The geometry of the cyclotomic trace

Aaron Mazel-Gee

with David Ayala and Nick Rozenblyum

1. *A naive approach to genuine G-spectra and cyclotomic spectra* (arXiv:1710.06416)
2. *Factorization homology of enriched ∞-categories* (arXiv:1710.06414)
3. *The geometry of the cyclotomic trace* (arXiv:1710.06409)

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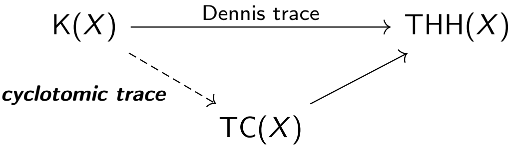
a refinement:



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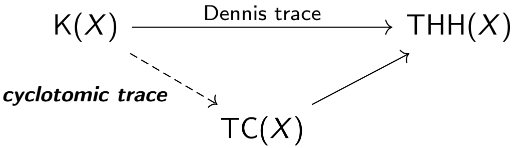

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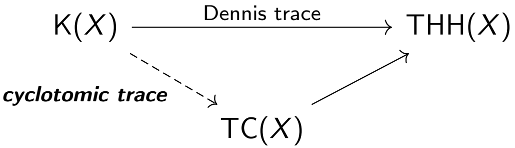

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[computationally accessible, but conceptually mysterious]

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Thm (Goodwillie/$\mathbb{Q}$ '86, McCarthy/$\mathbb{Z}$ '97, Dundas/$\mathbb{S}$ '97).

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"This is how people other than Quillen compute algebraic K-theory."
$\sim$ *A. Blumberg*, algebraic K-theorist

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- suggests higher-dim generalizations ($\rightsquigarrow$ "higher K-theory")

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$\mathbb{T}$-invariance is easy, but what does "sensitive" mean?

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$\rightsquigarrow$ become equal in the ***Tate construction***, the cofiber

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$\star$ for $\mathcal{C}$ a spectrally enriched ∞-category, a covering map

$$S^1_b \xleftarrow{r} S^1_a$$

of oriented circles induces a ***cyclotomic structure map***

$$\mathrm{THH}(\mathcal{C}) := \int_{S^1_b} \mathcal{C} \longrightarrow \left(\int_{S^1_a} \mathcal{C} \right)^{\tau C_r} =: \mathrm{THH}(\mathcal{C})^{\tau C_r}$$

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⋆ a ***generalized recollement*** over P_G; classical is over poset [1]

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⋆ a ***generalized recollement*** over P_G; classical is over poset [1]

⋆ hints at a DAG description of genuine G-spectra...

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Theorem 3 (A & M-G & R)

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$$\begin{array}{ccc} \mathrm{Cat}(\mathcal{S}) & \dashrightarrow & \mathrm{Cyc}^{\mathrm{h}}(\mathcal{S}) \\ & \searrow^{\mathrm{THH}_{\mathcal{S}}} & \downarrow \mathrm{fgt} \\ & & \mathcal{S} \end{array}$$

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Theorem 4 (A & M-G & R)

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the ***unstable cyclotomic trace***: for $\mathcal{C}$ a $\mathcal{S}$-enriched ∞-category,

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the ***cyclotomic trace***: for $\mathcal{C}$ a stable ∞-category,

$$\mathsf{K}(\mathcal{C}) \longrightarrow \mathsf{TC}(\mathcal{C}) := \mathsf{THH}(\mathcal{C})^{\mathsf{hCyc}}$$