$\pi_* L_{E(1)} S$ FOR $p \neq 2$

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We begin with the *fracture square*

$$\begin{array}{ccc} L_{E(1)}S & \xrightarrow{(\lambda^{E(1)}_{K(1)})_S} & L_{K(1)}S \\ {\scriptstyle (\lambda^{E(1)}_{E(0)})_S}\downarrow & & \downarrow{\scriptstyle (\eta_{E(0)})_{L_{K(1)}S}} \\ L_{E(0)}S & \xrightarrow[L_{E(0)}((\eta_{K(1)})_S)]{} & L_{E(0)}L_{K(1)}S, \end{array}$$

which is a homotopy pullback square.[1] Our goal is to compute the homotopy groups of everything in this picture.

First of all, $E(0) = H\mathbb{Q}$, and luckily arithmetic localization works the way we'd hope, so

$$\pi_i L_{E(0)} S = \begin{cases} \mathbb{Q}, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

We will use the fact that $E(0) = H\mathbb{Q}$ again later to compute $\pi_* L_{E(0)} L_{K(1)} S = \mathbb{Q} \otimes \pi_* L_{K(1)} S$.

Recall that E_n is the n^{th} *Lubin-Tate spectrum*, which has coefficient ring $\pi_* E_n \cong \mathbb{W}\mathbb{F}_{p^n}[\![u_1, \ldots, u_{n-1}]\!][u^{\pm}]$, where $|u_i| = 0$, $|u| = 2$, and $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\zeta]$ for ζ a primitive $(p^n - 1)^{st}$ root of unity.[2] This comes with a spectrum-level action of $\mathbb{G}_n$, the n^{th} *Morava stabilizer group*.[3] It is a fact that $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$, and it is a deep fact that $E_1^{h\mathbb{G}_1} \simeq L_{K(1)} S$. Moreover, given a group $G_1 \times G_2$ acting on a spectrum X, we have $X^{h(G_1 \times G_2)} = (X^{hG_1})^{hG_2}$; since

$$\mathbb{Z}_p^\times \cong \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}_2, & p = 2 \\ \mathbb{Z}/(p-1) \times \mathbb{Z}_p, & p \neq 2, \end{cases}$$

then to obtain $L_{K(1)} S$ we can first take homotopy fixed points of E_1 with respect to a cyclic group and then with respect to the p-adics.

This is a good idea for the following reason. Given a group G acting on a spectrum X, there is a *homotopy fixed point spectral sequence* running

$$H^{-s}(G, \pi_t X) \Rightarrow \pi_{s+t} X^{hG},$$

[1] For a spectrum F Bousfield guarantees a natural transformation $\eta_F : \mathrm{id} \to L_F$, and for spectra F_1 and F_2 with $\langle F_1 \rangle \geq \langle F_2 \rangle$, Bousfield guarantees a natural transformation $\lambda^{F_1}_{F_2} : L_{F_1} \to L_{F_2}$ which induces the equivalence $L_{F_2} L_{F_1} \simeq L_{F_2}$ of functors.

[2] There is a map $E(n) \to E_n$ which on homotopy induces $v_i \mapsto u_i u^{p^i - 1}$ for $1 \leq i \leq n-1$ and $v_n \mapsto u^{p^n - 1}$. This classifies the *Lubin-Tate formal group* over $(E_n)_*$, which is the universal deformation of the height-n *Honda formal group* over $\mathbb{F}_{p^n}$.

[3] This is the automorphism group of the Lubin-Tate formal group.

and in general, if M is a G-module and $|G| \cdot \mathrm{id}_M$ is an isomorphism, then $H^*(G,M) = H^0(G,M) = M^G$, the G-invariants of M. In our situation, the vanishing theorem above applies to the HFPSS for the first homotopy fixed point computation as long as we assume $p \neq 2$. We therefore make this assumption and continue along this route.

So, our HFPSS for $\pi_* E_1^{h\mathbb{Z}/(p-1)}$ has starting page

$$H^{-s}(\mathbb{Z}/(p-1), \pi_t E_1) = H^0(\mathbb{Z}/(p-1), \pi_t E_1) = (\pi_t E_1)^{\mathbb{Z}/(p-1)}.$$

This of course collapses for degree reasons, so we just need to compute these invariants. Now, there is an accidental equivalence $E_1 \simeq K_p^\wedge$, where $K_p^\wedge$ is p-completed complex K-theory, under which the action $\mathbb{G}_1 \to \mathrm{Aut}(E_1)$ extends the Adams operations $\mathbb{Z}\backslash\{0\} \to \mathrm{Aut}(K_p^\wedge)$. Explicitly, for $n \in \mathbb{Z}\backslash\{0\}$, the associated Adams operation ψ^n on $\pi_* K_p^\wedge$ is determined by $\psi^n(\beta^d) = n^d \cdot \beta^d$, where $\beta \in \pi_2 K_p^\wedge$ is (the image of) the Bott element. For $\alpha \in \mathbb{G}_1$, we will therefore write the associated action on $\pi_* E_1$ as ψ^α, which is determined by $\psi^\alpha(u^d) = \alpha^d \cdot u^d$. Our copy of $\mathbb{Z}/(p-1) \subset \mathbb{Z}_p^\times$ consists of the $(p-1)^{st}$ roots of unity, so it follows that $\pi_*(E_1^{h\mathbb{Z}/(p-1)}) = (\pi_* E_1)^{\mathbb{Z}/(p-1)} = \mathbb{Z}_p[u^{\pm(p-1)}]$.

Now, it turns out that when you take homotopy fixed points with respect to a continuous group action, you may as well be taking homotopy fixed points with respect to a dense subgroup.[4] So it suffices to choose a topological generator of $\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ and take homotopy fixed points with respect to the infinite cyclic subgroup that it generates. We'd like to use an element that's easy to work with, so let's verify that $1+p \in \mathbb{Z}_p^\times$ corresponds to a topological generator of $\mathbb{Z}_p$. This inclusion $\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ is as the subgroup of those p-adics beginning with 1, so we can take the inclusion $\mathbb{Z}_p \to \mathbb{Z}_p^\times$ to be given by $\alpha \mapsto \exp(p\alpha)$. This has a (partial) logarithm is defined by $\log(\alpha')/p \leftarrow\!\shortmid \alpha'$. So, $1+p \in \mathbb{Z}_p^\times$ corresponds to

$$A = \frac{\log(1+p)}{p} = \sum_{n=1}^{\infty} \frac{(-p)^{n-1}}{n} = 1 - \frac{p}{2} + \frac{p^2}{3} - \frac{p^3}{4} + \cdots .$$

This is indeed a topological generator: to approximate any given p-adic integer arbitrarily well, we can take an appropriate number of copies of A to get the 0^{th} digit, then add an appropriate number of copies of pA to correct the 1^{st} digit, then add an appropriate number of copies of p^2A to correct the 2^{nd} digit, continuing out as far as we like. In fact, it's not hard to see that in fact the additive topological generators are precisely the multiplicative units.

So, we obtain a fiber sequence[5]

$$E_1^{h\mathbb{G}_1} \to E_1^{h\mathbb{Z}/(p-1)} \xrightarrow{\psi^{1+p}-1} E_1^{h\mathbb{Z}/(p-1)} \to \Sigma E_1^{h\mathbb{G}_1} \to \cdots,$$

[4]This might be a slight lie, but it gives the right answer in this case at least.

[5]Recall that for any group G acting on a spectrum X, we can define the homotopy fixed points as $X^{hG} = F(\Sigma_+^\infty EG, X)^G$, the honest fixed points of the "freeified" function spectrum. If $G = \mathbb{Z}$, then we can take $EG = \mathbb{R}$. This admits a cellular filtration with $E\mathbb{Z}^{(0)} = \mathbb{Z}$ and $E\mathbb{Z}^{(1)} = \mathbb{R}$, which has $E\mathbb{Z}^{(1)}/E\mathbb{Z}^{(0)} = \Sigma\mathbb{Z}$. Upon applying $F(\Sigma_+^\infty -, X)^{\mathbb{Z}}$ to this cofiber sequence, we obtain the fiber sequence $\Sigma^{-1}X \to X^{h\mathbb{Z}} \to X$, which rotates to a fiber sequence $X^{h\mathbb{Z}} \to X \to X$ where the map $X \to X$ is the difference of the action of a generator of $\mathbb{Z}$ and the identity map. Setting $X = E_1^{h\mathbb{Z}/(p-1)}$ gives the fiber sequence that we use here.

and applying π_* gives us a long exact sequence. We therefore compute $(\psi^{1+p}-1)(u^{k(p-1)})$ (for $k \in \mathbb{Z}$) to determine $\pi_* E_1^{h\mathbb{G}_1}$.

- When $k = 0$, we have $(\psi^{1+p} - 1)(v^0) = 0$.
- When $k > 0$, we have

$$(\psi^{1+p} - 1)(u^{k(p-1)}) = ((1+p)^{k(p-1)} - 1)u^{k(p-1)} = \left(\sum_{i=1}^{k(p-1)} \binom{k(p-1)}{i} p^i \right) u^{k(p-1)}.$$

It turns out that up to a p-adic unit, this coefficient is just kp.[6]

- When $k < 0$, we know that $(1+p)^{-1} = \sum_{i=0}^{\infty} (-p)^i$, so we have

$$(\psi^{1+p} - 1)(u^{k(p-1)}) = ((1+p)^{k(p-1)} - 1)u^{k(p-1)} = \left(\left(\sum_{i=0}^{\infty} (-p)^i \right)^{(-k)(p-1)} - 1 \right) u^{k(p-1)}.$$

It turns out that up to a p-adic unit, this coefficient is also just kp.[7]

Since $\pi_* E_1^{h\mathbb{Z}/(p-1)}$ is even-concentrated, we can immediately deduce that

$$\pi_i L_{K(1)} S = \pi_i E_1^{h\mathbb{G}_1} = \begin{cases} \mathbb{Z}_p, & i = -1, 0 \\ \mathbb{Z}_p / kp, & i = 2k(p-1) - 1 \text{ for } k \in \mathbb{Z} \backslash \{0\} \\ 0, & \text{otherwise.} \end{cases}$$

This implies that

$$\pi_i L_{E(0)} L_{K(1)} S = \mathbb{Q} \otimes \pi_i L_{K(1)} S = \begin{cases} \mathbb{Q}_p, & i = -1, 0 \\ 0, & \text{otherwise.} \end{cases}$$

Finally, to compute $\pi_* L_{E(1)} S$, we have a Mayer-Vietoris long exact sequence in stable homotopy coming from the fracture square. Luckily this is quite easy because of the simplicity of $\pi_* L_{E(0)} S$ and $\pi_* L_{E(0)} L_{K(1)} S$ and because the maps on homotopy are the expected ones, and so we can read off that

$$\pi_i L_{E(1)} S = \begin{cases} \mathbb{Q}_p / \mathbb{Z}_p = \mathbb{Z}/p^\infty, & i = -2 \\ \mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}_{(p)}, & i = 0 \\ \mathbb{Z}_p / kp, & i = 2k(p-1) - 1 \text{ for } k \in \mathbb{Z} \backslash \{0\} \\ 0, & \text{otherwise.} \end{cases}$$

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[6] To see this, note that $1 + p \in \mathbb{Z}/p^n$ generates the second factor in $(\mathbb{Z}/p^n)^\times = \mathbb{Z}/(p-1) \times \mathbb{Z}/p^{n-1}$. So, $(1+p)^m \equiv 1 \pmod{p^n}$ iff $p^{n-1} | m$. Setting $m = k(p-1)$ gives that the p-adic valuation of our coefficient of $u^{k(p-1)}$ is $\nu_p((1+p)^{k(p-1)} - 1) = \nu_p(k(p-1)) + 1 = \nu_p(k) + 1$.

[7] Instead of using the previous method, another way to see this is just to replace our original choice of $1+p$ with $(1+p)^{-1}$ for the $k < 0$ calculation. (Note that this is also a topological generator.) This coefficient then becomes exactly the one we already saw above.