4 Fibered categories (Aaron Mazel-Gee)

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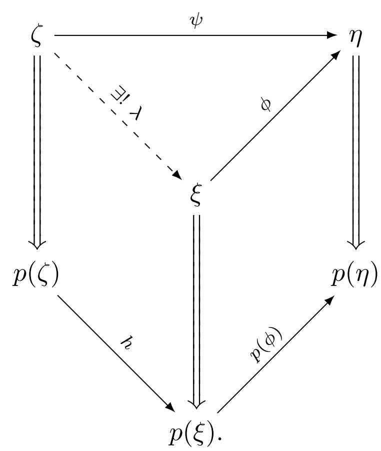
4.0 Introduction

In the same way that a sheaf is a special sort of functor, a stack will be a special sort of sheaf of groupoids (or a special special sort of groupoid-valued functor). It ends up being advantageous to think of the groupoid associated to an object X as living "above" X, in large part because this perspective makes it much easier to study the relationships between the groupoids associated to different objects. For this reason, we use the language of *fibered categories.*

We note here that throughout this exposition we will often say *equal* (as opposed to isomorphic), and we really will mean it.

4.1 Definitions and basic facts

Definition 1. Let $\mathcal{C}$ be a category. A *category over* $\mathcal{C}$ is a category $\mathcal{F}$ with a functor $p : \mathcal{F} \to \mathcal{C}$. A morphism $\xi \xrightarrow{\phi} \eta$ in $\mathcal{F}$ is called *cartesian* if for any other $\zeta \in \mathcal{F}$ with a morphism $\zeta \xrightarrow{\psi} \eta$ and a factorization $p(\zeta) \xrightarrow{h} p(\xi) \xrightarrow{p(\phi)} p(\eta)$ of $p(\psi)$ in $\mathcal{C}$, there is a unique morphism $\zeta \xrightarrow{\lambda} \eta$ giving a factorization $\zeta \xrightarrow{\lambda} \eta \xrightarrow{\phi} \eta$ of ψ such that $p(\lambda) = h$. Pictorially,



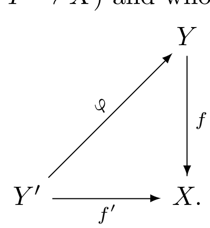
In this case, we call ξ a *pullback* of η along $p(\phi)$. The reason for this will be made clear(er) in a moment.

For each $U \in \mathcal{C}$, we have the category

$$\mathcal{F}(U) = \begin{cases} \mathrm{ob}(\mathcal{F}(U)) = \{\xi \in \mathcal{F} \colon p(\xi) = U\}, \\ \mathrm{Hom}_{\mathcal{F}(U)}(\xi', \xi) = \{f \in \mathrm{Hom}_{\mathcal{F}}(\xi', \xi) \colon p(f) = \mathrm{id}_U\}. \end{cases}$$

We call this the *fiber of* $\mathcal{F}$ *over* U.

Example 1. Let $\mathcal{C}$ be a category and choose $X \in \mathcal{C}$. Define the *over-category* $\mathcal{C}/X$, whose objects are objects-over-X (i.e. objects $Y \in \mathcal{C}$ with a specified map $Y \xrightarrow{f} X$) and whose morphisms are commutative diagrams



The functor $p : \mathcal{C}/X \to \mathcal{C}$ sending $Y \xrightarrow{f} X$ to Y and the above morphism to $Y' \xrightarrow{\varphi} Y$ makes $\mathcal{C}/X$ into a category over $\mathcal{C}$. The fiber $(\mathcal{C}/X)(Y)$ is just the discrete category $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Definition 2. A *fibered category over* $\mathcal{C}$ is a category $p : \mathcal{F} \to \mathcal{C}$ over $\mathcal{C}$ such that for every morphism $U \xrightarrow{f} V$ in $\mathcal{C}$ and object $\eta \in \mathcal{F}(V)$, there exists a cartesian arrow $\phi : \xi \to \eta$ such that $p(\phi) = f$ (i.e. $\xi \in \mathcal{F}(U)$). A morphism of fibered categories is a functor of categories-over-$\mathcal{C}$ sending cartesian arrows to cartesian arrows. Given two morphisms $g, g' : \mathcal{F} \to \mathcal{G}$ of fibered categories over $\mathcal{C}$, a *base-preserving natural transformation* $\alpha : g \to g'$ is a natural transformation of functors such that for every $\xi \in \mathcal{F}$, the morphism $\alpha_\xi : g(\xi) \to g'(\xi)$ projects to an identity arrow (i.e. is a morphism in $\mathcal{G}(p_F(\xi))$). We write $HOM_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ the category whose objects are morphisms of fibered categories over $\mathcal{C}$ and whose morphisms are base-preserving natural transformations.

Exercise 1. Figure out how pullbacks work in the fibered category $p : \mathcal{C}/X \to \mathcal{C}$.

Lemma 1. *If $p : \mathcal{F} \to \mathcal{C}$ is a fibered category, then every morphism $\psi : \zeta \to \eta$ factors as*

$$\zeta \xrightarrow{\lambda} \xi \xrightarrow{\phi} \eta,$$

where ϕ is cartesian and λ projects to an identity arrow (i.e. is a morphism in $\mathcal{F}(p(\zeta))$).

This is just saying that we can factor an arrow by first moving around inside our fiber and then applying a cartesian arrow. The idea here is that any map of bundles covering a map of bases factors through the pullback bundle.

Lemma 2. *Let $g : \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories over $\mathcal{C}$ such that for every object $U \in \mathcal{C}$ the induced functor $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful. Then g is a fully faithful functor.*

So, the full-faithfullness of a morphism of fibered categories can be checked fiberwise.

Proposition 1. *A morphism of fibered categories $g : \mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$ is an equivalence of fibered categories (with respect to $HOM_{\mathcal{C}}$, i.e. such that the isomorphisms between composite functors and identity functors are base-preserving) iff every $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories.*

So, whether a morphism of fibered categories is an equivalence can also be checked fiberwise.

4.2 The 2-Yoneda lemma

Recall the "strong" Yoneda lemma, which says that given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, the natural function

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set})}(h_X, F) &\to F(X) \\ \varphi &\mapsto \varphi(\mathrm{id}_X) \end{aligned}$$

is a bijection. This implies the "weak" Yoneda lemma, which we get when we replace F with h_Y for some $Y \in \mathcal{C}$. Another way of saying this is that the Yoneda functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set})$ given by $X \mapsto h^X$ is fully faithful, i.e. the Yoneda functor is an *embedding*.

In our fibered context, we have new, souped up versions of the Yoneda lemmas.

Theorem 1 (strong 2-Yoneda lemma)**.** *Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category. Then the natural functor*

$$\begin{aligned} HOM_{\mathcal{C}}((\mathcal{C}/X), \mathcal{F}) &\to \mathcal{F}(X) \\ (g : \mathcal{C}/X \to \mathcal{F}) &\mapsto g(\mathrm{id}_X) \end{aligned}$$

is an equivalence of categories.

Thus, to understand the fiber of $p : \mathcal{F} \to \mathcal{C}$ over X, all we have to do is probe $\mathcal{F}$ for sections over the entire over-category of X!

Corollary 1 (weak 2-Yoneda lemma)**.** *Let $X, Y \in \mathcal{C}$. Then*

$$\begin{aligned} HOM_{\mathcal{C}}((\mathcal{C}/X), (\mathcal{C}/Y)) &\to (\mathcal{C}/Y)(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ (g : \mathcal{C}/X \to \mathcal{C}/Y) &\mapsto g(\mathrm{id}_X) \end{aligned}$$

is an equivalence of categories.

Similarly, this corollary says that we can embed the category $\mathcal{C}$ into the 2-category of fibered categories over $\mathcal{C}$. Henceforth, we will not make a distinction between objects of $\mathcal{C}$ and fibered categories over $\mathcal{C}$ (i.e. we will simply write X for $\mathcal{C}/X$ when undue no confusion will arise); in a sense, we have *inflated* our category to include "objects" whose Yoneda functors take values in categories instead of just in sets.

4.3 Categories fibered in groupoids

4.3.1 ... coming from co/groupoid objects

Let us change gears for a moment. We will make a natural construction which will end up giving us a groupoid-valued functor, which will lead us to a source of many more groupoid-valued functors. We will then reinterpret these as fibered categories.

Given a ring R, we define the groupoid $Q(R)$ of (monic) quadratic expressions and changes of variable by

$$Q(R) = \begin{cases} \mathrm{ob}(Q(R)) = \{x^2 + bx + c : b, c \in R\} \cong R \times R, \\ \mathrm{Hom}_{Q(R)}((b', c'), (b, c)) = \{r \in R : (x + r)^2 + b'(x + r) + c' = x^2 + bx + c\}. \end{cases}$$

A ring homomorphism $R \to S$ determines a functor $Q(R) \to Q(S)$. So, these constructions assemble into a functor $Q : \mathbf{Rings} \to \mathbf{Groupoids}$.

Everyone loves a (co)representing object. Luckily for everyone, then, it is not hard to see that $\mathrm{ob}(Q(R)) = \mathrm{Hom}_{\mathbf{Rings}}(\mathbb{Z}[b, c], R)$: there is no choice about where to send the copy of $\mathbb{Z}$, and then the free generators b and c select, well, b and c (the linear and constant coefficients of our quadratic expressions). Let us write $A = \mathbb{Z}[b, c]$. Moreover, $\mathrm{mor}(Q(R)) = \mathrm{Hom}_{\mathbf{Rings}}(A[r], R)$: the copy of A picks out the source of our morphism, and the free generator r selects the change of coordinates. Let us write $\Gamma = A[r]$.

So, the pair of sets $(\mathrm{Hom}(A, R), \mathrm{Hom}(\Gamma, R))$ has the structure of a groupoid (the set of objects and the set of morphisms), and this is functorial in the ring R. By a general principle called "the method of the universal example" (which is really just an application of Yoneda's lemma), we can actually extract quite a bit of structure. For example, there are *source* and *target* functions $s, t : \mathrm{mor}(Q(R)) \to \mathrm{ob}(Q(R))$; by setting $R = \Gamma$, we obtain maps $s, t \in \mathrm{ob}(Q(\Gamma)) = \mathrm{Hom}(A, \Gamma)$ that are the images of $\mathrm{id}_\Gamma \in \mathrm{Hom}(\Gamma, \Gamma)$. What happens is that a morphism in $Q(R)$ is given by a map $\Gamma \to R$, and to pick out the source we precompose with s to get $A \xrightarrow{s} \Gamma \to R$ (and similarly for the target). Explicitly, $s : A \to \Gamma$ is the standard inclusion while $t : A \to \Gamma$ is given by $t(b) = b + 2r$ and $t(c) = c + br$.

In this way, all the axioms of a groupoid manifest themselves as maps within the pair (A, Γ):

groupoid axioms	structure maps
every arrow has a source and a target	$s, t : A \to \Gamma$
every object has an identity arrow	$\epsilon : \Gamma \to A$
every arrow has an inverse	$i : \Gamma \to \Gamma$
composable arrows compose	$m : \Gamma \to \Gamma \otimes_{s,A,t} \Gamma$

Exercise 2. Work out the rest of the structure maps.

These maps satisfy various identities dictated by what they're supposed to encode (e.g. $s \circ \epsilon = \mathrm{id}_A = t \circ \epsilon$, an associativity diagram, etc.), all of which makes the pair (A, Γ) into a *cogroupoid in* **Rings**.

Exercise 3. Determine the cogroupoids and their structure maps associated to the following constructions:

- objects are $\{x^2 + bx + c \colon b, c \in R\}$, morphisms are $\{x \mapsto ex + r \colon e, r \in R, e^2 = 1\}$;
- objects are $\{ax^2 + bx + c \colon a \in R^\times, b, c \in R\}$, morphisms are $\{x \mapsto ux + v \colon u \in R^\times, v \in R\}$;
- objects are $\{*\}$, morphisms are $\{f(t) \in R[[t]] \colon f(0) = 0, f'(t) \in R^\times\}$ (under composition of power series);
- objects are $\{F(x, y) \in R[[x, y]] \colon F(x, y) = F(y, x), F(x, 0) = x, F(F(x, y), z) = F(x, F(y, z))\}$, morphisms are $\{f(x) \text{ as above} \colon F_1(f(x), f(y)) = f(F_2(x, y)) \rightsquigarrow F_1 \xrightarrow{f} F_2\}$;
- objects are $\{y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \colon a_i \in R\}$, morphisms are $\{(x, y) \mapsto (\lambda^2 x + r, \lambda^3 y + s\lambda^2 x + t) \colon \lambda \in R^\times, r, s, t \in R\}$.

(The antepenultimate example is called *the group of infinitesimal automorphisms of the line fixing 0*, the penultimate example is called *the groupoid of (1-dimensional) (commutative) formal group laws and isomorphisms*, and the ultimate example is called *the groupoid of elliptic curves in Weierstraß form and transformations preserving the isomorphism type of the curve.*)

Aside 1. A cogroupoid (A, Γ) in **Rings** (which is actually called a *Hopf algebroid*) admits a notion of *comodule*: this is an A-module with a coaction of Γ satisfying counitality and coassociativity. These form an abelian category with enough injectives (assuming the source map $s : A \to \Gamma$ (or equivalently the target map) is flat). These should of course be thought of as something like sheaves on the Hopf algebroid (A, Γ). There is a cohomology functor given by $M \mapsto H^*((A,\Gamma); M) = \mathrm{Ext}^*_{\mathbf{Comod}_{(A,\Gamma)}}(A, M)$, and in particular one might care about $H^*((A,\Gamma); A)$ (since A is a nice and canonical (A,Γ)-comodule), which we simply call "the cohomology of (A,Γ)". Different Hopf algebroids can have the same cohomology, if the groupoids they represent are equivalent or even *locally* equivalent (in the flat topology). This gives some indication that these cogroupoid objects should not quite be our final object of study. Indeed, the category of comodules on a groupoid-valued functor is 2-categorically equivalent to the category of quasi-coherent sheaves on its stackification.

Incidentally, note that when applied to an algebraically closed field, the second construction in the above exercise may give an equivalent groupoid to the one (A,Γ) that we've been talking about this whole time: we might hope to retract the groupoid of quadratic expressions onto the full subgroupoid of monic quadratic expressions (although the choice of arrow may not be able to be made canonical, in which case this could very well fail). Of course, over an arbitrary field these groupoids will in general be inequivalent. This suggests that (depending on our goals) we may want to pass to algebraic closure before applying our groupoid-valued functor. The map $k \to \overline{k}$ is faithfully flat, and so Spec $\overline{k} \to$ Spec k is a cover in the flat topology. So this may be some indication that the flat topology is the right one for us, since in this case the stackifications of these two groupoid-valued functors will be equivalent.

Now, let's take what we've got and flip it and reverse it. Applying Spec everywhere, we get a *groupoid (pair)* (Spec A, Spec Γ) in **AffSch**. This takes us back to geometry, and is more like what we're used to anyways (representing instead of corepresenting). We get structure maps just as before, which we'll even call by the same names. More explicitly and more generally, any pair of objects (X_0, X_1) in a category $\mathcal{C}$ is called a *groupoid in* $\mathcal{C}$ if it has maps

$$s : X_1 \to X_0,\ t : X_1 \to X_0,\ \epsilon : X_0 \to X_1,\ i : X_1 \to X_1,\ m : X_1 \times_{s,X_0,t} X_1 \to X_1$$

that satisfy the obvious identities coming from the definition of a groupoid. (This is a generalization of the notion of a *group object* in a category, whose contravariant Yoneda functor lands in **Groups**.)

Now, given an object $U \in \mathcal{C}$, we define a category

$$\{X_0(U)/X_1(U)\} = \left\{ \begin{array}{l} \mathrm{ob}(\{X_0(U)/X_1(U)\}) = X_0(U) \\ \mathrm{Hom}_{\{X_0(U)/X_1(U)\}}(u, u') = \{\xi \in X_1(U) : s(\xi) = u,\ t(\xi) = u'\}. \end{array} \right.$$

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(V, U)$ gives a functor $f^* : \{X_0(U)/X_1(U)\} \to \{X_0(V)/X_1(V)\}$ by pullback, and $g^*f^* = (fg)^*$ (on the nose).

We finally define a category $\{X_0/X_1\}$. Its objects are given by pairs (U, u), where $U \in \mathcal{C}$ and $u \in \{X_0(U)/X_1(U)\}$. A morphism $(V, v) \to (U, u)$ in $\{X_0/X_1\}$ is given by a pair

$$(f \in \mathrm{Hom}_{\mathcal{C}}(V, U), \alpha \in \mathrm{Hom}_{\{X_0(V)/X_1(V)\}}(v, f^*u)).$$

We have a functor $p : \{X_0/X_1\} \to \mathcal{C}$ which make $\{X_0/X_1\}$ into a *category fibered in groupoids over* $\mathcal{C}$ (i.e. a fibered category over $\mathcal{C}$ where all fibers are groupoids).

4.3.2 ... and 2-categorical fiber product thereof

We first examine the situation for groupoids before generalizing to categories fibered in groupoids.

Definition 3. Let $\mathcal{G}_1 \xrightarrow{f} \mathcal{G} \xleftarrow{g} \mathcal{G}_2$ be a diagram of groupoids. We define the *2-categorical fiber product* $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ as follows. Its objects are triples (x, y, σ), where $x \in \mathcal{G}_\infty$, $y \in \mathcal{G}_2$, and $\sigma \in \mathrm{Hom}_{\mathcal{G}}(f(x), g(y))$. A morphism $(x', y', \sigma') \to (x, y, \sigma)$ is a pair of isomorphisms $a : x' \to x$ and $b : y' \to y$ such that the diagram

$$\begin{array}{ccc} f(x') & \xrightarrow{\sigma'} & g(y') \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ f(x) & \xrightarrow[\sigma]{} & g(y) \end{array}$$

commutes in $\mathcal{G}$.

There are projection functors $p_j : \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \to \mathcal{G}_j$ and a natural isomorphism $\Sigma : f \circ p_1 \to g \circ p_2$ (which is induced by the collection of maps σ). The above construction is universal with respect to this fact. More precisely, if $\mathcal{H}$ is a groupoid equipped with functors $\alpha : \mathcal{H} \to \mathcal{G}_1$, $\beta : \mathcal{H} \to \mathcal{G}_2$ and an isomorphism $\gamma : f \circ \alpha \to g \circ \beta$ of functors, then there is a triple

$$(h : \mathcal{H} \to \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2, \lambda_1 : \alpha \to p_1 \circ h, \lambda_2 : \beta \to p_2 \circ h)$$

making the diagram

$$\begin{array}{ccc} f \circ \alpha & \xrightarrow{f(\lambda_1)} & f \circ p_1 \circ h \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma \circ h} \\ g \circ \beta & \xrightarrow[g(\lambda_2)]{} & g \circ p_2 \circ h \end{array}$$

commute, and this triple is unique up to unique isomorphism. We may summarize by saying that there is a contractible 1-groupoid of 2-categorical fiber products.

We now turn to categories fibered in groupoids over $\mathcal{C}$. Let $\mathcal{F}_1 \xrightarrow{f} \mathcal{F} \xleftarrow{g} \mathcal{F}_2$ be a diagram of categories fibered in groupoids over $\mathcal{C}$. Any other category $\mathcal{G}$ fibered in groupoids over $\mathcal{C}$ gives us a groupoid $HOM_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_1) \times_{HOM_{\mathcal{C}}(\mathcal{G},\mathcal{F})} HOM_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_2)$ by our previous construction (note that these are all groupoids!). Then any morphism $\mathcal{H} \to \mathcal{G}$ of categories fibered in groupoids over $\mathcal{C}$ induces a morphism

$$HOM_{\mathcal{C}}(\mathcal{H}, \mathcal{G}) \longrightarrow HOM_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1) \times_{HOM_{\mathcal{C}}(\mathcal{H},\mathcal{F})} HOM_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_2)$$

of groupoids.

Theorem 2. *There exists a category $\mathcal{G}$ fibered in groupoids over $\mathcal{C}$ making the above functor an equivalence of categories for all $\mathcal{H}$. Any two such $\mathcal{G}$ are related by an isomorphism of categories fibered in groupoids over $\mathcal{C}$, and any two such isomorphisms are related by a unique 2-isomorphism.*

In short, there is a contractible 2-groupoid of 2-categorical pullbacks of categories fibered in groupoids over $\mathcal{C}$.

Exercise 4. Let G be a discrete group, $\mathcal{B}G$: **Top** $\to$ **Groupoids** be the groupoid of principal G-bundles. Check that

$$\text{pt} \times_{\mathcal{B}G} \text{pt} = G$$

(considered as a fibered category by the over-category construction, so in particular the resulting groupoid is always discrete).

Exercise 5. Let $X \to \mathcal{B}G$ classify $G \hookrightarrow E \twoheadrightarrow X$, and let $H \leq G$. Check that

$$X \times_{\mathcal{B}G} \mathcal{B}H = E \times_G G/H = E/H.$$