

FACTORIZATION HOMOLOGY AND LOW-DIMENSIONAL TOPOLOGY

PROJECT DESCRIPTION

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0. OVERVIEW AND BROADER IMPACTS

0.1. Context and main goal. The Jones polynomial [Jon85] is a fundamental invariant of knots and links in $\mathbb{R}^3$. The Jones polynomial was famously related to Chern–Simons theory by Witten in his seminal work [Wit89]. In particular, this extended the Jones polynomial to links in arbitrary 3-manifolds $L \subset Y$ at a physical level of rigor. This extension was soon made mathematically precise by Reshetikhin–Turaev [RT91], giving what is now called the **WRT invariant** (after Witten and Reshetikhin–Turaev).

From a more modern perspective [Wal, BJS], one may think of the WRT invariant as the 3d boundary theory of a 4d invertible TQFT known as *Crane–Yetter theory* [CY93, CKY97]. This latter theory assigns to a closed 3-manifold Y a 1-dimensional vector space, and to a link $L \subset Y$ it assigns a vector therein. Thereafter, the WRT invariant is recovered by trivializing this vector space, which gives a means of evaluating the given vector as a scalar. (For example, a 4-manifold X with $\partial X = Y$ provides such a trivialization, whose dependence on the choice of X is related to the *framing anomaly* of the theory [Wit89].)

In [Kho00], Khovanov categorified the Jones polynomial by introducing **Khovanov homology**, an invariant that assigns to each link $L \subset \mathbb{R}^3$ a graded chain complex $\text{Kh}(L) \in g\text{Ch}$ whose graded Euler characteristic recovers the Jones polynomial of L . This posed an immediate question:

What construction categorifies the WRT invariant?

Though fundamental to low-dimensional topology and extremely well-studied, the question of categorifying the WRT invariant remains open.

As above, this problem can be decomposed into two parts: (1) that of assigning to each closed 3-manifold Y a category and to each link $L \subset Y$ an object therein; and (2) that of constructing an evaluation functor to $g\text{Ch}$. Resolving part (1) is the main goal of the present proposal, work on which is already underway.

Problem A. *Construct a pointed braided $(\infty, 2)$ -category with duals enriched in graded chain complexes*

$$(V \in \text{Kh}) \in \text{Alg}_{\mathbb{E}_2}(\text{Cat}_2(g\text{Ch}))_*^{\text{dual}}$$

that categorifies the WRT invariant (and in particular recovers Kh) via factorization homology.

(I do not propose here to resolve part (2); this is related to the categorification of quantum groups at roots of unity, which is a highly active area of research (see e.g. [Kho16, KQ15, EQ16b, EQ16a, Qi14, EL20]).)

The term **factorization homology** appearing in the statement of Problem A refers to a categorified form of integration: whereas ordinary integration sums *numbers*, factorization homology sums *vector spaces* or *chain complexes*. This is a new and exciting homotopy-theoretic tool, which provides a fundamental link between higher category theory and manifold topology [AFT17a, AF15, AFR, AMGRa]. In the situation of Problem A, factorization homology would assign to each closed 3-manifold Y a *categorified skein module*, an $(\infty, 1)$ -category enriched in graded chain complexes

$$\text{Sk}_{\text{Kh}}(Y) := \int_Y \mathfrak{B}^2 \text{Kh} \in \text{Cat}(g\text{Ch}) ,$$

and to each link $L \subset Y$ it would assign an object

$$\text{Kh}(L \subset Y) := \left(\int_{L \subset Y} \mathfrak{B}^2(V \in \text{Kh}) \right) \in \text{Sk}_{\text{Kh}}(Y)$$

therein. (The distinguished object $V \in \text{Kh}$ may be thought of as “the fundamental 2-representation”, with which we label all components of L .)

In particular, a solution to Problem A would recover Khovanov homology via factorization homology: the skein module of $\mathbb{R}^3$ would be simply identified as $\text{Sk}_{\text{Kh}}(\mathbb{R}^3) \simeq g\text{Ch}$, and thereafter for any link $L \subset \mathbb{R}^3$ we would have an equivalence $\text{Kh}(L \subset \mathbb{R}^3) \simeq \text{Kh}(L)$. Even this consequence alone would be an exciting development in its own right. This is due to a number of attractive features of factorization homology, which may be summarized by the following analogy:

factorization homology : state-sum TQFT :: singular homology : simplicial homology .

In particular, factorization homology is manifestly well-defined (without reference to a triangulation, knot diagram, Heegaard splitting, etc.), and enjoys a continuous action of diffeomorphisms as well as homotopy-coherent functoriality for cobordisms. Hence, factorization homology would give a manifestly 3-dimensional construction of Khovanov homology: while highly sought, this has thus far remained elusive [Tur16, §9]. Indeed, all known constructions of $\mathrm{Kh}(L)$ require presenting the link L by a link diagram D : one defines a point-set chain complex $\mathrm{CKh}(D) \in \mathbf{Ch}$ and verifies that it changes by quasi-isomorphisms under Reidemeister moves, so that it gives a well-defined equivalence class in the underlying ∞ -category $\mathbf{Ch} := \mathbf{Ch}[\mathbf{W}_{\mathrm{q.i.}}^{-1}]$. By contrast, factorization homology proceeds in the underlying ∞ -category $\mathbf{Ch}$ from the outset.

As asserted above, the problem of categorifying the WRT invariant has generated a great deal of research at the interface of mathematics and physics. In particular, it has been studied extensively via both BPS states and gauge theory [GPV17, GPPV20, CGR17, GMb, NO16, GSV05, GIKV10, Wit12a, GW12, Wit12b, Wit18]. At least on a heuristic level, the approach via factorization homology (as proposed in Problem A) should be related to the gauge-theoretic approach by a form of Poincaré/Koszul duality [AF19]; indeed, the gauge-theoretic approach is itself motivated by S-duality [Wit12a].

0.2. Complementary goals. The other problems in this proposal all relate to Problem A, as follows.

First of all, in my joint work in progress on Problem A, we have found that it is in fact more natural from the point of view of topology to study a generalization thereof (Problem E).

An immediate goal following a resolution of Problem A is to compute the categorified skein modules $\mathrm{Sk}_{\mathrm{Kh}}(Y)$ and the objects $\mathrm{Kh}(L \subset Y) \in \mathrm{Sk}_{\mathrm{Kh}}(Y)$ therein (Problem F). A related goal is to investigate their functoriality and equivariance (Problem G), in connection with existing literature. I believe that this will rely on a comprehensive theory of *stratified ∞ -categories* that I have developed in recent joint work.

From a different direction, forthcoming joint work suggests that annular Khovanov homology may be thought of as a *cyclotomic link homology theory* (in the sense of THH, TC, and the cyclotomic trace from algebraic K-theory). This suggests a general form that a resolution of Problem G might take. Moreover, it comes packaged with a tantalizing conjecture of its own (Problem D), which augurs an exciting and productive connection between Khovanov homology and arithmetic geometry.

Finally, all of these investigations are in one way or another undergirded by factorization homology, in which area I propose to undertake some urgent fundamental work (Problems B-C).

0.3. Outline. I'll end this introductory section by discussing broader impacts in §0.4. The remaining sections of this proposal are organized as follows. In §1, I'll review the relevant background material and discuss my related work, in particular stating and discussing Problems B-D. The heart of this proposal is §2, where I'll discuss my joint work in progress on Problem A (via Problem E). In §3, I'll discuss computations as well as structural questions of equivariance and functoriality, in particular stating and discussing Problems F-G.

0.4. Broader impacts. Here I'll describe a number of past broader impacts I have made, as well as their continuations that the funding of this proposal would support.

0.4.1. Individual training of younger mathematicians. So far, I have supervised one research project, as well as five reading courses (with three students). The research project was conducted by Reuben Stern, an undergrad, who was funded in part by a generous internal grant from USC (a Zumberge award, \$30k). Together we proved a highly nontrivial result [MGS] that he has nearly finished writing up: a universal characterization of secondary algebraic K-theory, categorifying Blumberg–Gepner–Tabuada's work [BGT13]. Emboldened by our success, I plan to continue such mentorship: I have a long list of projects that are approachable by a motivated advanced undergrad. It is important to have ample funding for this endeavor, as it enables the participation of students whose financial needs require them to work over the summer.

0.4.2. General training of younger mathematicians. I have run a number of learning seminars (most recently [MGa]), especially surrounding two of my areas of particular expertise: factorization homology and higher category theory. I have also produced a number of expository works in these areas. My central goal is to bridge the gap between theory and practice, and to make these areas as approachable and appealing as possible to a wide audience. I plan to continue such activities. For instance, I am currently pursuing a joint project to generalize Gaitsgory–Lurie's proof [GL, §3] of nonabelian Poincaré duality in characteristic p to varieties of dimension greater than 1, which has led me to begin mapping out a corresponding graduate seminar that I would be very excited to deliver in the near future.

Double conferences. I was one of the lead co-organizers for the first-ever **double conference** in 2018 on *Higher algebra and mathematical physics* [MGeac]. This is a new eco-friendly conference format, which seeks to reduce long-distance travel (and hence carbon footprint) while still fostering long-distance interaction. (This format also enables participation by academics without access to travel funding, as well as those who cannot travel due to parental responsibilities.) Our goal is to catalyze a fundamental shift in mathematicians’ default expectations regarding conference travel. I also co-organized what was intended to be a second double conference on *Geometric representation theory* [MGeaa], although due to the pandemic it was moved entirely online. While this was a valuable experience, we are optimistic that the pandemic will eventually end, whereas we are quite certain that at that time the climate crisis will persist. Therefore, we have tentative plans to hold a third double conference in 2021 on *Homotopy theory with applications to arithmetic and geometry*, but we plan to postpone it if the event will not be able to be held as originally planned.

Public engagement. I have long been an avid promoter of the beauty of mathematics to the broader public, in both formal and informal capacities. For instance, I’ve given general-audience talks at San Quentin State Prison (on *The 2016 Nobel prize in physics: topological phases of matter*) and at an art & music festival (on *Beyond the 3rd dimension!*) via the Guerilla Science program. I plan to continue this fun and exciting work, in particular by continuing to promote mathematics in unexpected places through Guerilla Science.

Support for geometry & topology community. It is a high priority for me to maintain and strengthen the health of my research community. In addition to the usual journal refereeing, conference attendance, and research dissemination, I am an associate editor for a nascent journal supporting the African geometry & topology community [MGeab]. I also plan to organize an AMS special session on factorization homology in the near future (I have my eye on the October 2021 sectional meeting in Albuquerque, NM), and I am excited to participate in the upcoming MSRI ADJOINT workshop. At a “softer” level, I have been very active in providing support and empathy to my community during the pandemic. In particular, by vulnerably sharing my own fears and uncertainties in the MathOverflow homotopy theory chatroom (in which I am an active participant), I have worked to transform it into a more inclusive environment that welcomes discussion of all aspects of being a mathematician. I am proud and humbled to report that this has led to extended, heartfelt, one-on-one interactions with (so far) four other mathematicians, ranging over sensitive but essential topics such as work-life balance, jealousy and impostor syndrome, the job search, and the two-body problem.

1. BACKGROUND AND RELATED WORK

In this section, I’ll give some background and describe some relevant related work. I’ll begin by laying out some basic conventions in §1.1. In §1.2, I’ll briefly recall the various link homology theories that will be relevant in §2. I’ll discuss factorization homology in §1.3, indicating two foundational problems that I plan to solve (Problems B-C). The last two subsections describe related work that will be relevant in §3: in §1.4 I’ll discuss stratifications and the geometry of the cyclotomic trace, and in §1.5 I’ll discuss cyclotomic Khovanov spectra and state an outstanding conjecture that I would like to resolve (Problem D).

1.1. Basic conventions. In this subsection, I’ll lay out the notation and conventions that I’ll be using, organized roughly by topic. Throughout this proposal, I will allow myself to leave both certain adjectives (such as “graded”, “equivariant”, or “framed”) and their corresponding notations implicit when they are not directly pertinent to the discussion, in order to avoid awkward and repetitive language.

1.1.1. Homotopy theory and higher category theory. I will freely use the theory of ∞ -categories developed in [Lur09, Lura]. I’ll write $\mathcal{S}$, $\mathcal{Sp}$, and $\mathcal{Cat}$ for the ∞ -categories of spaces, spectra, and ∞ -categories, respectively. I’ll write $\mathcal{Ch}$ for the ordinary category of chain complexes of abelian groups and $\mathcal{Ch} := \mathcal{Ch}[\mathbf{W}_{q.i.}^{-1}]$ for its ∞ -categorical localization at the quasi-isomorphisms (i.e. the derived ∞ -category of $\mathbb{Z}$). Given an ∞ -category $\mathcal{C}$, I’ll write $f\mathcal{C}$ and $g\mathcal{C}$ for the ∞ -categories of filtered and graded objects in $\mathcal{C}$, respectively. When $\mathcal{C}$ has a zero object and cofibers, these are related by the associated-graded functor $f\mathcal{C} \xrightarrow{\text{gr}} g\mathcal{C}$.

1.1.2. Equivariant and cyclotomic spectra. For a compact Lie group G , I’ll write $\mathcal{Sp}^G$ for the ∞ -category of genuine G -spectra [LMSM86] and $\mathcal{Sp}^{hG} := \text{Fun}(BG, \mathcal{Sp})$ for the ∞ -category of homotopy G -spectra. Recall that for a closed subgroup $H \subseteq G$, a G -spectrum $E \in \mathcal{Sp}^G$ admits three sorts of H -fixedpoints: categorical (a.k.a. genuine) H -fixedpoints E^H , geometric H -fixedpoints $\Phi^H(E)$, and homotopy H -fixedpoints E^{hH} . These all admit residual homotopy-equivariance for the Weyl group $W(H) := N(H)/H$ (of G -equivariant

automorphisms of G/H). When G is finite, genuine G -spectra can be presented in terms of their categorical fixedpoints as *spectral Mackey functors* [GMa, Bar17]: writing Burn^G for the *Burnside* $(2, 1)$ -category of spans of finite G -sets, we have an equivalence $\mathcal{S}\mathfrak{p}^G \xrightarrow{\sim} \text{Mack}_G(\mathcal{S}\mathfrak{p}) := \text{Fun}^\oplus(\text{Burn}^G, \mathcal{S}\mathfrak{p})$ with the ∞ -category of additive functors. (The spectral Mackey functor corresponding to $E \in \mathcal{S}\mathfrak{p}^G$ acts as $(G/H) \mapsto E^H$.)

I'll write $\mathbb{T} := \text{U}(1)$ for the circle group, and $\mathcal{S}\mathfrak{p}^{<\mathbb{T}}$ for the ∞ -category of *proper-genuine* $\mathbb{T}$ -spectra (which by definition are equipped with categorical fixedpoints just for the proper closed (hence finite cyclic) subgroups of $\mathbb{T}$). Similarly, p -genuine $\mathbb{T}$ -spectra have categorical fixedpoints for the finite p -subgroups of $\mathbb{T}$.

The geometric fixedpoints functors $\mathcal{S}\mathfrak{p}^{<\mathbb{T}} \xrightarrow{\Phi^{C_r}} \mathcal{S}\mathfrak{p}^{<(\mathbb{T}/C_r)} \simeq \mathcal{S}\mathfrak{p}^{<\mathbb{T}}$ for $r \in \mathbb{N}^\times$ assemble into an action of the monoid $\mathbb{N}^\times$ on $\mathcal{S}\mathfrak{p}^{<\mathbb{T}}$, whose homotopy fixedpoints are by definition *cyclotomic spectra*: $\text{Cyc}(\mathcal{S}\mathfrak{p}) := (\mathcal{S}\mathfrak{p}^{<\mathbb{T}})^{\text{h}\mathbb{N}^\times}$ [BHM93, BM15, BG]. So, a cyclotomic spectrum is a proper-genuine $\mathbb{T}$ -spectrum E equipped with a compatible system of equivalences $E \simeq \Phi^{C_r}(E)$ for all $r \in \mathbb{N}^\times$. Similarly, *p-cyclotomic spectra* are the homotopy fixedpoints of the subcategory of p -genuine $\mathbb{T}$ -spectra for the submonoid $\mathbb{N}_p^\times := \{p^k\}_{k \geq 0} \subset \mathbb{N}^\times$.

1.1.3. Higher algebra. I'll write $\mathcal{V} := (\mathcal{V}, \boxtimes, 1_{\mathcal{V}})$ to denote a “nice” (i.e. “presentably”) symmetric monoidal ∞ -category, though note that it will play different roles in different discussions. I'll write $\text{Cat}_n(\mathcal{V})$ for the ∞ -category of $\mathcal{V}$ -enriched (∞, n) -categories [GH15]; I'll simply write $\text{Cat}(\mathcal{V}) := \text{Cat}_1(\mathcal{V})$, and refer to these as *$\mathcal{V}$ -enriched ∞ -categories*. (Here, $\text{Cat}(\mathcal{V})$ inherits a symmetric monoidal structure from $\mathcal{V}$, and we inductively define $\text{Cat}_n(\mathcal{V}) := \text{Cat}(\text{Cat}_{n-1}(\mathcal{V}))$.) A *pointed* $\mathcal{V}$ -enriched (∞, n) -category is one with a distinguished object; I'll write $\text{Cat}_n(\mathcal{V})_*$ for these. I'll write $\text{Alg}_{\mathbb{E}_n}(\mathcal{V})$ for the ∞ -category of $\mathbb{E}_n$ -algebras in $\mathcal{V}$; given $A \in \text{Alg}_{\mathbb{E}_n}(\mathcal{V})$, I'll write $\mathfrak{B}^n A \in \text{Cat}_n(\mathcal{V})_*$ for its *n-fold deloop*, which has just one i -morphism for all $i < n$. In categorical contexts, I'll use the terminology *braided* as a synonym for “ $\mathbb{E}_2$ -monoidal”.

Given $\mathcal{C} \in \text{Cat}(\mathcal{V})$, I'll write $\text{HH}(\mathcal{C}) := \int_{S^1} \mathcal{C} \in \mathcal{V}$ for its Hochschild homology (cf. §1.3.1). In the particular case where $\mathcal{V} = \mathcal{S}\mathfrak{p}$, I'll refer to this as *topological Hochschild homology* and write it as $\text{THH}(\mathcal{C})$. A fundamental fact is that THH is naturally a cyclotomic spectrum [BHM93].

1.1.4. Manifold topology. I'll write $\mathcal{A} := \mathbb{R}^2 \setminus \{0\} \subset \mathbb{R}^2$ for the (open) annulus and $\mathbb{A} := \mathcal{A} \times \mathbb{R} \subset \mathbb{R}^3$ for the (open) solid torus. By definition, an *annular link* is a link in $\mathbb{A}$. (The terminology comes from the fact that these are presented by link diagrams in $\mathcal{A}$.) I'll write $I := [0, 1]$ for the closed interval.

Given a manifold M , I'll write $\text{Cob}_n^d(M)$ for its *cobordism* (∞, n) -category of d -submanifolds, whose i -morphisms for any $0 \leq i \leq n$ are $(d + i)$ -dimensional submanifolds-with-corners of $M \times I^{\times i}$ (so that the codimension is always $\dim(M) - d$). (This can be constructed rigorously via factorization homology [AF].) A mnemonic for the notation is that the superscript d gives the dimension of objects (as is customary for manifolds) while the subscript n indicates the categorical dimension (as in §1.1.3).

For any $n \geq 0$, I'll write Br_n for the braid group on n strands. Given a braid $\mathbf{B} \in \text{Br}_n$, its *closure* defines a link $\widehat{\mathbf{B}}$, which may be considered either in $\mathbb{A}$ or in $\mathbb{R}^3$. I'll write $\sigma_1, \dots, \sigma_{n-1} \in \text{Br}_n$ for the standard generators. A *braid word* is a finite sequence in these generators and their inverses, i.e. an expression $w = (\sigma_{i_1}^{\varepsilon_1}, \dots, \sigma_{i_k}^{\varepsilon_k})$ for $k \geq 0$, $1 \leq i_j \leq n - 1$, and $\varepsilon_j \in \{\pm 1\}$; multiplication determines an element $\underline{w} \in \text{Br}_n$.

1.2. Link homology theories. Our approach to Problem A uses certain generalizations of Kh, which I'll recall in §1.2.1. I'll then recall (resp. annular) Khovanov spectra in §1.2.2, which appear in §§1.5 and 3.

1.2.1. Link homology theories beyond $\mathfrak{sl}_2$. Whereas Kh is governed by the higher representation theory of quantum $\mathfrak{sl}_2$ [LQR15, Web17], Khovanov–Rozansky introduced link homology theories KR_N associated to $\mathfrak{sl}_N$ for all $N \geq 1$ [Kho04, KR08a] (so that $\text{KR}_2 \simeq \text{Kh}$ [Hug14]). Thereafter, they introduced a triply-graded link homology theory HHH [KR08b] that categorifies the HOMFLYPT polynomial [FYH⁺85], which may be thought of as the “large- N limit” of the theories KR_N . (Whereas the HOMFLYPT polynomial recovers the $\mathfrak{sl}_N$ polynomial simply by specialization of variables, a spectral sequence relates HHH to KR_N [Ras15].)

Also relevant for us is the redefinition of HHH in [Kho07] via *Soergel bimodules* [Soe92, Lib19], or rather certain *Rouquier complexes* thereof [Rou]. I'll write SBim_n for the monoidal category of Soergel bimodules in type A_n . This categorifies the Hecke algebra [Soe07], and indeed its homotopy (not derived!) category $\mathbf{K}(\text{SBim}_n)$ is a miraculously simple presentation of the Hecke category: the latter is generated by IC sheaves on $B \backslash G/B$, and the equivalence is given by passing to (hyper)cohomology [Soe07, EW16].

The link invariant most directly relevant to our work is Kitchloo's *strict broken symmetries* spectrum $s\mathcal{B}$, introduced in [Kita] and studied further in [Kitb, Kite]. This is an equivalence class of filtered equivariant spectrum, as I'll explain further in §§2.3–2.4. Taking Borel-equivariant cohomology and passing to the E^2

page of the spectral sequence of the filtration (which is well-defined on equivalence classes) yields HHH, and computations suggest that a twisted version of this procedure yields KR_N [Kitb]. In particular, it is the filtrations at the level of topology that are responsible for the differentials (appearing in Rouquier complexes) at the level of algebra – both of which are avatars of the Hodge theory of the Hecke category [EW14].

1.2.2. (Annular) Khovanov spectra. In a different direction, Khovanov homology has been refined from a graded chain complex to a graded spectrum [LS14, LLS20, HKK16]. Rather than introduce separate notation, I’ll also denote this by $\text{Kh}(L) \in g\mathcal{S}\mathbf{p}$, as it will always be clear from context what is meant. Given an annular link $L \subset \mathbb{A} \subset \mathbb{R}^3$, its Khovanov spectrum acquires an additional (“ k ”) filtration given by counting winding numbers of generators [APS04], so that it is a (k)-filtered (q)-graded spectrum $\text{Kh}(L) \in fg\mathcal{S}\mathbf{p}$. Then, the *annular Khovanov spectrum* of L is by definition the associated-graded object $\text{Kh}_{\text{ann}}(L) := \text{gr}(\text{Kh}(L)) \in gg\mathcal{S}\mathbf{p}$, a bigraded spectrum. I’ll occasionally write $\text{Kh}_{(\text{ann})}$ to refer to both Kh and Kh_{ann} simultaneously.

1.3. Related work: factorization homology. In this subsection, I’ll briefly recall the features of factorization homology that are relevant to this proposal (including my own past work thereon). In §1.3.1, I’ll explain the basic setup, and state two foundational tasks (Problems B and C). I’ll then explain how factorization homology gives rise to link invariants in §1.3.2.

1.3.1. Foundations of factorization homology. In its simplest form, **factorization homology** is a pairing between a framed n -manifold M and an $\mathbb{E}_n$ -algebra $A \in \text{Alg}_{\mathbb{E}_n}(\mathcal{V})$, which returns an object $\int_M A \in \mathcal{V}$ [AFT17a, AF15]. By definition, this is a colimit over an ∞ -category of framed little n -disks in M : the disks are labeled by copies of A , using its $\mathbb{E}_n$ -multiplication when they collide. For instance, when $M = S^1$, this recovers the *Hochschild homology* of $\mathbb{E}_1$ -algebras (and in particular of associative algebras) [AF15, Thm. 3.19]. (The framing restriction on M can be relaxed, in exchange for requiring additional structure on the $\mathbb{E}_n$ -algebra. For example, one can take the factorization homology of an associative algebra with involution over an unoriented 1-manifold.)

The above is referred to informally as the “alpha” version of factorization homology; the “beta” version replaces the $\mathbb{E}_n$ -algebra with an (∞, n) -category $\mathcal{C}$, as established by my collaborators David Ayala, John Francis, and Nick Rozenblyum [AFR]. Here the manifold topology becomes substantially more involved, as it is a colimit over an ∞ -category of *variframed disk-stratifications* of M : i -dimensional disks are labeled by i -morphisms in $\mathcal{C}$, compatibly with source-target data. (To articulate this compatibility requires the variframing, which is roughly “framings of strata equipped with coherent compatibilities along links”.)

Although the “beta” version of factorization homology gives a means of proving the cobordism hypothesis [AF], it is still one step removed from constructions of physical interest, as it merely outputs a *space*. In order to obtain algebraic objects such as chain complexes, one must define the factorization homology of *enriched* (∞, n) -categories. I accomplished this with my collaborators in the case that $n = 1$.

Theorem 1 ([AMGRa]). *Factorization homology defines a pairing between a framed 1-manifold M and a $\mathcal{V}$ -enriched $(\infty, 1)$ -category $\mathcal{C}$, which returns an object $\int_M \mathcal{C} \in \mathcal{V}$.*

This leads to the following, which we have already begun to address.

Problem B. *Construct factorization homology for enriched (∞, n) -categories.*

It is worth immediately noting that it is mathematically meaningful to study Problem A even in the absence of a resolution to Problem B: for links in $\mathbb{R}^3$, the procedure described in §1.3.2 can be implemented by hand (as e.g. in [Saw96, §2]). A resolution to Problem B would then grant (i) well-definedness of the skein modules $\text{Sk}_{\text{Kh}}(Y)$ for general 3-manifolds Y and (ii) homotopy-coherent functoriality for link cobordisms.

We expect to solve Problem B in a manner akin to Thm. 1, which proceeds via the slogan: *enriched factorization homology via categorified factorization homology*. A fundamental issue arising in Problem B is that the usual definition $\text{Cat}_n(\mathcal{V}) := \text{Cat}(\text{Cat}_{n-1}(\mathcal{V}))$ of $\mathcal{V}$ -enriched (∞, n) -categories is essentially combinatorial. By contrast, we can already resolve a version of Problem B using an alternative definition $\text{Cat}'_n(\mathcal{V})$ that is more closely tailored to the topology of variframed n -manifolds. So to finish Problem B, it suffices to show that our alternative definition satisfies the same recursive relation $\text{Cat}'_n(\mathcal{V}) \simeq \text{Cat}(\text{Cat}'_{n-1}(\mathcal{V}))$, which would prove by induction that indeed $\text{Cat}'_n(\mathcal{V}) \simeq \text{Cat}_n(\mathcal{V})$. This conjectural equivalence is essentially a many-object version of the *Dunn additivity* [Dun88], which asserts that $\text{Alg}_{\mathbb{E}_n}(\mathcal{V}) \simeq \text{Alg}_{\mathbb{E}_1}(\text{Alg}_{\mathbb{E}_{n-1}}(\mathcal{V}))$.

As explained in §3, it will be important for computations to address the following.

Problem C. *Prove excision for factorization homology of enriched (∞, n) -categories.*

The term “excision” refers to a gluing formula (as in ordinary homology). For instance, this would articulate the manner in which factorization homology varies as one performs surgery along a link. More broadly, it would allow for the computation of factorization homology in terms of state-sums (recall the analogy of §0.1).

In essence, Problem C amounts to an assertion of *finality*, i.e. that a given colimit can be traded out for another. In the case that $n = 1$ it is relatively straightforward [AMGRa, Rmk. 1.28], but I expect that a full resolution of Problem C would be fairly difficult. However, it may be simpler in the particular relevant case that $n = 3$, due to the additional tools available.

1.3.2. Link invariants from factorization homology. Factorization homology provides invariants not just of manifolds, but of manifolds with embedded submanifolds – in particular, of links in 3-manifolds.

This proceeds as follows. Fix a pointed $\mathcal{V}$ -enriched $(\infty, 3)$ -category with adjoints $\mathcal{C} := (c \in \mathcal{C}) \in \text{Cat}_3(\mathcal{V})_*^{\text{adj}}$. Then, given a framed 3-manifold Y containing a (compatibly normally and tangentially framed) link $L \subset Y$, by the functoriality of factorization homology for fiber bundles (contravariantly) and open embeddings (covariantly), we obtain a composite morphism

$$\int_{\pi_0(L) \times \mathbb{R}^2} \mathcal{C} \longrightarrow \int_{L \times \mathbb{R}^2} \mathcal{C} \longrightarrow \int_Y \mathcal{C} \quad (1)$$

in $\mathcal{V}$. The source is (the tensoring of $1_{\mathcal{V}} \in \mathcal{V}$ over) the space of labelings of the components of L by 2-endomorphisms of $c \in \mathcal{C}$. Meanwhile, the target is by definition the **skein module** of Y (for $\mathcal{C}$), denoted

$$\text{Sk}_{\mathcal{C}}(Y) := \left(\int_Y \mathcal{C} \right) \in \mathcal{V};$$

in the case that $Y = \mathbb{R}^3$, this admits a canonical identification

$$\text{Sk}_{\mathcal{C}}(\mathbb{R}^3) := \left(\int_{\mathbb{R}^3} \mathcal{C} \right) \simeq 3\text{end}_{\mathcal{C}}(c) \in \mathcal{V}$$

with the $\mathcal{V}$ -object of 3-endomorphisms of $c \in \mathcal{C}$. In particular, if we choose a distinguished 2-endomorphism of $c \in \mathcal{C}$, then taking it as the default labeling gives a $\text{Sk}_{\mathcal{C}}(Y)$ -valued invariant of links in Y .

As a special case, given a braided $\mathcal{V}$ -enriched $(\infty, 1)$ -category with duals $\mathcal{D} \in \text{Alg}_{\mathbb{E}_2}(\text{Cat}(\mathcal{V}))^{\text{dual}}$, we may take $\mathcal{C} := \mathfrak{B}^2 \mathcal{D} \in \text{Cat}_3(\mathcal{V})_*^{\text{adj}}$ to be its 2-fold deloop. The canonical basepoint $*$ in $\mathfrak{B}^2 \mathcal{D}$ has 2-endomorphisms given by objects of $\mathcal{D}$ and 3-endomorphisms given by endomorphisms of the unit $1_{\mathcal{D}} \in \mathcal{D}$. (In particular, the duals in $\mathcal{D}$ become 2-adjoints in $\mathfrak{B}^2 \mathcal{D}$.) In this setting (e.g. for $\mathcal{V} = \text{Cat}(g\text{Ch})$ and $\mathcal{D} = \mathcal{K}\text{h}$), I’ll just write $\text{Sk}_{\mathcal{D}} := \text{Sk}_{\mathfrak{B}^2 \mathcal{D}} \in \mathcal{V}$ for the corresponding skein module, for simplicity.

Example 2. Let us take $\mathcal{D} := \overline{\text{Rep}}^{\text{f.d.}}(U_q(\mathfrak{sl}_2))$ to be the semisimplified category of finite-dimensional representations of quantum $\mathfrak{sl}_2$ at an appropriate root of unity [BW99, Saw06, EO].

- (1) In the case that $Y = \mathbb{R}^3$, the composite (1) is the colored Jones polynomial of L (evaluated at the chosen root of unity) [Saw96].
- (2) In general, the composite (1) is the WRT invariant of $L \subset Y$, in the sense articulated in §0.1; the WRT invariant of Y itself is recovered by taking $L = \emptyset$ to be the empty link.

1.4. Related work: stratifications, genuine G -spectra, and the geometry of the cyclotomic trace. In this subsection, I’ll describe some recent joint work with my collaborators David Ayala and Nick Rozenblyum, which will be relevant in §3.

Given a (presentable stable) ∞ -category $\mathcal{X}$ and a poset $\mathbf{P}$, we give a simple definition of a **stratification** of $\mathcal{X}$ over $\mathbf{P}$, which is closely patterned on (and in fact generalizes) the structure seen on quasicohherent sheaves over a stratified scheme [AMGRd, §§1.2-3]. The purpose of stratifications is that they allow for *reconstruction theorems*, in a way that may be summarized as follows.

Theorem 3 ([AMGRd, Thm. A]). *Suppose that $\mathcal{X}$ is stratified over $\mathbf{P}$. (1) We can reconstruct $\mathcal{X}$ from its strata $\{\mathcal{X}_p \subseteq \mathcal{X}\}_{p \in \mathbf{P}}$ along with gluing data therebetween. (2) We can reconstruct each object $X \in \mathcal{X}$ from its localizations $\{\Phi^p(X) \in \mathcal{X}_p\}_{p \in \mathbf{P}}$ to the strata of $\mathcal{X}$ along with gluing data therebetween.*

We refer to these as **macrocosm** and **microcosm** reconstruction theorems (inspired by the microcosm principle in category theory [BD98, §4.3]). As the base case, a stratification over the poset $[1] = \{0 \rightarrow 1\}$ is precisely the data of a *recollement* [BBD82].

Now, let G be a compact Lie group and let $\mathbf{P}_G$ denote its poset of closed subgroups ordered by subconjugacy. We provided the following fundamental source of stratifications.

Theorem 4 ([AMGRd, Thm. E]). *There is a canonical **geometric stratification** of Sp^G over P_G : for each $H \in P_G$, its H^{th} stratum is $\mathrm{Sp}^{\mathrm{hw}(H)}$, and its H^{th} localization functor is geometric H -fixedpoints: $\mathrm{Sp}^G \xrightarrow{\Phi^H} \mathrm{Sp}^{\mathrm{hw}(H)}$. (In particular, the notation is (not coincidentally) consistent with that of Thm. 3.)*

Hence, by Thm. 3(2), a genuine G -spectrum $E \in \mathrm{Sp}^G$ is equivalent data to its various geometric fixedpoints $\{\Phi^H(E) \in \mathrm{Sp}^{\mathrm{hw}(H)}\}_{H \in P_G}$ along with gluing data therebetween. In this case, the gluing data relates to the **proper Tate construction** [AMGRd, §5.2]. In brief, the proper Tate construction $(-)^{\tau G}$ takes the quotient of $(-)^{\mathrm{h}G}$ by norms from all proper subgroups of G .

Example 5. A genuine C_p -spectrum $E \in \mathrm{Sp}^{C_p}$ is completely specified by the data of (i) its underlying homotopy C_p -spectrum $E_0 := \Phi^e(E) \in \mathrm{Sp}^{\mathrm{h}C_p}$, (ii) its geometric C_p -fixedpoints spectrum $E_1 := \Phi^{C_p}(E) \in \mathrm{Sp}$, and (iii) a *gluing map* $E_1 \rightarrow (E_0)^{\mathrm{t}C_p}$ from E_1 to the (proper) C_p -Tate construction of E_0 . (Note that C_p has no nontrivial proper subgroups, so $(-)^{\tau C_p} = (-)^{\mathrm{t}C_p}$.)

So, the geometric stratification of Thm. 4 may be seen as giving an alternative presentation to genuine G -spectra to that by spectral Mackey functors: in terms of their geometric fixedpoints, as opposed to their categorical fixedpoints. (Results along these lines admit a long history, cf. [AMGRd, §0.2].)

We showed that the geometric stratification of $\mathrm{Sp}^{<\mathbb{T}}$ is suitably $\mathbb{N}^\times$ -equivariant, yielding the following.

Theorem 6 ([AMGRc, Thm. B]). *A cyclotomic spectrum is equivalent data to a homotopy $\mathbb{T}$ -spectrum $E \in \mathrm{Sp}^{\mathrm{h}\mathbb{T}}$ along with a compatible system of **cyclotomic structure maps** $E \xrightarrow{\sigma_r} E^{\tau C_r}$ (for all $r \in \mathbb{N}^\times$).*

This extends work of Nikolaus–Scholze, who proved a striking simplification of Thm. 6 for *connective* cyclotomic spectra [NS, Thm. 1.4] that is made possible by their vanishing result [NS, Lem. I.2.1].

Finally, we constructed the cyclotomic structure on THH in the terms of Thm. 6 [AMGRb, Thm. B] (incarnated as $\int_{S^1}$ via Thm. 1). This allowed us to give a precise conceptual description of TC in terms of derived algebraic geometry. Let X be a scheme, let $\mathcal{L}X$ denote its free loop space (a derived stack), and recall that $\mathrm{THH}(X) := \mathrm{THH}(\mathrm{Perf}_X) \simeq \mathcal{O}(\mathcal{L}X)$ is equivalent to functions on $\mathcal{L}X$ [BZFN10].

Theorem 7 ([AMGRb, Thm. C (cf. Rmk. 5)]). *$\mathrm{TC}(X)$ is obtained from $\mathrm{THH}(X) \simeq \mathcal{O}(\mathcal{L}X)$ by cutting out those precisely functions $\mathcal{L}X \xrightarrow{f} \mathbb{A}^1$ that are $\mathbb{T}$ -invariant and whose values at each point $(S^1 \xrightarrow{\gamma} X) \in \mathcal{L}X$ determines their values at all points $(S^1 \xrightarrow{\gamma} S^1 \xrightarrow{\gamma} X) \in \mathcal{L}X$ for all $r \in \mathbb{N}^\times$ “up to universal indeterminacy”: $f(r^* \gamma) \equiv f(\gamma)^r$ modulo norms.*

These are precisely the symmetries present on the trace-of-monodromy functions of vector bundles over X , which gives a conceptual description of the cyclotomic trace map $\mathrm{K}(X) \rightarrow \mathrm{TC}(X)$ [AMGRb, Thm. D]. (These symmetries can also be seen more simply in terms of traces of matrices [AMGRb, §0.4].)

For a spectral ∞ -category $\mathcal{C} \in \mathrm{Cat}(\mathrm{Sp})$, the cyclotomic structure on $\mathrm{THH}(\mathcal{C})$ constructed in this way derives from the **Tate diagonal** maps Δ_τ^r possessed by spectra, as indicated in Fig. 2, which arise from the *linearity* of the functor $((-)^{\otimes r})^{\tau C_r}$ [AMGRc, Lem. 2.19]. (This may be compared with the failure for an abelian group A of the function $A \xrightarrow{x \mapsto x \otimes x} (A^{\otimes 2})_{C_2}$ to be linear, which is captured by the norm elements $\mathrm{Nm}([x \otimes y]) := x \otimes y + y \otimes x$ for $[x \otimes y] \in (A^{\otimes 2})_{C_2}$.) Namely, given an r -fold cover $S^1 \xrightarrow{\gamma} S^1$, the corresponding cyclotomic structure map $\mathrm{THH}(\mathcal{C})^{\tau C_r} \xleftarrow{\sigma_r} \mathrm{THH}(\mathcal{C})$ (as in Thm. 6) is assembled from the Tate diagonal maps Δ_τ^r for the hom-spectra in $\mathcal{C}$.

1.5. Related work: cyclotomic Khovanov spectra. In this subsection, I’ll describe forthcoming joint work with my collaborators Matt Stoffregen and Mike Willis [MGSW] and state a conjecture that results therefrom (Problem D), all of which will be relevant in §3.

Fix any $n \geq 0$. I’ll write $\mathbf{B} \in \mathrm{Br}_n$ for an arbitrary braid; I’ll consider its closure as an annular link $\widehat{\mathbf{B}} \subset \mathbb{A}$, and I’ll simply write $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}) := \mathrm{Kh}_{(\mathrm{ann})}(\widehat{\mathbf{B}})$ to avoid awkward typography. I’ll write $\mathbf{T} := (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n \in \mathrm{Br}_n$ for the **full twist** element.

Our work concerns $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{T}^\infty)$, the (resp. **annular**) **Khovanov spectrum of the infinite twist**. One writes $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^\infty)$ for the colimit of the sequence $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}) \rightarrow \mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^2) \rightarrow \mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^3) \rightarrow \cdots$ of *cobordism maps*, which arise by recognizing the cube whose total cofiber is $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^i)$ as a subcube of that whose total cofiber is $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^{i+1})$. In the case that $\mathbf{B} = \mathbf{T}$, it has been shown that this sequences stabilizes in each q -degree [Sto07, Wil17], so that $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{T}^\infty)$ may be considered as a “categorified Laurent series” (i.e.

for all $k \in \mathbb{Z}$ the coefficient of q^k is finite). The Khovanov spectrum $\mathrm{Kh}(\mathbf{T}^\infty)$ is of longstanding interest: it categorifies the Jones–Wenzl idempotent in the Temperley–Lieb algebra (which represents the highest-weight projector for $U_q(\mathfrak{sl}_2)$ -representations) [CK12, Roz14], and has generated a wealth of fascinating conjectures and research activity [DGR06, Sto07, Wil17, GOR13, GORS14, Hog18, AH17].

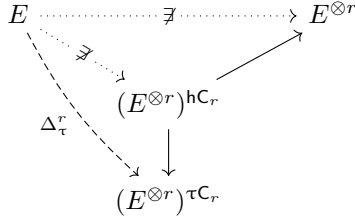


FIGURE 2

Meanwhile, for any $r \geq 1$, [SZ] endows $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^r)$ with a *genuine* C_r -structure, and establishes equivalences $\Phi^{C_r}(\mathrm{Kh}(\mathbf{B}^r)) \simeq \Phi^{C_r}(\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^r)) \simeq \mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B})$ (suitably interpreted: the filtration on the first term is canonically split). Combining these results with the above homotopy $\mathbb{T}$ -action should make plausible our main result, which reads as follows.

Theorem 8 ([MGSW]). (1) $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{T}^\infty)$ admits a cyclotomic structure, (2) $\mathrm{Kh}(\mathbf{T}^\infty)$ admits a proper-genuine $\mathbb{T}$ -structure, and (3) for any $r > 1$ we have $\Phi^{C_r}(\mathrm{Kh}(\mathbf{T}^\infty)) \simeq \mathrm{Kh}_{(\mathrm{ann})}(\mathbf{T}^\infty)$.

To my knowledge, Thm. 8(1) gives the first interesting example of a cyclotomic spectrum that is not constructed as THH , although it is certainly conceivable that it is in fact equivalent to THH of a well-chosen ring spectrum or spectral ∞ -category. (By [LLSa], $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B})$ may be constructed as THH with coefficients, which kills the $\mathbb{T}$ -action. And the colimit $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^\infty)$ currently admits no such description at all.)

Now, in their recent work [AN], Antieau–Nikolaus introduce and study the notion of a **topological Cartier module**: this may be defined as a proper-genuine $\mathbb{T}$ -spectrum E equipped with equivalences $E \simeq E^{C_r}$ for all $r \in \mathbb{N}^\times$. Working p -typically (as in [AN]) for simplicity, these participate in an adjunction $\Phi^p : \mathrm{TCart}_p \rightleftarrows \mathrm{Cyc}_p(\mathrm{Sp}) : \mathrm{TR}_p$ with p -typical cyclotomic spectra. In particular, for a p -typical topological Cartier module $E \in \mathrm{TCart}_p$, its geometric C_p -fixedpoints acquires a p -typical cyclotomic structure via the composite $\Phi^{C_p}(E) \simeq \Phi^{C_p}(E^{C_p}) \xrightarrow{\sim} \Phi^{C_p}(\Phi^{C_p}(E))$ (using a projection formula). Hence, Thm. 8 immediately suggests the following conjecture, which I am very keen to resolve.

Problem D. $\mathrm{Kh}(\mathbf{T}^\infty)$ admits a topological Cartier module structure.

A resolution of Problem D would undoubtedly shed new light on the mysterious nature of the Khovanov spectrum of the infinite twist. Relatedly, an examination of the resulting adjoint map $\mathrm{Kh}(\mathbf{T}^\infty) \rightarrow \mathrm{TR}_p(\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{T}^\infty))$ in TCart_p would likely give important insight on the question of functoriality of Khovanov homology for ramified covers (see Problem G and §3.2.2). More speculatively, especially building on Thm. 8, a resolution of Problem D could conceivably pave the way for a rich and fruitful interaction between Khovanov homology and recent computational and conceptual advances in the realm of arithmetic geometry such as [AN, NS, BMS18, BMS19, Hes18, BS, BLM], which would be an exciting outcome indeed.

It will be relevant for the discussion of §3.2 that our proof of Thm. 8(1) makes use of a novel **microcosm principle for cyclotomic objects**. I’ll give a brief synopsis of the proof for the interested reader (as [MGSW] is still forthcoming), which the reader willing to take this assertion on faith may safely skip.

We make use of the formalism of *equivariant ∞ -category theory*, particularly its attendant notion of *equivariant colimits* [BDG⁺, Sha]. For instance, given a C_2 -equivariant span $Y \xleftarrow{f} X \xrightarrow{f} Y$ of spectra where X itself is a C_2 -spectrum, it allows us to take the C_2 -colimit and obtain a C_2 -spectrum. In our setting, the cube whose total cofiber is $\mathrm{Kh}_{(\mathrm{ann})}(\mathbf{B}^r)$ carries a C_r -action, and this formalism yields the C_r -structure thereon. The general situation is this: the ∞ -category of G - ∞ -categories is $\mathrm{Cat}^G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat})$; the G - ∞ -category of G -spectra $\mathrm{Sp}_G \in \mathrm{Cat}^G$ is defined to have H -fixedpoints $(\mathrm{Sp}_G)^H := (\mathrm{Sp}_G)(G/H) := \mathrm{Sp}^H$; and the G -colimit of a G -functor $\mathcal{I} \rightarrow \mathrm{Sp}_G$ is a global section of Sp_G , which is simply an object of Sp^G – a microcosm principle for G -objects.

In particular, one has the ∞ -category $\text{Cat}^{<\mathbb{T}} := \text{Fun}(\mathcal{O}_{<\mathbb{T}}^{\text{op}}, \text{Cat})$ of proper $\mathbb{T}$ - ∞ -categories. This carries an action of $\mathbb{N}^\times$ (coming from its action on $\mathbb{T}$), and we define the ∞ -categories of **cyclotomic ∞ -categories** and **laxly cyclotomic ∞ -categories** to be its strict and lax limits: $\text{Cyc}(\text{Cat}) := (\text{Cat}^{<\mathbb{T}})^{\text{h}\mathbb{N}^\times}$ and $\text{Cyc}_{\text{lax}}(\text{Cat}) := (\text{Cat}^{<\mathbb{T}})^{\text{h}\mathbb{N}^\times}$. A key example of the former is given by [AMGRa, §4], which for any $(\infty, 2)$ -category $\mathcal{C} \in \text{Cat}_2 := \text{Cat}(\text{Cat})$ yields a cyclotomic ∞ -category structure $\text{HH}(\mathcal{C}) \in \text{Cyc}(\text{Cat})$ on its Hochschild homology (which by definition comes equipped with equivalences $\text{HH}(\mathcal{C})^{C_s} \simeq \text{HH}(\mathcal{C})$ for all $s \in \mathbb{N}^\times$). We construct a key example of the latter by hand: the laxly cyclotomic ∞ -category $\mathcal{S}p_{\text{Cyc}} := (\mathcal{S}p_{<\mathbb{T}}, \Phi^{\bullet}) \in \text{Cyc}_{\text{lax}}(\text{Cat})$ of cyclotomic spectra. Its underlying proper $\mathbb{T}$ - ∞ -category $\mathcal{S}p_{<\mathbb{T}}$ has C_r -fixedpoints given by $(\mathcal{S}p_{<\mathbb{T}})^{C_r} := \mathcal{S}p^{C_r}$, and for each $r \in \mathbb{N}^\times$, its r^{th} laxly cyclotomic structure map is the system $\{\mathcal{S}p^{C_{rs}} \xrightarrow{\Phi^{C_r}} \mathcal{S}p^{C_s}\}_{s \in \mathbb{N}^\times}$. The cyclotomic microcosm principle consists in the fact that **global sections of $\mathcal{S}p_{\text{Cyc}}$ are precise cyclotomic spectra**. We construct $\mathcal{S}p_{\text{Cyc}}$ from a similarly described laxly cyclotomic ∞ -category $\text{Burn}_{\text{Cyc}} := (\text{Burn}_{<\mathbb{T}}, (-)^{C_\bullet}) \in \text{Cyc}_{\text{lax}}(\text{Cat})$ by taking spectral Mackey functors.

We can now return to annular Khovanov spectra. Fix a braid word $w := (\sigma_{i_1}^{\varepsilon_1}, \dots, \sigma_{i_k}^{\varepsilon_k})$ that presents the chosen braid $\mathbf{B} \in \text{Br}_n$, and let $\underline{n} := (\{1, \dots, n\} \subset \mathbb{R}) \in \text{Cob}_2^0(\mathbb{R})$ denote the object consisting of n points in the real line. Assigning saddle cobordisms to braid generators $\sigma_j^{\varepsilon_j}$ yields a composite functor

$$F_w : [1]^{\times k} \xrightarrow{\prod_{j=1}^k (\text{saddle for } \sigma_{i_j}^{\varepsilon_j})} \text{end}_{\text{Cob}_2^0(\mathbb{R})}(\underline{n})^{\times k} \xrightarrow{\circ} \text{end}_{\text{Cob}_2^0(\mathbb{R})}(\underline{n})$$

to the ∞ -category of planar (n, n) -tangles (which is monoidal via stacking, i.e. composition of 1-morphisms in $\text{Cob}_2^0(\mathbb{R})$). Altogether, the cyclotomic structure on $\text{Kh}_{\text{ann}}(\mathbf{B}^\infty)$ is obtained by taking the parametrized colimit (or rather, parametrized total cofiber) of a resulting composite functor

$$([1]^{\times k})^{\times \bullet} \xrightarrow{F_w^{\times \bullet}} \text{end}_{\text{Cob}_2^0(\mathbb{R})}(\underline{n})^{\times \bullet} \longrightarrow \text{HH}(\text{Cob}_2^0(\mathbb{R})) \simeq \text{Cob}_1^1(\mathcal{A}) \xrightarrow{V} gg\text{Burn}_{\text{Cyc}} \xrightarrow{\Sigma_+^\infty} gg\mathcal{S}p_{\text{Cyc}}$$

among laxly cyclotomic ∞ -categories, in which the morphism V is the *Khovanov–Burnside 2-functor* constructed in [LLSb, §2.11] (or rather a cyclotomic upgrade of the annular version thereof, which incorporates the combinatorial results of [SZ]), and in which the equivalence follows from a hypercover argument.

Towards addressing Problem D, note that topological Cartier modules are the global sections of the laxly cyclotomic ∞ -category $\mathcal{S}p_{\text{TCart}} := (\mathcal{S}p_{<\mathbb{T}}, (-)^{C_\bullet}) \in \text{Cyc}_{\text{lax}}(\text{Cat})$. However, the above argument relies on the fact that the laxly cyclotomic structure maps $\Phi^{C_\bullet}$ of $\mathcal{S}p_{\text{Cyc}}$ are *parametrized left adjoints* (i.e. they satisfy a Beck–Chevalley condition), and hence commute with parametrized colimits. The laxly cyclotomic structure maps $(-)^{C_\bullet}$ of $\mathcal{S}p_{\text{TCart}}$ do not share this property. So, a resolution of Problem D will require a finer understanding of the situation at hand – optimistically, one resulting from an understanding of the functoriality of Khovanov homology for ramified covers (as discussed in §3.2.2).

2. KHOVANOV HOMOLOGY AS FACTORIZATION HOMOLOGY

I am currently investigating Problem A with my collaborators Leon Liu and David Reutter. We are moving surprisingly quickly, and our understanding of the situation is improving by the week. In this section, I will give a snapshot of our progress and explain how we expect to proceed.

In fact, we have found that from a topological point of view it is more natural to address the following refinement of Problem A.

Problem E. *Construct a pointed braided $(\infty, 2)$ -category with duals enriched in derived spectra*

$$(V \in \mathcal{S}\mathcal{B}) \in \text{Alg}_{\mathbb{E}_2}(\text{Cat}_2(\mathbf{D}(\mathcal{S}p)))_*^{\text{dual}}$$

that recovers $s\mathcal{B}$ via factorization homology.

(Of course, we have chosen the notation $\mathcal{S}\mathcal{B}$ so that it relates to $s\mathcal{B}$ as $\mathcal{K}h$ relates to Kh (recall §0.1). Coincidentally, however, it can *also* be read as an acronym for “Soergel bimodules”: as explained in §2.2, we expect $\mathcal{S}\mathcal{B}$ to encapsulate refinements of the categories SBim_n for all n .)

I’ll give a top-down explanation of our approach to Problem E, organized as follows. In §2.1, I’ll explain the machinery that we intend to use to build our braided $(\infty, 2)$ -category enriched in derived spectra $\mathcal{S}\mathcal{B}$. In particular this will involve actions of the braid groups Br_n (for all n), which I’ll discuss in §2.2. I’ll discuss derived spectra in §2.3. The heart of §2 is §2.4, where I’ll explain the details of our approach to Problem E within the context of the material described in §§2.1–2.3. I’ll discuss the relationship between $\mathcal{S}\mathcal{B}$ and $\mathcal{K}h$

(i.e. the relationship between Problems E and A) in §2.5, and I'll discuss the relationship between Problem A and the WRT invariant in §2.6.

Remark 9. Morrison–Walker–Wedrich have assembled the values of KR_N (for fixed N) into a braided 2-category [MWW, §6], which at first glance may appear to address Problem A. However, their construction applies KR_N to links in $\mathbb{R}^3$ *before* implementing (a variant of) factorization homology. This greatly obscures its relationship with links in arbitrary 3-manifolds, and at the same time renders completely tautological the fact that it recovers KR_N of links in $\mathbb{R}^3$. In particular, it seems to have little chance of categorifying the WRT invariant.

2.1. $\mathbb{E}_2$ -algebras. We intend to construct the braided $(\infty, 2)$ -category $\mathcal{SB}$ through Lurie's machinery [Lura, §5.5 and §A.6]. More specifically, there is a diagram

$$\text{Alg}_{\mathbb{E}_2}(\mathcal{V}) \simeq \text{coShv}_{\mathcal{V}}^{\text{fact}}(\text{Ran}(\mathbb{R}^2)) \subseteq \text{coShv}_{\mathcal{V}}^{\text{cbl}}(\text{Ran}(\mathbb{R}^2)) \simeq \text{Fun}(\text{Enter}(\text{Ran}(\mathbb{R}^2)), \mathcal{V}) \quad (3)$$

(which we'll apply in the case that $\mathcal{V} = \text{Cat}_2(\mathbf{D}(\text{Sp}))$) that I'll now explain.

First of all, $\text{Ran}(\mathbb{R}^2)$ denotes the **Ran space** of $\mathbb{R}^2$: the space of nonempty finite subsets of $\mathbb{R}^2$, topologized so that points can collide. This is stratified by cardinality over the poset $\mathbb{N}$ of natural numbers, so that its n^{th} stratum is the subspace $\text{Conf}_n(\mathbb{R}^2) \subset \text{Ran}(\mathbb{R}^2)$ of unordered configurations of n points in $\mathbb{R}^2$.

Next, a cosheaf over a stratified space T is called **constructible** if it is locally constant over each stratum. So if T is trivially stratified, a constructible cosheaf is simply a local system, and we have an equivalence $\text{coShv}_{\mathcal{V}}^{\text{cbl}}(T) \simeq \text{Fun}(\Pi_{\infty}(T), \mathcal{V})$ with the ∞ -category of functors from its fundamental ∞ -groupoid. The right-hand equivalence in (3) generalizes this (taking $T = \text{Ran}(\mathbb{R}^2)$). Namely, $\text{Enter}(T)$ denotes the **enter-path ∞ -category** of T , whose objects are the points of T and whose morphisms are *entering paths* in T (i.e. they can enter deeper strata but cannot leave); and it participates in an equivalence $\text{coShv}_{\mathcal{V}}^{\text{cbl}}(T) \simeq \text{Fun}(\text{Enter}(T), \mathcal{V})$.

Now, the left-hand equivalence in (3) equates $\mathbb{E}_2$ -algebras in $\mathcal{V}$ with **factorizable** constructible cosheaves over $\text{Ran}(\mathbb{R}^2)$. Through the equivalence $\text{coShv}_{\mathcal{V}}^{\text{cbl}}(\text{Ran}(\mathbb{R}^2)) \simeq \text{Fun}(\text{Enter}(\text{Ran}(\mathbb{R}^2)), \mathcal{V})$, we may approximately define *factorizability* as the condition on a functor $\text{Enter}(\text{Ran}(\mathbb{R}^2)) \xrightarrow{\mathcal{F}} \mathcal{V}$ that for points $p, q \in \text{Ran}(\mathbb{R}^2)$ that have disjoint support in $\mathbb{R}^2$, the induced map $\mathcal{F}(p) \boxtimes \mathcal{F}(q) \rightarrow \mathcal{F}(p \cup q)$ is an equivalence. (Disjointness is not well-defined in the ∞ -category $\text{Enter}(\text{Ran}(\mathbb{R}^2))$ (since it is not invariant under equivalences), which is why this is only an approximation to the true definition.) So, an $\mathbb{E}_2$ -algebra $A \in \text{Alg}_{\mathbb{E}_2}(\mathcal{V})$ corresponds to a factorizable functor $\text{Enter}(\text{Ran}(\mathbb{R}^2)) \xrightarrow{\mathcal{F}_A} \mathcal{V}$ carrying a nonempty finite subset $S \subset \mathbb{R}^2$ to the object $A^{\boxtimes S} \in \mathcal{V}$; entering paths in $\text{Ran}(\mathbb{R}^2)$ correspond to collisions of points, and these record the $\mathbb{E}_2$ -structure of A .

2.2. Br_n -actions. The n^{th} tensor power of any object of a braided ∞ -category carries a canonical action of the n^{th} braid group Br_n . In the terms of §2.1, this Br_n -action is incarnated as a local system on the n^{th} stratum $\text{Ran}(\mathbb{R}^2)_n \simeq \text{Conf}_n(\mathbb{R}^2) \simeq \mathfrak{B}\text{Br}_n$. We view constructing the Br_n -action on the object $V^{\otimes n} \in \mathcal{SB}$ as an important stepping stone towards Problem E. Moreover, we expect the $\mathbf{D}(\text{Sp})$ -enriched monoidal ∞ -category $\text{end}_{\mathcal{SB}}(V^{\otimes n})$ to refine the graded monoidal ∞ -category $\mathbf{K}(\text{SBim}_n)$, and hence we expect our Br_n -action on $V^{\otimes n}$ to refine the monoidal functor $\text{Br}_n \rightarrow \mathbf{K}(\text{SBim}_n)$ obtained in [Bez06, TT].

As in those works (see also [Del97, Dob06]), we expect to obtain our braid group action $\text{Br}_n \curvearrowright V^{\otimes n} \in \mathcal{SB}$ from an action of the braid monoid Br_n^+ (i.e. the submonoid of Br_n with the same generators and relations but without inverses), as explained in §2.4. However, whereas past works proceed combinatorially, we envision a topological manifestation of our Br_n^+ -action. Namely, $\text{Ran}(\mathbb{R}^2)$ admits a **vertical stratification** (first studied in [Cep]), which I'll denote by $\text{Ran}(\mathbb{R}^2)^v$. This refines its cardinality stratification: it records the x -coordinates of points, giving a stratification over the ∞ -category (which is equivalent to a poset) of nonempty finite subsets-with-multiplicity of $\mathbb{R}$ ordered by anticollision and repetition. For instance, Fig. 4 (left) depicts some exiting paths (definitionally opposite to entering paths, $\text{Exit}(T) := \text{Enter}(T)^{\text{op}}$) between the four distinct strata of the stratified subspace $\text{Conf}_3(\mathbb{R}^2)^v \subset \text{Ran}(\mathbb{R}^2)^v$, which is stratified over the poset of compositions of 3 ordered by refinement. In fact, we believe that there exists a localization

$$\text{Exit}(\text{Conf}_n(\mathbb{R}^2)^v) \longrightarrow \mathfrak{B}\text{Br}_n^+; \quad (5)$$

roughly speaking, it inverts those exiting paths in $\text{Conf}_n(\mathbb{R}^2)^v$ that enact clockwise half-rotations, so that what remains are counterclockwise half-rotations and the inverses of clockwise half-rotations – i.e., all counterclockwise rotations. For instance, Fig. 4 (right) schematically depicts the category $\text{Exit}(\text{Conf}_3(\mathbb{R}^2)^v)$, with the diagram to its left consisting of the dashed arrows: leftward/rightward arrows correspond to a dot moving to the left/right, with vertical placement indicating which dot is moving. In this example, it

is not hard to verify directly that inverting these morphisms yields the category $\mathfrak{BBr}_3^+$; for instance, the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ results from an evident equality between certain composite morphisms $(3) \rightarrow (2+1) \rightarrow (1+1+1)$ and $(3) \rightarrow (1+2) \rightarrow (1+1+1)$. More systematically, every configuration of n points in $\mathbb{R}^2$ admits a lexicographic ordering (checking x -coordinates before y -coordinates), and we expect that the functor (5) inverts precisely those exiting paths that preserve this ordering. Thereafter, a Br_n^+ -action would be equivalent to a constructible sheaf over $\text{Conf}_n(\mathbb{R}^2)^v$ such that the monodromy morphisms over these maps are equivalences. On the other hand, if a stratified space T is equipped with a refined stratification T' , then the induced functor $\text{Exit}(T') \rightarrow \text{Exit}(T)$ is a localization [AFT17b, Prop. 1.2.13]. Hence, we obtain a localization $\text{Exit}(\text{Conf}_n(\mathbb{R}^2)^v) \rightarrow \text{Exit}(\text{Conf}_n(\mathbb{R}^2)) \simeq \Pi_\infty(\text{Conf}_n(\mathbb{R}^2)) \simeq \mathfrak{BBr}_n$, so that a Br_n -action is equivalent to a constructible sheaf over $\text{Conf}_n(\mathbb{R}^2)^v$ such that *all* monodromy morphisms are equivalences.

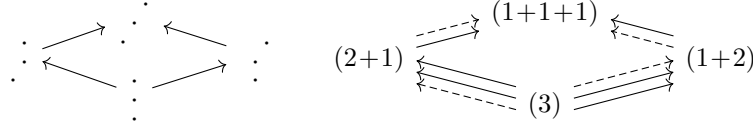


FIGURE 4

2.3. From $f\mathcal{S}\mathfrak{p}$ to $\mathbf{D}(\mathcal{S}\mathfrak{p})$. Kitchloo associates to each braid word w a filtered spectrum $s\mathcal{B}(w) \in f\mathcal{S}\mathfrak{p}$, but this is only invariant under Markov moves up to a notion he calls *quasi-equivalence* [Kita, Def. 3.4]. We believe $s\mathcal{B}(w)$ is most naturally an object of $\mathbf{D}(\mathcal{S}\mathfrak{p})$, the *derived ∞ -category of spectra*, as I'll now explain.

Filtered spectra determine spectral sequences; I'll call a morphism in $f\mathcal{S}\mathfrak{p}$ an E^k -equivalence if it induces an equivalence on E^k pages. Then, quasi-equivalences are E^2 -equivalences (and the converse holds up to convergence issues). On the other hand, the Lurie–Dold–Kan correspondence [Lura, Thm. 1.2.4.1] provides an equivalence $s\mathcal{S}\mathfrak{p} \xrightarrow{\sim} f_{\geq 0}\mathcal{S}\mathfrak{p}$ between simplicial spectra and nonnegatively-filtered spectra, carrying a simplicial spectrum to the skeletal filtration of its geometric realization. Now, localizing $s\mathcal{S}\mathfrak{p}$ at the E^2 -equivalences yields $\mathbf{D}_{\geq 0}(\mathcal{S}\mathfrak{p})$, the *nonnegatively-graded (nonabelian) derived ∞ -category of spectra*. Concretely, this admits an identification $\mathbf{D}_{\geq 0}(\mathcal{S}\mathfrak{p}) := \text{Fun}^\times(\mathcal{S}\mathfrak{p}_{\text{sph}}^{\text{op}}, \mathcal{S})$ with the ∞ -category of product-preserving presheaves of spaces on the full subcategory $\mathcal{S}\mathfrak{p}_{\text{sph}} \subset \mathcal{S}\mathfrak{p}$ of finite sums of sphere spectra [Lur09, §5.5.8]; the localization functor $s\mathcal{S}\mathfrak{p} \rightarrow \mathbf{D}_{\geq 0}(\mathcal{S}\mathfrak{p})$ is then given by the formula $Y_\bullet \mapsto (X \mapsto |\text{hom}_{\mathcal{S}\mathfrak{p}}(X, Y_\bullet)|)$ [MGb, Thm. 2.22]. (This construction goes back to [Qui70], where it was first used to study the nonabelian derived functor of Kähler differentials for commutative rings; when applied to an abelian category $\mathcal{C}$, it recovers $\mathbf{D}_{\geq 0}(\mathcal{C})$.) We expect that a stable analog of this construction yields an analogous presentation of the stabilization $\mathbf{D}(\mathcal{S}\mathfrak{p}) := \text{Stab}(\mathbf{D}_{\geq 0}(\mathcal{S}\mathfrak{p})) \simeq \text{Fun}^\times(\mathcal{S}\mathfrak{p}_{\text{sph}}^{\text{op}}, \mathcal{S}\mathfrak{p})$ (cf. [Lurb, Prop. C.1.5.7 and Cor. C.1.2.3]), which is analogously a localization of the ∞ -category $f\mathcal{S}\mathfrak{p}$ of (unboundedly-)filtered spectra.

I studied $\mathbf{D}_{\geq 0}(\mathcal{S}\mathfrak{p})$ extensively in my thesis [MGc]. Specifically, I equipped $s\mathcal{S}\mathfrak{p}$ with a *model ∞ -category* structure [MGd, MGf, MGb], which provides a robust toolkit for making computations in $\mathbf{D}_{\geq 0}(\mathcal{S}\mathfrak{p})$ using (generally much simpler) computations in $s\mathcal{S}\mathfrak{p}$. We expect these techniques to be helpful in our work described in §2.4.

2.4. $\mathcal{S}\mathcal{B}$ via Bruhat stratifications, spherical functors, and perverse schobers. We are currently working to unpack the constructions of [Kita] in the terms of §§2.1–2.3. I'll first recall those constructions, and then explain our approach.

Fix any $n \geq 0$. I'll write $G := \text{U}(n)$ for the unitary group, $T := \text{U}(1)^{\times n} \subset G$ for the standard maximal torus, and $G_i \subset G$ for the Levi subgroup corresponding to σ_i ; I'll consider these all as (T, T) -(bimodule) spaces via left and right multiplication, and thereafter as (T, T) -spectra by suspension (suppressed from the notation for simplicity). For reasons that will become clear presently, I'll write $G_{i+}^{-1} := \text{Th}(-N_T(G_i))$ for the Thom (T, T) -spectrum of the negative normal bundle of $T \subset G_i$. (Note that this normal bundle has a nontrivial G -action, which is what necessitates the use of *genuine* G -spectra.)

Given a length- k braid word $w = (\sigma_{i_1}^{\varepsilon_1}, \dots, \sigma_{i_k}^{\varepsilon_k})$, Kitchloo defines a k -cube $[1]^{\times k} \xrightarrow{\mathcal{B}_T(w_\bullet)} \mathcal{S}\mathfrak{p}^{(T, T)}$ of (T, T) -spectra: identifying objects of $[1]^{\times k}$ with subwords of w , it is defined by assigning σ_i^ε to G_{i+}^ε (for $\varepsilon \in \{\pm 1\}$), so that e.g. the subword $(\sigma_2, \sigma_5^{-1}, \sigma_3)$ is assigned to the (T, T) -spectrum $G_{2+} \wedge_T G_{5+}^{-1} \wedge_T G_{3+} := G_{2+} \wedge_T \text{Th}(-N_T(G_5)) \wedge_T G_{3+}$. Induction (on the left) determines a k -cube $\mathcal{B}(w_\bullet) := \text{Ind}_T^G(\mathcal{B}_T(w_\bullet))$ of G -spectra, which in turn determines the filtered G -spectrum $s\mathcal{B}(w) \in f\mathcal{S}\mathfrak{p}^G$ as follows: its underlying G -spectrum $s\mathcal{B}(w)_\infty \in \mathcal{S}\mathfrak{p}^G$ is the total cofiber of $\mathcal{B}(w_\bullet)$, and its filtration arises from a filtration of $[1]^{\times k}$.

It is immediate from the definition that the filtered spectrum $s\mathcal{B}(w) \in f\mathcal{Sp}$ is supported in $k+1$ consecutive degrees. Hence, it cannot possibly define an invariant of the braid $\underline{w} \in \text{Br}_n$, let alone of the link $\widehat{\underline{w}} \subset \mathbb{R}^3$. Rather, Kitchloo shows that it descends to a braid invariant when considered up to quasi-equivalence [Kita, Thms. 5.1 and 6.1]. (I'll implicitly discuss the resulting link invariant (i.e. Markov II) in §2.4.3.) As discussed in §2.3, we intend to interpret this as guaranteeing a well-defined *derived* spectrum $s\mathcal{B}(\underline{w}) := [s\mathcal{B}(w)] \in \mathbf{D}(\mathcal{Sp})$. With $(V \in \mathcal{SB}) \in \text{Alg}_{\mathbb{E}_2}(\text{Cat}_2(\mathbf{D}(\mathcal{Sp})))_*$ in hand, and writing $\text{Br}_n \xrightarrow{\alpha_n} \text{end}_{\mathcal{SB}}(V^{\otimes n})$ for the Br_n -action on $V^{\otimes n} \in \mathcal{SB}$, this should arise as the derived spectrum

$$s\mathcal{B}(\underline{w}) \simeq \text{hom}_{\text{end}_{\mathcal{SB}}(V^{\otimes n})}(\text{id}_{V^{\otimes n}}, \alpha_n(\underline{w})) \in \mathbf{D}(\mathcal{Sp})$$

of 2-morphisms from the identity 1-endomorphism of $V^{\otimes n}$ to $\alpha_n(\underline{w})$, with residual equivariance coming from endomorphisms of $\text{id}_{V^{\otimes n}}$. (Note that HHH can also be defined using Hochschild cohomology [Kho07].)

We intend to construct $\mathcal{SB}$ in three steps, as I'll now describe. We are currently in the first step, so of course my descriptions of the latter two steps will be relatively sketchy and brief.

2.4.1. The Br_n^+ -action. When w is a *positive* braid word, the filtered G -spectrum $s\mathcal{B}(w) \in f\mathcal{Sp}^G$ is definitionally the suspension spectrum of a filtered G -space. Hence, we expect to encode our Br_n^+ -action as a constructible sheaf of spaces over $\text{Conf}_n(\mathbb{R}^2)^v$ (recall §2.2). Indeed, we expect this to arise directly from the **Bruhat stratification** $\text{U}(n)^{\mathbb{B}}$ of $\text{U}(n)$ over $\mathfrak{S}_n^{\mathbb{B}}$ (the Bruhat order on (the underlying set of) the symmetric group): not merely the decomposition into strata, but also the data of how they are glued together via *links*. I'll briefly recall the formalism of stratified spaces and their links, describe the situation when $n = 2$, and then indicate our expectations for the general case.

As the simplest example of a stratified space, let us consider a closed-open decomposition $Z \hookrightarrow M \hookleftarrow U$ among manifolds as a stratification $M \rightarrow [1]$ (with 0^{th} stratum Z and 1^{st} stratum U), and write $\text{Link}_{[1]}(M)$ for the link (i.e. the space of exiting paths that start in Z and end in U). Then, the commutative square in Fig. 6 is a pushout on underlying spaces. In more familiar terms, the map $\text{Link}_{[1]}(M) \rightarrow Z$ is equivalent to the projection $S(N_Z(M)) \rightarrow Z$ from the sphere bundle of the normal bundle of $Z \subset M$, so that both horizontal cofibers in Fig. 6 are simply the Thom space $\text{Th}(N_Z(M))$. More generally, a stratified space $X \rightarrow \mathbf{P}$ has for each conservative functor $[n] \xrightarrow{\varphi} \mathbf{P}$ a corresponding iterated link $\text{Link}_{\varphi}(X) := \Gamma_{[n]}(\varphi^* X)$, and from these we may recover X as the colimit $\text{colim}_{\varphi \in \text{sd}(\mathbf{P})^{\text{op}}}(\text{Link}_{\varphi}(X))$ [BGH, Thm. 2.7.4].

$$\begin{array}{ccc} \text{Link}_{[1]}(M) & \xrightarrow{\text{ev}_0} & Z \\ \text{ev}_1 \downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

FIGURE 6

The Bruhat stratification of $G = \text{U}(2)$ is simply the closed-open decomposition $T \hookrightarrow G \hookleftarrow G \setminus T$ into diagonal and non-diagonal unitary matrices. Via the product decomposition $\text{U}(2) \cong \text{U}(1) \times \text{SU}(2)$, the resulting pushout as in Fig. 6 is essentially given by crossing $\text{U}(1)$ with the “bagel-bagel” Heegaard splitting $\text{SU}(2) \cong S^3 \cong (S^1 \times D^2) \sqcup_{(S^1 \times S^1)} (D^2 \times S^1)$; more precisely, crossing $\text{U}(1)$ with one core circle in $\text{SU}(2)$ gives the standard torus $T = \text{U}(1)^{\times 2}$, and crossing it with the other core circle in $\text{SU}(2)$ gives the subset $T' := \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\} \subset \text{U}(2)$ of off-diagonal matrices, a torus onto which $G \setminus T$ deformation retracts (T, T) -equivariantly.

Now, the length-1 word $w = (\sigma_1)$ is assigned to the two-step filtered (T, T) -spectrum $G_+ \rightarrow \text{cofib}(T \rightarrow G)_+$. (It is instructive to verify that this gives the corresponding Rouquier complex of Soergel bimodules on (T, T) -equivariant cohomology; the key observation is that the target gives the twisted $\mathbb{k}[x_1, x_2]$ -bimodule $\mathbb{k}[x_1, x_2] \xrightarrow{(-) \cdot (-)} \mathbb{k}[x_1, x_2] \xrightarrow{(-) \cdot \sigma(-)} \mathbb{k}[x_1, x_2]$.) Because the square of Fig. 6 is a pushout, we can identify the second term with $\text{cofib}(\text{Link}_{[1]}(G) \xrightarrow{\text{ev}_1} G \setminus T)_+$; in turn, ev_1 is equivalent to the projection $S(N_{T'}(G)) \rightarrow T'$, and hence we identify the filtered (T, T) -spectrum as the Thom collapse map $G_+ \rightarrow \text{Th}(N_{T'}(G))_+$.

In the general case, we believe that the strata of $\text{U}(n)^{\mathbb{B}}$ all retract onto n -dimensional tori, which together assemble into the subgroup $(\text{U}(1) \wr \mathfrak{S}_n) \subset \text{U}(n)$ of “permutation matrices with phases” (the normalizer of $T \subset \text{U}(n)$). More broadly, we expect that the iterated links of $\text{U}(n)^{\mathbb{B}}$ are closely related to the Bott–Samelson resolutions of its closed Schubert subvarieties [BS55], which are the space-level avatars of the Bott–Samelson bimodules (which generate SBim_n under sums and retracts) [Lib19]. A lingering question is how the group structure of $\text{U}(n)$ interacts with its Bruhat stratification. Inspired by forthcoming work of Ayala–Francis on orthogonal groups [Aya], we believe that $\text{Exit}(\text{U}(n)^{\mathbb{B}})$ admits a monoid structure in *spans* of ∞ -categories (refining the group structure on the ∞ -groupoid $\text{U}(n)$), which contains the ordinary group $\text{U}(1) \wr \mathfrak{S}_n$ as its maximal subgroupoid. (In particular, we observe that the poset $\mathfrak{S}_n^{\mathbb{B}}$ admits a monoid structure in spans of

finite sets, which linearizes to give the Hecke algebra.) Here, spans are necessitated by the fact that lengths are merely *sub*-additive: for instance, the product of non-diagonal elements of $U(2)$ need not be diagonal.

2.4.2. The Br_n -action. It is clear that to pass from a Br_n^+ -action to a Br_n -action we must stabilize from spaces to spectra, so that all monodromy functors over $\text{Conf}_n(\mathbb{R}^2)^v$ become invertible. We expect the Thom spectra $G_{i+}^{-1} := \text{Th}(-N_T(G_i))$ will then arise entirely through formal considerations: namely, via the theory of **spherical functors** [ST01, AL17], applied in the Morita $(\infty, 2)$ -category of derived ring spectra (as in §2.3). Specifically, it appears that G_{i+} arises as a *spherical twist* (a certain automorphism determined by a spherical adjunction), with inverse G_{i+}^{-1} (cf. [Kita, Theorem 6.1]). Indeed, spherical functors in a Morita 2-category play exactly this role in the definition of Rouquier complexes [Rou].

2.4.3. From $\text{Conf}_n(\mathbb{R}^2)$ to $\text{Ran}(\mathbb{R}^2)$. Ranging over the strata of $\text{Ran}(\mathbb{R}^2)$ amounts to studying the $\mathbb{E}_2$ -structure on $\mathcal{SB}$ (recall §2.1), which relates endomorphisms of $V^{\otimes m}$ and $V^{\otimes n}$ with those of $V^{\otimes(m+n)}$. In the setting of [Kita], this relationship mediates the stabilization of braids (i.e. the Markov II move), and so [Kita, §7] should serve as a guidepost in this third and final step. We expect that the $(\mathbf{D}(\mathbf{Sp})\text{-enriched})$ stable ∞ -categories $\text{end}_{\mathcal{SB}}(V^{\otimes n})$ will assemble into a **perverse schober** on $\text{Ran}(\mathbb{R}^2)$ (i.e. a categorified perverse sheaf [KSb]), which for each $n \geq 1$ pulls back along the map $\mathbb{C}^n/\mathfrak{S}_n \xrightarrow{\text{image}} \text{Ran}(\mathbb{C}) = \text{Ran}(\mathbb{R}^2)$ to give the perverse schober suggested in [KSb, §4] and [KSa, §8.C]. (Perverse schobers are defined in terms of spherical functors, which provides further evidence for their relevance here.) This should record the data of the object $\mathcal{SB} \in \text{Cat}_2(\mathbf{D}(\mathbf{Sp}))$, whose $\mathbb{E}_2$ -structure we may then record via the paradigm laid out in §2.1.

In fact, we are optimistic that in defining $\mathcal{SB}$ we will ultimately be able to dispense with Bruhat-stratified unitary groups and replace them with stratified partial flag varieties, whose additional symmetries allow for a more systematic description of their relevant structures. (This may be compared with the passage from matrices to abstract linear transformations; we are inspired by forthcoming work of Clausen–Jansen [Jan].) Indeed, in parallel with the considerations of §2.4.1, we have been investigating a diagrammatic calculus for computing $s\mathcal{B}$ by labeling braid diagrams by flags of $\mathbb{C}$ -vector spaces: at a crossing one must change the labeling subspace and record an exiting path. In fact, this is what alerted us to the similarity between Bott–Samelson resolutions and the iterated links of $U(n)^{\mathbb{B}}$ in the first place.

2.5. From $\mathcal{SB}$ to $\mathcal{Kh}$. In [Kitb], Kitchloo shows how $s\mathcal{B}$ recovers HHH and conjectures how it recovers KR_N : both procedures amount to taking equivariant cohomology and passing to the E^2 page of a spectral sequence arising from the filtration. We believe that our construction may provide a more topological method of passing from $s\mathcal{B}$ to KR_N . For instance, recall that the $(1+1)$ -dimensional TQFT that defines $\mathcal{Kh} \simeq KR_2$ corresponds to $H^*(\mathbb{CP}^1) \cong \mathbb{Z}[x]/x^2$ (which is a commutative Frobenius algebra via Poincaré duality). Hence, it seems possible that passing from $s\mathcal{B}$ to KR_2 amounts to truncating the partial flag varieties arising in §2.4.3, by length and/or by ambient dimension, in order to recover $\mathbb{CP}^1 = \text{Gr}_{1,2}(\mathbb{C})$ (see also e.g. [Lau12]). Moreover, this connection generalizes, as KR_N is likewise related to the cohomology rings of partial flag varieties [MSV09, RW20]. At the same time, we intend to derive inspiration from (and possibly even recover) the rich algebraic structure on HHH allowing for the recovery of KR_N therefrom [DGR06, Ras15]; this suggests the existence of various relevant filtrations and symmetries enjoyed by $\mathcal{SB}$ (or perhaps its 2hom-wise associated-graded).

2.6. From $\mathcal{Kh}$ to the WRT invariant. In the theory of spherical functors, the *twist* and *cotwist* functors are by definition the co/fibers of certain co/unit maps. It should follow formally that the link homology theory defined by $\mathcal{SB}$ – and thereafter, that defined by $\mathcal{Kh}$ – adheres to a “categorified skein relation” consisting of co/fiber sequences in the corresponding categorified skein modules (generalizing the usual categorified skein relation for Khovanov homology, which is responsible for its skein long exact sequence). Given this, I would hope to prove that the factorization homology of $\mathcal{Kh}$ categorifies the WRT invariant by induction on the number of crossings in a given link diagram. Indeed, this is precisely how one verifies that $\mathcal{Kh}$ categorifies the Jones polynomial.

3. EQUIVARIANCE AND FUNCTORIALITY FOR KHOVANOV HOMOLOGY VIA STRATIFICATIONS

In this section, I’ll discuss computations and general structural features that I intend to explore following a resolution of Problem A. I’ll phrase my discussion in terms of Khovanov spectra, as it appears that they may enjoy much more functoriality than their $\mathbb{Z}$ -linear counterparts (just as how only *topological* Hochschild homology carries a cyclotomic structure). For simplicity, in this section I’ll write $\text{Sk} := \text{Sk}_{\mathcal{Kh}}$. In §3.1 I’ll

discuss basic computations, and in §3.2 I'll speculate on the general structural features, particularly as they relate to equivariance. The latter discussion relies heavily on the related work surveyed in §§1.4-1.5.

3.1. Basic computations. Following a resolution of Problem A, the following would constitute an important first task.

Problem F. *Compute the categorified skein module $\text{Sk}(Y) \in \text{Cat}(g\text{Sp})$ for various 3-manifolds Y , and give an algorithm for computing $\text{Kh}(L \subset Y) \in \text{Sk}(Y)$.*

Given a solid understanding of $\text{Sk}(Y)$, I expect that local-to-global principles will provide a robust means of computing $\text{Kh}(L \subset Y)$ via local categorified skein relations (as in §2.6).

Regarding the computation of skein modules, a good first case is the solid torus $Y = \mathbb{A}$. Recalling the definition of Kh_{ann} from §1.2.2, a natural guess would be that $\text{Sk}(\mathbb{A}) \simeq fg\text{Sp}$ is the ∞ -category of filtered graded spectra, such that for any annular link $L \subset \mathbb{A}$ the situation is as indicated in Fig. 7. It should be relatively easy to compute $\text{Sk}(\mathbb{A})$ by hand: for any coefficients $(c \in \mathbb{C}) \in \text{Cat}_3(\mathcal{V})_*^{\text{adj}}$, the product decomposition $\mathbb{A} \cong S^1 \times \mathbb{R}^2$ identifies $\int_{\mathbb{A}}(c \in \mathbb{C})$ simply as the Hochschild homology of $\text{3end}_{\mathbb{C}}(c) \in \text{Alg}_{\mathbb{E}_3}(\mathcal{V})$ (with a residual $\mathbb{E}_2$ -structure). I would expect a similar relationship with annular $\mathfrak{sl}_N$ -homology [GHW].

$$\begin{array}{ccccc}
& & \text{Sk}(\mathbb{A}) & \xrightarrow{\text{Sk}(\mathbb{A} \hookrightarrow \mathbb{R}^3)} & \text{Sk}(\mathbb{R}^3) \\
& & \wr \downarrow & & \wr \downarrow \\
gg\text{Sp} & \xleftarrow{gr} & fg\text{Sp} & \xrightarrow{fgt} & g\text{Sp} \\
\wr \downarrow & & \wr \downarrow & & \wr \downarrow \\
\text{Kh}_{\text{ann}}(L) & \longleftarrow & \text{Kh}(L \subset \mathbb{A}) & \longrightarrow & \text{Kh}(L \subset \mathbb{R}^3) \simeq \text{Kh}(L)
\end{array}$$

FIGURE 7

More broadly, I would expect to compute skein modules (and thereafter, the values of Khovanov homology therein) using excision (Problem C). For instance, it should be possible to derive a general (albeit noncanonical) formula for $\text{Sk}(Y)$ using a Heegaard decomposition of Y ; note that this would be given by a finite colimit of copies of $\text{Sk}(\mathbb{R}^3)$ and $\text{Sk}(\mathbb{A})$. It would be interesting to compare such computations with the general formalism of [APS04], which extends Khovanov homology to links in I -bundles over a surface F . Beyond the

usual q - and homological gradings, their invariant comes with a third grading over the free abelian group on (equivalence classes of certain) free loops in F , and it would be very plausible for this to generalize along the lines of Fig. 7 (which is the case that $F = \mathbb{A}$ is the annulus).

3.2. Equivariance and functoriality for Kh via stratifications. We strongly expect that the object $V \in \mathcal{Kh}$ will be (not just 1- but) 2-dualizable, so that the assignment $(L \subset Y) \mapsto \text{Kh}(L \subset Y)$ defines a functor $\text{Cob}_1^1(Y) \rightarrow \text{Sk}(Y)$. Indeed, Kh in $\mathbb{R}^3$ is already known to enjoy such functoriality at the level of the homotopy category $\text{ho}(\mathcal{Ch})$, as much work has been done to show [CMW09, Cap07, Vog20, Bla10, Jac04, BN05, Kho06]. (So in particular, a resolution of Problem A would upgrade those results to the level of the ∞ -category $\mathcal{Ch}$.) Moreover, it follows directly from the construction that given any link $L \subset Y$ and any embedding $Y \xrightarrow{\varphi} Y'$, we obtain a functor $\text{Sk}(Y) \xrightarrow{\varphi_*} \text{Sk}(Y')$ and an equivalence $\text{Kh}(\varphi(L) \subset Y') \simeq \varphi_!(\text{Kh}(L \subset Y))$ (as already implied in Fig. 7). However, it appears that there is in fact a great deal more functoriality at play.

Problem G. *Codify the equivariance and functoriality of $\text{Kh}(L \subset Y) \in \text{Sk}(Y)$ in the contexts of (i) unramified covers, (ii) ramified covers, and (iii) involutive links.*

These are listed in order of increasing complexity, and I'll discuss them each in turn. As a general remark, I'll note that geometric fixedpoints play a central role in equivariant factorization homology [Hor], which provides further evidence (beyond the following discussion) that the theory of stratifications described in §1.4 may be expected to play an essential role here.

3.2.1. Unramified covers. The first case to consider is that of ordinary (unramified) covers among 3-manifolds.

A tantalizing picture is suggested by the results of [SZ] (cf. §1.5). Namely, consider the r -fold cover $\mathbb{A} \xrightarrow{r} \mathbb{A}$. Recall that $\text{Kh}_{\text{ann}}(r^*L)$ admits a genuine C_r -structure. This is certainly not entirely determined by $\text{Kh}_{\text{ann}}(L)$, but it is partly so: namely, we have an equivalence $\Phi^{C_r}(\text{Kh}_{\text{ann}}(r^*L)) \simeq \text{Kh}_{\text{ann}}(L)$. This appears to be a *categorification* of the formula $f(r^*\gamma) \equiv f(\gamma)^r$ of Thm. 7. (For a more parallel description, by adjunction this equivalence is the data of a morphism $\text{Kh}_{\text{ann}}(r^*L) \rightarrow (\Phi^{C_r})^R(\text{Kh}_{\text{ann}}(L))$ that becomes an equivalence upon applying Φ^{C_r} .) More specifically, we may view functions $\mathcal{L}X \rightarrow \mathbb{A}^1$ as being categorified by functors $\text{Cob}_1^1(\mathbb{A}) \rightarrow fg\text{Sp}$ (apologies for double-booking the symbol $\mathbb{A}$), and the particular functor Kh_{ann} enjoys the

property of being *cyclotomically invariant*: it is invariant under the $\mathbb{T}$ -action on $\text{Cob}_1^1(\mathbb{A})$, and moreover its value at an object $L \subset \mathbb{A}$ determines for all $r \in \mathbb{N}^\times$ its value at the object $r^*L \subset \mathbb{A}$ modulo universal indeterminacy. (I will resist the temptation to speculate on what might be the “categorified vector bundle” of which Kh_{ann} is the trace-of-monodromy functor.)

Abstracting slightly, suppose that $\tilde{Y} \xrightarrow{\varphi} Y$ is a G -torsor among 3-manifolds for some finite group G . In analogy with the Tate diagonal map of Fig. 2, I expect that $\text{Sk}(Y)$ and $\text{Sk}(\tilde{Y})$ are related by a diagram

$$\text{Sk}(Y) \longrightarrow \text{Sk}^{\tau^G}(\tilde{Y}) \longleftarrow \text{Sk}^{h^G}(\tilde{Y}) \longrightarrow \text{Sk}(\tilde{Y}) : \quad (8)$$

the two terms in the middle have yet to be defined, but $\text{Sk}^{h^G}(\tilde{Y})$ may be thought of as the skein module for G -equivariant links in $\tilde{Y}$ (i.e. links pulled back from Y) while $\text{Sk}^{\tau^G}(\tilde{Y})$ may be thought of as the skein module for “the component of $\text{Kh}(\varphi^*L \subset \tilde{Y})$ that is determined by $\text{Kh}(L \subset Y)$ ”. More specifically, I would expect these data to assemble in such a way that $\text{Kh}(L \subset Y)$ participates in a *microcosm reconstruction* of $\text{Kh}(\varphi^*L \subset \tilde{Y})$. Indeed, for $G = \mathbb{C}_p$ I would expect a *macrocosm reconstruction* of $\text{Sk}^{h^G}(\tilde{Y})$ from $\text{Sk}^{\tau^G}(\tilde{Y})$ and $\text{Sk}(\tilde{Y})$ (recall Ex. 5); in the general case, intermediate covers of Y should play a role as well.

Returning to the case of solid tori, in light of the proof of Thm. 8 outlined in §1.5, it seems exceedingly likely that such categorified Tate diagonals would endow $\text{Sk}(\mathbb{A})$ with the structure of a *laxly cyclotomic* ∞ -category: the proposed diagram (8) for $\mathbb{A} \xrightarrow{r} \mathbb{A}$ may well be nothing other than

$$fg\mathcal{S}p \xrightarrow{\sim} fg\mathcal{S}p \xleftarrow{\Phi^{C_r}} fg\mathcal{S}p^{C_r} \xrightarrow{\text{fgt}} fg\mathcal{S}p .$$

Once this case is understood, it should be relatively straightforward to extend these structures to incorporate all unramified covers among closed 3-manifolds.

3.2.2. Ramified covers. The next case to consider is that of a *ramified* cover $\tilde{Y} \xrightarrow{\varphi} Y$. Here, I’ll restrict to links $L \subset Y$ that are disjoint from the ramification locus, so that they pull back to links in $\tilde{Y}$. In the case of the r -fold ramified cover $\mathbb{R}^3 \xrightarrow{r} \mathbb{R}^3$ (i.e. $(\rho, \theta, z) \mapsto (\rho, r\theta, z)$ in cylindrical coordinates), the Khovanov homology of links of the form $r^*L \subset \mathbb{R}^3$ (said to have *periodic symmetry*) is well-studied [Mus19, Pol19, Zha18, BP, BPS].

Again, the results of [SZ] are suggestive: for any annular link $L \subset \mathbb{A}$ we have an equivalence $\Phi^{C_r}(\text{Kh}(r^*L)) \simeq \text{Kh}_{\text{ann}}(L)$. Namely, this suggests that $\text{Kh}(L \subset \mathbb{A})$ may again be viewed as a component in a microcosm reconstruction for $\text{Kh}(r^*L \subset \mathbb{R}^3)$ (as in §3.2.1), but here it would appear that the latter carries more than a mere genuine $\mathbb{C}_r$ -structure – this is a shadow of what is predicted by Problem D. Indeed, whereas unramified G -bundles are classified by BG , ramified G -bundles are classified by $BG^\triangleleft$, the left cone thereon (as a stratified space). Now, [BH, §9.2] develops a formalism of “globally equivariant” homotopy theory, which functorially assigns to each (unpointed) groupoid BG the ∞ -category $\mathcal{S}p^G$ of genuine G -spectra. I expect $\text{Kh}(r^*L \subset \mathbb{R}^3)$ to carry a “genuine $BG^\triangleleft$ -spectrum structure” (defined thereby); it should be reasonably straightforward to extend the geometric stratification of Thm. 4 to this slightly more exotic situation.

Now, observe that the situation just considered (along with that of §3.2.1) constitutes the unique local model for arbitrary ramified covers $\tilde{Y} \xrightarrow{\varphi} Y$. Hence, I would hope to obtain a global understanding of the behavior of skein modules and Khovanov homology for ramified covers by means of excision (Problem C).

3.2.3. Involutive links and beyond. Perhaps the most mysterious case to consider is that where the pair $L \subset Y$ carries a G -action such that L intersects the G -fixedpoints of Y nontrivially. This situation seems too far off to even speculate upon, so I will just describe the situation and leave it at that.

First of all, note that this situation cannot even be considered as a ramified cover among 3-manifolds with embedded links; rather, one would have to pass to the world of orbifolds. To my knowledge, it has only been studied for *involutive* links, i.e. in the case that $G = \mathbb{C}_2$. Given an involutive link diagram D , Lobb–Watson endow the usual Khovanov complex $\text{CKh}(D)$ with a perturbed differential, and show that the resulting homology theory is well-defined on isotopy classes of involutive links [LW]. Notably, despite working over $\mathbb{F}_2$, they show that this invariant is able to detect mutation, which sets it apart from $\mathbb{F}_2$ -linear ordinary Khovanov homology [Blo10, Weh10]. It would be interesting to see how this situation might generalize beyond the case of $G = \mathbb{C}_2$, and beyond the case of $Y = \mathbb{R}^3$.

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