x^\pm], -)$

$\text{Mod}_{k[x]}^{(x)\text{-tors}} \subset \text{Mod}_{k[x]}$ the full stable cocomplete subcategory generated by $\{k[x]/x^n\}_{n \geq 1}$

$\text{Mod}_{k[[x]]}^{(x)\text{-compl}} \simeq \lim(\cdots \rightarrow \text{Mod}_{k[x]/x^2} \rightarrow \text{Mod}_{k[x]/x} \rightarrow \cdots) \rightsquigarrow \{- \otimes_{k[x]} k[x]/x^n\}_{n \geq 1} \dashv$ "lim"

add pix here!!!

so, get a *recollement* of $\mathrm{QC}(X)$: $\mathrm{QC}(X_Z^\Delta) \leftarrow \hat{i}_* \dashv \mathrm{QC}(X) \leftarrow j_* \dashv \mathrm{QC}(U)$



Microcosm thm for recollements: for any $F \in \mathrm{QC}(X)$, writing $F_0 := \hat{i}^* F$ and $F_1 := j^* F$,

$$\begin{array}{ccc} F & \xrightarrow{\eta_j(F)} & j_* F_1 \\ \text{pullback } \eta_{\hat{i}}(F) \downarrow & & \downarrow j_* j^*(\eta_{\hat{i}}(F)) = j_*(\mu) \\ \hat{i}_* F_0 & \xrightarrow{\eta_j(\hat{i}_* F_0)} & j_* j^* \hat{i}_* F_0 \end{array}$$

...and this pullback square is *unique*, in the sense of the

Macrocsm thm for recollements:

$\mu = \text{monodromy} = \text{"gluing data"}$

$$\mathrm{QC}(X) \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\mathrm{QC}(X_Z^\Delta) \xrightarrow{j^* \hat{i}_*} \mathrm{QC}(U) \right) := \left\{ \left(F_0 \in \mathrm{QC}(X_Z^\Delta), F_1 \in \mathrm{QC}(U), \begin{array}{c} F_1 \\ \mu \downarrow \\ j^* \hat{i}_* F_0 \end{array} \right) \right\}.$$

given $\mathcal{J} \xrightarrow{\mathcal{C}^\bullet} \mathrm{Cat}$, an obj of $\begin{cases} \text{strict} \\ \text{left-lax} \\ \text{right-lax} \end{cases}$ limit is: $\{x_i \in \mathcal{C}_i\}_{i \in \mathcal{J}} + \forall i \xrightarrow{\varphi} j \text{ in } \mathcal{J}, \begin{cases} \mathcal{C}_\varphi(x_i) \simeq x_j \\ \mathcal{C}_\varphi(x_i) \rightarrow x_j \\ \mathcal{C}_\varphi(x_i) \leftarrow x_j \end{cases}$
+ higher coherence data

Ex. 2

$$Z \xrightarrow{\text{closed}} Y \xrightarrow{\text{closed}} X \rightsquigarrow [2] \xrightarrow{\text{lax}} \text{Cat}$$

Thm (A-M-G-R).

$$\text{QC}(X) \xrightarrow{\sim} \lim^{\text{r.lax}} \left(\begin{array}{ccc} & \text{QC}((X \setminus Z)_{(Y \wedge Z)}) & \\ \alpha \nearrow & & \searrow \beta \\ \text{QC}(X_Z^\wedge) & \xrightarrow{\gamma} & \text{QC}(X \setminus Y) \end{array} \right) \cdot$$

$\uparrow \eta$

$$\begin{array}{c} F_1 \\ \downarrow \mu_{01} \\ \alpha F_0 \end{array} \quad F_0 \quad \begin{array}{ccc} F_2 & \xrightarrow{\mu_{12}} & \beta F_1 \\ \mu_{02} \downarrow & \circlearrowleft \mu_{012} & \downarrow \beta(\mu_{01}) \\ \gamma F_0 & \xrightarrow{\eta(F_0)} & \beta \alpha F_0 \end{array}$$

add pix here!!!

$$\text{simpler: } \lim^{\text{l.lax}} \left(\begin{array}{ccc} & c_1 & \\ \alpha \nearrow & & \searrow \beta \\ c_0 & \xrightarrow{\gamma} & c_2 \end{array} \right) \simeq \left\{ \left(\begin{array}{ccc} & x_1 & \\ \uparrow \mu_{01} & & \\ \alpha x_0 & & \end{array} \right) \quad \left(\begin{array}{ccc} & & \\ & x_2 & \xrightarrow{\mu_{12}} \beta x_1 \\ \gamma x_0 & \xrightarrow{\eta(x_0)} & \beta \alpha x_0 \end{array} \right) \right\}$$

because now $\exists! \mu_{02}$

* above, need μ_{012} because laxness of diagram \neq laxness of limit

the general picture:

fix any scheme X and any set of closed subschemes including X .

order by inclusion \rightsquigarrow determines a poset P

\rightsquigarrow a *stratification* of X over P : a map of posets

$$P \xrightarrow{X_\bullet} \{\text{closed subschemes of } X\}.$$

notation: $X_{<p} := \bigcup_{q < p} X_q$.

assumption: P satisfies *descending chain condition*.

Thm (A-M-G-R). $\text{QC}(X) \xrightarrow{\sim} \lim^{r.lax} \left(\begin{array}{ccc} P & \xrightarrow{\text{I.lax}} & \text{Cat} \\ p & \longmapsto & \text{QC}((X \setminus X_{<p})_{(X_p \hat{\setminus} X_{<p})}) \end{array} \right).$

§2.2: THE GEOMETRY OF TC

recall: $\mathrm{THH} \in \mathrm{Cyc}(\mathcal{S}p)$ a *cyclotomic spectrum*, $\mathrm{TC} := \mathrm{THH}^{\mathrm{hCyc}}$.

cyclotomic spectra defined in terms of

cyclonic spectra := genuine-proper \mathbb{T} -spectra := $\mathcal{S}p^{\mathfrak{g}^{<\mathbb{T}}}$.

remember "genuine" fixedpoints for proper subgroups of \mathbb{T} . *genuine*: e.g. $EG \xrightarrow{\sim} \mathrm{pt}$.

$$\text{Thm (A-M-G-R). } \mathcal{S}p^{\mathfrak{g}^{<\mathbb{T}}} \xrightarrow{\sim} \lim^{\mathrm{r.lax}} \left(\begin{array}{ccc} \mathbb{N}^{\mathrm{div}} & \xrightarrow{\mathrm{l.lax}} & \mathrm{Cat} \\ \cup & & \cup \\ i & & \mathcal{S}p^{\mathrm{h}(\mathbb{T}/C_i)} \xleftarrow{\sim} \mathcal{S}p^{\mathrm{h}\mathbb{T}} \\ j \downarrow & \mapsto & \downarrow (-)^{\tau C_j} \\ ij & & \mathcal{S}p^{\mathrm{h}(\mathbb{T}/C_{ij})} \xrightarrow{\sim} \mathcal{S}p^{\mathrm{h}\mathbb{T}} \end{array} \right).$$

$\mathbb{N}^{\mathrm{div}} \cong$ poset of proper subgroups of \mathbb{T}

$$i \leftrightarrow C_i$$

Larger thm. $\mathcal{S}p^{\mathfrak{g}^G}$ is stratified over poset of closed subgroups of G up to subconjugacy.

Conj. there's an $\mathbb{N}^{\mathrm{div}}$ -stratified stack $\mathcal{B}\mathbb{T}$, a "mapping telescope" of copies of $B\mathbb{T}$, such that $\mathcal{S}p^{\mathfrak{g}^{<\mathbb{T}}} \simeq \mathrm{QC}(\mathcal{B}\mathbb{T})$ compatibly with the above generalized recollement.

the geometry of TC: recall: $\mathrm{THH}(X) \simeq \mathcal{O}(\mathcal{L}X) = \mathcal{O}(\mathrm{map}(S^1, X))$

$$\begin{array}{ccc} \mathrm{Cyc}(\mathrm{Sp}) \simeq \mathrm{QC}(\mathcal{B}\mathbb{T})^{\mathrm{hN}^\times} & & \\ \downarrow (-)^{\mathrm{hCyc}} & \swarrow \downarrow \Gamma^{\mathrm{hN}^\times} & \\ \mathrm{Sp} & & \end{array}$$

the \mathbb{T} -action on THH: using universal circle bundle $E\mathbb{T} \downarrow B\mathbb{T}$,

$$\begin{array}{ccc} \mathrm{map}_{/B\mathbb{T}}^{\mathrm{rel}}(E\mathbb{T}, X) & & E\mathbb{T} \longleftarrow E\mathbb{T} \times_{B\mathbb{T}} K \dashrightarrow X \\ \downarrow & \nearrow \text{---} & \downarrow \\ B\mathbb{T} \longleftarrow K & & B\mathbb{T} \longleftarrow K \end{array}$$

$\rightsquigarrow \mathcal{O}_{\mathrm{map}_{/B\mathbb{T}}^{\mathrm{rel}}(E\mathbb{T}, X)} \in \mathrm{QC}(B\mathbb{T})$ exhibits \mathbb{T} -action on fiber = $\mathcal{O}_{\mathrm{map}(S^1, X)} = \mathcal{O}_{\mathcal{L}X}$

the cyclotomic structure on THH: using universal circle bundle $\mathcal{E}\mathbb{T} \downarrow \mathcal{B}\mathbb{T}$, get

$$\begin{array}{ccc} \mathcal{L}X := \mathrm{map}_{/\mathcal{B}\mathbb{T}}^{\mathrm{rel}}(\mathcal{E}\mathbb{T}, X) \xrightarrow{\mathrm{N}^\times\text{-eq}} \mathcal{B}\mathbb{T} & \rightsquigarrow & \begin{array}{ccc} \mathrm{Cyc}(\mathrm{Sp}) \simeq \mathrm{QC}(\mathcal{B}\mathbb{T})^{\mathrm{hN}^\times} & & \\ \downarrow \Psi & & \downarrow \Psi \\ \mathrm{THH}(X) \longleftrightarrow \mathcal{O}_{\mathcal{L}X} & & \end{array} \end{array}$$

the $\mathbb{N}^{\mathrm{div}}$ -stratified stack $\mathcal{L}X \downarrow \mathcal{B}X$ records the symmetries of $\mathcal{L}X$ coming from rotations and covering maps of circles, and $\mathrm{TC}(X)$ is its \mathbb{T} - and \mathbb{N}^\times -equivariant global functions.

recall §1.3: $\mathrm{TC}(X)$ = functions on $\mathcal{L}X$ that are \mathbb{T} -invariant and "sensitive" to covering maps of circles.

§2.3

the *Witt monoid* := $W := \mathbb{T} \rtimes \mathbb{N}^\times$ (captures "Frobenius" and "Verschiebung" on THH)

§2.3: factorization homology of enriched ∞ -categories

Q1: where does the cyclotomic structure on THH come from?

Q2: where does the cyclotomic trace $K \rightarrow TC$ come from?

mechanism 1: factorization homology....

recall: $\mathrm{THH}(\mathcal{C}) := \int_{S^1}^{\mathcal{S}\mathbf{P}} \mathcal{C} :=$ spectrally-enriched factorization homology of \mathcal{C} over S^1 .

cobordism hypothesis (Baez–Dolan, Lurie, Ayala–Francis): factorization homology \leftrightarrow TQFTs.

local-to-global philosophy (Beilinson–Drinfeld, Costello–Gwilliam, ...): for $\mathcal{U} = \{U \rightarrow M\}$ a covering, $\int_M \mathcal{C} \in \mathcal{D}(\mathrm{Vect})$ glued from $\{\int_U \mathcal{C}\}_{\mathcal{U}}$ and $Z(M) \in \int_M \mathcal{C}$ glued from $\{Z(U) \in \int_U \mathcal{C}\}_{\mathcal{U}}$.

mechanism 2: linearization....

differential calculus: $(M, p) \rightsquigarrow T_p M \quad \rightsquigarrow \quad$ Goodwillie calculus: $(\mathcal{S}, X) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{P}}(X)$
 $(\mathcal{S}, \mathrm{pt}) \rightsquigarrow \mathrm{Shv}_{\mathcal{S}\mathbf{P}}(\mathrm{pt}) \simeq \mathcal{S}\mathbf{P}$

...as applied to *fiber bundles* among compact framed manifolds:

A1: covering maps $S^1 \xrightarrow{r} S^1$ (for $r \in \mathbb{N}^\times$).

A2: the fiber bundle $S^1 \rightarrow \mathbb{D}^0$, and its invariance:

$$\begin{array}{ccc} S^1 & \xrightarrow{r} & S^1 \\ & \searrow & \swarrow \\ & \mathbb{D}^0 & \end{array} .$$

Q1: where does the cyclotomic structure on THH come from?

A1: factorization homology + linearization + covering maps $S^1 \downarrow S^1$.

Thm (A-M-G-R).

①

diagonal package for spaces \rightsquigarrow

$$\begin{array}{ccc} \text{Cat}(\mathcal{S}) & \dashrightarrow & \text{Cyc}^h(\mathcal{S}) \\ & \searrow \text{THH}_{\mathcal{S}} & \downarrow \text{fgt} \\ & & \mathcal{S} \end{array}$$

packages *diagonal maps* for spaces: $X \rightarrow (X^{\times r})^{hC_r}$

$\text{Cyc}^h(\mathcal{S}) := \text{Fun}(\text{BW}, \mathcal{S}) :=$ "unstable cyclotomic spaces"

$\text{BW} :=$ (objects: framed S^1 's , morphisms: covering maps)

②

diagonal package for spaces

linearization \rightsquigarrow

Tate package for spectra \rightsquigarrow

$$\begin{array}{ccc} \text{Cat}(\text{Sp}) & \dashrightarrow & \text{Cyc}(\text{Sp}) \\ & \searrow \text{THH} & \downarrow \text{fgt} \\ & & \text{Sp} \end{array}$$

packages *Tate diagonal maps* for spectra: $E \rightarrow (E^{\otimes r})^{\tau C_r}$

$\text{Cyc}(\text{Sp}) :=$ cyclotomic spectra

Q2: where does the cyclotomic trace $K \rightarrow TC$ come from?

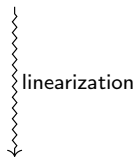
A2: factorization homology + linearization + the fiber bundle $S^1 \downarrow \mathbb{D}^0$.

Thm (A-M-G-R).

the *unstable cyclotomic trace*: for \mathcal{C} a \mathcal{S} -enriched ∞ -category,

$$\max^{\ell} \text{ subgpd of } \mathcal{C} := \iota \mathcal{C} \simeq \int_{\mathbb{D}^0} \mathcal{C} \longrightarrow \left(\int_{S^1} \mathcal{C} \right)^{h\mathbb{W}} =: \mathrm{THH}_{\mathcal{S}}(\mathcal{C})^{h\mathbb{W}} =: \mathrm{TC}_{\mathcal{S}}^h(\mathcal{C})$$

input: the fiber bundle $S^1 \downarrow \mathbb{D}^0$ is invariant for the \mathbb{W} -action on S^1



the *cyclotomic trace*: for \mathcal{C} a stable ∞ -category,

$$K(\mathcal{C}) \longrightarrow \mathrm{THH}(\mathcal{C})^{h\mathrm{Cyc}} =: \mathrm{TC}(\mathcal{C})$$

input: K-theory is the *universal additive invariant* of stable ∞ -categories (Blumberg–Gepner–Tabuada)

enriched factorization homology:

given:

- $M \in \mathbf{Mfld}_1$ a 1-manifold
- $\mathcal{C} \in \mathbf{Cat}(\mathcal{V})$ a \mathcal{V} -enriched ∞ -category

want to define: $\int_M \mathcal{C} \in \mathcal{V}$. first define both of these terms...

idea: this will be a colimit in \mathcal{V}

- over *disk-refinements* R of M
- of compatible labelings λ
 - of the vertices $R^{(0)}$ by objects of \mathcal{C} and
 - of the edges $R^{(1)}$ by morphisms in \mathcal{C} .

$$\int_M \mathcal{C} := \operatorname{colim}_{(R \rightarrow M) \in \mathcal{D}(M)} \left(\operatorname{colim}_{R^{(0)} \xrightarrow{\lambda} \mathcal{C}} \left(\bigotimes_{e \in R^{(1)}} \underline{\operatorname{hom}}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right).$$

slogan: enriched factorization homology via *categorified* factorization homology.

$$M \in \mathcal{M}, c \in \text{Cat}(\mathcal{V}) \rightsquigarrow \int_M c \in \mathcal{V}$$

Defs (Ayala–Francis–Rozenblyum). the ∞ -category $\mathcal{M} := \text{Mfld}_1$ of *compact variframed 1-manifolds* has

- obj: $\coprod_{\text{finite}} (\text{finite directed graphs}) \amalg (\text{framed } S^1\text{'s})$,
- mor: TBD (to be described).

given $M \in \mathcal{M}$, its category $\mathcal{D}(M)$ of *disk-refinements* has:

- obj: $\text{config}^{\text{ns}}$ of dots in 1-dim^ℓ strata of M s.t. $\not\exists$ smooth S^1 's;
- mor: disappearances, anticollisions, and isotopies.

add pix here!!! $\mathcal{D}(S^1) = \Delta_{\circ}^{\text{op}}$, full diagram

$$\begin{array}{ccccccc}
 \Delta_{\circ}^{\text{op}} & \longrightarrow & \Lambda^{\text{op}} & \longleftarrow & \tilde{\Lambda}^{\text{op}} & \xleftarrow{\text{ff}} & \mathcal{D} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \{S^1\} & \longrightarrow & BT & \longleftarrow & BW & \xleftarrow{\text{ff}} & \mathcal{M} \\
 & & \downarrow & & \downarrow & & \\
 & & \text{pt} & \longleftarrow & \text{BN}^{\times} & &
 \end{array}$$

$$M \in \mathcal{M}, c \in \text{Cat}(\mathcal{V}) \rightsquigarrow \int_M c \in \mathcal{V}$$

$$(\mathcal{V}, \boxtimes) \in \text{Alg}(\text{Cat}) \quad \longleftrightarrow \quad \text{bar construction } \mathfrak{BV} : \Delta^{\text{op}} \longrightarrow \text{Cat}$$

$$[n] \longmapsto \mathcal{V}^{\times n}$$

\mathfrak{BV} is a **category object** (a.k.a. *Segal object*) in Cat :

for all $n \geq 2$, $\mathfrak{BV}_n \xrightarrow{\sim} \mathfrak{BV}_1 \times_{\mathfrak{BV}_0} \cdots \times_{\mathfrak{BV}_0} \mathfrak{BV}_1$.

- ∞ -category of objects: $\mathfrak{BV}_0 = \mathcal{V}^{\times 0} \simeq \text{pt}$;
- ∞ -category of morphisms: $\underline{\text{hom}}_{\mathfrak{BV}}(\text{pt}, \text{pt}) \simeq \mathcal{V}$;
- composition: $\mathcal{V} \times \mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$.

another category object in Cat (in fact in \mathcal{S}): for any space X , the **codiscrete** category object $\text{cd}(X) : \Delta^{\text{op}} \longrightarrow \mathcal{S}$.

$$[n] \longmapsto X^{\times(n+1)}$$

- space of objects: $\text{cd}(X)_0 = X^{\times 1} = X$;
- space of morphisms: $\underline{\text{hom}}_{\text{cd}(X)}(x, y) = \text{pt}$; hence “codiscrete”
- composition: $\text{pt} \times \text{pt} \xrightarrow{\sim} \text{pt}$.

$$M \in \mathcal{M}, \mathcal{C} \in \text{Cat}(\mathcal{V}) \rightsquigarrow \int_M \mathcal{C} \in \mathcal{V}$$

Def (Gepner–Haugseeng). a \mathcal{V} -enriched ∞ -category

$\mathcal{C} \in \text{Cat}(\mathcal{V})$ is specified by:

- its space of objects $\iota\mathcal{C}$;
- a *right-lax functor* of category objects

$$\text{cd}(\iota\mathcal{C}) \xrightarrow{\text{hom}_{\mathcal{C}}} \mathfrak{B}\mathcal{V} \quad \Delta^{\text{op}}$$

$$\begin{array}{ccc}
 \iota\mathcal{C} & \xrightarrow{\quad ! \quad} & \text{pt} & [0] \\
 \downarrow x \mapsto (x,x) & \swarrow \lrcorner & \downarrow \mathbb{1}_{\mathcal{V}} & \downarrow \sigma_0 \\
 (\iota\mathcal{C})^{\times 2} & \xrightarrow{(x,y) \mapsto \text{hom}_{\mathcal{C}}(x,y)} & \mathcal{V} & [1] \\
 \uparrow (x,y,z) \mapsto (x,z) & \swarrow \lrcorner & \uparrow \boxtimes & \uparrow \delta_1 \\
 (\iota\mathcal{C})^{\times 3} & \xrightarrow{(x,y,z) \mapsto (\text{hom}_{\mathcal{C}}(x,y), \text{hom}_{\mathcal{C}}(y,z))} & \mathcal{V}^{\times 2} & [2]
 \end{array}$$

$$\mathbb{1}_{\mathcal{V}} \rightarrow \text{hom}_{\mathcal{C}}(x, x)$$

$$\text{hom}_{\mathcal{C}}(x, y) \boxtimes \text{hom}_{\mathcal{C}}(y, z) \rightarrow \text{hom}_{\mathcal{C}}(x, z)$$

$$M \in \mathcal{M}, \mathcal{C} \in \text{Cat}(\mathcal{V}) \rightsquigarrow \int_M \mathcal{C} \in \mathcal{V}$$

recall slogan: *enriched* factorization homology via *categorified* factorization homology.

the *Grothendieck construction*: for a category object \mathcal{Y} in Cat ,

$$\mathcal{M} \xrightarrow{\int_{(-)} \mathcal{Y}} \text{Cat} \quad \rightsquigarrow \quad \begin{array}{c} \int_{|\mathcal{M}} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \end{array} \quad \begin{array}{l} \text{"covariant bundle"} \\ \text{of } \infty\text{-categories"} \end{array}$$

$$\int_M \mathcal{C} := \text{colim} \left(\int_{|\mathcal{D}(M)} \text{cd}(\iota \mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \text{hom}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right)$$

in general, requires \mathcal{V} to be *symmetric monoidal*
(or e.g. only *cyclically monoidal* if $M = S^1$)

add pix here!!!

rigorizes heuristic formula
$$\int_M \mathcal{C} := \text{colim}_{(R \rightarrow M) \in \mathcal{D}(M)} \left(\text{colim}_{R(\mathbf{0}) \xrightarrow{\lambda} \iota \mathcal{C}} \left(\boxtimes_{e \in R(\mathbf{1})} \text{hom}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right)$$

$$\int_M \mathcal{C} := \operatorname{colim} \left(\int_{|\mathcal{D}(M)} \operatorname{cd}(\iota \mathcal{C}) \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\operatorname{hom}}_{\mathcal{C}}} \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right)$$

equivalently,

$$\begin{array}{ccccc}
 \int_{|\mathcal{D}(M)} \operatorname{cd}(\iota \mathcal{C}) & \xrightarrow{\int_{|\mathcal{D}(M)} \underline{\operatorname{hom}}_{\mathcal{C}}} & \int_{|\mathcal{D}(M)} \mathfrak{B}\mathcal{V} & \xrightarrow{\boxtimes} & \mathcal{V} \\
 \downarrow & & \nearrow \text{fiberwise colim} & & \uparrow \\
 \mathcal{D}(M) & & & & \\
 \downarrow & & \nearrow \text{(fiberwise) colim} & & \\
 \operatorname{pt} & & \int_M \mathcal{C} & &
 \end{array}$$

e.g. $\operatorname{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} \simeq \operatorname{colim}(\Delta_{\circlearrowleft}^{\operatorname{op}} \rightarrow \mathcal{V}) \simeq \operatorname{colim}(B^{\operatorname{cyc}} \mathcal{C} : \Delta^{\operatorname{op}} \rightarrow \Delta_{\circlearrowleft}^{\operatorname{op}} \rightarrow \mathcal{V})$

recall: $\Delta_{\circlearrowleft}^{\operatorname{op}} := \mathcal{D}(S^1)$, $\Delta^{\operatorname{op}} \simeq \mathcal{D}(S_*^1)$

cyclic bar construction (classical defⁿ of HH)

fact: $\Delta^{\operatorname{op}} := \mathcal{D}(S_*^1) \rightarrow \mathcal{D}(S^1) =: \Delta_{\circlearrowleft}^{\operatorname{op}}$ is final

$$\begin{array}{ccccc}
 \int_{|\Delta_{\circ}^{\text{op}}|} \text{cd}(\iota\mathcal{C}) & \xrightarrow{\int_{|\Delta_{\circ}^{\text{op}}|} \underline{\text{hom}}_{\mathcal{C}}} & \int_{|\Delta_{\circ}^{\text{op}}|} \mathfrak{B}\mathcal{V} & \xrightarrow{\boxtimes} & \mathcal{V} \\
 \downarrow & & \swarrow \text{fiberwise colim} & & \nearrow \\
 \Delta_{\circ}^{\text{op}} & & & & \\
 \downarrow & & \searrow \text{(fiberwise colim)} & & \\
 \text{pt} & & & & \text{HH}(\mathcal{C})
 \end{array}$$

build in functoriality for rotations of S^1 :

$$\begin{array}{ccccccc}
 \int_{|\Delta_{\circ}^{\text{op}}|} \text{cd}(\iota\mathcal{C}) & \longrightarrow & \int_{|\Lambda^{\text{op}}|} \text{cd}(\iota\mathcal{C}) & \xrightarrow{\int_{|\Lambda^{\text{op}}|} \underline{\text{hom}}_{\mathcal{C}}} & \int_{|\Lambda^{\text{op}}|} \mathfrak{B}\mathcal{V} & \xrightarrow{\boxtimes} & \mathcal{V} \\
 \downarrow & & \downarrow & & \swarrow \text{fiberwise colim} & & \nearrow \\
 \Delta_{\circ}^{\text{op}} & \longrightarrow & \Lambda^{\text{op}} & & & & \\
 \downarrow & & \downarrow & & \searrow \text{fiberwise colim} & & \\
 \{S^1\} & \longrightarrow & B\mathbb{T} & & & & \text{T} \curvearrowright \text{HH}(\mathcal{C})
 \end{array}$$

pullbacks

$$\begin{array}{ccc}
 \int_{|\Lambda^{\text{op}}|} \text{cd}(\iota\mathcal{C}) & \xrightarrow{\int_{|\Lambda^{\text{op}}|} \text{hom}_{\mathcal{C}}} & \int_{|\Lambda^{\text{op}}|} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \\
 \downarrow & \nearrow \text{fiberwise colim} & \\
 \Lambda^{\text{op}} & & \\
 \downarrow & \nearrow \text{fiberwise colim} & \\
 B\mathbb{T} & & \text{Tr} \sim \text{HH}(\mathcal{C})
 \end{array}$$

attempt to build in functoriality for self-coverings of S^1 :

$$\begin{array}{ccc}
 \int_{|\tilde{\Lambda}^{\text{op}}|} \text{cd}(\iota\mathcal{C}) & \xrightarrow{\int_{|\tilde{\Lambda}^{\text{op}}|} \text{hom}_{\mathcal{C}}} & \int_{|\tilde{\Lambda}^{\text{op}}|} \mathfrak{B}\mathcal{V} \xrightarrow{\begin{array}{c} \boxtimes \\ \mathbb{W} \end{array}} \mathcal{V} \\
 \downarrow & \nearrow \text{fiberwise colim} & \\
 \tilde{\Lambda}^{\text{op}} & & \\
 \downarrow & \nearrow \text{fiberwise colim} & \\
 B\mathbb{W} & & \mathbb{W} \curvearrowright \text{HH}^{\times}(\mathcal{C})
 \end{array}$$

only exists when $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal! notation: HH^{\times}

e.g. asks for maps $V \rightarrow V^{\boxtimes r}$, or more generally $\prod_i V_i \rightarrow \prod_i (V_i)^{\boxtimes r}$

add pix here!!!

this functor \prod is the *diagonal package*, and the \mathbb{W} -action on $\text{HH}^{\times}(\mathcal{C})$ is its *unstable cyclotomic structure*. $\text{Cyc}^h(\mathcal{V}) := \text{Fun}(B\mathbb{W}, \mathcal{V})$

$$\begin{array}{ccc}
 \int_{|\tilde{\Lambda}^{\text{op}}} \text{cd}(r\mathcal{C}) & \xrightarrow{f_{|\tilde{\Lambda}^{\text{op}}} \text{hom}_c} & \int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{\Pi} \mathcal{V} \\
 \downarrow & \searrow^{\text{fiberwise colim}} & \uparrow \\
 \tilde{\Lambda}^{\text{op}} & & \\
 \downarrow & \searrow^{\text{fiberwise colim}} & \\
 B\mathbb{W} & & W \circ \text{HH}^*(c)
 \end{array}$$

$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$

to understand the **diagonal package**, work “locally”: equivalently,

$$\begin{array}{ccc}
 \int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V} & \xrightarrow{\text{---}\Pi\text{---}} & \underline{\mathcal{V}} := \mathcal{V} \times B\mathbb{N}^{\times} \\
 \swarrow & & \downarrow \\
 \tilde{\Lambda}^{\text{op}} & & B\mathbb{N}^{\times} \\
 \downarrow & \xrightarrow{\hspace{10em}} & \\
 B\mathbb{W} & &
 \end{array}
 \quad \Bigg| \quad
 \begin{array}{c}
 \text{Fun}_{/B\mathbb{N}^{\times}}^{\text{rel}}(\int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}) \\
 \downarrow \text{---}\Pi\text{---} \\
 B\mathbb{N}^{\times}
 \end{array}$$

contravariant bundle of ∞ -categories: defines right \mathbb{N}^{\times} -action on fiber $\text{Fun}(\int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}})$, where $r \in \mathbb{N}^{\times}$ acts as $F \mapsto (F(\pi_r^*(-)))^{\text{hC}_r}$

$$\int_{|\Delta_{\circ}^{\text{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{\pi_r^*} \int_{|\Delta_{\circ}^{\text{op}}} \mathfrak{B}\mathcal{V} \text{ induced by } S^1 \xleftarrow{r} S^1 \quad \text{add pix here!!!}$$

action by *homotopy fixed points*, so denote by $\text{Fun}(\int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}) \curvearrowright_{\text{h}} \mathbb{N}^{\times}$;
 $(\Pi, \text{h}) := \Pi$ an object of its (in fact strict!) limit **strict because $\mathcal{V} \xrightarrow{\sim} (\mathcal{V}^{\times r})^{\text{hC}_r}$**

$$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) \quad \left| \quad \begin{array}{c} \int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{\text{---}\Pi\text{---}} \underline{\mathcal{V}} := \mathcal{V} \times \mathbb{B}\mathbb{N}^{\times} \\ \tilde{\Lambda}^{\text{op}} \swarrow \quad \downarrow \\ \mathbb{B}\mathbb{W} \xrightarrow{\quad\quad\quad} \mathbb{B}\mathbb{N}^{\times} \end{array} \right| \begin{array}{c} \text{Fun}^{\text{rel}}_{/\mathbb{B}\mathbb{N}^{\times}}(\int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}) \\ \Pi \downarrow \\ \mathbb{B}\mathbb{N}^{\times} \end{array} \quad \left. \vphantom{\int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V}} \right\} (\Pi, \mathfrak{h}) \in \lim \left(\text{Fun}(\int_{|\Lambda^{\text{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}) \curvearrowright_{\mathfrak{h}} \mathbb{N}^{\times} \right)$$

recall construction of unstable cyclotomic structure $\mathbb{W} \curvearrowright \mathbb{H}\mathbb{H}^{\times}$:

$$\begin{array}{ccccc}
 \int_{|\tilde{\Lambda}^{\text{op}}} \text{cd}(\iota\mathcal{C}) & \xrightarrow{\int_{|\tilde{\Lambda}^{\text{op}}} \text{hom}_{\mathcal{C}}} & \int_{|\tilde{\Lambda}^{\text{op}}} \mathfrak{B}\mathcal{V} & \xrightarrow{\text{---}\Pi\text{---}} & \underline{\mathcal{V}} \\
 \downarrow & & & & \downarrow \\
 \tilde{\Lambda}^{\text{op}} & & & & \mathbb{B}\mathbb{N}^{\times} \\
 \downarrow & \nearrow \text{fiberwise colim} & & & \\
 \mathbb{B}\mathbb{W} & \xrightarrow{\quad\quad\quad} & & & \mathbb{B}\mathbb{N}^{\times}
 \end{array}$$

$\mathbb{W} \curvearrowright \mathbb{H}\mathbb{H}^{\times}(\mathcal{C})$

rephrase in language of \mathbb{N}^{\times} -actions:

$$\begin{array}{ccc}
 (\text{Fun}(\int_{|\Lambda^{\text{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}) \curvearrowright_{\mathfrak{h}} \mathbb{N}^{\times}) & \xrightarrow{\text{hom}_{\mathcal{C}}^*} & (\text{Fun}(\int_{|\Lambda^{\text{op}}} \text{cd}(\iota\mathcal{C}), \mathcal{V}) \curvearrowright_{\mathfrak{h}} \mathbb{N}^{\times}) & \xleftarrow{\text{r.lax}} & (\text{Fun}(\mathbb{B}\mathbb{T}, \mathcal{V}) \curvearrowright_{\mathfrak{h}} \mathbb{N}^{\times}) \\
 \lim(\bullet) & & & & \lim^{\text{r.lax}}(\bullet) =: \text{Cyc}^{\mathfrak{h}}(\mathcal{V}) \\
 \psi & & & & \psi \\
 (\Pi, \mathfrak{h}) & \xrightarrow{\quad\quad\quad} & & & (\mathbb{W} \curvearrowright \mathbb{H}\mathbb{H}^{\times}(\mathcal{C}))
 \end{array}$$

$$(\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}S, S) \underset{h}{\frown} \mathbb{N}^{\times}) \xrightarrow{\mathrm{hom}_{\mathbb{C}}^*} (\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), S) \underset{h}{\frown} \mathbb{N}^{\times}) \xleftarrow{\mathrm{r.lax}_{\mathbb{C}}^*} (\mathrm{Fun}(BT, S) \underset{h}{\frown} \mathbb{N}^{\times})$$

$$(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times) = (\mathcal{S}, \times)$$

$$\lim(\bullet)$$

$$\lim^{\mathrm{r.lax}}(\bullet) =: \mathrm{Cyc}^h(S)$$

$$\Psi$$

$$\Psi$$

$$(\amalg, h) \longrightarrow (\mathbb{W} \curvearrowright \mathrm{HH}^{\times}(\mathcal{C}))$$

so, **diagonal package** for spaces induces **unstable cyclotomic structure**.

completely analogously, **Tate package** for spectra induces **cyclotomic structure**:

$$(\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathfrak{B}Sp, Sp) \underset{\tau}{\frown}^{\mathrm{l.lax}} \mathbb{N}^{\times}) \xrightarrow{\mathrm{hom}_{\mathbb{C}}^*} (\mathrm{Fun}(\int_{|\Lambda^{\mathrm{op}}} \mathrm{cd}(\iota\mathcal{C}), Sp) \underset{\tau}{\frown}^{\mathrm{l.lax}} \mathbb{N}^{\times}) \xleftarrow{\mathrm{r.lax}_{\mathbb{C}}^*} (\mathrm{Fun}(BT, Sp) \underset{\tau}{\frown}^{\mathrm{l.lax}} \mathbb{N}^{\times})$$

$$\lim^{\mathrm{r.lax}}(\bullet)$$

$$\lim^{\mathrm{r.lax}}(\bullet) =: \mathrm{Cyc}(Sp)$$

$$\Psi$$

$$\Psi$$

$$(\otimes, \tau) \longrightarrow (\mathrm{Cyc} \curvearrowright \mathrm{THH}(\mathcal{C}))$$

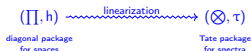
Q.: where does the Tate package come from?

recall **A**:

$$(\amalg, h) \xrightarrow{\text{linearization}} (\otimes, \tau)$$

diagonal package
for spaces

Tate package
for spectra



three ingredients:

(1) an action $\text{Fun}(-, \mathcal{S}p) \underset{\tau}{\overset{l.lax}{\curvearrowright}} \mathbb{N}^\times$

(2) a functor $(\text{Fun}(-, \mathcal{S}p) \underset{h}{\curvearrowright} \mathbb{N}^\times) \xrightarrow{r.lax} (\text{Fun}(-, \mathcal{S}p) \underset{\tau}{\overset{l.lax}{\curvearrowright}} \mathbb{N}^\times)$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}p$, a subcat $\text{Lin}(\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}, \mathcal{S}p) \subset \text{Fun}(\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}, \mathcal{S}p)$ of *fiberwise multilinear functors*



which is *stable* under action $\text{Fun}(\int_{|\Lambda^{op}} \mathfrak{B}\mathcal{V}, \mathcal{S}p) \underset{\tau}{\overset{l.lax}{\curvearrowright}} \mathbb{N}^\times$

* sweeping minor issues under the rug (\mathcal{S} vs. \mathcal{S}^{fin} , $\mathcal{S}p$ vs. $\mathcal{S}p^{fin}$)

$$\lim \left(\text{Fun}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S})_h \right) \xrightarrow{\text{linearization}} \lim^{r.\text{lax}} \left(\text{Fun}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}p, \mathcal{S}p)_\tau \right)$$

Ψ Ψ
 diagonal package Tate package
 for spaces for spectra

(1) an action $\text{Fun}(-, \mathcal{S}p) \xrightarrow{\text{Lax}} \mathbb{N}^\times$

(2) a functor $\left(\text{Fun}(-, \mathcal{S}p) \frown_h \mathbb{N}^\times \right) \xrightarrow{r.\text{lax}} \left(\text{Fun}(-, \mathcal{S}p) \xrightarrow{\text{Lax}} \mathbb{N}^\times \right)$

(3) for $\mathcal{V} = \mathcal{S}$ or $\mathcal{S}p$, a subcat $\text{Lin}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}p) \subset \text{Fun}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{V}, \mathcal{S}p)$ of **fiberwise multilinear functors** which is stable under $\xrightarrow{\text{Lax}} \mathbb{N}^\times$ sweeping minor issues under the rug (\mathcal{S} vs. \mathcal{S}^{fin} , $\mathcal{S}p$ vs. $\mathcal{S}p^{\text{fin}}$)

$$\begin{array}{c} \left(\text{Fun}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}) \frown_h \mathbb{N}^\times \right) \xrightarrow[r.\text{lax}]{\Sigma_+^\infty \circ -} \left(\text{Fun}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}p) \frown_h \mathbb{N}^\times \right) \xrightarrow[r.\text{lax}]{(2)} \left(\text{Fun}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}p) \xrightarrow{\text{Lax}} \mathbb{N}^\times \right) \xleftarrow{-\circ(\int_{\Lambda^{\text{op}}} \mathfrak{B}\Sigma_+^\infty)} \left(\text{Fun}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}p, \mathcal{S}p) \xrightarrow{\text{Lax}} \mathbb{N}^\times \right) \\ \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\ \left(\text{Lin}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}, \mathcal{S}p) \xrightarrow{\text{Lax}} \mathbb{N}^\times \right) \xleftarrow{\sim} \left(\text{Lin}(\int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}p, \mathcal{S}p) \xrightarrow{\text{Lax}} \mathbb{N}^\times \right) \\ \left(\Pi, h \right) \xrightarrow{\quad} \left(\otimes \circ \int_{\Lambda^{\text{op}}} \mathfrak{B}\Sigma_+^\infty, h \right) \xrightarrow{\quad} \left(\otimes \circ \int_{\Lambda^{\text{op}}} \mathfrak{B}\Sigma_+^\infty, \tau \right) \xrightarrow{\quad} \left(\otimes, \tau \right) \end{array}$$

$$\mathcal{S} \xrightarrow{\Sigma_+^\infty} \mathcal{S}p \text{ is symmetric monoidal} \iff \begin{array}{ccc} \int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S} & \xrightarrow{\Pi} & \mathcal{S} \\ \int_{\Lambda^{\text{op}}} \mathfrak{B}\Sigma_+^\infty \downarrow & \circlearrowleft & \downarrow \Sigma_+^\infty \\ \int_{\Lambda^{\text{op}}} \mathfrak{B}\mathcal{S}p & \xrightarrow{\quad} & \mathcal{S}p \\ & & \otimes \end{array}$$

universal property: $\mathcal{S}p$ is the **stabilization** of \mathcal{S} \iff $\text{Lin}(\mathcal{S}, \mathcal{S}p) \xleftarrow{\sim} \text{Lin}(\mathcal{S}p, \mathcal{S}p)$

summary

talk 1

§1.1 traces in differential geometry

the Chern character $KU \xrightarrow{\text{mdrmy}} H_{dR}^*$

§1.2 traces in algebraic geometry

$K(X) \xrightarrow{\text{mdrmy}} \text{THH}(X) = \mathcal{O}(\mathcal{L}X)$, $K \rightarrow \text{THC}^-$ a local \mathbb{Q} -equiv^{ce}, $K \rightarrow \text{TC}$ a local equiv^{ce}

§1.3 the geometry of the cyclotomic trace

$\text{TC}(X) = \mathbb{T}\text{-inv}^t$ functions on $\mathcal{L}X$ that are also "sensitive" to $\gamma \mapsto r^*\gamma$ (e.g. $\text{tr}(M^{\otimes r}) \equiv \text{tr}(M)^{\otimes r}$)

talk 2

§2.1 stratified schemes and generalized recollements

for X a stratified scheme, $\text{QC}(X) \xrightarrow{\sim} \lim^{r.\text{lax}}(\dots.l.\text{lax}\dots)$

§2.2 the geometry of TC

$\mathbb{N}^\times\text{-eq } \mathbb{N}^{\text{div-strat}} \mathcal{L}X \downarrow \mathcal{B}\mathbb{T}$ such that $\text{THH}(X) \leftrightarrow \mathcal{O}_{\mathcal{L}X}$ and $\text{TC}(X) = \text{W-eq fns on } \mathcal{L}X$

§2.3 factorization homology of enriched ∞ -categories

via fact hlgy and linearizⁿ: $S^1 \downarrow S^1 \rightsquigarrow \text{cyclo str on THH}$ and $S^1 \downarrow \mathbb{D}^0 \rightsquigarrow \text{cyclo trace}$