Every love story is a GHOsT story: Goerss–Hopkins obstruction theory for ∞-categories

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November 21, 2013

1. Introduction

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For example, the p-adic K-theory of an $\mathscr{E}_\infty$-ring spectrum naturally carries the structure of a *θ-algebra*.

Conversely, given a θ-algebra A, Goerss–Hopkins obstruction theory allows us to search for an $\mathscr{E}_\infty$-ring spectrum R with $(K_p^\wedge)_* R \cong A$.

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In fact, Goerss–Hopkins obstruction theory allows us compute *all* its homotopy groups!

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These all ultimately rely on the E^2*-model structure* of Dwyer–Kan–Stover for simplicial based spaces. Also called the *resolution* model structure, this is meant to give "projective resolutions" in a nonabelian setting.

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We'll start with nothing more than the desired algebraic image A, and we'll find a moduli space $\mathscr{M}_\infty(A)$ of simplicial objects which is equivalent to the moduli space $\mathscr{M}(A)$ of realizations of A.

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In fact, we'll have such theories of Postnikov towers on both the topology side and the algebra side, and the obstruction theory will be based on very a tight relationship between them.

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The space $BG(M, n+1)$ represents G-twisted cohomology for spaces over $BG \simeq P_1 X$.

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The group actions keep track of two different ways of getting equivalences of induced n-types.

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(This actually began as a joint project with Markus Spitzweck to construct a motivic GHOsT, but it eventually became clear that we weren't doing anything inherently motivic anyways.)

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So, this could give a basis for doing "naive" derived algebraic geometry in other contexts, too.

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And if nothing else: a guy's gotta write a thesis, right??!?

2. Blanc–Dwyer–Goerss obstruction theory

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This is easier for two reasons:

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- There's no operad in the game.

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We then have the functor $\pi_* : \mathcal{C} \to \mathcal{A}$ given by

$$(\pi_* X)(S^\beta) = [S^\beta, X]_{\mathcal{C}}.$$

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But more generally, if we have a functor $\varphi : \mathcal{G} \to \mathcal{M}$ to a model category $\mathcal{M}$ which preserves coproducts up to weak equivalence, then for any $X \in \mathcal{M}$ we can also obtain $\pi_*^{\varphi} X \in \mathcal{A}$ by setting

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(This reidentification is due to the *Hilton–Milnor theorem* for computing the homotopy type of $\Omega(\Sigma X_1 \vee \Sigma X_2)$.)

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Thus, collecting over all S^β, this displays $\pi_n Y$ as the kernel of the structure map of an abelian group object in $\mathcal{A}_{/\pi_0 Y}$.

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Now that we know what modules are, we're ready to talk about...

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Using the E^2*-model structure* on $s\mathcal{C}$ (which we'll define in a moment), we define $\mathscr{M}_\infty(A) \subset s\mathcal{C}_{E^2}$ to be the moduli space of those $Y \in s\mathcal{C}$ with $\pi_0 \pi_* Y \cong A$ and $\pi_i \pi_* Y = 0$ for $i > 0$.

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Note that for $Y \in \mathscr{M}_\infty(A)$, the above spectral sequence collapses, so $|Y| \in \mathscr{M}(A)$. In fact, geometric realization defines an equivalence

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$$\begin{tikzcd} \mathscr{M}_n(A) \arrow[r] \arrow[d] & B\mathrm{Aut}(A, \Omega^n A) \arrow[d] \\ \mathscr{M}_{n-1}(A) \arrow[r] & \widehat{\mathscr{H}}^{n+2}(A, \Omega^n A). \end{tikzcd}$$

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Taking the homotopy orbits of this basepoint map yields the map in question (for $X = A$, $M = \Omega^n A$, and $i = n + 2$).

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As we make our way up the tower for $\mathscr{M}_\infty(A) \simeq \mathscr{M}(A)$, this recovers the obstructions to existence and uniqueness asserted earlier!

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But even just to describe the exact couple, we'll need to talk about...

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The E^2-equivalences are exactly those maps that induce an isomorphism of E^2-pages of the spectral sequence; in fact, they automatically induce isomorphisms of E^2-exact couples.

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In fact, we're secretly using the E^2-model structure on the algebra side, too: our model structure on $s\mathcal{A}$ is just the E^2-model structure with respect to the full subcategory

$$\mathcal{G}_{\mathcal{A}} = \pi_* \mathcal{G} \subset \mathcal{A}.$$

Recall that we've defined the functors $\pi_n : s\mathcal{A} \to \mathcal{A}$ for all $n \geq 0$, and in fact we've seen that $\pi_n \in \mathrm{Mod}_{\pi_0}(\mathcal{A})$ for $n \geq 1$.

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By definition, the E^2-equivalences are created by the classical homotopy groups.

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These actually take part in the two-variable adjunctions

$$\begin{aligned} - \boxtimes - &: s\mathcal{S}\mathrm{et} \times \mathcal{C} \to s\mathcal{C} \\ \mathrm{map}(-,-)_0 &: s\mathcal{S}\mathrm{et}^{op} \times s\mathcal{C} \to \mathcal{C} \\ \hom^{\mathrm{lw}}_{\mathcal{C}}(-,-) &: \mathcal{C} \times s\mathcal{C} \to s\mathcal{S}\mathrm{et} \end{aligned} \qquad \begin{aligned} - \wedge - &: s\mathcal{S}\mathrm{et}_* \times \mathcal{C}_\emptyset \to s\mathcal{C} \\ \mathrm{map}_*(-,-)_0 &: s\mathcal{S}\mathrm{et}_*^{op} \times s\mathcal{C} \to \mathcal{C}_\emptyset \\ \hom^{\mathrm{lw}}_{\mathcal{C}}(-,-) &: \mathcal{C}_\emptyset \times s\mathcal{C} \to s\mathcal{S}\mathrm{et}_*. \end{aligned}$$

In this language, the classical homotopy groups are given by

$$\pi_n \pi_\beta Y = \pi_n [S^\beta, Y]^{\mathrm{lw}}_{\mathcal{C}} = \pi_n(\pi_0^{\mathrm{lw}} \hom^{\mathrm{lw}}_{\mathcal{C}}(S^\beta, Y)).$$

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Of course, the best way to understand an exact couple is to unroll it into a long exact sequence, and when we do this we get...

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But by a stroke of divine providence, the spiral exact sequence is $2/3$ natural and $1/3$ classical! Thus, we can control the natural homotopy groups in order to control the classical homotopy groups, which are what show up on the E^2-page.

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When we collect over all $S^\beta \in \mathcal{G}$ and range over all $n \geq 1$, these splice together into the E^1-exact couple.

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Applying $[S^\beta, -]_{\mathcal{C}_\emptyset}$ yields the same long exact sequences as before.

In either case, the key fact that identifies the derived exact couple is that we have an exact sequence

$$[D^{n+1}_{\mathbf{\Delta}} \wedge S^{\beta}, Y]_{s\mathcal{C}_{\mathrm{Reedy}}} \to [S^n_{\mathbf{\Delta}} \wedge S^{\beta}, Y]_{s\mathcal{C}_{\mathrm{Reedy}}} \to [S^n_{\mathbf{\Delta}} \wedge S^{\beta}, Y]_{s\mathcal{C}_{E^2}} \to 0.$$

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That is, it's an E^2-cofibration into an E^2-contractible object.

Finally, the spiral exact sequence runs

$$\cdots \longrightarrow \Omega\pi_{n-1,*}Y \longrightarrow \pi_{n,*}Y \longrightarrow \pi_n\pi_*Y \longrightarrow \cdots$$
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Here, for any $A \in \mathcal{A}$, we define $\Omega A \in \mathcal{A}$ by $(\Omega A)(S^\beta) = A(\Sigma S^\beta)$.

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If we denote their common value by $\pi_0 Y \in \mathcal{A}$, then the spiral exact sequence (excluding this last little bit) actually takes place in $\mathrm{Mod}_{\pi_0 Y}(\mathcal{A})$.

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To prove that the spiral exact sequence supports this module structure requires lots and lots of fiddling around with the interplay between the Reedy and E^2-model structures on $s\mathcal{C}$.

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Anyways, now that we have the spiral exact sequence in hand, we can finally talk about...

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i	0	1	2	$\cdots$	$n-1$	n	$n+1$	$n+2$	$n+3$	$\cdots$
$\pi_i\pi_* Y$	A	0	0	$\cdots$	0	0	0	$\Omega^{n+1}A$	0	$\cdots$
$\pi_{i,*} Y$	A	ΩA	$\Omega^2 A$	$\cdots$	$\Omega^{n-1}A$	$\Omega^n A$	0	0	0	$\cdots$

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By construction, $\pi_{-i} \mathscr{H} \cong H^i$ for $i \geq 0$ and $\pi_j \mathscr{H} = 0$ for $j > 0$.

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which for any $Y \in s\mathcal{C}_{E^2}$ yields an equivalence

$$\hom_{(s\mathcal{C}_{E^2})_{/K_A^{\mathrm{top}}}}(Y, K_A^{\mathrm{top}}(M, n)) \xrightarrow{\sim} \hom_{s\mathcal{A}_{/K_A^{\mathrm{alg}}}}(\pi_* Y, K_A^{\mathrm{alg}}(M, n)) = \mathscr{H}_A^n(\pi_* Y, M).$$

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...and *that's* how you get based spaces from Π-algebras!

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That is, in some sense, Y "knows" how to thread together any $n+1$ consecutive homotopy groups encoded in $A \in \Pi\texttt{-alg}$.

3. From Blanc–Dwyer–Goerss to Goerss–Hopkins

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Together, these make for a nearly endless supply of new twists and subtleties.

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To achieve this, we enlarge our $\mathcal{G}$ using this One Weird Old Trick.

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So, we assume that E satisfies Adams's condition.

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Then, the functor

$$E_* : s\mathcal{C}_{E^2_{\mathcal{G}^E_{\mathcal{C}}}} \to s\mathcal{A}_{E^2_{\mathcal{G}_{\mathcal{A}}}}$$

takes $\mathcal{G}^E_{\mathcal{C}}$-free objects to $\mathcal{G}_{\mathcal{A}}$-free objects!

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This gives us an adjunction $\mathrm{Alg}_T(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_O(\mathcal{C})$.

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- the operads $\mathcal{T}_n$ are all Σ-*free*, and
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This gives us an adjunction $\mathrm{Alg}_{\mathcal{T}}(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

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(Note that when $\mathcal{O}$ is an $\mathscr{A}_\infty$-operad, we can simply take $\mathcal{T} = \mathrm{const}(\mathcal{O})$. This is the source of the comparative simplicity of the Hopkins–Miller obstruction theory.)

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So, we can now resolve any O-algebra $X \in \mathrm{Alg}_O(\mathcal{C})$ by a free T-algebra $Y \in \mathrm{Alg}_T(s\mathcal{C})$ with $E_* Y \in \mathrm{Alg}_{E_* T}(s\mathcal{A})$ cofibrant!

However, note that we are now using the functor $\{[P, -]_{\mathcal{C}}\}_{P \in \mathcal{G}^E_{\mathcal{C}}}$ to determined our equivalences in $s\mathcal{C}_{E^2_{\mathcal{G}^E_{\mathcal{C}}}}$.

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So, we must further localize $s\mathcal{C}_{E^2_{\mathcal{G}^E_{\mathcal{C}}}}$ and $\mathrm{Alg}_T(s\mathcal{C})_{E^2_{\mathcal{G}^E_{\mathcal{C}}}}$ so that $\pi_* E_*$ creates equivalences.

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Of course, we could go on for days about all of the...

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But let's not.

Especially because the next section is so much cleaner!

4. To ∞-categories and beyond!

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And then, they also need to fuss around with *semi*-model structures in order to perform the Bousfield E-localizations we just mentioned.

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But perhaps most importantly, when we pass to ∞-categories, the underlying mathematical ideas – which are extremely beautiful! – come through a lot more clearly.

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But perhaps most importantly, when we pass to ∞-categories, the underlying mathematical ideas – which are extremely beautiful! – come through a lot more clearly.

So: let's get involved!

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(Recall that $\mathcal{P}^{\delta}_{\Sigma}$ denotes the category of discrete, product-preserving presheaves.)

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We write $\mathcal{A}$ for the target of the functor $E_* : \mathcal{C} \to \mathcal{A}$ given by

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In fact, we have an identification $\mathcal{A} \simeq \mathrm{Shv}^\delta_\Sigma(\mathcal{G}_{\mathcal{A}})$ with the category of *sheaves* for the topology generated by the epimorphisms.

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The main examples of sifted colimits are geometric realizations and filtered colimits; in fact, these generate all sifted colimits, in the sense that the functor

$$\begin{array}{ccc} s(\mathrm{Ind}(\mathcal{D})) & \longrightarrow & \mathcal{P}_\Sigma(\mathcal{D}) \\ Y & \longmapsto & \left(d \mapsto |\hom^{\mathrm{lw}}_{\mathcal{D}}(d, Y)|\right) \end{array}$$

is essentially surjective.

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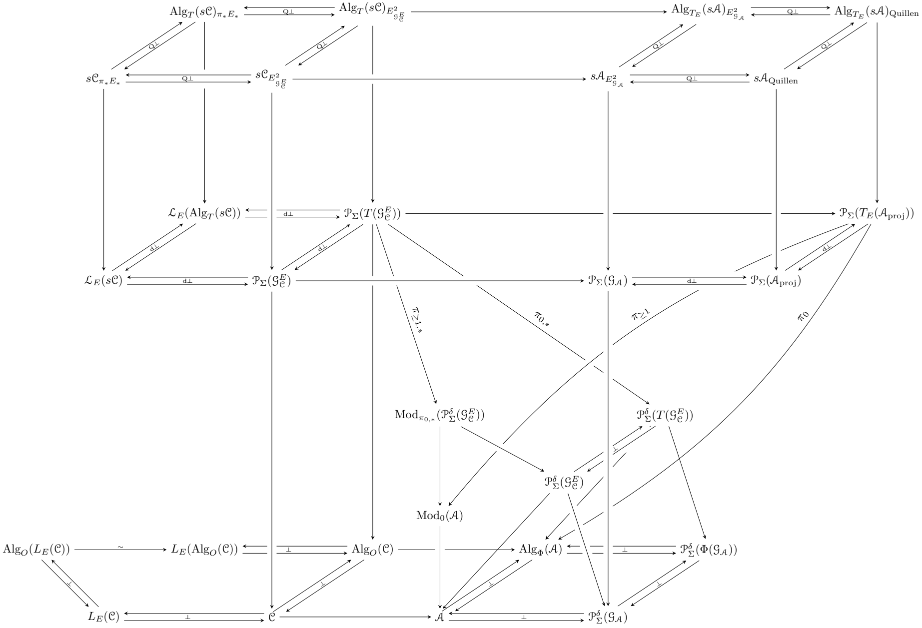

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But how are these related?

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Some care must be taken with the lifting axiom, which now asserts that a certain map of spaces is an *effective epimorphism* (that is, an epimorphism on π_0).

We also assume that $\mathbf{C}$ and $\mathbf{W} \cap \mathbf{C}$ are preserved under tensoring with $\mathcal{S}$, and dually that $\mathbf{F}$ and $\mathbf{W} \cap \mathbf{F}$ are preserved under cotensoring with $\mathcal{S}$. (These are "continuous" versions of the analogous facts for model 1-categories when $\mathcal{S}$ is replaced by $\mathcal{S}et$.)

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On the other hand, since in general there is a complicated interplay between co/limits in $\mathcal{D}$ and co/limits in $\mathrm{ho}(\mathcal{D})$, we are nervous about drawing any conclusions about the relationship between these two model ∞-categories.

(Note that the functor $\mathcal{D} \to \mathrm{ho}(\mathcal{D})$ is the unit of the adjunction $\mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{C}\mathrm{at}_1$, but it is not itself an adjoint.)

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These are both a lot cleaner than their classical counterparts, because everything is already homotopically well-behaved in the ∞-category $\mathcal{D}$.

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Instead, we will *remember* the relations, and we will build them into a *space* of "homotopy classes of maps".

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These should be thought of as a "cofibrant **W**-cohypercover" and a "fibrant **W**-hypercover", respectively. They also very closely resemble Dwyer–Kan's co/simplicial resolutions.

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(One of the keys to passing from 1-topos theory to ∞-topos theory is replacing *quotients by equivalence relations* with *geometric realizations of simplicial objects*.)

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$$\begin{tikzcd} \hom_{\mathcal{D}}^{\overset{l}{\sim}}(d_1, d_2) \arrow[r, "\sim"] & \left\|\hom_{\mathcal{D}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(d_1), \mathrm{path}_\bullet(d_2))\right\| \arrow[d, "\wr"] & \hom_{\mathcal{D}}^{\overset{r}{\sim}}(d_1, d_2) \arrow[l, "\sim"'] \\ & \hom_{\mathcal{D}[\mathbf{W}^{-1}]}(d_1, d_2) & \end{tikzcd}$$

(where $\|-\|$ denotes the colimit of a bisimplicial space).

Proving for instance that the right horizontal map

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In fact, this is *not* an equivalence in $s\mathcal{S}$ itself...but it's an E^2-equivalence! (To prove this, use the lifting axiom for the acyclic cofibrations $S^n \otimes d \overset{\approx}{\rightarrowtail} S^n \otimes d'$ against the various fibrations contained in $\mathrm{path}_\bullet(d_2)$.)

On the other hand, to show that the map

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In fact, there's an extremely elegant method for inverting a class of maps in an ∞-category – which looks a whole lot like a "high-dimensional hammock construction" – using Rezk's theory of *complete Segal spaces*.

Recall that the *complete Segal space* functor $CSS : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to s\mathcal{S}$ takes a relative ∞-category $(\mathcal{D}, \mathbf{W})$ and returns the simplicial space $CSS(\mathcal{D}, \mathbf{W})_\bullet$ given by

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Then, we have the simple formula

$$\hom_{\mathcal{D}[\mathbf{W}^{-1}]}(d_1, d_2) = \lim \left(\begin{tikzcd} & CSS(\mathcal{D}, \mathbf{W})_1 \arrow[d, "{(\delta_0, \delta_1)}"] \\ \mathrm{pt} \arrow[r, "{(d_1, d_2)}"'] & CSS(\mathcal{D}, \mathbf{W})_0 \times CSS(\mathcal{D}, \mathbf{W})_0 \end{tikzcd} \right).$$

This construction makes it clear that $\hom_{\mathcal{D}[\mathbf{W}^{-1}]}(d_1, d_2)$ is indeed some sort of "space of zig-zags", subject to the usual relations.

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From here, we can mimic the analogous classical proof by Dwyer–Kan to show that we have an equivalence

$$\left\|\hom^{\mathrm{lw}}_{\mathcal{D}}(\mathrm{cyl}^\bullet(d_1), \mathrm{path}_\bullet(d_2))\right\| \xrightarrow{\sim} \hom_{\mathcal{D}[\mathbf{W}^{-1}]}(d_1, d_2).$$

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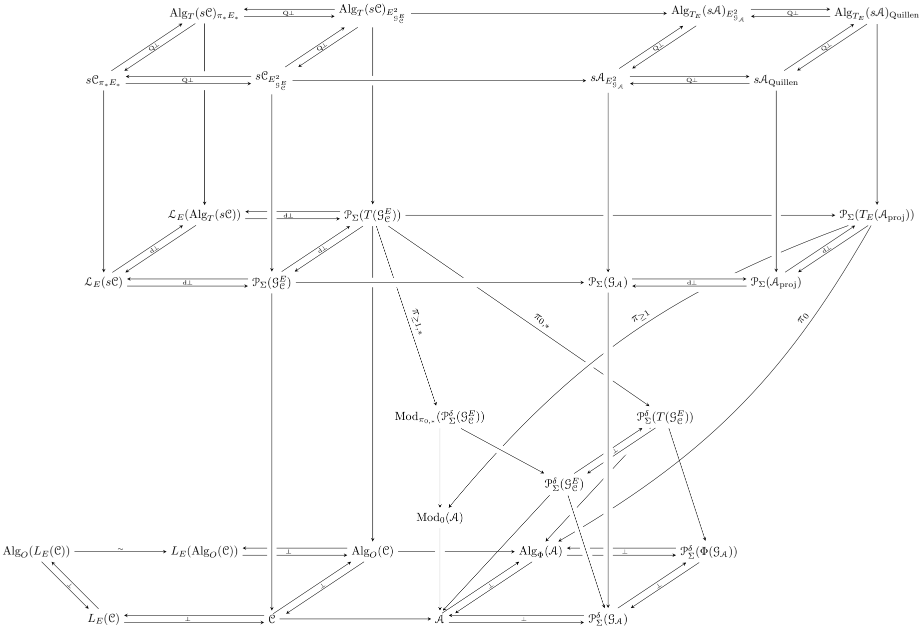
And now once again, for our collective viewing pleasure, here is The Diagram.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$\mathrm{Q}\perp$
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$\mathrm{d}\perp$
$\pi_{\geq 1,*}$
$\pi_{0,*}$
$\pi_{\geq 1}$
π_0
$\mathrm{Mod}_{\pi_{0,*}}(\mathcal{P}^\delta_\Sigma(\mathcal{G}^E_{\mathcal{C}}))$
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$\sim$
$L_E(\mathrm{Alg}_O(\mathcal{C}))$
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As a sample application, let's attempt to reimagine the construction of *tmf*.

In Behrens's *Notes on the construction of tmf*, for $p > 2$ he constructs the $K(1)$-local sheaf $\mathscr{O}^{\mathrm{top}}_{K(1)}$ over $\overline{\mathscr{M}}^{ord}_{\mathrm{ell},p}$ "one spectrum at a time":

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This latter condition is automatic for elliptic cohomology theories: by fiat, $\mathscr{O}^{\mathrm{top}}$ is locally required to have that $\pi_{2n}\mathscr{O}^{\mathrm{top}} \cong \omega^{\otimes n}$ and $\pi_{2n+1}\mathscr{O}^{\mathrm{top}} = 0$.

So, we can hope to construct $\mathscr{O}^{\text{top}}_{K(1)}$ (or maybe even $\mathscr{O}^{\text{top}}$!) in one swoop by doing obstruction theory in $\mathcal{C} = \text{Fun}(\mathscr{X}_{\text{ét,aff}}, \mathcal{S}\text{p})$.

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Then, the obstructions will live in André–Quillen cohomology *group objects* in its subcategory of discrete objects.

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So, among cellular equivariant spectra, to detect equivalences it suffices to check $RO(G)$-graded homotopy groups (instead of Mackey functors).

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In particular, if X is constant at an object $X \in \mathcal{C}$, then this deserves to be written $X^{\otimes T}$. (When T has trivial G-action, this is just $X^{\otimes |T|}$.)

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For instance, if $B = \operatorname{Spec} k$, then this must be given by $T = \operatorname{Spec} A$ for A an *étale k-algebra*:

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This provides for an analogous notion of genuine commutative motivic objects.

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Equivariant genuine symmetric operads are explored extensively in recent work of Blumberg–Hill on what they call $\mathscr{N}_\infty$*-operads*, and we expect to recover these as a special case.

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In any case, this is probably still light-years away.

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