# SEMIDIRECT PRODUCTS ARE HOMOTOPY QUOTIENTS

AARON MAZEL-GEE

ABSTRACT. We compute the homotopy quotient of the $G$-action on a category of $G$-objects and nonequivariant morphisms. We begin with a 1-category; a priori this process may yield a higher category, but this turns out not to be the case. Though we don't explicitly pursue it, the argument generalizes readily to $\infty$-categories enriched in $G$-spaces. As a corollary, it follows immediately that when the category is a one-object groupoid, the homotopy quotient constructs the semidirect product for the action of $G$ on the automorphism group.

## 1. GROUPS AND CATEGORIES

For the record, we discuss a few different possible interactions of a discrete group $G$ with a category $\mathscr{C}$.

The strongest thing we might ask for is that $G$ act on each object of $\mathscr{C}$. This is equivalent to giving a homomorphism $G \to \mathrm{End}(c) = \hom(c,c)$ for every $c \in \mathscr{C}$.[1] We refer to $\mathscr{C}$ as a *category of G-objects* (and nonequivariant morphisms).

In fact, this automatically yields a left action of $G$ on the entire category $\mathscr{C}$, in which case we call $\mathscr{C}$ a *G-category*. To define this, for any $g \in G$ and $f \in \hom(c_1, c_2)$ we must have a commutative diagram

$$\begin{array}{ccc} c_1 & \xrightarrow{f} & c_2 \\ {\scriptstyle g\cdot -}\downarrow & & \downarrow{\scriptstyle g\cdot -} \\ g\cdot c_1 & \xrightarrow[g\star f]{} & g\cdot c_2 \end{array}$$

(where we use a star to visually distinguish the $G$-actions on morphisms from those on objects). Of course, $G$ fixes the objects of $\mathscr{C}$, so that $g \cdot c_1 = c_1$ and $g \cdot c_2 = c_2$. So in fact we have $g \star f \in \hom(c_1, c_2)$ as well – that is, each individual hom-set carries a natural $G$-action – and this is given by $g \star f = g \cdot (f(g^{-1} \cdot -))$.[2] Moreover, the action distributes over composition of morphisms. This may all be summarized by saying that $\mathscr{C}$ is *enriched in G-sets*.

Thus, we have taken a category $\mathscr{C}$ of $G$-objects, attempted to extract the data of a $G$-category, and noticed that in fact $\mathscr{C}$ is also enriched in $G$-sets. The general situation is described by the following sequence of irreversible implications:

$$\left\{ \begin{array}{c} \mathscr{C} \text{ is a category} \\ \text{of } G\text{-objects} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \mathscr{C} \text{ is enriched} \\ \text{in } G\text{-sets} \end{array} \right\} \Longrightarrow \left\{ \ \mathscr{C} \text{ is a } G\text{-category} \ \right\}.$$

The second is given by defining the $G$-action on $\mathscr{C}$ to fix objects. Of course, this suggests an obvious example of its irreversibility. For any right $G$-set $T$, we can build the associated *action groupoid*, denoted $T/\!\!/G$: its objects are the elements of $T$, and its morphisms are given by $\hom(t_1, t_2) = \{g \in G : t_1 \cdot g = t_2\}$.[3] Obviously this cannot come from a category enriched in $G$-sets, because in that case the objects would be fixed.

The irreversibility of the first implication is slightly more subtle. To construct a minimal example, let $G'$ be any group and set $\mathscr{C} = \mathrm{pt}/\!\!/G'$. Then, to say that $\mathscr{C}$ is enriched in $G$-sets is to give an action of $G$ on $G'$, i.e. a

[1] Note that this corresponds to a *left* action of $G$ on $c$, which is in this sense more natural than a *right* action (which would be given by a homomorphism $G^{op} \to \mathrm{End}(c)$) – inasmuch as we follow the standard notation for composition of morphisms, at least.

[2] Note that the conjugation $G$-action on $\hom(c_1, c_2)$ is forced upon us.

[3] We choose $T$ to have a *right* action so that the category $T/\!\!/G$ admits an action of $G$ on the *left*.

homomorphism $G \to \mathrm{Aut}(G')$. This arises as a homomorphism $G \to G'$ (i.e. a $G$-action on the unique object of $\mathscr{C}$) exactly when this homomorphism factors through the subgroup of *inner* automorphisms of $G'$.[4]

## 2. Model categorical nonsense

Suppose that $\mathscr{C}$ is a category of $G$-objects. Then we can consider $\mathscr{C}$ as a $G$-category, and we would like to compute the homotopy quotient $\mathscr{C}_{hG}$. To make sense of this notion, we consider $\mathscr{C}$ as a quasicategory (by identifying it with its nerve) and work in simplicial sets with the Joyal model structure. That is, we'd like to extract a homotopy colimit of the diagram $\mathscr{C} : BG \to \mathsf{sSet}_{\mathrm{Joyal}}$, where $BG = \mathrm{pt}/\!\!/G$. We work in the projective model structure, so that we can compute this as the left derived functor of the colimit, which is a left Quillen functor. Explicitly, we claim that in this model structure, $\mathscr{C} \times EG \to \mathscr{C}$ is a cofibrant replacement, where $EG = G/\!\!/G$.

First of all, recall that the identity functor induces a Quillen adjunction $\mathsf{sSet}_{\mathrm{Joyal}} \leftrightarrows \mathsf{sSet}_{\mathrm{Quillen}}$. Now, our map is an acyclic fibration in $\mathsf{sSet}_{\mathrm{Quillen}}$ since these are preserved under products, and $EG \to \mathrm{pt}$ is an acyclic fibration since $EG$ is the nerve of a contractible groupoid, while every identity map is an acyclic fibration. Thus, considered in $\mathsf{sSet}_{\mathrm{Joyal}}$, this map is again an acyclic fibration; in particular, it's a weak equivalence.

Then, to see that $\mathscr{C} \times EG$ is cofibrant, note that the projective model structure on a category of $BG$-diagrams has its generating cofibrations given by the product of $G$ with the generating cofibrations of the base model category. Since the generating cofibrations of $\mathsf{sSet}_{\mathrm{Joyal}}$ are the same as those of $\mathsf{sSet}_{\mathrm{Quillen}}$, we see that the cofibrant $BG$-diagrams are exactly those simplicial $G$-sets which are levelwise free. This property is of course the defining characteristic of $EG$, and so $\mathscr{C} \times EG$ has this property too. Thus, we can compute $\mathscr{C}_{hG} \simeq (\mathscr{C} \times EG)/G$.

## 3. The computation

We begin by studying $\mathscr{C} \times EG$. Its objects are pairs $(c, g)$ of an object $c \in \mathscr{C}$ and an object $g \in EG$, and $\hom((c_1, g_1), (c_2, g_2)) = \hom(c_1, c_2) \times \{g_1^{-1} g_2\}$. Composition is determined by the commutative diagram

$$\begin{array}{ccc} \hom((c, g_1),(c_2, g_2)) \times \hom((c_2, g_2),(c_3, g_3)) & \longrightarrow & \hom((c_1, g_1),(c_3, g_3)) \\ \| & & \| \\ \left(\hom(c_1, c_2) \times \{g_1^{-1} g_2\}\right) \times \left(\hom(c_2, c_3) \times \{g_2^{-1} g_3\}\right) & \longrightarrow & \hom(c_1, c_3) \times \{g_1^{-1} g_3\}, \end{array}$$

in which the lower map is just composition in each factor.

Now, the $G$-action on $\mathscr{C} \times EG$ is given by $g \cdot (c_1, g_1) = (c_1, g g_1)$. Thus, $g$ acts on morphisms as the upper map in the commutative diagram

$$\begin{array}{ccc} \hom((c_1, g_1),(c_2, g_2)) & \xrightarrow{g \cdot -} & \hom((c_1, g g_1),(c_2, g g_2)) \\ \| & & \| \\ \hom(c_1, c_2) \times \{g_1^{-1} g_2\} & & \hom(c_1, c_2) \times \{(g g_1)^{-1}(g g_2)\} \\ & \searrow^{(g \star -) \times \mathrm{id}} & \| \\ & & \hom(c_1, c_2) \times \{g_1^{-1} g_2\}. \end{array}$$

[4] Surely there's a general obstruction theory governing this question of descent. (There's probably even a name for it.)

From here, we can see that the objects of $(\mathscr{C} \times EG)/G$ are precisely the objects of $\mathscr{C}$, although we'll denote the equivalence class $\{(c, g) : g \in G\}$ by $[c]$ for clarity. Moreover, from this we can also compute that

$$\begin{aligned}\hom([c_1],[c_2]) &= \left( \coprod_{g_1,g_2 \in G} \hom((c_1,g_1),(c_2,g_2)) \right) \Big/ G \\ &= \left( \coprod_{g_1,g_2 \in G} \hom(c_1,c_2) \times \{g_1^{-1} g_2\} \right) \Big/ G.\end{aligned}$$

By what we have just seen, in the latter identification $G$ fixes the second coordinate, and so this decomposes as

$$\hom([c_1],[c_2]) = \coprod_{g_0 \in G} \left( \left( \coprod_{g_0 = g_1^{-1} g_2} \hom(c_1,c_2) \times \{g_1^{-1} g_2\} \right) \Big/ G \right).$$

That is, the $G$-action permutes the various $\hom((c_1,g_1),(c_2,g_2))$ with the same "$G$-slope" $g_0 = g_1^{-1} g_2$; we can equivalently write these collectively as $\hom((c_1,g_1),(c_2,g_1 g_0))$, where $g_0$ is fixed and $g_1$ varies. As stated above, if we identify these all with $\hom(c_1,c_2)$, the action becomes precisely the $G$-action; that is, the diagram

$$\begin{array}{ccc} \hom((c_1,g_1),(c_2,g_1 g_0)) & \xrightarrow{(g_1' g_1^{-1}) \cdot -} & \hom((c_1,g_1'),(c_2,g_1' g_0)) \\ \| & & \| \\ \hom(c_1,c_2) \times \{g_0\} & \xrightarrow[((g_1' g_1^{-1}) \star -) \times \mathrm{id}]{} & \hom(c_1,c_2) \times \{g_0\} \end{array}$$

commutes. So finally, we simply have that

$$\hom([c_1],[c_2]) = \hom(c_1,c_2) \times G,$$

where the second coordinate records the $G$-slope $g_0$ and where we choose the distinguished representative $\hom((c_1,e),(c_2,e g_0))$ for the first coordinate.

However, composition is where things get interesting. We begin to unwind this as

$$\begin{array}{ccc} \hom([c_1],[c_2]) \times \hom([c_2],[c_3]) & \longrightarrow & \hom([c_1],[c_3]) \\ \| & & \| \\ (\hom(c_1,c_2) \times G) \times (\hom(c_2,c_3) \times G) & \longrightarrow & \hom(c_1,c_3) \times G. \end{array}$$

Now, suppose we would like to compose the elements $(f_1,h_1)$ and $(f_2,h_2)$. We can consider $(f_1,h_1) \in \hom((c_1,g_1),(c_2,g_1 h_1))$ for any $g_1$ and $(f_2,h_2) \in \hom((c_2,g_2),(c_3,g_2 h_2))$ for any $g_2$. For these to be composable in $\mathscr{C} \times EG$, we therefore take $g_2 = g_1 h_1$. Then, we have the composition

$$\begin{array}{ccc} \hom((c_1,g_1),(c_2,g_1 h_1)) \times \hom((c_2,g_1 h_1),(c_3,g_1 h_1 h_2)) & \longrightarrow & \hom((c_1,g_1),(c_3,g_1 h_1 h_2)) \\ \| & & \| \\ (\hom(c_1,c_2) \times \{h_1\}) \times (\hom(c_2,c_3) \times \{h_2\}) & \longrightarrow & \hom(c_1,c_3) \times \{h_1 h_2\}, \end{array}$$

where the lower arrow is given by composition in each factor. But here's the (literal) twist: given the way we've chosen to represent the first coordinate of $\hom([c_1],[c_2])$, in that coordinate we're now composing the elements $g_1 \star f_1$ and $(g_1 h_1) \star f_2$. This simplifies as

$$((g_1 h_1) \star f_2) \circ (g_1 \star f_1) = (g_1 \star (h_1 \star f_2)) \circ (g_1 \star f_1) = g_1 \star ((h_1 \star f_2) \circ f_1),$$

which corresponds to the element $((h_1 \star f_2) \circ f_1, h_1 h_2)$ in the target. (As expected, this is independent of $g_1$ and $g_2$.)

## 4. REMARKS

Note that, preserving the set of objects as it does, this presentation of the Borel construction (for 1-types) is far, far smaller than the usual one (in topological spaces). If one were feeling philosophical, one might say that this construction trades off fattening up the space for introducing (more) local monodromy. But we aren't, so we won't.[5]

Note also that, as promised, this argument generalizes almost immediately to the case where $\mathscr{C}$ is an $\infty$-category enriched in $G$-spaces; it only remains to remove any reference to specific morphisms.

[5]This might be a little cavalier anyways, since the induced $G$-action on the fundamental groupoid of a space with a nontrivial $G$-action won't fix objects anyways. Maybe the real point is just that groupoids are *always* a cleaner way of presenting 1-types. But in any case, the fact remains that it's much clearer what the Borel construction is *actually* doing here.