The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

1 A naive approach to genuine $G$-spectra and cyclotomic spectra (arXiv:1710.06416)
2 Factorization homology of enriched $\infty$-categories (arXiv:1710.06414)
3 The geometry of the cyclotomic trace (arXiv:1710.06409)
a scheme (derived) \( K(X) = \) algebraic K-theory of \( X \) \( \approx \) group-completion of \( \text{VBdl}(X) / \text{iso}., \oplus \) [hard to compute!]

\( \text{THH}(X) = \) topological Hochschild homology of \( X \) \( \approx \int S^1 \text{Perf} X \approx O(L_X) \), functions on the free loopspace of \( X \) [easier to compute]

the Dennis trace map \( K(X) \to \text{THH}(X) \)

\[
\begin{pmatrix}
S^1 \gamma \to X \to \text{trace of monodromy of } E\mid_\gamma
\end{pmatrix}
\]
$X$ a scheme (derived)
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$:= K(\text{Perf}_X)$
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\[\approx \int S^1 \text{Perf}_X \]

\[\approx O(L_X), \text{ functions on the free loopspace of } X\]

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[hard to compute!]

\[ \text{Dennis trace map} \]

\[ K(X) \stackrel{\text{Dennis}}{\longrightarrow} \text{THH}(X) \]

\[ (E \downarrow X) \mapsto \text{trace of monodromy of } E|_\gamma \]

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$$K(X) \longrightarrow \text{THH}(X)$$

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a refinement:

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\[ \simeq .....??!!?? \]
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\[ \approx \ldots ???!!?? \]

[computationally accessible, but conceptually mysterious]
why we care about $TC$

*Thm (Goodwillie/$\mathbb{Q}$ '86, McCarthy/$\mathbb{Z}$ '97, Dundas/$\mathbb{S}$ '97).*
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$$
\begin{array}{ccc}
K(\tilde{A}) & \to & K(A) \\
\downarrow & & \downarrow \\
\text{TC}(\tilde{A}) & \to & \text{TC}(A)
\end{array}
$$

is a pullback.

"This is how people other than Quillen compute algebraic K-theory."  
$\sim$ A. Blumberg, algebraic K-theorist
why we care about TC

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\[
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\downarrow & \downarrow \\
TC(\tilde{\mathbb{A}}) & \longrightarrow TC(\mathbb{A})
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“This is how people other than Quillen compute algebraic K-theory.”

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...but what is $\text{TC}(X)$, really?
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intermediate factorization through *negative cyclic homology*:

$$K(X) \rightarrow \text{TC}(X) \rightarrow \text{THC}^{-}(X) \rightarrow \text{THH}(X)$$
...but what is $\text{TC}(X)$, really?

Intermediate factorization through negative cyclic homology:

$$K(X) \to \text{TC}(X) \to \text{THC}^{-}(X) \to \text{THH}(X)$$

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...but what *is* $\text{TC}(X)$, really?

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Here, $\mathcal{K}(X)$ is the $K$-hypothesis of $X$, $\text{TC}(X)$ is the Tate construction of $X$, $\text{THC}^-(X)$ is the negative cyclic homology of $X$, and $\text{THH}(X)$ is the topological Hochschild homology of $X$. The notation $\Omega^*_\text{dR}(X)$ refers to the de Rham cohomology of $X$, and $\text{H}^*_\text{dR}(X)$ refers to the de Rham cohomology of the inertia space $\mathcal{L}X$. $\mathcal{L}X$ denotes the inertia stack of $X$. The $h\mathbb{T}$ notation indicates the $h$-completion of $\text{THH}(X)$ with respect to the Tate cohomology.
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original definition (Bökstedt–Hsiang–Madsen '93):

- Uses genuine-equivariant stable homotopy theory useful (e.g. equivariant Poincaré duality).
- Not conceptual (no DAG interpretation known).
- Used opaque point-set manipulations based on vague analogy with free loopspaces.

firmer categorical footing (Blumberg–Mandell '13):
- Define homotopy theory of “cyclotomic spectra”

more recent definition (Nikolaus–Scholze '17?):
- Removes genuine-equivariance.
- Restricts to connective ring spectra.

This talk, inspired by Nikolaus–Scholze:
- Applies to any spectrally-enriched $\infty$-category.
- Uses factorization homology to keep track of symmetries.
- Admits direct interpretation in DAG via $L_X$.
- Suggests higher-dim generalizations ($\Rightarrow$ “higher K-theory”).

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  - define homotopy theory of “cyclotomic spectra”

more recent definition (Nikolaus–Scholze ’17?):
  - removes genuine-equivariance
  - restricts to *connective* ring spectra

this talk, inspired by Nikolaus–Scholze:
  - applies to any spectrally-enriched $\infty$-category
  - uses *factorization homology* to keep track of symmetries
  - admits direct interpretation in DAG via $LX$
  - suggests higher-dim generalizations (⇝ “higher K-theory”)
overview
overview

\[ \text{Sp} \xleftarrow{\text{triv}} \text{Cyc}(\text{Sp}) \xrightarrow{\downarrow} \text{TC}(X) \xleftarrow{\cup} \text{THH}(X) \]
overview

\[
\begin{align*}
\text{Sp} & \xrightarrow{\text{triv}} \text{Cyc(Sp)} \\
\cup & \quad \cup \\
\text{TC}(X) & \leftarrow \text{THH}(X)
\end{align*}
\]

\[
\text{TC}(X) := \text{fixedpoints of cyclotomic structure on THH}(X)
\]

\[\rightsquigarrow \text{built by “imposing conditions” on functions on } \mathcal{L}X\]
overview

\[
\begin{align*}
\text{Sp} & \xrightarrow{\text{triv}} \text{Cyc(Sp)} \\
\cup & \xrightarrow{(-)^{h\text{Cyc}}} \\
\text{TC}(X) & \leftarrow \text{THH}(X)
\end{align*}
\]

\[\text{TC}(X) := \text{fixedpoints of cyclotomic structure on } \text{THH}(X)\]

\[\Rightarrow \text{built by “imposing conditions” on functions on } \mathcal{L}X\]

**main idea:** \(\text{TC}(X) \approx \text{functions on } \mathcal{L}X \text{ that are...}\)
overview

\[
\begin{array}{c}
\text{Sp} \quad \xrightarrow{\text{triv}} \quad \downarrow \quad \xrightarrow{(-)^{h\text{Cyc}}} \quad \text{Cyc(Sp)} \\
\cup \quad \quad \quad \quad \quad \quad \cup \\
\text{TC}(X) \quad \xleftarrow{\quad \quad \quad \quad \quad \quad} \quad \text{THH}(X)
\end{array}
\]

\(\text{TC}(X) := \) fixedpoints of \textit{cyclo\-tomic structure} on \(\text{THH}(X)\)

\(\leadsto \) built by “imposing conditions” on functions on \(\mathcal{L}X\)

main idea: \(\text{TC}(X) \approx \) functions on \(\mathcal{L}X\) that are...

- invariant under the \(\mathbb{T}\)-action on \(\mathcal{L}X\);
overview

\[ \text{Sp} \quad \xrightarrow{\text{triv}} \quad \text{Cyc(Sp)} \quad \sqcup \quad \xrightarrow{(-)^{h\text{Cyc}}} \quad \sqcup \]

\[ \text{TC}(X) \quad \sqcup \quad \text{THH}(X) \]

\[ \text{TC}(X) := \text{fixedpoints of cyclotomic structure on } \text{THH}(X) \]

\[ \leadsto \text{built by “imposing conditions” on functions on } \mathcal{L}X \]

**main idea:** \( \text{TC}(X) \approx \text{functions on } \mathcal{L}X \) that are...

- invariant under the \( \mathbb{T} \)-action on \( \mathcal{L}X \);
- “sensitive” to the relationship between \( S^1 \xrightarrow{\gamma} X \) and \( S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X \).
overview

\[
\begin{align*}
\text{Sp} \quad & \xrightarrow{\text{triv}} \quad \text{Cyc}(\text{Sp}) \\
\cup \quad & \quad (\_)^{h\text{Cyc}} \quad \cup \\
\text{TC}(X) \quad & \leftarrow \quad \text{THH}(X)
\end{align*}
\]

\(\text{TC}(X) := \text{fixedpoints of cyclotomic structure on THH}(X)\)

\(\leadsto \text{built by “imposing conditions” on functions on } \mathcal{L}X\)

**main idea:** \(\text{TC}(X) \approx \text{functions on } \mathcal{L}X \text{ that are...}\)

- invariant under the \(\mathbb{T}\)-action on \(\mathcal{L}X\);
- “sensitive” to the relationship between \(S^1 \xrightarrow{\gamma} X\) and \(S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X\).

\(\mathbb{T}\)-invariance is easy, but what does “sensitive” mean?
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{f} S^1 \xrightarrow{\gamma} X$
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \text{diag}(m_1, \ldots, m_n) \in M_{n \times n}(R)$
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \text{diag}(m_1, \ldots, m_n) \in M_{n \times n}(R)$

$$\text{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j,$$
relationship between $S^1 \to X$ and $S^1 \to S^1 \to X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \text{diag}(m_1, \ldots, m_n) \in M_{n \times n}(R)$

$$\text{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j , \quad \text{tr}(M^2) = \sum_k m_k \cdot m_k$$
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

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\[ \text{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j, \quad \text{tr}(M^2) = \sum_k m_k \cdot m_k \]

- both \textit{cyclically invariant}, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 1: $r = 2$, $M = \text{diag}(m_1, \ldots, m_n) \in M_{n \times n}(R)$

$$\text{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j \quad , \quad \text{tr}(M^2) = \sum_{k} m_k \cdot m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^C_2$
- difference is *norms*:
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

**Q.:** $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

**Ex. 1:** $r = 2$, $M = \text{diag}(m_1, \ldots, m_n) \in M_{n \times n}(R)$

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- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is *norms*: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$(R \otimes R)^{C_2} \xrightarrow{Nm} (R \otimes R)^{C_2}$$

$$[x \otimes y] \mapsto \sum_{\sigma \in C_2} \sigma(x \otimes y)$$
relationship between $S^1 \overset{\gamma}{\to} X$ and $S^1 \overset{r}{\to} S^1 \overset{\gamma}{\to} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

**Ex. 1:** $r = 2$, $M = \text{diag}(m_1, \ldots, m_n) \in M_{n \times n}(R)$

$$\text{tr}(M)^2 = \sum_{i,j} m_i \cdot m_j, \quad \text{tr}(M^2) = \sum_k m_k \cdot m_k$$

- both *cyclically invariant*, i.e. lie in the fixedpoints $(R \otimes R)^{C_2}$
- difference is *norms*: image of $\sum_{i<j}[m_i \otimes m_j]$ under

$$\begin{align*}
(R \otimes R)^{C_2} &\xrightarrow{Nm} (R \otimes R)^{C_2} \\
[x \otimes y] &\mapsto \sum_{\sigma \in C_2} \sigma(x \otimes y)
\end{align*}$$

$\sim$ become equal in the *Tate construction*, the cofiber

$$\begin{array}{c}
(R \otimes R)^{C_2} \xrightarrow{Nm} (R \otimes R)^{C_2} \longrightarrow (R \otimes R)^{tC_2}
\end{array}$$
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 2:
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 2: $M = \text{diag}(m_1, m_2)$, $r \in \mathbb{N}^\times$ arbitrary
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

Ex. 2: $M = \text{diag}(m_1, m_2)$, $r \in \mathbb{N}^\times$ arbitrary

now, difference between

$$\text{tr}(M)^r = (m_1 + m_2)^r, \quad \text{tr}(M^r) = ((m_1)^r + (m_2)^r)$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$
relationship between $S^1 \overset{\gamma}{\to} X$ and $S^1 \overset{r}{\to} S^1 \overset{\gamma}{\to} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

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now, difference between

$$\text{tr}(M)^r = (m_1 + m_2)^r, \ \text{tr}(M^r) = ((m_1)^r + (m_2)^r)$$

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fact: these are never coprime to $r$
relationship between $S^1 \xrightarrow{\gamma} X$ and $S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M^r)$?

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fact: these are never coprime to $r$

$\leadsto$ quotient $(R^{\otimes r})^{C_r}$ by norms from all proper subgroups of $C_r$
relationship between $S^1 \gamma X$ and $S^1 \gamma S^1 \gamma X$

Q.: $M$ an $n \times n$ matrix, compare $\text{tr}(M)^r$ and $\text{tr}(M'^r)$?

Ex. 2: $M = \text{diag}(m_1, m_2)$, $r \in \mathbb{N}^\times$ arbitrary

now, difference between

$$\text{tr}(M)^r = (m_1 + m_2)^r, \quad \text{tr}(M'^r) = ((m_1)^r + (m_2)^r)$$

governed by binomial coefficients $\binom{r}{i}$ for $0 < i < r$

fact: these are never coprime to $r$

$\hookrightarrow$ quotient $(R^\otimes r)^C_r$ by norms from all proper subgroups of $C_r$

$\hookrightarrow$ $\text{tr}(M'^r) \equiv \text{tr}(M)^r$ in the generalized Tate construction $(R^\otimes r)^{\tau C_r}$
relationship between \( S^1 \xrightarrow{\gamma} X \) and \( S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X \)

**Q.:** \( M \) an \( n \times n \) matrix, compare \( \text{tr}(M)^r \) and \( \text{tr}(M^r) \)?

**Ex. 2:** \( M = \text{diag}(m_1, m_2) \), \( r \in \mathbb{N}^\times \) arbitrary

now, difference between

\[
\text{tr}(M)^r = (m_1 + m_2)^r, \quad \text{tr}(M^r) = ((m_1)^r + (m_2)^r)
\]
governed by binomial coefficients \( \binom{r}{i} \) for \( 0 < i < r \)

fact: these are never coprime to \( r \)

\( \leadsto \) quotient \((R \otimes r)^{Cr}\) by norms from all proper subgroups of \( Cr \)

\( \leadsto \) \( \text{tr}(M^r) \equiv \text{tr}(M)^r \) in the **generalized** Tate construction \((R \otimes r)^{\tau Cr}\)

\( \star \) for \( C \) a spectrally enriched \( \infty \)-category, a covering map

\[
S^1_b \xleftarrow{r} S^1_a
\]
of oriented circles induces a **cyclotomic structure map**

\[
\text{THH}(C) := \int_{S^1_b} C \rightarrow \left( \int_{S^1_a} C \right)^{\tau Cr} =: \text{THH}(C)^{\tau Cr}
\]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\mathcal{S}) \cong \lim_{r \to \infty} \text{lax}(\mathcal{S}_{h T}) \]

\[ \mathcal{S}_{h T} := \text{Fun}(B_T, \mathcal{S}) \]

\[ \text{lax} \circ \tau \]

\[ \tau_{N \times r} \]

\[ \sigma_{r \to \tau C} \]

\[ \text{slogan: } \tau C(X) \text{ is built from } \text{THH}(X) \cong \mathcal{O}(L X) \text{ by selecting just those functions: } \]

\[ \text{that are } T \text{-invariant; } \]

\[ \text{whose values on } S^1 \gamma \to X \text{ determine their values on } S^1 r \to S^1 \gamma \to X \text{ "to the greatest extent possible", subject to all possible coherences between these determinations.} \]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \cong \text{Sp}^hT := \text{Fun}(B\mathbb{T}, \text{Sp}) \]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \text{Sp}^{hT} \]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \left( \text{Sp}^h_T \overset{\tau}{\otimes} \mathbb{N}^\times \right) \]
Theorem 1 (A & M-G & R)

$$\text{Cyc}(\text{Sp}) \simeq \left( \text{Sp}^h T \overset{\text{lax}}{\leftarrow} T \otimes \mathbb{N}^\times \right)$$
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim \left( \text{Sp}^{h_T} \overset{\text{lax}}{\underset{\tau}{\cap}} \mathbb{N}^\times \right) \]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim^{r.\text{lax}} \left( \text{Sp}^h_{\mathbb{T}} \underset{\tau}{\otimes} \mathbb{N}^\times \right) \]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim^{r.lax} \left( \text{Sp}^h \mathbb{T} \xleftarrow{lax} \mathbb{N}^X \right). \]

A slogan: $\text{TC}(X)$ is built from $\text{THH}(X) \approx \mathcal{O}(L_X)$ by selecting just those functions: that are $T$-invariant; whose values on $S^1 \gamma \to X$ determine their values on $S^1 r \to S^1 \gamma \to X$ "to the greatest extent possible", subject to all possible coherences between these determinations.
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \cong \lim^{r \cdot \text{lax}} \left( \text{Sp}^\text{hT} \, \text{lax} \, \tau \, \mathbb{N}^\times \right). \]

\( \star \) an object of \( \lim^{r \cdot \text{lax}} \) is given by \( T \in \text{Sp}^\text{hT} \) equipped with:

\( \star \cdot \) slogan: \( \text{TC}(X) \) is built from \( \text{THH}(X) \approx O(L_X) \) by selecting just those functions:

- that are \( T \)-invariant;
- whose values on \( S_1 \stackrel{\gamma}{\to} X \) determine their values on \( S_1 \stackrel{r}{\to} S_1 \stackrel{\gamma}{\to} X \) "to the greatest extent possible," subject to all possible coherences between these determinations.

Aaron Mazel-Gee
The geometry of the cyclotomic trace
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim^{r.\text{lax}} \left( \text{Sp}^{hT} \underset{\tau}{\wedge} N^\times \right). \]

★ an object of \( \lim^{r.\text{lax}} \) is given by \( T \in \text{Sp}^{hT} \) equipped with:

- for each \( r \in N^\times \),

[bullet points and continuation]
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim^{r.lax} \left( \text{Sp}^{h\mathbb{T}} \circlearrowleft_{\tau} \mathbb{N}^{\times} \right). \]

\[ \star \text{ an object of } \lim^{r.lax} \text{ is given by } T \in \text{Sp}^{h\mathbb{T}} \text{ equipped with:} \]

- for each \( r \in \mathbb{N}^{\times} \), a cyclotomic structure map \( T \xrightarrow{\sigma_r} T^{\tau C_r} \);
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \cong \lim^{r,lax} \left( \text{Sp}^h\mathbb{T} \vee_{\tau} \mathbb{N}^\times \right). \]

\* an object of \( \lim^{r,lax} \) is given by \( T \in \text{Sp}^h\mathbb{T} \) equipped with:
  - for each \( r \in \mathbb{N}^\times \), a cyclotomic structure map \( T \overset{\sigma_r}{\to} T\tau C_r \);
  - for each \( r, s \in \mathbb{N}^\times \),...
Theorem 1 (A & M-G & R)

\[
\text{Cyc}(\text{Sp}) \simeq \lim^{r, \text{lax}} \left( \text{Sp}^h \mathbb{T} \underset{\tau}{\vee} \mathbb{N}^\times \right).
\]

\(\star\) an object of \(\lim^{r, \text{lax}}\) is given by \(T \in \text{Sp}^h \mathbb{T}\) equipped with:

- for each \(r \in \mathbb{N}^\times\), a cyclotomic structure map \(T \xrightarrow{\sigma_r} T\tau C_r\);
- for each \(r, s \in \mathbb{N}^\times\), the data of a commutative square

\[
\begin{array}{ccc}
T & \xrightarrow{\sigma_r} & T\tau C_r \\
\downarrow{\sigma_{rs}} & & \downarrow{(\sigma_s)\tau C_r} \\
T\tau C_{rs} & \xrightarrow{\text{can.}} & (T\tau C_s)\tau C_r
\end{array}
\]

\(\star\) slogan: \(T\text{C})(X)\) is built from \(\text{THH}(X) \approx O(L_X)\) by selecting just those functions:

- that are \(T\)-invariant;
- whose values on \(S^1_\gamma \to X\) determine their values on \(S^1_r \to S^1_\gamma \to X\) "to the greatest extent possible", subject to all possible coherences between these determinations.
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim^{r.lax} \left( \text{Sp}^{h\mathbb{T}} \text{lax}_{\tau} \mathbb{N}^{\times} \right). \]

* an object of \( \lim^{r.lax} \) is given by \( T \in \text{Sp}^{h\mathbb{T}} \) equipped with:
  - for each \( r \in \mathbb{N}^{\times} \), a cyclotomic structure map \( T \xrightarrow{\sigma_r} T^{\tau C_r} \);
  - for each \( r, s \in \mathbb{N}^{\times} \), the data of a commutative square
    \[
    \begin{array}{ccc}
    T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\
    \downarrow{\sigma_{rs}} & & \downarrow{(\sigma_s)^{\tau C_r}} \\
    T^{\tau C_{rs}} & \xrightarrow{\text{can.}} & (T^{\tau C_s})^{\tau C_r}
    \end{array}
    \]
  - for each \( r_1, \ldots, r_n \in \mathbb{N}^{\times} \),
Theorem 1 (A & M-G & R)

$$\text{Cyc}(\text{Sp}) \simeq \lim_{r, \text{lax}} \left( \text{Sp}^{h\mathbb{T}} \underset{\tau}{\left(\bigwedge_{\mathbb{N}^\times}\right)} \right).$$

an object of \(\lim_{r, \text{lax}}\) is given by \(T \in \text{Sp}^{h\mathbb{T}}\) equipped with:

- for each \(r \in \mathbb{N}^\times\), a cyclotomic structure map \(T \xrightarrow{\sigma_r} T^\tau C_r\);
- for each \(r, s \in \mathbb{N}^\times\), the data of a commutative square

\[
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T & \xrightarrow{\sigma_r} & T^\tau C_r \\
\downarrow{\sigma_{rs}} & & \downarrow{(\sigma_s)^\tau C_r} \\
T^\tau C_{rs} & \xrightarrow{\text{can.}} & (T^\tau C_s)^\tau C_r
\end{array}
\]

- for each \(r_1, \ldots, r_n \in \mathbb{N}^\times\), the data of a commutative \(n\)-cube...
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \cong \lim^{r, \text{lax}} \left( \text{Sp}^{h \mathbb{T}} \Leftrightarrow \text{lax} \bigtriangleup_{\tau} \mathbb{N}^{\times} \right). \]

* an object of \( \lim^{r, \text{lax}} \) is given by \( T \in \text{Sp}^{h \mathbb{T}} \) equipped with:
  - for each \( r \in \mathbb{N}^{\times} \), a cyclotomic structure map \( T \xrightarrow{\sigma_r} T\tau C_r \); 
  - for each \( r, s \in \mathbb{N}^{\times} \), the data of a commutative square

\[
\begin{array}{ccc}
T & \xrightarrow{\sigma_r} & T\tau C_r \\
\downarrow{\sigma_{rs}} & & \downarrow{(\sigma_s)^{\tau C_r}} \\
T\tau C_{rs} & \xrightarrow{\sim\text{ can.}} & (T\tau C_s)^{\tau C_r}
\end{array}
\]

for each \( r_1, \ldots, r_n \in \mathbb{N}^{\times} \), the data of a commutative \( n \)-cube...

Thm. [Nikolaus–Scholze]
for \( T \) connective and \( r = s = p \) prime
Cor.: suff to specify just \( \sigma_p \)
(since \( \sigma_{pn} = (\sigma_p)^{on} \), and \( n \)-cubes canonically commute \( \forall \ n \geq 2 \))
Theorem 1 (A & M-G & R)

\[
\text{Cyc} (\text{Sp}) \simeq \lim_{r,\text{lax}} \left( \text{Sp}^h \underbrace{\vdash}_{\tau} \mathbb{N} \right).
\]

- An object of \( \lim_{r,\text{lax}} \) is given by \( T \in \text{Sp}^h \) equipped with:
  - For each \( r \in \mathbb{N}^\times \), a cyclotomic structure map \( T \xrightarrow{\sigma_r} T \tau \mathcal{C}_r \);
  - For each \( r, s \in \mathbb{N}^\times \), the data of a commutative square
    \[
    \begin{array}{ccc}
    T & \xrightarrow{\sigma_r} & T \tau \mathcal{C}_r \\
    \sigma_{rs} \downarrow & & \downarrow (\sigma_s) \tau \mathcal{C}_r \\
    T \tau \mathcal{C}_{rs} & \overset{\sim}{\xrightarrow{\text{can.}}} & (T \tau \mathcal{C}_s) \tau \mathcal{C}_r
    \end{array}
    \]
  - For each \( r_1, \ldots, r_n \in \mathbb{N}^\times \), the data of a commutative \( n \)-cube...

- Slogan: \( \text{TC}(X) \) is built from \( \text{THH}(X) \approx \mathcal{O}(LX) \) by selecting just those functions:

Thm. [Nikolaus–Scholze]
for \( T \) connective and \( r = s = p \) prime
Cor.: suff to specify just \( \sigma_p \)
(since \( \sigma_p^n = (\sigma_p)^\circ n \), and \( n \)-cubes canonically commute \( \forall n \geq 2 \))
Theorem 1 (A & M-G & R)

\[
\text{Cyc}(\text{Sp}) \simeq \lim^{r,\text{lax}} \left( \text{Sp}^{h\mathbb{T}} \overset{\tau}{\rightleftarrows} N^\times \right).
\]

★ an object of \( \lim^{r,\text{lax}} \) is given by \( T \in \text{Sp}^{h\mathbb{T}} \) equipped with:

- for each \( r \in N^\times \), a cyclotomic structure map \( T \xrightarrow{\sigma_r} T^{\tau C_r} \);
- for each \( r, s \in N^\times \), the data of a commutative square

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T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\
\downarrow{\sigma_{rs}} & & \downarrow{(\sigma_s)^{\tau C_r}} \\
T^{\tau C_{rs}} & \xrightarrow{\sim \text{can.}} & (T^{\tau C_s})^{\tau C_r}
\end{array}
\]

- for each \( r_1, \ldots, r_n \in N^\times \), the data of a commutative \( n \)-cube...

★ slogan: \( TC(X) \) is built from \( \text{THH}(X) \approx O(\mathcal{L}X) \) by selecting just those functions:

- that are \( \mathbb{T} \)-invariant;

Thm. [Nikolaus–Scholze]
for \( T \) connective and \( r = s = p \) prime

Cor.: suff to specify just \( \sigma_p \)

(since \( \sigma_{pn} = (\sigma_p)^n \), and \( n \)-cubes canonically commute \( \forall n \geq 2 \))
Theorem 1 (A & M-G & R)

\[ \text{Cyc}(\text{Sp}) \simeq \lim^{r.\text{lax}} \left( \text{Sp}^{h\mathbb{T}} \lax{\tau} \mathbb{N}^\times \right). \]

\[ \star \text{ an object of } \lim^{r.\text{lax}} \text{ is given by } T \in \text{Sp}^{h\mathbb{T}} \text{ equipped with:} \]

\begin{itemize}
  \item for each \( r \in \mathbb{N}^\times \), a cyclotomic structure map \( T \xrightarrow{\sigma_r} T\tau C_r \);
  \item for each \( r, s \in \mathbb{N}^\times \), the \textit{data} of a commutative square
\end{itemize}

\[ \begin{array}{ccc}
  T & \xrightarrow{\sigma_r} & T\tau C_r \\
  \sigma_{rs} \downarrow & & \downarrow (\sigma_s)\tau C_r \\
  T\tau C_{rs} & \xrightarrow{\sim \text{can.}} & (T\tau C_s)\tau C_r
\end{array} \]

\[ \star \text{ slogan: } TC(X) \text{ is built from } \text{THH}(X) \simeq \mathcal{O}(\mathcal{L}X) \text{ by selecting just those functions:} \]

\begin{itemize}
  \item that are \( \mathbb{T} \)-invariant;
  \item whose values on \( S^1 \xrightarrow{\gamma} X \) determine their values on \( S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X \) “to the greatest extent possible”,
\end{itemize}

Thm. [Nikolaus–Scholze]
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\[ \star \] an object of \( \lim_{r.\text{lax}} \) is given by \( T \in \text{Sp}^{h\mathbb{T}} \) equipped with:

- for each \( r \in \mathbb{N}^\times \), a cyclotomic structure map \( T \xrightarrow{\sigma_r} T^{\tau C_r} \);
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\[
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T & \xrightarrow{\sigma_r} & T^{\tau C_r} \\
\downarrow{\sigma_{rs}} & & \downarrow{(\sigma_s)^{\tau C_r}} \\
T^{\tau C_{rs}} & \xrightarrow{\sim_{\text{can.}}} & (T^{\tau C_s})^{\tau C_r}
\end{array}
\]

- for each \( r_1, \ldots, r_n \in \mathbb{N}^\times \), the data of a commutative \( n \)-cube...

\[ \star \text{slogan: } \text{TC}(X) \text{ is built from } \text{THH}(X) \cong \mathcal{O}(\mathcal{L}X) \text{ by selecting just those functions:} \]

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- whose values on \( S^1 \xrightarrow{\gamma} X \) determine their values on \( S^1 \overset{r}{\to} S^1 \xrightarrow{\gamma} X \) “to the greatest extent possible”, subject to all possible coherences between these determinations.

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main input (inspired by Glasman & many others)...

\[ \star \text{over } H \in P_G, \text{ functor is } \text{Sp}^G \Phi H \longrightarrow \text{Sp}^W(H). \]
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notation: \( G \) a compact Lie group, \( P_G \) its poset of closed subgroups under subconjugacy.

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\[ \star \text{ a generalized recollement over } P_G; \text{ classical is over poset } [1] \]

\[ \star \text{ hints at a DAG description of genuine } G \text{-spectra...} \]
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There's a canonical left-lax left \( P_G \)-module \( \text{Sp}^{gG} \), whose value on \( H \in P_G \) is \( \text{Sp}^{hW(H)} \), with

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Theorem 3 (A & M-G & R)
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1

*diagonal package* for spaces
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Theorem 3 (A & M-G & R)

1. \( \text{diagonal package for spaces} \rightarrow \text{Cat}(S) \rightarrow \text{Cyc}^h(S) \)

\( THH_S \rightarrow S \)
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\[ \text{diagonal package for spaces} \quad \xrightarrow{\sim} \quad \text{Cyc}^h(\mathcal{S}) \]

\[ \text{Cat}(\mathcal{S}) \xrightarrow{\sim} \text{Cyc}^h(\mathcal{S}) \]

\[ \text{THH}_S \xrightarrow{\text{fgt}} S \]

\[ \text{Cyc}^h(\mathcal{S}) := \text{Fun}(B\mathcal{W}, \mathcal{S}) := \text{“unstable cyclotomic spaces”} \]

\[ \mathcal{W} \cong T \times N^\times \text{ the “Witt monoid”} \]
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Theorem 3 (A & M-G & R)

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\[ \text{THH}_S \longrightarrow S \]

\[ \text{THH}_S \] is the “Witt monoid”

2. **diagonal package** for spaces

**linearization** (à la Goodwillie calculus)

**Tate package** for spectra
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Theorem 4 (A & M-G & R)
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Theorem 4 (A & M-G & R)

the unstable cyclotomic trace: for $\mathcal{C}$ a $\mathcal{S}$-enriched $\infty$-category,

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Theorem 4 (A & M-G & R)

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Linearization (à la Goodwillie calculus)

the **cyclotomic trace**: for $\mathcal{C}$ a stable $\infty$-category,

$$K(\mathcal{C}) \longrightarrow TC(\mathcal{C}) := \text{THH}(\mathcal{C})^{h\text{Cyc}}$$